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Gravitational scattering of zero-rest-mass plane waves*

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We have used the Feynman-diagram technique to calculate the differential cross sections $d\sigma/d\Omega$ for the scattering of zero-rest-mass plane waves of spin 0, 1, and 2 by linearized Schwarzschild and Kerr geometries in the long-wavelength, weak-field limit (wavelength of incident radiation $\gg$ radius of scatterer $\gg$ mass of scatterer). We find that the polarization of right (or left) circularly polarized electromagnetic waves is unaffected by the scattering process (i.e., helicity is conserved), and that the two helicity (polarization) states of the photon are scattered differently by the Kerr geometry. This coupling between the photon helicity and the angular momentum of the scatterer also leads to a partial polarization of unpolarized incident light. For gravitational waves, on the other hand, there is neither helicity conservation nor helicity-dependent scattering; and the angular momentum of the scatterer has no polarizing effect on incident, unpolarized gravitational waves.

I. INTRODUCTION

Recent observations by Harwit *et al.*[1] have placed an upper limit on the difference of deflection between left and right circularly polarized radio beams passing near the limb of the sun. Whereas previous electromagnetic tests of general relativity (light bending near the sun, Shapiro time delay of radar signals, gravitational red-shift[2]) probe only the geometric-optics limit of electromagnetic-gravitational coupling, this experiment goes beyond geometric optics. The deflection is independent of polarization in the geometric-optics limit; but for real, physical waves the helicity of the wave should couple to the angular momentum of the deflecting object ("magnetic-type" gravitational effect) to produce helicity-dependent deflection—helicity dependence which, for the sun, is below the accuracy of Harwit *et al.*, but which should exist nevertheless.

A number of recent papers have used general relativity theory to investigate this helicity dependence and other aspects of the interaction between incoming waves and a gravitating body.[3-13] Gradually the full picture of such interactions is emerging, but there remain as yet a number of gaps in the picture. The purpose of this paper is to fill in one of those gaps: the full details of the long-wavelength limit for rotating and weakly gravitating bodies

$$\text{(wavelength)} \equiv 2\pi/\omega \gg \text{(size of body)} \equiv L \gg \text{(gravitational radius)} \equiv M \tag{1.1}$$

for scalar and gravitational waves as well as electromagnetic.

In the regime $2\pi/\omega \gg L \gg M$ it is better to speak of a "scattering" of the waves than a "deflection"; and it is most useful to calculate the amplitude T_{fi} for scattering of an incoming plane wave $|i\rangle$ into an outgoing (final) plane wave $|f\rangle$. From this scattering amplitude one can derive everything of interest—the explicit form of the scattered wave, the differential scattering cross section $d\sigma/d\Omega$, the amount of focusing, the deflection angle in the regime where it has meaning, i.e., (wavelength) $\ll$ (impact parameter), etc.

We, like some others before us,[14,15] have found the Feynman-diagram technique to be far more powerful than partial-wave analyses for studying the long-wavelength limit of classical scattering. Historically the Feynman technique was first used in conjunction with quantum-electrodynamical processes.[16-18] Its efficiency as a problem-solving tool soon led to its widespread use in many aspects of quantum interactions, including quantum gravity.[19-21] However, since classical scattering is the long-wavelength limit of quantum scattering, one can perfectly well use the technique to solve our type of classical problem.

Our paper is in six sections. Section II gives the Lagrangians, vertex rules, and diagrams needed for each type of wave (scalar, electromagnetic, and gravitational), as well as the formula for the differential scattering cross section in terms of the transition amplitude. In Secs. III, IV, and V we treat the scattering of scalar, electromagnetic, and gravitational waves, respectively. Section VI discusses and contrasts our results with those of other authors.

II. FEYNMAN DIAGRAMS FOR SCATTERING

The classical problem of the scattering of a massless field propagating in a slightly curved spacetime may be treated by quantizing both the gravitational background and the scattered field.

In this scenario both fields evolve in a Minkowski spacetime and couple according to the Feynman vertex rules. This approach may be contrasted to the work of Peters,[13] in which the gravitational background is considered to be a passive nondynamical entity, whose influence on the propagating field is embodied in a curved-spacetime Green's function. In this section we summarize the relevant Feynman rules.

The wave equation for source-free scalar waves

$$\Box\psi - uR\psi = 0 \tag{2.1}$$

may be obtained from the Lagrangian density

$$\mathfrak{L}_S = -\tfrac{1}{2}\sqrt{-g}\,(g^{\alpha\beta}\psi_{,\alpha}\psi_{,\beta} + uR\psi^2)\,, \tag{2.2}$$

where u is a constant, R is the curvature scalar, $g = \det\| g_{\alpha\beta}\|$, and $\Box \equiv (-g)^{-1/2}\partial_\alpha(g^{\alpha\beta}\sqrt{-g}\,\partial_\beta)$. For $u = \frac{1}{6}$, ψ represents conformally invariant waves.

Following Feynman[19-21] and Gupta,[22] and since we require that $|\bar{h}^{\alpha\beta}| \ll 1$ everywhere, we expand the gravitational field about the flat Minkowski background:

$$\sqrt{-g}\; g^{\alpha\beta} \equiv \mathfrak{g}^{\alpha\beta} \equiv \eta^{\alpha\beta} - 2\lambda\bar{h}^{\alpha\beta}\,, \tag{2.3}$$

where the gravitational coupling constant $\lambda = \sqrt{8\pi}$ and we use units in which $G = \hbar = c = 1$. Indices are raised and lowered using the Minkowski metric $\eta_{\alpha\beta} = \eta^{\alpha\beta} = \mathrm{diag}(-1,1,1,1)$, commas denote partial derivatives, semicolons denote covariant derivatives with respect to the metric, and $\bar{h}^{\alpha\beta}$ is the trace-reversed metric perturbation.

