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Strong-interaction effects on the $K_L \to 2\mu$ decay and K_L-K_S mass difference. A reply.

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The strong-interaction corrections to the matrix element of the $K_L \to 2\mu$ decay and the K_L-K_S mass difference arising from gluon exchanges at short distances are considered. We discuss the controversy existing in the literature on the problem and try to elucidate our approach proposed earlier.

I. INTRODUCTION

The processes of the second order in weak interactions, namely, $K_L \to 2\mu$ decay and the K_L-K_S mass difference, have been considered within the Weinberg-Salam model in a number of papers.[1-9] In Refs. 1–5, 9 the calculations were performed within the free-quark approximation, while in Refs. 7–9 the gluon corrections to the matrix element are included.

There existed a controversy over the results obtained in various papers which has been only partially settled at the present time. According to Ref. 1 the $K_L \to 2\mu$ branching ratio in the free-quark approximation is equal to

$$\frac{\Gamma(K_L \to 2\mu)}{\Gamma(K^+ \to \mu\nu)} = \frac{G^2 m_c^4 \cos^2\theta_C}{2\pi^4} , \qquad (1)$$

where θ_C is the Cabibbo angle and m_c is the mass of the charmed quark. On the other hand, according to Refs. 2, 6 the branching ratio (1) is just equal to zero in the approximation considered. Because of this difference the problem has been reanalyzed in Refs. 3–5 and the final result obtained by these authors coincides with Eq. (1). Moreover, the most recent calculation by the authors of Ref. 2 also agrees[9] with Eq. (1), so that this matter seems to be settled. However, the controversy over the calculation of the gluon corrections still exists.

The calculation of the effects of the strong interactions on the processes under discussion was tried first in Ref. 7. The problem has been reconsidered in our previous paper,[8] in which we obtained results which differ rather radically from those of Ref. 7 and indicated our understanding of the origin of the difference. More recently, there appeared a paper[9] in which new results for the same effects are obtained and which accounts partially for the critical remarks of our paper.[8] However, some essential points of our treatment are not yet included, so that, to the best of our understanding, the final result of Ref. 9 is not correct. There is no direct critique of our approach in Ref. 9 although it contains an indirect argument which, according to the opinion of the authors of Ref. 9, shows that our results are not self-consistent.

We feel that it might be helpful under the circumstances to present a more detailed derivation of our results and to try to elucidate the reasons for the differences. We will consider also the indirect argument mentioned above and present a critique of the assertions of the authors of Ref. 9.

It is worth noting that the primary motivation for the consideration of the processes of the second order in weak interactions was the derivation of upper bounds on the mass of the charmed quark. In particular, it was claimed in Ref. 7 that the strong-interaction corrections affect these bounds essentially. However, according to our calculations the account of strong interactions is not important numerically.[8] The same conclusion is reached in the most recent publication[9] despite the remaining difference in the results obtained. Thus, the whole problem seems to be of academic rather than practical interest. Still, we believe that it is worth having a correct theoretical formula for the gluon corrections to the matrix elements considered, no matter how important numerically these corrections are.

The organization of the paper is as follows. In Sec. II-V we consider the amplitude of the $K_L \to 2\mu$ decay. In particular, Sec. II contains a discussion of the gluon corrections to the box graph (see Figs. 1, 2), or, more generally, to the T product of two hadronic currents. In Sec. III the gluon corrections to the T product of three currents or to the triangle graph (see Fig. 4) are evaluated. In Sec. IV we will present the final result for the matrix element of the $K_L \to 2\mu$ decay taking into account the strong interactions at short distances. In Sec. V we will discuss the problem raised by Gaillard *et al.*[9] namely, the independence of the results obtained on the choice of the gauge of the W boson field and strong-interaction effects on the gauge-dependent terms. Finally, in Sec. VI we will discuss briefly the case of the K_L-

K_S mass difference. Although this matrix element is not discussed in Ref. 9, we will present the results of the corresponding calculation for the sake of completeness.

