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Particle emission rates from a black hole. III. Charged leptons from a nonrotating hole*

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The Hawking emission rates from a nonrotating black hole of small charge are calculated for electrons and muons and their antiparticles. During the stochastic emission of these charged leptons, the charge of the hole fluctuates. Assuming that the only type of charged particle emitted significantly by a hole of mass M is one of these spin-1/2 species with mass μ and charge e, the probability distribution for the charge of the hole is computed for $0 \leq GM\mu/\hbar c \leq 0.4$. The rms value varies from $6.14e$ for $GM\mu/\hbar c = 0$ to $2.76e$ for $GM\mu/\hbar c = 0.4$ and is predicted to be $2.34e$ for $GM\mu/\hbar c \gg 1$. The electrostatic attraction between the emitted particle and its antiparticle, along with the charge fluctuations, causes the average emission rate and power to be lower than for otherwise-similar uncharged particles. This effect of the charge is calculated (ignoring radiative and self-energy corrections, which are of the same order in e) to be a few percent, depending upon $GM\mu/\hbar c$. The particle rest mass μ also impedes the emission, but by factors which can become much greater for a large enough hole: The average power is reduced to 50% of its value for massless spin-1/2 particles at $GM\mu/\hbar c = 0.160$ ($M = 8.33\times10^{16}$ g for electrons, 4.03×10^{14} g for muons) and to 10% at $GM\mu/\hbar c = 0.271$ ($M = 1.41\times10^{17}$ g for electrons, 6.82×10^{14} g for muons). It is estimated that muons and heavier particles would contribute about 14% of the power of a nonrotating black hole of $M = 5\times10^{14}$ g, helping it to decay away in nearly 16×10^{9} yr, roughly the present age of the universe.

I. INTRODUCTION

Previous papers in this series[1,2] have made numerical calculations of the rates at which black holes emit neutral massless particles by Hawking's quantum process.[3] It was found that a primordial black hole of initial mass around 5×10^{14} g (if nonrotating) to 7×10^{14} g (if created maximally rotating) would just decay away within the present age of the universe. A black hole in this interesting mass range has a temperature of order 20 MeV, so in addition to the massless radiation there will be prolific emission of electrons and positrons and some emission of muons and heavier particles. The present paper investigates the effects of the charge and mass of these spin-$\frac{1}{2}$ particles on the emission.

Gibbons,[4] Zaumen,[5] and Carter[6] have shown that a small black hole will quickly give up most of its electric charge, though paper I noted that there should remain random charge fluctuations of order unity [i.e., of order $(\hbar c)^{1/2} = 11.7e$, where e is the positron charge, using here and henceforth the Planck units spelled out in paper I]. Before doing any numerical calculations, let us estimate the size of these fluctuations.

For a nonrotating black hole with mass

$$M \gg 2.18\times10^{-5}\ \text{g} = \text{Planck mass} \equiv (\hbar c/G)^{1/2} \equiv 1, \tag{1}$$

the charge neutralization will rapidly make

$$Q_* \equiv Q/M \ll 1, \tag{2}$$

as explained in Refs. 4–6 and paper I.

Therefore, the Hawking emission rate[3] for charged leptons will be

$$\frac{dN_\pm}{dt\,d\omega} = \sum_i \frac{\Gamma_i/2\pi}{e^{8\pi M\omega \mp 4\pi\alpha Z}+1}, \tag{3}$$

where the sum is over all angular modes i, Γ_i is the absorption probability for each mode i at frequency ω, $\alpha = e^2$ is the fine-structure constant, and Z is the charge of the hole after the particle is emitted, in positron units ($Q = Ze$ in Planck units).

Paper I showed that the peak in the neutrino power spectrum occurred at $M\omega = 0.18$, where $e^{-8\pi M\omega} \approx 0.01 \ll 1$, so we might expect that over the dominant part of the spectrum

$$\frac{dN_\pm}{dt\,d\omega} \approx \frac{1}{2\pi} \sum_i \Gamma_i e^{-8\pi M\omega \pm 4\pi\alpha Z}. \tag{4}$$

It is not obvious how Γ_i should vary with Z. However, one might expect electrostatic repulsion to produce a decrease with Z (for positive particles) by a fractional amount of order $Z\alpha$, partially canceling the effect of Z in the exponential thermal factor in Eq. (4). To get an order-of-magnitude estimate, one may ignore the variation of Γ_i with Z and integrate Eq. (4) to obtain

$$dN_\pm/dt \approx C e^{\pm 4\pi\alpha Z}. \tag{5}$$

Then as the black hole emits particles and antiparticles stochastically and possibly builds up a net charge of one sign or the other, it will tend to emit more charges of that same sign than of the opposite sign, pushing the black hole back toward neutrality. Over a time long enough for many particles to be emitted, the fluctuations will result in a certain stationary probability distribution for the charge of the black hole.

Later in this paper the actual probability distribution will be computed numerically, but first let us see what Eq. (5) as a crude guess gives us. The probability distribution, $P(Z)$, will be stationary if the number of black holes going from Z to $Z-1$ balances the number going from $Z-1$ to Z. Let $R(Z)$ be the rate for a positive charge to come from a hole with a net Z left behind (i.e., $Z+1$ excess charges originally). Under the assumption of CP invariance for the emission process, $R(-Z)$ is the corresponding rate for the antiparticle to be emitted leaving a hole with charge Z behind (charge $Z-1$ before the emission). Therefore, the balance in $Z \leftrightarrow Z-1$ requires

$$P(Z)R(Z-1)=P(Z-1)R(-Z). \quad (6)$$

One can solve this inductively from $Z=0$ to obtain

$$P(Z)=\frac{R(-Z)R(-Z+1)\cdots R(-2)R(-1)}{R(Z-1)R(Z-2)\cdots R(1)R(0)}P(0), \quad (7)$$

where $P(0)$ is chosen so as to normalize the distribution. Now if we use Eq. (5) as an approximation for $R(Z)$, we obtain

$$P(Z)\approx e^{-4\pi\alpha Z^2}P(0), \quad (8)$$

which is like a normal distribution with standard derivation

$$Z_{\rm rms}\approx(8\pi\alpha)^{-1/2}=2.335\,06, \quad (9)$$

except that Z is not continuous but discrete, though the rms value for the discrete distribution is the same to nine decimal places. This is the same probability distribution that Bekenstein[7] predicts from information-theory arguments.

