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# Relativistic Klein-Gordon systems

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A static spherical distribution of incoherent matter which is a source of the Klein-Gordon field is considered in equilibrium under its gravitational attraction and short-range repulsion. Numerical solutions of the full Einstein-scalar equations are obtained. The solutions are regular everywhere, and have simple physical interpretation. The stability of the system under various degrees of concentration is discussed. The impossibility of a bounded, static configuration of massless Klein-Gordon charges under its self-gravitation is deduced; furthermore, it is shown that unbounded configurations are exact solutions of the vacuum equations with cosmological constant.

## I. INTRODUCTION

It is a general belief that gravity is the only interaction present in every physical system; however, its attractive effect has to be balanced by some kind of repulsive interaction in order to prevent collapse. In the case of a conventional Schwarzschild internal solution, collapse is prevented by pressure; however, in order to describe the structure of microscopic systems one should avoid the concept of pressure, which is essentially a macroscopic quantity. Long-range fields have been tried,[1,2] such as the Coulomb and repulsive scalar fields, but the resulting systems proved either unstable or insensitive to each other. Quantum effects are of course essential for microscopic objects, or even for macroscopic objects such as neutron stars, where the kinetic energy of the constituents due to the Pauli exclusion principle plays an important role, even though it seems worthwhile to investigate from the purely classical viewpoint the role of the interaction between gravitation and short-range fields in the formation and stability of elementary systems.[3]

The simplest short-range substitute for the scalar pressure of the Schwarzschild solution is a repulsive short-range scalar field. In this connection a simple nonsingular physical system was recently discussed.[4] It is a static sphere of incoherent dust, which is assumed to be, at the same time, a source of gravitation and of a short-range scalar field. A static equilibrium situation is obtained under the combined effects of the long-range, attractive gravitation and the short-range repulsion of the scalar field. It is shown that the linearized solutions of the Einstein-scalar equations are free from the gravitational instability. Since an important feature of general relativity is exactly its nonlinear character, more interesting results can be expected in the limit of strong nonlinear fields. It is the purpose of this paper to study that system in its nonlinear limits.

Analytic solutions of Einstein equations involving short-range scalar fields have not been obtained. We then look for numerical solutions, taking advantage of the fact that only two dimensionless parameters are sufficient for characterizing our system; the parameters are related to the central density of matter and the ratio of the scalar charge density to the matter density. It is shown that the radius of the distribution should be determined by the boundary condition to the scalar field which consists of the eigenvalue problem for the radius for a given set of the parameters. Some results are demonstrated and discussed. It is also shown that the pure Klein-Gordon field cannot have a stable static configuration under the gravitation.

## II. FIELD EQUATIONS

We start from the Einstein-scalar equations[4]

$$R^\mu_\nu - \tfrac{1}{2}R\delta^\mu_\nu = -2\epsilon c^{-2}T^\mu_\nu\ , \quad \epsilon = 4\pi G/c^2 \tag{1}$$

$$S^{;\mu}_{;\mu} + S/l^2 = \epsilon\sigma\ , \tag{2}$$

$$T^\mu_\nu = c^2\rho u^\mu u_\nu - c^2\epsilon^{-1}[S^{;\mu}S_{;\nu} + \tfrac{1}{2}\delta^\mu_\nu(S^2/l^2 - S^{;\alpha}S_{;\alpha})]\ , \tag{3}$$

where $\rho$ is the matter density with velocity field $u^\mu$ and $S$ is a repulsive scalar field with range $l$ and source density $\sigma$; as usual, the semicolon means the covariant derivative. We assume a homogeneity in the material, which we express by $\sigma/\rho = f = \text{const}$, with $f^2 > 1$. This latter requirement implies the predominance of the repulsive forces at short distances: This is a necessary requirement for avoiding the collapse.

As we consider a static, spherically symmetric distribution of matter, we may write the line element as

$$ds^2 = e^{2\eta}(dx^0)^2 - e^{2\alpha}dr^2 - r^2d\theta^2 - r^2\sin^2\theta d\phi^2\ ; \tag{4}$$

the functions $\rho$, $S$, $\eta$, $\alpha$ depend only on $r$.

In the internal region ($r \leq R$) the above equations reduce to

$$\eta' = -fS' , \tag{5}$$

$$\epsilon l^2 \rho = S(S+f)(f^2-1)^{-1} , \tag{6}$$

$$xS' = f - [f^2 - 1 + (1+x^2S^2)e^{2\alpha}]^{1/2} , \tag{7}$$

$$\alpha' = S'(f - xS') + xS(f+S)(f^2-1)^{-1}e^{2\alpha} , \tag{8}$$

where $x = r/l$ and a prime means $d/dx$. For definiteness we consider $f > 1$; a change of sign in $S$ and $S'$ is required for $f < -1$, in these equations.