II. $K_L \to 2\mu$ DECAY. THE BOX GRAPH

In this section we will consider in detail the calculation of the box graph represented in Fig. 1 taking into account the gluon exchanges (see Fig. 2). This is the simplest calculation in the series, and for this reason the difference between various papers is not so drastic as in the other cases. However, the difference does exist, and we will elucidate the reasons for this. It is worth noting that there are some common points in all the derivations and the difference appears only at the last step of the derivation.

The matrix element associated with the graph of Fig. 1(a) is of the form

$$M_{\Box} = -\frac{1}{4\pi^2} G^2 \sin\theta_C \cos\theta_C \bar{\lambda}\gamma_\rho\gamma_5 \mathfrak{N}\, \bar{\mu}\gamma_\rho\gamma_5\mu I \ , \quad (2)$$

where

$$I = \tfrac{1}{2} m_c^2 \left[m_W^4 \int \frac{dp^2}{(p^2+m_W^2)^2} \frac{1}{(p^2+m_c^2)} \right] ,$$

and m_W is the W-boson mass, m_c is the mass of the charmed quark, and we have chosen the Feynman gauge for the W-boson propagator.

The calculation of the type (2) we will call the free-quark approximation since the gluon corrections are not taken into account. The calculation can be easily performed to the very end, and the result is as follows:

$$I = \tfrac{1}{2} m_c^2 \left(\ln \frac{m_W^2}{m_c^2} - 1 \right) . \quad (3)$$

Let us proceed now with the calculation of the effects of strong interactions at short distances. Hereafter, we will assume the validity of the "standard" model according to which the strong interactions are mediated by exchanges of an octet of color gluons coupled to the color degrees of freedom of the four quarks $\mathcal{P}, \mathfrak{N}, \lambda, c$.

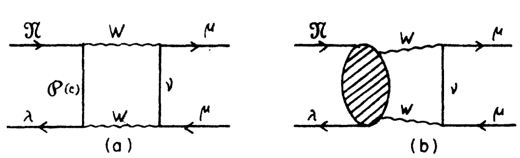

FIG. 1. The box graph for the transition $\bar{\lambda}\mathfrak{N}\to\mu^+\mu^-$ ($K_L \to 2\mu$ decay). (a) The bare-quark approximation, (b) gluon corrections included; here the blob stands for the amplitude of the transition with the strong interactions taken into account.

Because of the gluon exchanges there arises an additional dependence on p^2 of the amplitude of the Compton scattering of the W boson with momentum p on the quark, $W\mathfrak{N} \to W\lambda$ (see Fig. 2). We absorb this dependence into the definition of the factor $F(p^2)$ so that the matrix element (2) is replaced now by

$$I \to \tfrac{1}{2} m_c^2 m_W^4 \int \frac{dp^2}{(p^2+m_W^2)^2} \frac{F(p^2)}{(p^2+m_c^2)}$$
$$\approx \tfrac{1}{2} m_c^2 \int_{m_c^2}^{m_W^2} \frac{dp^2}{p^2} F(p^2) , \quad (4)$$

where the latter equality is valid only in the logarithmic approximation, to which we will confine ourselves in this section.

The factor $F(p^2)$ introduced above can be readily calculated in the leading logarithmic approximation in which all the terms $(g^2 \ln p^2)^n$ are summed up and the terms of the order g^2 are neglected, g being the coupling constant of the quark-gluon interaction. As is well known, the use of the renormalization-group technique allows the summation to be performed once the calculation in the lowest order in g^2 is explicitly made.

In the lowest order in g^2 there exist several graphs contributing to the process under discussion, and these graphs are represented in Fig. 2. If the Landau gauge is used for the gluon field, then only the graph of Fig. 2(c) gives the logarithmic factor and the effect of the strong interactions reduces to the mass renormalization. Thus, in the lowest order in g^2 we have for $p^2 \gg m_c^2$:

$$F(p^2) = \left(1 - 8 \frac{g^2}{16\pi^2} \ln \frac{p^2}{m_c^2} \right) . \quad (5)$$

If the summation of all the leading log terms is performed, then

$$F(p^2) = \left(1 + \frac{25}{3} \frac{g^2(m_c^2)}{16\pi^2} \ln \frac{p^2}{m_c^2} \right)^{-24/25} . \quad (6)$$

This result can be obtained either by a direct summation of the graphs or by using the renormalization-group technique.