We may also ask how the average emission rate is affected by the charge fluctuations. If we write Eq. (5) in terms of the charge of the hole before emission and then average over the probability distribution (8) for the charge, we get

$$\langle dN_\pm/dt\rangle \approx CP(0)\sum_{Z=-\infty}^{\infty} e^{4\pi\alpha(Z-1-Z^2)}$$
$$\approx e^{-3\pi\alpha}C=0.933\,536C. \quad (10)$$

Thus we might expect the average emission rate of charged particles to be about 7% less than it would be for otherwise-similar uncharged particles. Under our crude assumptions, the charge will cause a fractional correction of about 7% to the average power emitted also. It will be shown below that numerical calculations imply that the reduction in average power due to the charge varies between 2% and 5%, for black holes of mass up to 2×10^{17} g, depending upon the mass of the hole. However, the numerical calculations ignore certain interactions with photons, discussed below, which should also give a fractional correction of order α to the power emitted.

The rest masses of the particles emitted can have a much larger effect upon the emission rates. Basically, the particle mass μ provides a lower cutoff on the energy that an emitted particle may have, thus eliminating the part of the spectrum at lower energies that a massless particle would have. For example, the neutrino power spectrum peaks at $M\omega=0.18$,[1] so if $M\mu \gtrsim 0.18$, one would expect the particle power spectrum to be decreasing for all $\omega>\mu$ and to include less than about half of the total neutrino power. It also turns out that the rest mass reduces the absorption probabilities in the various angular modes at a given energy, but this is a smaller effect.

II. METHOD OF CALCULATING THE EMISSION RATES

The emission rates will be calculated from Eq. (3) after solving the Dirac equation in the field of the black hole to find the absorption probabilities Γ_i. This calculation actually involves the approximation that each particle is emitted individually and is coupled only to the stationary gravitational and electromagnetic field that represents the black hole after the emission. Since the average time between the emission of successive leptons will turn out to be greater than 10^3M, it should be a very good approximation to ignore the interactions between different leptons emitted. However, it will not be so accurate to ignore the interactions between the leptons and photons, by which are meant nonstationary electromagnetic fields in contrast to the stationary field used as a potential in the Dirac equation to be solved. For example, the emission of a free photon will give a radiative correction of order α. Furthermore, each charged lepton will be surrounded by a cloud of virtual photons and so will not propagate exactly the same in the curved spacetime as the point Dirac particle model used here. Viewed classically, tidal forces act on the extended electric field of the particle. This self-energy correction will also be of order α. There will be additional corrections from the vacuum polarization of the black hole, but they will be smaller than the ones above by factors of $Z\alpha$, which is small for a typical value of the fluctuating hole charge.

As a result of this approximation, the results will be physically accurate only to order α. In particular, the differences in the average emission rates between charged and neutral particles need radiative and self-energy corrections of the same order as their magnitude given here. The numerical results are listed to more significant figures than their physical accuracy warrants in order that they need not be recalculated when the radiative and self-energy corrections can be made.

The Dirac equation in a spherically symmetrical gravitational and electromagnetic field has been given by Brill and Wheeler[8]; they reduce it to a pair of coupled first-order radial equations, Eq. (39) of their paper. However, for illustrative purposes let us show how these same equations can be gotten from Chandrasekhar's separation[9] of the Dirac equation in the Kerr geometry and Page's extension[10] of this result to the Kerr-Newman fields.

Let

$$R_{-1/2} \equiv R_1 \text{ and } R_{+1/2} \equiv (2/\Delta)^{1/2} R_2 \tag{11}$$

in Eqs. (40) of Ref. 9 and substitute K from Eq. (14) of Ref. 10 into Eqs. (18) of Ref. 9 to generalize it to a Kerr-Newman field. Also, for simplicity, replace $\sqrt{2}\,\mu_e$ and $\sqrt{2}\lambda$ in Ref. 9 by μ and λ, as in Ref. 10—then μ becomes the particle mass and λ is an angular eigenvalue which reduces to an integer when the hole is nonrotating. Then Eqs. (40) of Ref. 9 take on the symmetrical form

$$\left(\frac{d}{dr} + \frac{iK}{\Delta}\right) R_1 = (\lambda + i\mu r)\Delta^{-1/2} R_2, \tag{12}$$

$$\left(\frac{d}{dr} - \frac{iK}{\Delta}\right) R_2 = (\lambda - i\mu r)\Delta^{-1/2} R_1. \tag{13}$$

To reduce the coefficients in these equations to real quantities, let

$$R_1 = G + iF \text{ and } R_2 = G - iF. \tag{14}$$

Then adding and subtracting $\frac{1}{2}\Delta^{1/2}$ times Eqs. (12) and (13) result in

$$(\Delta^{1/2} d/dr - \lambda)G + (-K\Delta^{-1/2} - \mu r)F = 0, \tag{15}$$

$$(\Delta^{1/2} d/dr + \lambda)F + (+K\Delta^{-1/2} - \mu r)G = 0. \tag{16}$$

These are the general radial equations for a charged Dirac particle in the Kerr-Newman field of a charged, rotating black hole.

When the black hole is not rotating,

$$\Delta = r^2 - 2Mr + Q^2, \quad K = \omega r^2 - eQr, \tag{17}$$

and the angular eigenvalue becomes

$$\lambda = j(j+1) - l(l+1) + \tfrac{1}{4} = (j + \tfrac{1}{2})\,\mathrm{sgn}(j - l), \tag{18}$$

where j is the total angular momentum of the particle (unfortunately called l in papers I and II) and $l = j \pm \frac{1}{2}$ is the orbital angular momentum, so that λ is a nonzero integer. Then Eqs. (15) and (16) become proportional to Eq. (39) of Ref. 8, where k is the same as my $-\lambda$.