For the external region($r > R$), where $\rho = 0$, the equations are

$$\eta' = -xS'^2 - \alpha' , \tag{9}$$

$$S'' = Se^{2\alpha} - x^{-1}S'[1+(1+x^2S^2)e^{2\alpha}] , \tag{10}$$

$$2x\alpha' = 1 - x^2S'^2 - (1+x^2S^2)e^{2\alpha} . \tag{11}$$

## III. SOLUTION OF EQUATIONS

The coupled ordinary differential equations (7) and (8) for $S(x)$ and $\alpha(x)$ can be numerically integrated when the initial conditions are given. A simple analysis shows that $\alpha = \alpha' = S' = 0$ at $x = 0$; we then fix a value for the parameter $f$ and also an initial value $S_0$ for $S(0)$ and start the integration from the origin outward.

For a given initial condition, the radius $R$ should be determined uniquely. To find the value of $R$ we first proceed with the numerical integration of the interior equations (7) and (8) up to a certain test radius $r = r_t$; for $r > r_t$ we switch to the exterior equations (10) and (11). We impose the continuity of $\alpha$, $S$, and $S'$ through $r = r_t$, and we also impose the requirement that $S$ vanish at infinity. We look for the correct value $r_t = R$ by iteration, which satisfies the above condition.

Having $S(x)$ and $\alpha(x)$ we can now obtain the external solution for $\eta(x)$ from (9); we impose the asymptotic condition $\eta(\infty) = 0$ which determines the integral constant. The internal solution for $\eta(x)$ is obtained from (5) and must be continuous at $r = R$; the continuity of its radial derivative through $r = R$ follows automatically from the continuity of $\alpha$, $S$, and $S'$, as can be seen from the following general expression, valid for both internal and external regions:

$$2x\eta' = (1+x^2S^2)e^{2\alpha} - (1+x^2S'^2) . \tag{12}$$

## IV. RESULT AND DISCUSSION

We numerically integrated the equations, with values for the parameter $f$ ranging from 1.1 to 5, and with $S_0$ ranging from $10^{-5}$ to $10^1$. The main interesting features found in all solutions obtained are present in the three cases described next.

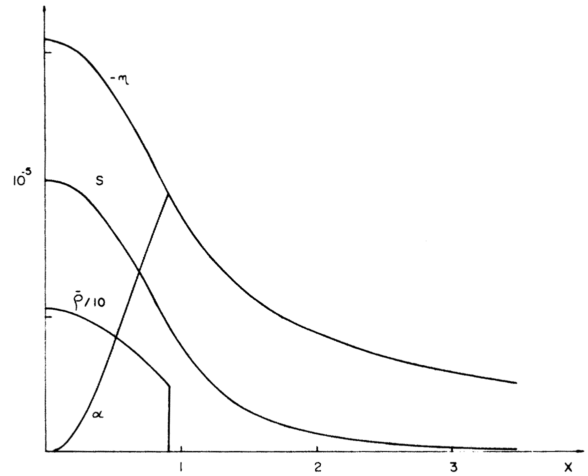


FIG. 1. Case $f = 1.1$, $S_0 = 10^{-5}$. The scalar field $S$, dimensionless matter density $\bar{\rho} = \epsilon l^2 \rho$, and gravitational potentials $\alpha$ and the negative of $\eta$ as functions of radial variable $x = r/l$.

In case $f = 1.1$ and $S_0 = 10^{-5}$ (Fig. 1) the solution almost coincides with that of the linearized equations as expected. The matter density $\rho(x)$ is essentially $x^{-1}\sin x$, which shows a maximum finite value around the center and decreases monotonically to the boundary $R \simeq 0.9l$ of the sphere; we plotted $\bar{\rho}/10$, where

$$\bar{\rho}(x) = \epsilon l^2 \rho(x) \tag{13}$$

is a dimensionless quantity. The also dimensionless scalar field $S(x)$ starts from the maximum preassigned value $S_0 = 10^{-5}$ on the origin and decreases monotonically to zero at infinity; in the external region ($x > 0.9$) it presents the usual Yukawa behavior $x^{-1}e^{-x}$. The dimensionless gravitational potential $\eta(x)$ shows the well-known pattern (we plotted its negative in order to save space) with maximum slope close to the boundary $x = 0.9$; in this weak-field limit ($S^2 \ll \bar{\rho} \ll 1$) we can relate $\eta(x)$ to the Newtonian potential $\Phi(x)$ produced by the matter density $\rho(x)$,