The determinant factor $\sqrt{-g}$, $g^{\alpha\beta}$, and R now become infinite series in λ,

$$\sqrt{-g} = (-\det\| g_{\alpha\beta}\|)^{1/2} = (-\det\|\mathfrak{g}^{\alpha\beta}\|)^{1/2} = 1 - \lambda\bar{h} + O(\lambda^2)\,, \tag{2.4}$$

$$g^{\alpha\beta} = \eta^{\alpha\beta} - 2\lambda(\bar{h}^{\alpha\beta} - \tfrac{1}{2}\bar{h}\eta^{\alpha\beta}) + O(\lambda^2)\,, \tag{2.5}$$

$$R = 2\lambda(\bar{h}^{\alpha\beta}{}_{,\alpha\beta} + \tfrac{1}{2}\bar{h}_{,\alpha}{}^{\alpha}) + O(\lambda^2)\,, \tag{2.6}$$

where the trace of the metric perturbation is denoted by $\bar{h} = \bar{h}_\mu{}^\mu$. Expanding (2.2) in powers of λ we find that

$$\mathfrak{L}_S = \sum_{n=0}^{\infty} \lambda^n \mathfrak{L}_n\,, \tag{2.7}$$

where

$$\mathfrak{L}_0 = -\tfrac{1}{2}\eta^{\alpha\beta}\psi_{,\alpha}\psi_{,\beta}\,, \tag{2.8}$$

$$\mathfrak{L}_1 = \bar{h}^{\alpha\beta}\psi_{,\alpha}\psi_{,\beta} - u(\bar{h}^{\alpha\beta}{}_{,\alpha\beta} + \tfrac{1}{2}\bar{h}_{,\alpha}{}^{\alpha})\psi^2\,. \tag{2.9}$$

The free (i.e., noninteraction) Lagrangian $\mathfrak{L}_0$ describes the free propagation of the scalar field ψ in Minkowski space, whereas the terms proportional to λ, λ^2, etc. represent the interaction parts of $\mathfrak{L}$, i.e., they determine how the gravitational field $\bar{h}^{\alpha\beta}$ couples to the scalar field ψ. In this formalism, quantization of the Lagrangian density is equivalent to treating $\bar{h}^{\alpha\beta}$ and ψ as quantum field operators.

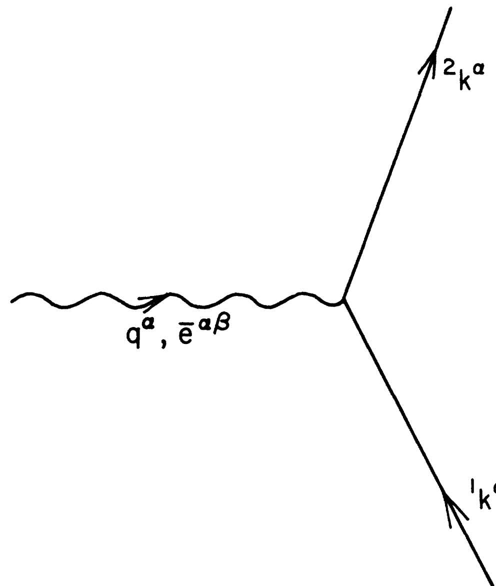

FIG. 1. The graviton–(zero-rest-mass field)–(zero-rest-mass field) vertex. The wavy line represents a graviton. The solid lines represent either scalar, electromagnetic, or gravitational quanta.

From $\mathfrak{L}_1$ we may derive the amplitude T_{21} for a transition of the scalar field from an initial plane-wave state with wave vector ("momentum") $^1k^\alpha$ to a final state with "momentum" $^2k^\alpha$ while absorbing a graviton with momentum q^α and polarization $\bar{e}^{\alpha\beta}$ (Fig. 1):

$$T_{21} = 2\lambda\bar{e}^{\alpha\beta}[\,{}^1k_{(\alpha}{}^2k_{\beta)} + u(q_\alpha q_\beta + \tfrac{1}{2}\eta_{\alpha\beta}q^2)]\,. \tag{2.10}$$

Here we have used the notation $A_{(\alpha}B_{\beta)} = \frac{1}{2}(A_\alpha B_\beta + B_\alpha A_\beta)$ and the superscript 1 (2) denotes the initial (final) state. Conservation of 4-momentum requires that

$$\underline{{}^2k} = \underline{{}^1k} + \underline{q}\,. \tag{2.11}$$

In this calculation we shall limit ourselves to interactions proportional to λ^2 (single-graviton exchange); in other words, we shall calculate the scattering cross sections in the first Born approximation. In the classical limit for the scattering of waves with angular frequency ω by a mass M with angular momentum J, this corresponds to calculating at first order in the dimensionless quantities $M\omega$ and $J\omega^2$. Since our interest is restricted to a gravitational-background geometry generated by classical energy-momentum distributions which are not affected appreciably by the scattering process, we may replace the virtual graviton by an external field.[23] In particular we consider only *static* fields; hence in the vertex rule (2.10) $\bar{e}^{\alpha\beta}$ stands for the 3-dimensional Fourier transform of $\bar{h}^{\alpha\beta}$ and the graviton 4-momentum is pure spacelike ($q^0=0$).