In particular, we are interested in black holes obeying Eq. (2), so that the Reisner-Nordström geometry becomes indistinguishable from the Schwarzschild geometry even though the electromagnetic coupling to the charge may not be negligible. In that case, Eqs. (15) and (16) may be written out as

$$\left[\left(1 - \frac{2M}{r}\right)^{1/2} \frac{d}{dr} - \frac{\lambda}{r}\right] G + \left[-\left(1 - \frac{2M}{r}\right)^{-1/2} \left(\omega - \frac{Z\alpha}{r}\right) - \mu\right] F = 0, \tag{19}$$

$$\left[\left(1 - \frac{2M}{r}\right)^{1/2} \frac{d}{dr} + \frac{\lambda}{r}\right] F + \left[+\left(1 - \frac{2M}{r}\right)^{-1/2} \left(\omega - \frac{Z\alpha}{r}\right) - \mu\right] G = 0. \tag{20}$$

To calculate the emission rates, we must solve Eqs. (19) and (20) to get the absorption probabilities Γ_i to insert into Eq. (3). We start with a purely ingoing wave near the horizon, integrate the equations outward, and then resolve the solution into incoming and outgoing waves at large radii to see what fraction Γ_i of the incoming waves have gone down the hole and what fraction R_i have been reflected back to become the outgoing waves. The difference between the calculated $\Gamma_i + R_i$ and unity gives a measure of the numerical error in solving the radial equations.

The Appendix describes a numerical method developed to solve the general Eqs. (15) and (16) for a black hole of arbitrary charge and rotation. Once the angular eigenvalues become available for a rotating hole,[11] calculations of the emission from a general Kerr-Newman hole can be done by the methods of the Appendix. However, the present paper gives the results only for a weakly charged nonrotating hole. In this case fewer calculations are needed, since all the angular modes with different axial angular momenta m but the same j and l give the same emission, and the contributions of different modes decrease rapidly with j, so that only a few angular modes need be calculated.

After the computer program was developed for calculating the absorption probability Γ for a given $M\mu$, $M\omega$, $eQ = Z\alpha$, and angular mode, the Γ's were summed over the dominant angular modes, and Eq. (3) was used to give the emission rate per frequency interval. This was then integrated over frequencies to give the total emission rate and power for various values of $M\mu$ and Z.

Equation (7) was used to calculate the probability distributions for the charge Z (in positron units) of black holes of various masses. The emission rate and power as a function of Z for each $M\mu$ was then averaged over each distribution to give the average emission rate and power as a function of $M\mu$, the black-hole characteristic size (half its Schwarzschild radius) in units of the particle's reduced Compton wavelength.

III. RESULTS

To illustrate how the black-hole cross section σ, emission rate, and power depend upon $M\omega$, $M\mu$, and Z, Figs. 1–3 show $\sigma v^2/27\pi M^2$, $dN_+/dtd\omega$, and $M\,dE_+/dtd\omega$ vs $M\omega$ for $M\mu=0$, 0.2, and 0.4 and $Z=-20$, 0, and 20. These were calculated accurate to roughly one part in 10^3 or better. The quantity

$$\frac{\sigma v^2}{27\pi M^2}=\frac{1}{27(M\omega)^2}\sum_\lambda |\lambda|\Gamma_\lambda, \qquad (21)$$

where the eigenvalue λ is summed over all positive ($l=j-\frac{1}{2}=\lambda-1$) and negative ($l=j+\frac{1}{2}=-\lambda$) integers which give a significant contribution, was calculated rather than the cross section itself or its di-

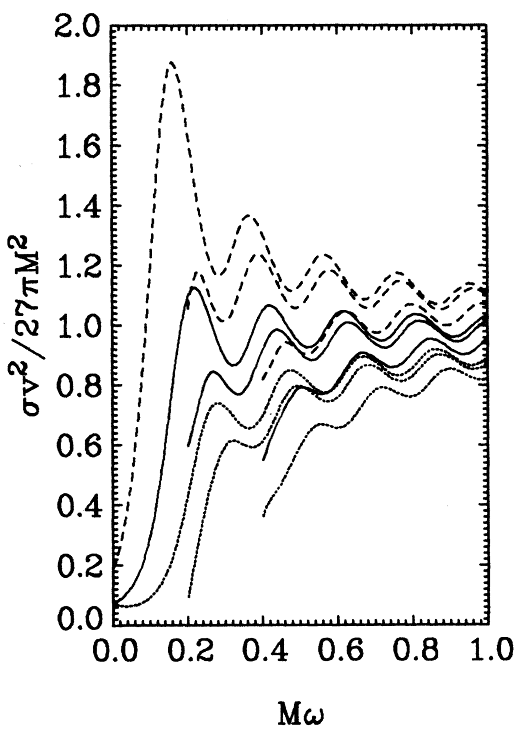

FIG. 1. Cross section σ of a black hole of mass M to spin-$\frac{1}{2}$ particles of charge e, mass μ, and energy ω, expressed in dimensionless form [Eq. (21)] with the singularity at zero velocity v factored out. Dashed curves are for a hole with $Z=-20$; solid curves, $Z=0$; dotted curves, $Z=20$. The three curves for each Z correspond to different black-hole masses and start at $M\omega=M\mu=0$, 0.2, and 0.4, respectively, for black holes of $M=0$, 1.04×10^{17} g, and 2.08×10^{17} g for electron emission or $M=0$, 5.03×10^{14} g, and 1.01×10^{15} g for muons. Tabular values of the data represented by this figure (at intervals of 0.01 in $M\omega$) are available upon request.

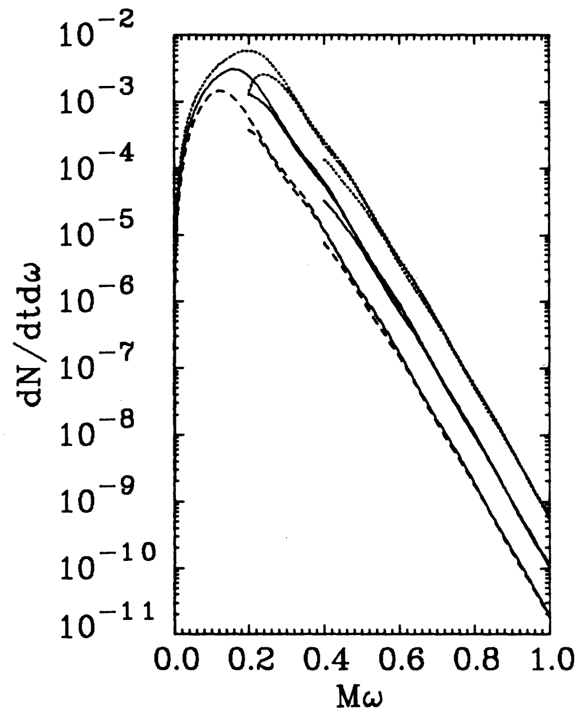

FIG. 2. Emission number rate in spin-$\frac{1}{2}$ positively charged particles from a hole of $Z=20$ (dotted curves), 0 (solid curves), and -20 (dashed curves). Curves for each Z have $M\mu=0$, 0.2, and 0.4 and start at that value of $M\omega$.