It is worth noting that all the authors agree that in the case considered the effect of strong interactions reduces to the corrections to the c-quark

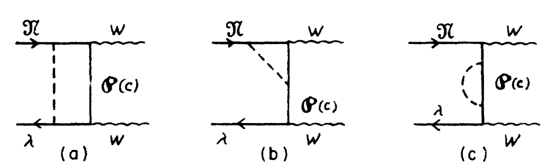

FIG. 2. The graphs describing the lowest-order gluon corrections to the amplitude of the $\bar{\lambda}\mathfrak{N}\to W^+W^-$ transition.

effective mass. However, the authors of Refs. 7, 9 use in Eq. (6) $g^2(m^2)\ln(p^2/m^2)$ instead of $g^2(m_c^2)\times\ln(p^2/m_c^2)$, where m is some characteristic hadronic mass, and we will return to the discussion of this point later. Moreover, in our paper[8] we substituted $F(p^2)$ into Eq. (4) and performed the integration over p^2 explicitly to find

$$\int_{m_c^2}^{m_W^2} F(p^2)\,\frac{dp^2}{p^2} = \frac{1}{1-\frac{24}{25}}\left(\ln\frac{m_W^2}{m_c^2}F(m_W^2)+\frac{F(m_W^2)-1}{\frac{24}{25}g^2(m_c^2)/16\pi^2}\right), \tag{7}$$

while the authors of Refs. 7, 9 assume that it is sufficient to multiply Eq. (3) by $F(m_W^2)$ to get the final answer.

In other words, the difference can be summarized as follows. We calculate the integrals of the type

$$\int_{m^2}^{m_W^2}\frac{dp^2}{p^2}\left(\ln\frac{p^2}{m^2}\right)^{\alpha} = \frac{1}{\alpha+1}\left(\ln\frac{m_W^2}{m^2}\right)^{\alpha+1}$$

explicitly, while the authors of Refs. 7, 9 replace the true integration by the following procedure:

$$\int_{m^2}^{m_W^2}\frac{dp^2}{p^2}\left(\ln\frac{p^2}{m^2}\right)^{\alpha} \to \left(\ln\frac{m_W^2}{m^2}\right)^{\alpha}\int_{m^2}^{m_W^2}\frac{dp^2}{p^2},$$

which is certainly not correct since the integrals of the type considered cannot be evaluated by substituting for the integrand its value on the upper limit of integration.

In the case considered the difference reduces to the numerical factor and not to a power of the logarithmic factor. Thus, in this case the result of Ref. 7, 9 can be used for the sake of an estimate, but not as an exact answer (as it is assumed to be) since all the multiplicative numerical factors must be kept in the case considered. In the case of the triangle graph (see the next section) the same neglect of the explicit integration leads to the result which differs from the correct one by the powers of the logarithmic factor as well.

Let us return now to the discussion of the other point of difference mentioned above. We would like to insist that in the logarithmic approximation considered here the lower limit of integration in Eq. (7) is m_c^2 and not m^2 as is assumed in Refs. 7, 9. Moreover, for $p^2 \lesssim m_c^2$ there is no logarithmic factor in the gluon corrections to the mass operator, as is readily seen from the consideration of the graphs in the perturbation theory [see, e.g., Eq. (5)]. If m_c is not much larger than m then the difference is not so important numerically, but here we have aimed at deriving a correct theoretical formula and for this reason we would like to emphasize this difference as well.

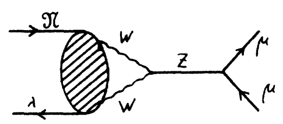

FIG. 3. The triangle graph corresponding to the T product of two hadronic currents.

In conclusion of this section let us notice that all the calculations above are performed in momentum space, while the calculations of Refs. 7, 9 are made in coordinate space. This makes in fact no difference, and in the next section we will try the coordinate space to make the process of comparing various papers more straightforward.