The transition amplitude T_{21} above has been normalized by the definition

$$S_{21}=\delta_{21}+i(2\pi)^4\delta^4({}^2\underline{k}-{}^1\underline{k}-\underline{q})T_{21}, \tag{2.12}$$

where S_{21} is the S matrix connecting the initial to the final state. With this normalization for T_{21}, the differential cross section for the scattering of a zero-rest-mass wave with frequency ω into a solid angle $d\Omega$ is

$$d\sigma=\frac{2\pi}{2\omega 2\omega}|T_{21}|^2D, \tag{2.13}$$

where D denotes the density of final states,

$$D=\frac{\omega^2}{(2\pi)^3}d\Omega. \tag{2.14}$$

Turn now to the scattering of electromagnetic waves off a slightly curved background. The manifestly covariant photon Lagrangian density, obtained by minimal coupling to gravity, is

$$\mathcal{L}_{EM}=-\tfrac{1}{4}\sqrt{-g}\,(g^{\mu\alpha}g^{\nu\beta}F_{\mu\nu}F_{\alpha\beta}), \tag{2.15}$$

where $F_{\mu\nu}$ is the electromagnetic field tensor computed from the Maxwell vector potential A_μ by

$$F_{\mu\nu}=A_{\nu,\mu}-A_{\mu,\nu}. \tag{2.16}$$

From (2.15) and (2.16) one obtains the field equations for the source-free electromagnetic field:

$$F_{\mu\nu;\lambda}+F_{\nu\lambda;\mu}+F_{\lambda\mu;\nu}=0, \tag{2.17}$$

$$F_{\mu\nu}{}^{;\nu}=0. \tag{2.18}$$

We expand the photon Lagrangian density in powers of λ according to (2.7) and obtain

$$\mathcal{L}_0=-\tfrac{1}{4}\eta^{\mu\alpha}\eta^{\nu\beta}F_{\mu\nu}F_{\alpha\beta}, \tag{2.19}$$

$$\mathcal{L}_1=(\bar{h}^{\mu\alpha}\eta^{\nu\beta}-\tfrac{1}{4}\bar{h}\eta^{\mu\alpha}\eta^{\nu\beta})F_{\mu\nu}F_{\alpha\beta}. \tag{2.20}$$

After proper permutation of the photon labels, $\mathcal{L}_1$ provides the graviton-photon-photon vertex rule (see Fig. 1)

$$T_{21}=2\lambda\bar{e}^{\alpha\beta}\{{}^1k_{(\alpha}{}^2k_{\beta)}({}^1\underline{\epsilon}\cdot{}^2\underline{\epsilon}^*)+{}^1\epsilon_{(\alpha}{}^2\epsilon^*_{\beta)}({}^1\underline{k}\cdot{}^2\underline{k})-{}^1k_{(\alpha}{}^2\epsilon^*_{\beta)}({}^2\underline{k}\cdot{}^1\underline{\epsilon})$$
$$-{}^2k_{(\alpha}{}^1\epsilon_{\beta)}({}^1\underline{k}\cdot{}^2\underline{\epsilon}^*)-\tfrac{1}{2}\eta_{\alpha\beta}[({}^1\underline{k}\cdot{}^2\underline{k})({}^1\underline{\epsilon}\cdot{}^2\underline{\epsilon}^*)-({}^1\underline{k}\cdot{}^2\underline{\epsilon}^*)({}^2\underline{k}\cdot{}^1\underline{\epsilon})]\}. \tag{2.21}$$

Here ${}^1k^\alpha$ and ${}^1\epsilon^\alpha$ are the 4-momentum and polarization vector of the ingoing photon, whereas ${}^2k^\alpha$ and ${}^2\epsilon^\alpha$ denote the respective properties of the outgoing photon. In accordance with the external-field approximation $\bar{e}^{\alpha\beta}$ denotes the Fourier transform of $\bar{h}^{\alpha\beta}$. Note that the transition amplitude (2.21) is invariant under a gauge transformation of the form

$${}^i\epsilon_\alpha\to{}^i\epsilon_\alpha+\gamma^i k_\alpha\quad(i=1,2), \tag{2.22}$$

where γ is an arbitrary scalar.

Finally we turn to the scattering of gravitational waves by the gravitational background. One arrives at the matter-free Einstein field equations

$$R_{\mu\nu}=0 \tag{2.23}$$

by varying the Lagrangian density

$$\mathcal{L}_G=\frac{1}{2\lambda^2}\sqrt{-g}R. \tag{2.24}$$

Taking for our basic fields $\mathfrak{g}^{\mu\nu}=\sqrt{-g}g^{\mu\nu}$ and $\mathfrak{g}_{\mu\nu}=g_{\mu\nu}/\sqrt{-g}$ rather than the metric itself, we can express the Einstein gravitational Lagrangian density (2.24) in the particularly convenient Goldberg[24] form:

$$\mathcal{L}_G=-\frac{1}{16\lambda^2}(2\mathfrak{g}^{\alpha\beta}\mathfrak{g}_{\sigma\mu}\mathfrak{g}_{\tau\nu}-\mathfrak{g}^{\alpha\beta}\mathfrak{g}_{\mu\tau}\mathfrak{g}_{\sigma\nu}-4\eta^\alpha{}_\sigma\eta^\beta{}_\tau\mathfrak{g}_{\mu\nu})\mathfrak{g}^{\mu\tau}{}_{,\alpha}\mathfrak{g}^{\sigma\nu}{}_{,\beta}. \tag{2.25}$$