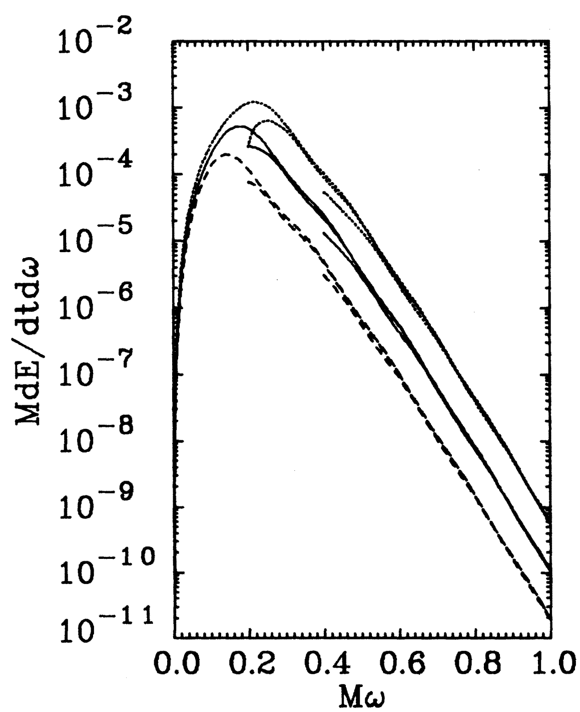

FIG. 3. Power in spin-$\frac{1}{2}$ positively charged particles, with the same notation as in Fig. 2.

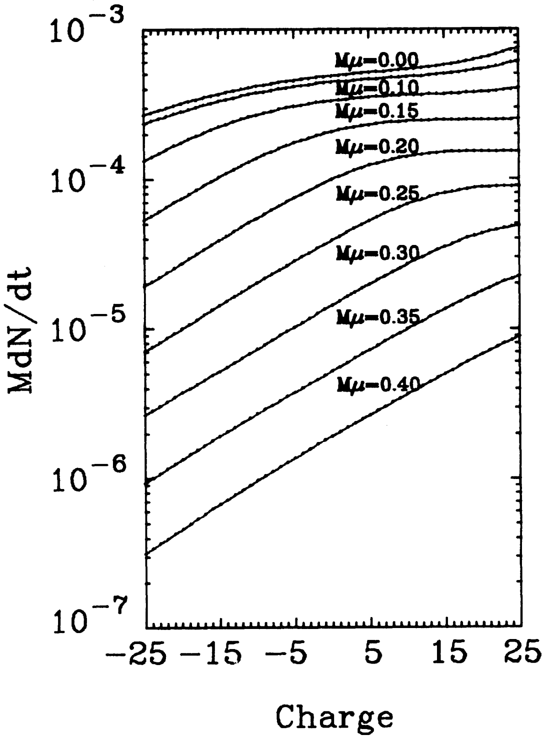

FIG. 4. Emission rates in spin-$\frac{1}{2}$ particles of charge e from a black hole with charge Ze after the emission. The abscissa gives the charge Z in positron units. The dots at each integral value of Z are all that are significant; the lines merely connect the dots with the same $M\mu$.

mensionless form $M^{-2}\sigma$, since the cross section diverges as v^{-2} at low velocity $v=[1-(M\mu/M\omega)^2]^{1/2}$,[12] whereas the quantity above remains finite. For $M\omega \gg \max(1, Z\alpha)$ the quantity above varies from $\frac{16}{27}$ at $v=0$ to 1 at $v=1$, as shown by Eq. (95) of Ref. 12, and as hinted by the convergence of the curves in Fig. 1 toward 1 at high energy. The oscillations of the curves with energy occurs as successively higher angular modes become strongly absorbed to contribute in Eq. (21). In other words, the quantization of angular momentum causes the maximum effective impact parameter $b=j/(\omega v)$ that is strongly absorbed to be discrete and increase in jumps.

From the values of $\sigma v^2/27\pi M^2$, the emission rate and power data shown in Figs. 2, 3 can be computed by the following formulas:

$$\frac{dN_+}{dt d\omega}=\frac{27(M\omega)^2/\pi}{e^{8\pi M\omega-4\pi\alpha Z}+1}\left(\frac{\sigma v^2}{27\pi M^2}\right), \tag{22}$$

$$\frac{M\,dE_+}{dt d\omega}=\frac{27(M\omega)^3/\pi}{e^{8\pi M\omega-4\pi\alpha Z}+1}\left(\frac{\sigma v^2}{27\pi M^2}\right). \tag{23}$$

For $M\mu \gg \max(1, Z\alpha)$, $\sigma v^2 \approx 16\pi M^2$ at $v \ll 1$, as noted above, so then the spectra become

$$\frac{dN_+}{dt d\omega}\approx\frac{16}{\pi}(M\omega)^2 e^{-8\pi M\omega+4\pi\alpha Z}, \tag{24}$$

$$\frac{MdE_+}{dt d\omega}\approx\frac{16}{\pi}(M\omega)^3 e^{-8\pi M\omega+4\pi\alpha Z}, \tag{25}$$

over $v \ll 1$, which includes the dominant part of the spectra. Thus one may integrate the spectra to get the total emission rate and power from a large hole:

$$M\frac{dN_+}{dt}\approx\frac{2}{\pi^2}(M\mu)^2 e^{-8\pi M\mu+4\pi\alpha Z}, \tag{26}$$

$$M^2\frac{dE_+}{dt}\approx\frac{2}{\pi^2}(M\mu)^3 e^{-8\pi M\mu+4\pi\alpha Z}. \tag{27}$$