$$\eta(x) = c^{-2}\Phi(x) ; \tag{14}$$

the potentials $\eta$ and $\Phi$ then approximately present the usual $x^{-1}$ behavior for $x > 0.9$. Finally the metric potential $\alpha(x)$ is related to the ratio between radial physical lengths and the corresponding radial coordinate intervals, $dl_{\text{phys}} = e^{\alpha}dr$; since $\alpha(x)$ is positive from the center to infinity, all physical radial distances are numerically larger than the corresponding radial coordinate intervals. One finds that $\alpha(x)$ has a parabolic ($x^2$) behavior near the origin followed by a slight bending rightward before reaching the boundary of the sphere. On

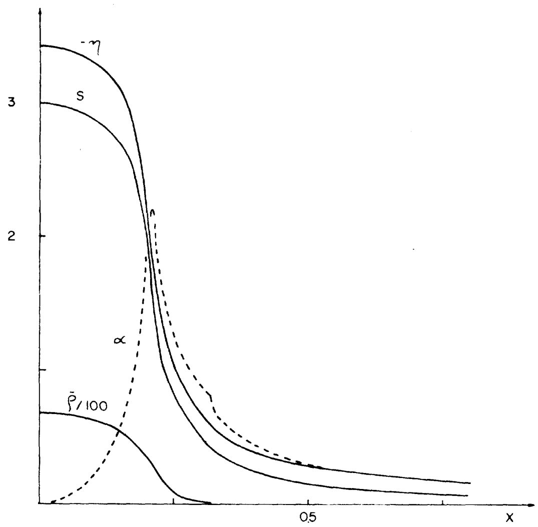


FIG. 2. Case $f=1.1$, $S_0=3$. The scalar field $S$, matter density $\bar{\rho}$, and gravitational potentials $\alpha$ and $-\eta$ as functions of $x$.

this boundary $\alpha$ is continuous, but its $x$ derivative has a discontinuity

$$\alpha'_{\text{int}} - \alpha'_{\text{ext}} = \epsilon l R \rho(R) e^{2\alpha(R)}, \tag{15}$$

as can be shown from Eqs. (6), (7), (8), and (11). For $x > 0.9$ one finds that $\alpha(x)$ closely follows $-\eta(x)$ in this weak-field limit, as it should in a Schwarzschild external solution. The gravitational mass of the whole system, as defined by

$$M = -\frac{c^2}{G} \lim_{r\to\infty} [r\eta(r)], \tag{16}$$

is $0.87 \times 10^{-5}$ in units $c^2 l/G$.

We next consider the case $f=1.1$ as before, but with $S_0=3$ (Fig. 2); this is no longer a weak-field solution. The density of matter $\rho(x)$ still shows a larger concentration on the origin, and dilutes monotonically to a nonzero value at the boundary $R=0.32l$. We note the diffused property of the distribution near the surface region which is not observed in the weak-field limit. Potentials $\eta(x)$ and $S(x)$ have a behavior similar to that of the previous case. A somewhat different pattern, however, is presented by the metric potential $\alpha(x)$: It still has a parabolic ($x^2$) behavior near the center and reaches a maximum in the region of maximum radial derivative of the gravitational potential ($\eta'$), but it now decreases in the tail region of the matter distribution. There is a discontinuity of slope (15) on the boundary of the sphere; for increasing $x$, the potential $\alpha(x)$ gradually approaches the Newton-Schwarzschild hyperbolic ($x^{-1}$) behavior, since the scalar-field density $S^2(x)$ tends to zero exponentially. It is worthwhile to stress the coincidence of regions in which the material system presents maximum gravitation (as given by $\eta'$) and maximum dilatation of the physical radial distances (as given by $\alpha$). It is also interesting to remark that the maximum gravitation occurs in the interior ($x \simeq 0.2 < R/l$) of the sphere; this is a consequence of the faint concentration of the outermost shells. The gravitational mass of the system, as defined by (16), is 0.11 $c^2 l/G$.