After we expand (2.24) in powers of λ, the components of $\mathcal{L}_G$ become

$$\mathcal{L}_0=-\tfrac{1}{4}(2\bar{h}^{\alpha\beta,\mu}\bar{h}_{\alpha\beta,\mu}-\bar{h}^{,\mu}\bar{h}_{,\mu}-4\bar{h}^{\alpha\beta,\mu}\bar{h}_{\mu\beta,\alpha}), \tag{2.26}$$

$$\mathcal{L}_1=-\bar{h}^{\mu\nu}(\bar{h}_{\alpha\beta,\mu}\bar{h}^{\alpha\beta}{}_{,\nu}+2\bar{h}_{\mu\alpha}{}^{,\beta}\bar{h}_{\nu\beta}{}^{,\alpha}-2\bar{h}_{\mu\beta,\alpha}\bar{h}_\nu{}^{\beta,\alpha}+\bar{h}^{,\alpha}\bar{h}_{\mu\nu,\alpha}-\tfrac{1}{2}\bar{h}_{,\mu}\bar{h}_{,\nu}). \tag{2.27}$$

The interaction part $\mathcal{L}_1$, appropriately symmetrized with respect to the graviton labels, provides the expression for the three-graviton vertex (see Fig. 1):

$$
\begin{aligned}
T_{21} = \lambda \bar{e}^{\mu\nu}\{&-2({}^1\bar{\mathfrak{e}} : {}^2\bar{\mathfrak{e}}^* \, {}^1k_\mu \, {}^2k_\nu - {}^1\bar{e}_{\mu\nu} \, {}^2\bar{e}^{*\alpha\beta} q_\alpha \, {}^1k_\beta + {}^2\bar{e}^*_{\mu\nu} \, {}^1\bar{e}^{\alpha\beta} q_\alpha \, {}^2k_\beta) \\
&-4[{}^1\bar{e}_{\mu\alpha} \, {}^2\bar{e}^*_{\nu\beta} \, {}^1k^\beta \, {}^2k^\alpha - q^\beta({}^2\bar{e}^*_{\mu\alpha} \, {}^1\bar{e}^\alpha{}_\beta \, {}^1k_\nu - {}^1\bar{e}_{\mu\alpha} \, {}^2\bar{e}^{*\alpha}{}_\beta \, {}^2k_\nu)] \\
&+4({}^1\underline{k} \circ {}^2\underline{k} \, {}^1\bar{e}_{\mu\beta} \, {}^2\bar{e}^{*\beta}_\nu - \underline{q} \cdot {}^1\underline{k} \, {}^1\bar{e}_{\beta\nu} \, {}^2\bar{e}^{*\beta}_\mu + \underline{q} \cdot {}^2\underline{k} \, {}^2\bar{e}^*_{\beta\nu} \, {}^1\bar{e}_\mu{}^\beta) \\
&-[{}^1\underline{k} \cdot {}^2\underline{k}({}^1\bar{e}_{\mu\nu} \, {}^2\bar{e}^* + {}^2\bar{e}^*_{\mu\nu} \, {}^1\bar{e}) - \underline{q} \cdot {}^1\underline{k} \, {}^1\bar{e} \, {}^2\bar{e}^*_{\mu\nu} + \underline{q} \cdot {}^2\underline{k} \, {}^2\bar{e}^* \, {}^1\bar{e}_{\mu\nu} + \eta_{\mu\nu}(\underline{q} \cdot {}^2\underline{k} - \underline{q} \cdot {}^1\underline{k}) \, {}^1\bar{\mathfrak{e}} : {}^2\bar{\mathfrak{e}}^*] \\
&+[{}^1k_\mu \, {}^1k_\nu \, {}^1\bar{e} \, {}^2\bar{e}^* + \eta_{\mu\nu}(q_\alpha \, {}^2k_\beta \, {}^2\bar{e}^* \, {}^1\bar{e}^{\alpha\beta} - q_\alpha \, {}^1k_\beta \, {}^1\bar{e} \, {}^2\bar{e}^{*\alpha\beta})]\} \,.
\end{aligned}
\tag{2.28}
$$

where ${}^1k^\alpha, {}^1\bar{e}^{\alpha\beta}$; ${}^2k^\alpha, {}^2\bar{e}^{\alpha\beta}$; and $q^\alpha, \bar{e}^{\alpha\beta}$ refer to the momenta and polarizations of the gravitons and ${}^1\bar{\mathfrak{e}} : {}^2\bar{\mathfrak{e}}$ denotes the tensor inner product. Unlike the graviton-photon-photon transition amplitude (2.21), the three-graviton transition amplitude is *not* invariant under the analogous gauge transformation, which in this instance is of the form

$$ {}^i\bar{e}^{\alpha\beta} \to {}^i\bar{e}^{\alpha\beta} + {}^ik^\alpha \chi^\beta + {}^ik^\beta \chi^\alpha \quad (i = 1, 2)\,, \tag{2.29}$$

where χ^α represents an arbitrary vector.