The formulas are the same for negative particles, except that Z is replaced by $-Z$. These formulas agree with Eq. (5), so the charge probability distribution should be the same as given in Eq. (8) if $M\mu \gg \max(1, Z\alpha)$. Also, the emission averaged over the charge distribution should be, as in Eq. (10),

$$\langle M\,dN_+/dt\rangle \approx (2/\pi^2)(M\mu)^2 e^{-8\pi M\mu-3\pi\alpha}, \tag{28}$$

$$\langle M^2\,dE_+/dt\rangle \approx (2/\pi^2)(M\mu)^3 e^{-8\pi M\mu-3\pi\alpha}. \tag{29}$$

For other values of $M\mu$ and Z than those shown in Figs. 1–3, the emission was calculated only at those values of $M\omega$ (generally $\lesssim 0.6$) needed to give accurate results for the numerical integrations over frequency. For $M\mu=0$, the calculations were made at all integral values of Z from −25 to 25 to an accuracy of roughly one part in 10^4. For $M\mu = 0.05$, 0.10, 0.15, 0.20, 0.25, 0.30, 0.35, and 0.40, calculations were made at every fifth value of Z from −25 to 25 to an accuracy of roughly one part in 10^3 at low $M\mu$ and 10^2 at high $M\mu$, and cubic spline interpolations[13] were made for the other integral values of Z. When the interpolation scheme was tested with every fifth value calculated for $M\mu = 0$, it reproduced the other values with an average error of about one part in 10^4.

The results for the total emission rate in positive spin-$\frac{1}{2}$ particles vs Z is shown in Fig. 4, where Z is the charge of the black hole after it emits the particle. If the curves were straight lines on the semilog plot, then Eq. (5) would apply (possibly with a different coefficient of Z in the exponent—the slope of the semilog curve), and the charge probability distributions would have Gaussian shapes as in Eq. (8). The actual probability distributions, calculated from Eq. (7) with $R(Z)$ being the values of $M\,dN_+/dt$ shown in Fig. 4, are shown in Fig. 5. The rms values of Z and a comparison of the higher moments with those of normal distributions having the same standard deviations is given in Table I.

The total power emitted in positive particles at

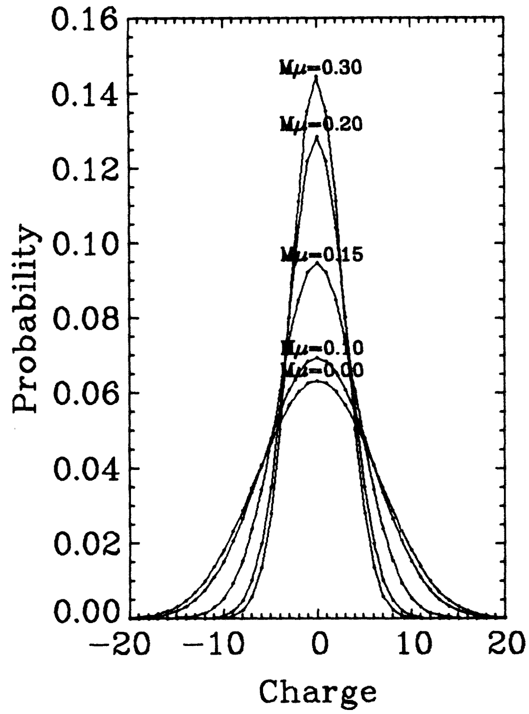

FIG. 5. Probability distribution for the charge Z (in positron units) of a black hole that emits charged particles predominantly of mass μ. The curve for $M\mu=0.05$ (not shown) is very near the one for $M\mu=0$; those for $M\mu=0.35$ and 0.40 would be very near that for $M\mu=0.30$.

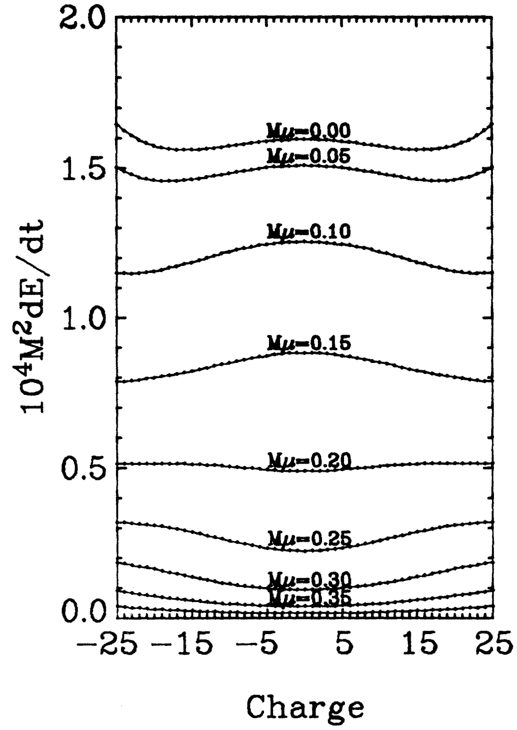

FIG. 6. Sum of the power in spin-$\frac{1}{2}$ particles of charge e and $-e$ from a hole with charge Ze before the emission. (The labeled value of the charge is Z rather than Ze.) The bottom curve, not labeled, is for $M\mu=0.40$.

all energies has a similar dependence on Z as the total emission rate shown in Fig. 4. Therefore, it is more interesting to plot the sum of the power in both positive and negative particles vs Z, which is shown in Fig. 6. Here, unlike in Fig. 4, Z is taken as the charge of the black hole before emission; i.e., the curves are the power in emitting a positive charge to leave a hole with charge $Z-1$, plus the power in emitting a negative charge to leave the hole with $Z+1$. Contrary to the naive expectation that one would get if the power obeyed a formula like Eq. (5), the sum in positive and negative charges does not always rise with $|Z|$ but instead falls when $M\mu \leq 0.15$, at least at low $|Z|$.

The power in positive and negative particles as a function of Z may be averaged over the charge probability distribution to get the average power emitted as the black hole fluctuates in charge. This quantity, as well as the average emission rate in positive and negative particles, are listed in

TABLE I. Root-mean-square charge of a nonrotating black hole and ratios of the higher moments of the charge probability distribution to those of a Gaussian with the same rms value. Angular brackets denote an average over the probability distribution.