We finally consider the case $f=5$ and $S_0=1$ (Fig. 3). A few preliminary words are necessary to understand the peculiar situation found in this case. It is known[5] that the "effective energy density" that produces a static gravitational field is

$$2T_0^0 - T = (c^2/\epsilon l^2)(\bar{\rho} + S^2), \tag{17}$$

which in our system is proportional to $\bar{\rho} + S^2$. In the previous two cases, the major contribution to the attractive gravitational effects came from the dimensionless matter density $\bar{\rho}(x)$, but in the present case the main contribution comes from $S^2(x)$. A trivial calculation starting from (6) shows that one always has $\bar{\rho}(x) \le S^2(x)$ in situations where $f/S(x) \le f^2 - 2$; in the present case ($f=5$) we then have predominance of the $S^2$ contribution in regions where $S(x) > 0.22$, that is, from the center of symmetry, where $\bar{\rho} = 0.25$ and $S_0^{\,2} = 1$, to the radius given by $x = 13.7$, as can be seen in Fig. 3. Another interesting feature concerns the metric potential $\alpha(x)$; in the previous two cases we found a positive parabolic behavior near the origin. Indeed, a few calculations starting from (7) and (8) show that one always has near the center

$$\lim_{x\to 0} [x^{-2}\alpha(x)] = \tfrac{1}{6} S_0 (f^2-1)^{-1} [2f - (f^2-3)S_0]; \tag{18}$$

we then have negative values for $\alpha(x)$ in the innermost shells when $2f/S_0 < f^2 - 3$. This is what hap-

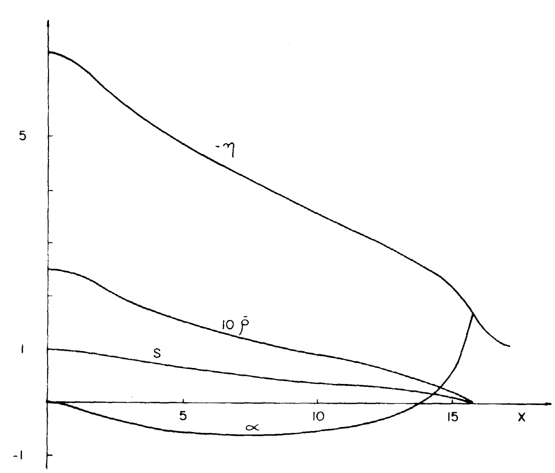


FIG. 3. Case $f=5$, $S_0=1$. The quantities $S$, $\bar{\rho}$, $\alpha$, and $-\eta$ as functions of $x$.

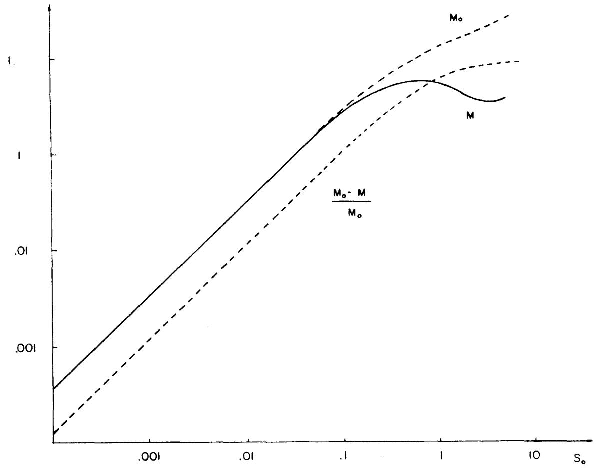


FIG. 4. The proper mass $M_0$, the Schwarzschild mass $M$, and the ratio of the binding energy $M_0 - M$ to the proper mass (in units $c = 1$), as functions of the central value $S_0$ of the scalar field, for $f = 1.2$. A $\log_{10}$-$\log_{10}$ scale is used.

pens in the present case. The metric potential $\alpha(x)$ starts from the zero value on the origin and assumes negative values with increasing $x$, with a minimum in the region where the gravitational potential $\eta(x)$ shows a minimum derivative. For $x > 13.7$ one finds positive values for $\alpha(x)$, with a maximum near the boundary $x = 15.8$ of the sphere, a region where $\eta(x)$ presents a maximum radial derivative; with increasing $x$ in the external region, the two functions $\alpha(x)$ and $-\eta(x)$ asymptotically coalesce, as originated by a mass $M = 7.63c^2 l/G$.

For a better understanding of the behavior of a system under various degrees of concentration, we plotted in Fig. 4 the gravitational mass $M$ (Schwarzschild mass), the proper mass $M_0$, and the ratio of the binding energy to the total proper mass, as functions of the central value of the scalar field. The value 1.2 is arbitrarily chosen for $f$. The main characteristics described in the following are quite analogous for other values of $f$. The proper mass (invariant mass) $M_0$ is defined[6] by

$$M_0 = 4\pi \int_0^R \rho e^{\alpha} r^2 dr , \tag{19}$$

and the binding energy is

$$B = (M_0 - M)c^2 , \tag{20}$$

where $M$ is the gravitational mass defined in (16).