In general, the gauge invariance of the amplitudes is guaranteed by the Feynman-diagram formalism as long as all the diagrams of the same order in the coupling constant are included. Owing to our ignorance of the propagator for an object of mass M and very high quantum-mechanical spin, we omit all diagrams but the graviton-pole diagram. (This difficulty in formulating the quantum problem could probably be avoided by a classical analysis.) In the external-field approximation (no recoil of scatterer) the amplitude corresponding to this diagram is given by (2.28), where $\bar{e}^{\mu\nu}$ stands for the 3-dimensional Fourier transform of $\bar{h}^{\mu\nu}$. The external-field approximation serves to simplify the algebra, but the effect of the omitted diagrams is to yield an amplitude (2.28) that is not gauge invariant, and is valid only for small scattering angles.

III. SCALAR WAVES

Since the waves have wavelength much larger than the scatterer, they cannot probe (at first order) either the scatterer's internal structure or the quadrupole and higher-order moments of its gravitational field. For this reason, and because we calculate only to lowest order in λ, we can approximate the scatterer's gravitational field by the linearized metric for the exterior of a spherical body endowed with angular momentum:

$$
\begin{aligned}
g_{00} &= -\left(1 - \frac{2M}{r}\right), \\
g_{0j} &= g_{j0} = -\frac{2M}{r^3}(\vec{a} \times \vec{r})_j\,, \\
g_{jk} &= \left(1 + \frac{2M}{r}\right)\delta_{jk}\,.
\end{aligned}
\tag{3.1}
$$

Here M is the mass of the body and $M\vec{a} = \vec{J}$ is its angular momentum. The Fourier transforms of the $\bar{h}_{\alpha\beta}$ are given by

$$
\begin{aligned}
\bar{e}_{00} &= \frac{\lambda M}{q^2}\,, \\
\bar{e}_{0j} &= \bar{e}_{j0} = \frac{i\lambda M}{2q^2}(\vec{a} \times \vec{q})_j\,, \\
\bar{e}_{jk} &= 0\,,
\end{aligned}
\tag{3.2}
$$

where $\vec{q}$ is the (pure spacelike) momentum transfer $\vec{q} = {}^2\vec{k} - {}^1\vec{k}$ ($q^0 = 0$). Permitting the angular momentum per unit mass $\vec{a}$ to vanish in (3.1) or (3.2), we recover the linearized Schwarzschild geometry. Using Eqs. (2.10), (2.13), (2.14), and (3.2), the differential scattering cross section becomes

$$
\frac{d\sigma}{d\Omega} = \frac{M^2}{\sin^4(\frac{1}{2}\theta)} \{[1 - 2u \sin^2(\tfrac{1}{2}\theta)]^2 + \omega^2[\hat{a} \cdot ({}^1\hat{k} \times {}^2\hat{k})]^2\}\,. \tag{3.3}
$$

In the above ω is the angular frequency of the scalar wave, ${}^1\hat{k}$ and ${}^2\hat{k}$ are unit 3-vectors along the propagation directions of the incident and scattered fields, respectively, and θ is the angle between ${}^1\hat{k}$ and ${}^2\hat{k}$. Allowing $\vec{a}$ to vanish (linearized Schwarzschild geometry) one recovers the result previously obtained by Peters[13]:

$$
\left(\frac{d\sigma}{d\Omega}\right)_{\rm Schw} = \frac{M^2}{\sin^4(\frac{1}{2}\theta)}[1 - 2u \sin^2(\tfrac{1}{2}\theta)]^2\,. \tag{3.4}
$$

Owing to the r^{-1} dependence of the Newtonian potential, for the case of minimal coupling ($u = 0$), Eq. (3.5) reduces to the usual $1/\sin^4(\frac{1}{2}\theta)$ Rutherford-type cross section. For nonminimal coupling ($u \neq 0$), the cross section still exhibits the Rutherford-type angular dependence, but only for $\theta \ll 1$. This is not surprising, since it is the scalar curvature R which gives rise to u-dependent terms in the cross section. Considering that R is nonzero only along the world line of the scatterer, we see that for large impact parameters (i.e., small scattering angles) the scalar curvature cannot significantly contribute to the differential cross section. One may rewrite the scattering cross section for rotating bodies (3.3) in the suggestive form

$$\frac{d\sigma}{d\Omega}=\left(\frac{d\sigma}{d\Omega}\right)_{\rm Schw}+\frac{M^2a^2\omega^2}{\sin^4(\frac{1}{2}\theta)}\sin^2\alpha\sin^2\theta\sin^2\varphi \qquad (3.5)$$

with α, θ, and φ as shown in Fig. 2.