$M\mu$	$\langle Z^2\rangle^{1/2}$	$\langle Z^4\rangle/3!!\langle Z^2\rangle^2$	$\langle Z^6\rangle/5!!\langle Z^2\rangle^3$	$\langle Z^8\rangle/7!!\langle Z^2\rangle^4$	$\langle Z^{10}\rangle/9!!\langle Z^2\rangle^5$	$\langle Z^{12}\rangle/11!!\langle Z^2\rangle^6$
0	6.1428	0.9396	0.8361	0.7088	0.5751	0.4481
0.05	6.2527	0.9444	0.8476	0.7263	0.5961	0.4695
0.10	5.6466	0.9573	0.8808	0.7818	0.6711	0.5584
0.15	4.1791	0.9864	0.9623	0.9303	0.8925	0.8506
0.20	3.1196	1.0149	1.0441	1.0878	1.1464	1.2211
0.25	2.8247	1.0046	1.0141	1.0291	1.0501	1.0777
0.30	2.7589	1.0017	1.0052	1.0105	1.0177	1.0270
0.35	2.7608	1.0016	1.0047	1.0090	1.0144	1.0208
0.40	2.7578	0.9977	0.9934	0.9873	0.9794	0.9701

TABLE II. Total emission rate and power in spin-$\frac{1}{2}$ particles and antiparticles of mass μ from a nonrotating black hole of mass M. "Charged particles" means particles of charge $\pm e$, with the emission averaged over the probability distribution for the charge of the hole. "Neutral particles" means particles of zero charge whose emission does not depend upon the charge of the hole. "Charged/neutral" means the ratio of the average emission of charged particles to that of neutral ones. Radiative and self-energy corrections have been ignored, so these numbers are physically significant only to order α.

	Emission rate, $M\,dN/dt$			Emission power, M^2dE/dt		
$M\mu$	Charged particles	Neutral particles	Charged/neutral	Charged particles	Neutral particles	Charged/neutral
0	9.516×10^{-4}	9.714×10^{-4}	0.9797	1.589×10^{-4}	1.637×10^{-4}	0.9706
0.05	8.581×10^{-4}	8.788×10^{-4}	0.9765	1.499×10^{-4}	1.546×10^{-4}	0.9696
0.10	6.430×10^{-4}	6.642×10^{-4}	0.9680	1.243×10^{-4}	1.288×10^{-4}	0.9657
0.15	3.974×10^{-4}	4.151×10^{-4}	0.9571	8.770×10^{-5}	9.163×10^{-5}	0.9571
0.20	1.892×10^{-4}	1.993×10^{-4}	0.9495	4.909×10^{-5}	5.165×10^{-5}	0.9505
0.25	7.396×10^{-5}	7.768×10^{-5}	0.9521	2.264×10^{-5}	2.378×10^{-5}	0.9519
0.30	2.729×10^{-5}	2.859×10^{-5}	0.9543	9.692×10^{-6}	1.016×10^{-5}	0.9537
0.35	9.965×10^{-6}	1.029×10^{-5}	0.9684	4.029×10^{-6}	4.171×10^{-6}	0.9661
0.40	3.682×10^{-6}	3.735×10^{-6}	0.9859	1.664×10^{-6}	1.695×10^{-6}	0.9818

Table II at the nine values of $M\mu$ used. For comparison, the power and emission rates for neutral Dirac particles are also listed, as well as the ratios of the charged-to-uncharged results. The charged particles always have a lower average emission, though not 7% lower as naively predicted above. This effect would persist even if the curves in Fig. 6 were flat so that the average power were the same as the power from an initially uncharged hole, for when a charged particle is emitted from such a hole, it leaves behind a charge of the opposite sign which tends to hold back the emitted particle. Again it should be noted that these calculations do not take into account other effects which should give fractional corrections of order α.

In order to get a smooth plot of the dependence of the average emission rate and power on $M\mu$, crude functional fits were found, and then the correction factors were approximated by cubic splines. The crude functional fits were chosen to be

$$M\left\langle\frac{dN_+}{dt}+\frac{dN_-}{dt}\right\rangle=\frac{4+2z+z^2}{16\pi^4(e^z+1)}\,e^{-3\pi\alpha}R_f(z), \qquad (30)$$

$$M^2\left\langle\frac{dE_+}{dt}+\frac{dE_-}{dt}\right\rangle=\frac{12+6z+3z^2+z^3}{128\pi^5(e^z+1)}\,e^{-3\pi\alpha}P_f(z), \qquad (31)$$

where $z\equiv 8\pi M\mu$ and where $R_f(z)$ and $P_f(z)$ are the correction factors. The functions in Eqs. (30) and (31) were chosen so as to guarantee that R_f and P_f have no low-order derivatives at $z=0$ and that they reduce to unity as z becomes infinite. Indeed, the correction factors did turn out to be fairly constant over the nine values of $M\mu$ calculated, with R_f increasing from 0.794 at $M\mu=0$ to 1.232 at $M\mu=0.2$ and then generally decreasing to 1.141 at $M\mu=0.4$, and P_f increasing from 1.111 at $M\mu=0$ to 1.290 at $M\mu=0.2$ and then generally decreasing to 1.166 at $M\mu=0.4$. Therefore, one might expect cubic spline fits of $R_f(z)$ and $P_f(z)$ to be fairly accurate. The results of these fits were inserted back into Eqs. (30) and (31) to get the approximate average emission rate and power vs $M\mu$, and then