All these quantities increase almost linearly in a $\log_{10}$-$\log_{10}$ scale up to $\log_{10} S_0 \simeq 0.5 \times 10^{-2}$ (nonrelativistic region), then bend down in the relativistic region. For very high central values of $S_0$ ($\log_{10} S_0 > 5$), the numerical procedure fails due to the computational difficulty.

It is interesting to note that the gravitational mass $M$ has a maximum at $\log_{10} S_0 \simeq 0.8$. A similar situation is well known in the case of neutron-star models when the gravitational mass is plotted against the central density of the neutron star.[7] In the latter case the existence of maximum mass is related to the gravitational instability, and solutions with the central density higher than this maximum point are unstable against collapse. In analogy to the above, it seems that the solutions of our system with $\log_{10} S_0 > 0.8$ are unstable for the value $f = 1.2$. However, we should note that the proper mass does not have a maximum, in contrast to the neutron-star models. The ratio of the binding energy to the proper mass seems to increase monotonically with $S_0$, tending to unity. Around the maximum of $M$ the binding energy reaches about 60% of the total proper mass.

We found, in (17), that the scalar field $S$ contributes positively to the effective energy density. One might then conjecture whether a bounded, massless scalar source could have static equilibrium under the combined effects of its inherent short-range repulsion and the gravitational attraction produced solely by the term $S^2(x)$. This situation could be formally obtained from the previous results by making $\rho \to 0$ and $|f| \to \infty$ in such a way that $\sigma = f\rho$ is finite. However, we prefer to revert to the original Eqs. (1) to (3), and make $\rho = 0$. We then obtain

$$\sigma S' = 0 , \tag{21}$$

$$S'' = (S - \epsilon l^2 \sigma) e^{2\alpha} - x^{-1} S'[1 + (1 + x^2 S^2)] e^{2\alpha} , \tag{22}$$

$$\eta' = -\alpha' - x S'^2 , \tag{23}$$

$$2x\alpha' = 1 - x^2 S'^2 - (1 + x^2 S^2) e^{2\alpha} . \tag{24}$$

The solution of (24) which is regular in the origin is

$$\exp[-2\alpha(x)] = 1 + \frac{2}{x} \int_0^x \{S^2(\xi) + S'^2(\xi) \exp[-2\alpha(\xi)]\} \xi^2 d\xi , \tag{25}$$

which implies that $\alpha(x) \leq 0$. Since in gravitational solutions of bounded systems one must have the asymptotic Schwarzschild behavior, in which $\alpha(x) \geq 0$, we conclude that it is impossible to have bounded distributions of massless charges in static equilibrium.

An unbounded exact solution exists, however, in (21) to (24). It is

$$S = \text{const} = S_0 , \quad \sigma = S_0 / \epsilon l^2 = \text{const} , \tag{26}$$

$$\exp(2\eta) = \exp(-2\alpha) = 1 + S_0{}^2 x^2 / 3 . \tag{27}$$

This is the static, spherically symmetric solution regular in the origin of Einstein's equations

$$R^\mu_\nu - \tfrac{1}{2}\delta^\mu_\nu R - \delta^\mu_\nu S_0{}^2/l^2 = 0\,, \tag{28}$$

as can be seen from (1) and (3), with $S = \text{const} = S_0$. Since (28) is a vacuum equation with cosmological constant $\Lambda = S_0{}^2/l^2$, we are allowed to identify the vacuum pressure produced by $\Lambda$ to the repulsive effects created by a constant scalar field of short range.

Our scalar field has no self-interaction. It may be of great interest to study the system of a pure Yukawa field with self-interaction under gravitation. Investigations along this line are in progress.

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[1]U. K. De and A. K. Raychaudhuri, Proc. R. Soc. London A303, 47 (1968).

[2]A. F. da F. Teixeira, I. Wolk, and M. M. Som, J. Phys. A 9, 53 (1976); 9, 1267 (1976).

[3]J. A. Souza and A. F. da F. Teixeira, Int. J. Theor. Phys. and Can. J. Phys. (to be published).

[4]A. F. da F. Teixeira, I. Wolk, and M. M. Som, Phys. Rev. D 12, 319 (1975).

[5]R. C. Tolman, *Relativity, Thermodynamics and Cosmology* (Oxford Univ. Press, New York, 1934), Sec. 92.

[6]T. Kodama and M. Yamada, Prog. Theor. Phys. 47, 444 (1971).

[7]B. K. Harrison, K. S. Thorne, M. Wakano, and J. A. Wheeler, *Gravitation Theory and Gravitational Collapse* (Univ. of Chicago, Press, Chicago, Illinois, 1965), p. 46.