Equation (3.5) shows that the effect of angular momentum is to add a positive-semidefinite term to $(d\sigma/d\Omega)_{\rm Schw}$. For small scattering angles this angular momentum term is negligible with respect to $(d\sigma/d\Omega)_{\rm Schw}$. This can be easily understood by noticing that for large impact parameters the r^{-1} dependence of the Newtonian potential $\bar{h}_{00}$ dominates the r^{-2} dependence of the magnetic-type gravitational field $\bar{h}_{0i}$, which is the source of the angular momentum term. Another interesting feature of (3.5) is that the scattering in the backward direction is finite and independent of the angular momentum $\vec{a}$:

$$\left.\frac{d\sigma}{d\Omega}\right|_{\theta=\pi}=M^2(1-2u)^2\,. \qquad (3.6)$$

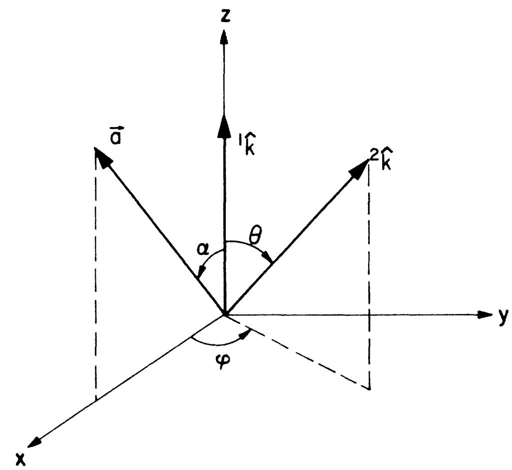

FIG. 2. The spatial orientation of the angular momentum $\vec{a}$ and the scattered direction $^2\hat{k}$ relative to the incident direction $^1\hat{k}$.

IV. ELECTROMAGNETIC WAVES

Theoretically more interesting and of possible observational importance is the gravitational scattering of electromagnetic waves. We choose the polarizations of the photons to be purely spacelike $[{}^1\underline{\epsilon}=(0,{}^1\vec{\epsilon}),\ {}^2\underline{\epsilon}=(0,{}^2\vec{\epsilon})]$ and use Eqs. (2.13), (2.14), and (2.21). The result for the scattering of electromagnetic waves with initial polarization ${}^1\vec{\epsilon}$ into some polarization ${}^2\vec{\epsilon}$ is

$$\frac{d\sigma}{d\Omega}=\frac{M^2}{4\sin^4(\frac{1}{2}\theta)}\,|(1+\cos\theta)({}^1\vec{\epsilon}\cdot{}^2\vec{\epsilon}^*)-({}^1\hat{k}\cdot{}^2\vec{\epsilon}^*)({}^2\hat{k}\cdot{}^1\vec{\epsilon})$$
$$+i\omega\{2({}^1\hat{k}\times{}^2\hat{k})\cdot\vec{a}({}^1\vec{\epsilon}\cdot{}^2\vec{\epsilon}^*)+[({}^2\hat{k}-{}^1\hat{k})\times{}^2\vec{\epsilon}^*]\cdot\vec{a}({}^2\hat{k}\cdot{}^1\vec{\epsilon})+[({}^2\hat{k}-{}^1\hat{k})\times{}^1\vec{\epsilon}]\cdot\vec{a}({}^1\hat{k}\cdot{}^2\vec{\epsilon}^*)\}|^2\,. \qquad (4.1)$$

For linear polarizations (${}^1\vec{\epsilon}$ and ${}^2\vec{\epsilon}$ real) the contribution of the angular momentum $\vec{a}$ to the cross section (4.1) will be proportional to $a^2\omega^2$, whereas for circular polarizations (${}^1\vec{\epsilon}$ and ${}^2\vec{\epsilon}$ complex) the contribution will include an $a\omega$ term. We first consider circular polarizations (i.e., pure helicity states) and we choose for the photon basis states

$${}^1\vec{\epsilon}^{\,R}_{L}=\frac{1}{\sqrt{2}}(\hat{e}_x\pm i\hat{e}_y)\,,\quad {}^2\vec{\epsilon}^{\,R}_{L}=\frac{1}{\sqrt{2}}(\hat{e}_\theta\pm i\hat{e}_\varphi)\,, \qquad (4.2)$$

where $\hat{e}_x,\hat{e}_y,\hat{e}_\theta,\hat{e}_\varphi$ are unit vectors in the x, y, θ, and φ directions. After some algebraic manipulations (4.1) yields

$$\left(\frac{d\sigma}{d\Omega}\right)_{RL}=\left(\frac{d\sigma}{d\Omega}\right)_{LR}=0\,, \qquad (4.3)$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{RR,LL}=M^2\{[\cot^2(\tfrac{1}{2}\theta)\pm 2a\omega\cos\tfrac{1}{2}\theta(\cos\alpha\cos\tfrac{1}{2}\theta+\sin\alpha\sin\tfrac{1}{2}\theta\cos\varphi)]^2+4a^2\omega^2\sin^2\alpha\cot^2(\tfrac{1}{2}\theta)\sin^2\varphi\}\,, \qquad (4.4)$$

where the first (second) subscript denotes the initial (final) polarization and the upper (lower) sign in (4.4) refers to the RR (LL) case. For the linearized Schwarzschild geometry (4.4) reduces to recent results obtained by Peters[13]:

$$\left(\frac{d\sigma}{d\Omega}\right)^{\rm Schw}_{RR}=\left(\frac{d\sigma}{d\Omega}\right)^{\rm Schw}_{LL}=M^2\cot^4(\tfrac{1}{2}\theta)\,. \qquad (4.5)$$

In the circular polarization basis the scattering matrix is diagonal, which explicitly shows that helicity is conserved by the scattering process. This is not restricted only to our situation, but rather is a general property of electromagnetic wave propagation in *any* orientable spacetime manifold.[11,25] Moreover, for the Schwarzschild geometry the scattering cross section is helicity independent, whereas for a rotating scatterer it is helicity dependent. This results in a differential gravitational deflection of right and left circularly polarized electromagnetic radiation by a rotating object. For a given impact parameter b of the incident beam, we define the angular splitting as

$$\delta = (\text{angle by which } R \text{ helicity photon is scattered minus angle by which } L \text{ helicity photon is scattered}). \tag{4.6}$$