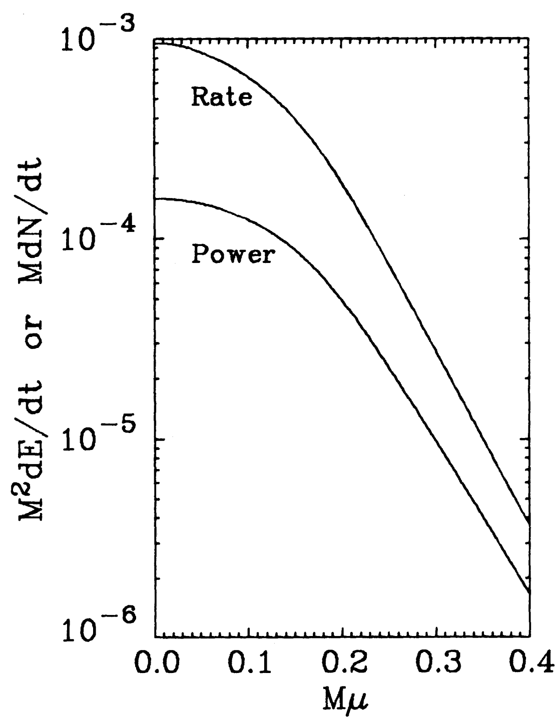

FIG. 7. Average emission rate and power of spin-$\frac{1}{2}$ particles of charge e and mass μ from a black hole of mass M. The curves are the results of a fit to the calculated values at nine values of $M\mu$ spaced evenly across the graph. $M\mu=1$ corresponds to $M=5.20\times10^{17}$ g or to 2.52×10^{15} g for μ being the mass of the electron or muon, respectively.

these values were plotted in Fig. 7. The power is down a factor of 2 from the $M\mu = 0$ power when $M\mu = 0.160$, down a factor of 4 when $M\mu = 0.215$, and down a factor of 10 when $M\mu = 0.271$.

IV. IMPLICATIONS FOR ELECTRONS AND MUONS

Any black hole smaller than about 10^{17} g will be emitting electrons and positrons prolifically and so have rapid charge fluctuations of a few positron units, with the rms value approaching $6.14e$ as the black hole gets much hotter than the electron rest energy. The average power in electrons and positrons of both helicities becomes greater than 50% of the power in all four kinds of neutrinos as a black hole gets smaller than 8.33×10^{16} g, and it approaches 97% of the neutrino power as the mass gets much smaller (e.g., within 1% of this limit for $M \leq 1.1 \times 10^{16}$ g). The average power in muons and antimuons similarly reaches 50% of the neutrino power at 4.03×10^{14} g and becomes greater at smaller masses.

Now we can try to estimate the total emission in all kinds of particles from a black hole with $M = 5 \times 10^{14}$ g. The muon-antimuon power is about 31% of the neutrino power. At this black-hole mass, pions have $M\mu = 0.26$ ($\pi^{\pm}$) and 0.25 (π^0). If their power were the same as an equal number of spin states of spin-$\frac{1}{2}$ particles at the same $M\mu$, they would contribute about 9% as much power as neutrinos. Known heavier particles would contribute only negligibly, so the total power would be that in gravitons and photons (whose emission was calculated in paper I), plus roughly 2.37 times the neutrino contribution, or

$$dE/dt \approx 4.25 \times 10^{-4} M^{-2} = 2.93 \times 10^{17} \text{ erg sec}^{-1} \tag{32}$$

at

$$M = 5 \times 10^{14} \text{ g} = 2.30 \times 10^{19} \text{ Planck units.} \tag{33}$$

If the power went as M^{-2} at lower masses, the total lifetime of such a hole would be 16 billion years, roughly the age of the universe. Muons and pions will contribute power which will increase somewhat faster than M^{-2} at lower M as their rest masses become less of a hindrance to their emission, but since they make up only about 14% of the power at 5×10^{14} g, the lifetime should not be much shorter than 16 billion years. This result strengthens the estimate of paper I that a primordial black hole will have just decayed away within the present age of the universe if its initial mass were about 5×10^{14} g.

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APPENDIX

For the purpose of calculating the emission rates of charged leptons from black holes of arbitrary charge and rotation, a computer program was wtitten to solve the general Eqs. (15) and (16). It proved convenient to define

$$x \equiv \frac{r - r_+}{r_+ - r_-} \tag{A1}$$

and

$$y \equiv \ln(e^x - 1) \tag{A2}$$

as dimensionless radial variables, where

$$\Delta \equiv (r - r_+)(r - r_-) \tag{A3}$$

defines r_+ and r_-. The waves have an infinite number of oscillations near the horizon at $x = 0$; $y \to -\infty$ there and stretches out the waves so that they remain well behaved in that variable. By defining

$$K_0 = \frac{(r_+^2 + a^2)\omega - eQr_+ - am}{r_+ - r_-}$$

$$K_1 = 2r_+\omega - eQ, \tag{A4}$$

$$K_2 = (r_+ - r_-)\omega,$$

$$M_0 = r_+\mu, \quad M_1 = (r_+ - r_-)\mu \tag{A5}$$

[where I have replaced σ and m in Eq. (14) of Ref. 9 by ω and $-m$, respectively, in order to return to the conventions of Teukolsky's original separation of the neutrino equation in the Kerr[14] field], we may write the general radial equations in the explicit form

$$\frac{dG}{dy} = \frac{\lambda(1 - e^{-x})}{x^{1/2}(1+x)^{1/2}} G + \left(\frac{M_0 + M_1 x}{x^{1/2}(1+x)^{1/2}} + \frac{K_0 + K_1 x + K_2 x^2}{x(1+x)} \right)(1 - e^{-x})F, \tag{A6}$$

$$\frac{dF}{dy} = \frac{-\lambda(1 - e^{-x})}{x^{1/2}(1+x)^{1/2}} F + \left(\frac{M_0 + M_1 x}{x^{1/2}(1+x)^{1/2}} - \frac{K_0 + K_1 x + K_2 x^2}{x(1+x)} \right)(1 - e^{-x})G. \tag{A7}$$

Here y is taken as the independent variable, and x is defined implicitly by Eq. (A2), which has the explicit inverse

$$x = \ln(1 + e^y), \tag{A8}$$

a slight numerical advantage over the Regge-Wheeler tortoise coordinate[15] or the analogous radial coordinates used in the Kerr geometry by various people (see Ref. 14).

At the horizon ($y \to -\infty$, $x \to 0$), an ingoing wave has the form

$$G \propto e^{iK_0 y}, \quad F = iG. \tag{A9}$$

At small nonzero values of x, the solution was expanded in a power series of the form

$$R_1 \approx e^{iK_0 y} x^{1/2}(a_1 + b_1 x + c_1 x^2),$$
$$R_2 \approx e^{iK_0 y}(1 + b_2 x + c_2 x^2), \tag{A10}$$

where a_1, b_1, c_1, b_2, and c_2 were found explicitly in terms of λ, K_0, K_1, K_2, M_0, and M_1 and used to give highly accurate starting values for R_1 and R_2 [and hence G and F by Eq. (14)]. Then Eqs. (A6) and (A7) were integrated outward to $y \gg 1$ by the Bulirsch-Stoer method[16] with variable step sizes for controlled accuracy.