We then solve the inverse scattering problem[26] and find, to lowest order in $a\omega$,

$$\delta = 2a\omega \cos\alpha (4M/b)^3[\ln(b/2M) - \tfrac{3}{4}] . \tag{4.7}$$

To obtain this result we have used the constraint that

$$\delta \ll 4M/b \ll 1 . \tag{4.7'}$$

It must be stressed that so far we have only discussed pure helicity states. For any linearly polarized or unpolarized incident wave the scattering cross section summed over final polarization states becomes

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{tot}} = M^2\{\cot^4(\tfrac{1}{2}\theta) + 4a^2\omega^2[\cos^2(\tfrac{1}{2}\theta)(\cos\alpha\cos\tfrac{1}{2}\theta + \sin\alpha\sin\tfrac{1}{2}\theta\cos\varphi)^2 + (\sin\alpha\cot\tfrac{1}{2}\theta\sin\varphi)^2]\}, \tag{4.8}$$

which for the Schwarzschild case reduces to (4.5). We therefore conclude that all linearly polarized incident beams are deflected through the same angle. However, since the diagonal elements of the scattering matrix in the circular-polarization basis are unequal, linearly polarized incident waves become elliptically polarized. For an unpolarized wave packet, on the other hand, the paths of different-helicity photons are split by an amount given by (4.7). In addition, the angular momentum $\vec{a}$ induces a partial polarization of the scattered waves. We define the amount of this polarization by

$$p = \left| \frac{\left(\frac{d\sigma}{d\Omega}\right)_{RR} - \left(\frac{d\sigma}{d\Omega}\right)_{LL}}{\left(\frac{d\sigma}{d\Omega}\right)_{RR} + \left(\frac{d\sigma}{d\Omega}\right)_{LL}} \right| , \tag{4.9}$$

and we find, to lowest order in a,

$$p = 4a\omega\,|\cos\alpha\cos\tfrac{1}{2}\theta + \sin\alpha\sin\tfrac{1}{2}\theta\cos\varphi| \times \sin\tfrac{1}{2}\theta\tan\tfrac{1}{2}\theta . \tag{4.10}$$

In concluding this section we note that, independent of $\vec{a}$ or the initial polarization, the cross section for scattering in the backward direction vanishes.

V. GRAVITATIONAL WAVES

Using (2.13), (2.14), and (2.28) we compute the differential cross section for the scattering of gravitational waves from an initial polarization ${}^1\vec{\epsilon}$ into some final polarization ${}^2\vec{\epsilon}$:

$$\frac{d\sigma}{d\Omega} = \frac{M^2}{\sin^4(\frac{1}{2}\theta)}\{\cos^2\theta + \omega^2[({}^1\hat{k}\times{}^2\hat{k})\cdot\vec{a}]^2\}\,|{}^1\vec{\epsilon} : {}^2\vec{\epsilon}^*|^2 . \tag{5.1}$$

This result was derived in the transverse-traceless (TT) gauge.[2] Although the transition amplitude (2.28) is not gauge invariant by itself, (5.1) yields reliable results for small momentum transfers, i.e., for small scattering angles. By analogy with the photon case, we choose for the graviton-basis states the circular polarizations given by

$$\begin{aligned} {}^1\vec{\epsilon}^R_L &= \tfrac{1}{2}[\hat{e}_x\hat{e}_x - \hat{e}_y\hat{e}_y \pm i(\hat{e}_x\hat{e}_y + \hat{e}_y\hat{e}_x)] , \\ {}^2\vec{\epsilon}^R_L &= \tfrac{1}{2}[\hat{e}_\theta\hat{e}_\theta - \hat{e}_\varphi\hat{e}_\varphi \pm i(\hat{e}_\theta\hat{e}_\varphi + \hat{e}_\varphi\hat{e}_\theta)] . \end{aligned} \tag{5.2}$$

Substitution of the initial and final states into (5.1) yields

$$\begin{aligned} \left(\frac{d\sigma}{d\Omega}\right)_{RL} = \left(\frac{d\sigma}{d\Omega}\right)_{LR} &= \frac{M^2}{16\sin^4(\frac{1}{2}\theta)}(\cos^2\theta + a^2\omega^2\sin^2\alpha\sin^2\theta\sin^2\varphi) \\ &\times (1-\cos\theta)^4 \end{aligned} \tag{5.3a}$$

$$\begin{aligned} \left(\frac{d\sigma}{d\Omega}\right)_{RR} = \left(\frac{d\sigma}{d\Omega}\right)_{LL} &= \frac{M^2}{16\sin^4(\frac{1}{2}\theta)}(\cos^2\theta + a^2\omega^2\sin^2\alpha\sin^2\theta\sin^2\varphi) \\ &\times (1+\cos\theta)^4 . \end{aligned} \tag{5.3b}$$