At large radii, where the difference between x and y is negligible, the solution may be expressed in the form

$$2G = Z_{\rm in} e^{\alpha + \beta + i\gamma + i\varphi} + Z_{\rm out} e^{\alpha + \beta - i\gamma - i\varphi}, \tag{A11}$$

$$2F = iZ_{\rm in} e^{-\alpha + \delta + i\epsilon + i\varphi} - iZ_{\rm out} e^{-\alpha + \delta + i\epsilon - i\varphi}, \tag{A12}$$

where $|Z_{\rm in}|^2$ and $|Z_{\rm out}|^2$ are the fluxes of incoming and outgoing waves, relative to the purely ingoing flux down the hole, and where

$$\alpha = \frac{1}{4} \ln\left(\frac{\kappa + \mathfrak{M}}{\kappa - \mathfrak{M}}\right), \tag{A13}$$

$$\varphi = \int (\kappa^2 - \mathfrak{M}^2)^{1/2} dx = \text{const} + kx + c \ln x + O(x^{-1}), \tag{A14}$$

$$\beta = \frac{\lambda(\lambda - 1)}{4k^2 x^2} + O(x^{-3}), \tag{A15}$$

$$\delta = \frac{\lambda(\lambda + 1)}{4k^2 x^2} + O(x^{-3}), \tag{A16}$$

$$\gamma = \frac{\lambda(\lambda - 1)}{2kx} - \frac{(\lambda - 1)(k\lambda + c\lambda + 2kv) - \lambda c}{4k^2 x^2} + O(x^{-3}), \tag{A17}$$

$$\epsilon = \frac{\lambda(\lambda + 1)}{2kx} - \frac{(\lambda + 1)(k\lambda + c\lambda + 2kv) + \lambda c}{4k^2 x^2} + O(x^{-3}), \tag{A18}$$

with

$$\kappa \equiv \frac{(r_+ - r_-)K}{\Delta} = \left(K_2 + \frac{K_1}{x} + \frac{K_0}{x^2}\right)\left(1 + \frac{1}{x}\right)^{-1}, \tag{A19}$$

$$\mathfrak{M} \equiv \frac{(r_+ - r_-)\mu r}{\Delta^{1/2}} = \left(M_1 + \frac{M_0}{x}\right)\left(1 + \frac{1}{x}\right)^{-1/2}, \tag{A20}$$

$$k \equiv \lim_{x \to \infty} (\kappa^2 - \mathfrak{M}^2)^{1/2} = (K_2{}^2 - M_1{}^2)^{1/2}, \tag{A21}$$

$$c \equiv (2K_1K_2 - 2K_2{}^2 - 2M_0M_1 + M_1{}^2)/(2k), \tag{A22}$$

$$v \equiv (2K_1M_1 - K_2M_1 - 2K_2M_0)/(2k)^2. \tag{A23}$$

By using these expressions, we can calculate

$$E \equiv \frac{|Z_{\rm in}|^2 + |Z_{\rm out}|^2}{|Z_{\rm in}|^2 - |Z_{\rm out}|^2} = \frac{\frac{1}{2}e^{-2\alpha + 2\delta}|2G|^2 + \frac{1}{2}e^{2\alpha + 2\beta}|2F|^2 - \tan(\gamma - \epsilon)\,{\rm Re}(2G^* 2F)}{{\rm Im}(2G^* 2F)}. \tag{A24}$$

Then the absorption probability is

$$\Gamma = 1 - \left|\frac{Z_{\rm out}}{Z_{\rm in}}\right|^2 = \frac{2}{E + 1}. \tag{A25}$$

This method of calculating Γ from the solution at large radii is accurate through $O((2kx)^{-2})$ (assuming the solution is accurate) and is independent of the normalization of the solution at the horizon (so long as it is purely ingoing there). However, the differential equations give rise to a conserved quantity (analogous to a Wronskian), proportional to the net inward flux of particles:

$$W = |R_2|^2 - |R_1|^2 = {\rm Im}(2G^* 2F) = \text{const.} \tag{A26}$$

This constant is normalized to unity at the horizon, so inasmuch as the numerical scheme is successful at keeping it constant, the solution at large radii should give $|Z_{\rm in}|^2 - |Z_{\rm out}|^2 = 1$. Therefore, another value of E can be obtained from the numerator alone in Eq. (A24) and the corresponding value of Γ obtained from Eq. (A25). The difference between this and the previous value for Γ gives a measure of the error incurred in numerically solving the radial equations.

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[1]D. N. Page, Phys. Rev. D 13, 198 (1976), hereinafter denoted paper I.

[2]D. N. Page, Phys. Rev. D 14, 3260 (1976), hereinafter denoted paper II.

[3]S. W. Hawking, Nature (London) 248, 30 (1974);

Commun. Math. Phys. 43, 199 (1975).
[4]G. W. Gibbons, Commun. Math. Phys. 44, 245 (1975).
[5]W. T. Zaumen, Nature (London) 247, 530 (1974).
[6]B. Carter, Phys. Rev. Lett. 33, 558 (1974).
[7]J. D. Bekenstein, Phys. Rev. D 12, 3077 (1975).
[8]D. R. Brill and J. A. Wheeler, Rev. Mod. Phys. 29, 465 (1957).
[9]S. Chandrasekhar, Proc. R. Soc. London A349, 571 (1976).
[10]D. N. Page, Phys. Rev. D 14, 1509 (1976).
[11]D. N. Page (unpublished).
[12]W. G. Unruh, Phys. Rev. D 14, 3251 (1976).
[13]I. J. Schoenberg, Q. Appl. Math. 4, 45 (1946); 4, 112 (1946).
[14]S. A. Teukolsky, Astrophys. J. 185, 635 (1973).
[15]T. Regge and J. A. Wheeler, Phys. Rev. 108, 1063 (1957).
[16]R. Bulirsch and J. Stoer, Numerische Mathematik 8, 1 (1966).