The nonvanishing of (5.3a) clearly illustrates that here, unlike the electromagnetic case, helicity is *not* conserved. Moreover, there is neither different scattering of opposite helicity states [see (5.3)] nor partial polarization of unpolarized incident gravitational radiation. The latter is easily seen by noting that the scattering cross section for *either* helicity state is given by [adding (5.3a) and (5.3b)]

$$\left(\frac{d\sigma}{d\Omega}\right)_R = \left(\frac{d\sigma}{d\Omega}\right)_L = \frac{M^2}{\sin^4(\frac{1}{2}\theta)}\left(\cos^2\theta + a^2\omega^2 \sin^2\alpha \sin^2\theta \sin^2\varphi\right) \times \left(\cos^2\theta + \tfrac{1}{8}\sin^4\theta\right). \tag{5.4}$$

Similarly, for the scattering of orthogonal linear polarizations denoted by

$$\begin{aligned} {}^1\overline{\epsilon}_+ &= \frac{1}{\sqrt{2}}(\hat{e}_x\hat{e}_x - \hat{e}_y\hat{e}_y), \\ {}^1\overline{\epsilon}_\times &= \frac{1}{\sqrt{2}}(\hat{e}_x\hat{e}_y + \hat{e}_y\hat{e}_x), \end{aligned} \tag{5.5}$$

one finds, after summing over the final polarizations and use of (5.1),

$$\left(\frac{d\sigma}{d\Omega}\right)_+ = \frac{M^2}{\sin^4(\frac{1}{2}\theta)}\left(\cos^2\theta + a^2\omega^2 \sin^2\alpha \sin^2\theta \sin^2\varphi\right) \times \left(\cos^2\theta + \tfrac{1}{4}\sin^4\theta \cos^2 2\varphi\right), \tag{5.6a}$$

$$\left(\frac{d\sigma}{d\Omega}\right)_\times = \frac{M^2}{\sin^4(\frac{1}{2}\theta)}\left(\cos^2\theta + a^2\omega^2 \sin^2\alpha \sin^2\theta \sin^2\varphi\right) \times \left(\cos^2\theta + \tfrac{1}{4}\sin^4\theta \sin^2 2\varphi\right). \tag{5.6b}$$

For unpolarized incident gravitational waves [i.e., averaging over φ in (5.6a) and (5.6b) and summing], the differential scattering cross section is given by (5.4). Allowing $\vec{a}\to 0$, we recover Peters's results apart from a factor of $\cos^2\theta$:

$$\left(\frac{d\sigma}{d\Omega}\right)^{\text{Schw}}_{\text{this paper}} = \cos^2\theta \left(\frac{d\sigma}{d\Omega}\right)^{\text{Schw}}_{\text{Peters}}. \tag{5.7}$$

For small-angle scattering there is good agreement. One may recover Peters's result exactly by calculating the scattering of gravitational waves off a massive spin-0 meson. Inclusion of all the relevant Feynman diagrams then leads to a gauge-invariant transition amplitude. Actually, for the choice of the TT gauge only the graviton-pole and seagull diagrams survive, and one obtains Peters's results exactly, i.e.,

$$\left(\frac{d\sigma}{d\Omega}\right)^{\text{Schw}}_{{}^1\overline{\epsilon}\to{}^2\overline{\epsilon}} = \frac{M^2}{\sin^4(\frac{1}{2}\theta)}\left|{}^1\overline{\epsilon} : {}^2\overline{\epsilon}^*\right|^2. \tag{5.8}$$

As a concluding remark we note that, independent of the polarization of the incident gravitational wave and the angular momentum $\vec{a}$, the cross section for backscatter is nonzero. Whereas the exact dependence of $(d\sigma/d\Omega)_{\theta=\pi}$ on the angular momentum $\vec{a}$ cannot be inferred from the cross sections derived above (they are valid only for *small* scattering angles), one finds from (5.8) that the gravitational backscatter in a linearized Schwarzschild geometry is given by

$$\left(\frac{d\sigma}{d\Omega}\right)^{\text{Schw}}_{\theta=\pi} = M^2. \tag{5.9}$$

In addition, if the incident radiation is in a pure helicity state, the backscattered radiation must have the opposite helicity.

VI. SUMMARY AND CONCLUSIONS

The differential cross sections for the weak-field gravitational scattering of long-wavelength scalar, electromagnetic, and gravitational waves have been calculated using Feynman perturbation methods.

For the linearized Schwarzschild geometry, we have recovered the results obtained by Peters,[13] although he used a Green's-function formalism. In particular, for electromagnetic waves helicity is conserved, whereas for gravitational waves it is not. Endowing the scatterer with an angular momentum $\vec{a}$ leads to helicity-dependent effects in electromagnetic wave scattering. Although the photon helicity is still conserved, the coupling between this helicity and the angular momentum of the scatterer results in (a) different scattering of right and left circularly polarized photons and (b) partial polarization of unpolarized incident electromagnetic radiation. The high-frequency limits of these effects have been discussed before by Mashhoon.[5,11] Whereas in the high-frequency limit ($\omega M \gg 1$), the angular split δ [defined by (4.6)], and polarization p [defined by (4.9)] are proportional to $a\omega^{-1}$, in the low-frequency limit ($\omega M \ll 1$) they are proportional to $a\omega$. This confirms the belief that the magnetic-type gravitational field of a rotating body distinquishes between the helicity states of a photon only in the diffraction limit, i.e., when the wavelength of the incident photon is of the same order as the Schwarzschild radius of the scatterer.

Gravitational waves do not exhibit any of these angular-momentum–induced effects.

As a final comment, we note that this method may easily be applied to the gravitational scattering of noninteger spin or massive fields.

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