III. $K_L \to 2\mu$ DECAY. THE TRIANGLE GRAPH

In the previous section we considered the strong-interaction corrections to the matrix element of T product of two weak currents (Figs. 1 and 3). In what follows we will find the gluon corrections to the triangle graph of Fig. 4 which corresponds to the matrix element from the T product of three hadronic currents. Therefore, there exist now two independent distances instead of one distance in the case of the box graph. To be more precise, let us notice that the Ward identity for the $Z\lambda\mathfrak{N}$ vertex allows us to reduce the calculation of the matrix element of the $K_L\to 2\mu$ decay to that of the T product of two currents and the matrix element of the T product of two currents and pseudoscalar density $\bar{c}\gamma_5 c$. The former case was discussed in detail in the previous section, while the latter case is dealt with below.

Let us denote the appropriate T product of the three operators by $T_{\mu\nu}(x,y)$:

$$T_{\mu\nu}(x,y) = m_c T\{j_\mu^+(x)j_\nu^-(y)\bar{c}(0)\gamma_5 c(0)\}, \tag{8a}$$

where $j^\pm$ are the sources of the W-boson field. The part of the $Z\lambda\mathfrak{N}$ vertex contributing to the $K_L\to 2\mu$ decay is then given by the following expression:

$$\Gamma_\lambda \sim \langle 0\,|\int d^4x\,d^4y\,D_{\mu\nu}^W(x-y)T_{\mu\nu}(x,y)(x+y)_\lambda\,|\mathfrak{N}\bar{\lambda}\rangle, \tag{8b}$$

where $D_{\mu\nu}^W$ is the Green's function of the W boson.

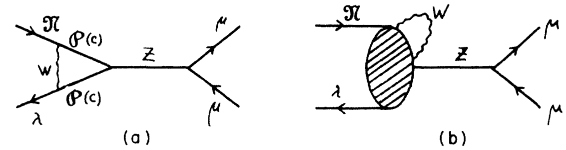

FIG. 4. The triangle graphs corresponding to the T product of three hadronic currents: (a) bare-quark approximation, (b) gluon exchanges included.

The authors of Refs. 7, 9 assumed that the distances x, y of the order m_W^{-1} are essential for the calculation of the $Z\lambda\mathfrak{N}$ vertex. Our point is that the whole region of x, y

$$m_W^{-1} \leq x, y \leq m_c^{-1}, \quad x - y \sim m_W^{-1}$$

contributes to the matrix element considered. This observation changes the result completely.

To substantiate the point let us consider first the free-quark case. The integral describing the $Z\lambda\mathfrak{N}$ vertex is then of the form

$$\Gamma_\lambda \sim m_c \int d^4x d^4y (x+y)_\lambda \bar{\lambda}[\gamma_\mu \tfrac{1}{2}(1+\gamma_5) S(y, m_c) \gamma_5 S(-x, m_c) \gamma_\mu \tfrac{1}{2}(1+\gamma_5)] \mathfrak{N} D(x-y, m_W^2)$$
$$= m_c^2 \int_{x, y < m_c^{-1}} d^4x d^4y (x+y)_\lambda \left(\frac{\partial}{\partial x_\rho} + \frac{\partial}{\partial y_\rho}\right) \frac{1}{x^2 y^2} \bar{\lambda}_L \gamma_\rho \mathfrak{N}_L D(x-y, m_W^2), \tag{9}$$

where $S(x, m)$ is the Green's function of the fermion field with mass m, and we use the Feynman gauge for the W-boson propagator $D^W_{\mu\nu}(x-y)$ so that

$$D^W_{\mu\nu}(x-y) = g_{\mu\nu} D(x-y, m_W^2).$$

It is convenient to introduce the variables ρ, z so that $(x-y) = \rho$ and $(x+y)/2 = z$. Then we get

$$\Gamma_\lambda \sim m_c^2 \int_{z < m_c^{-1}} d^4z d^4\rho D(\rho, m_W^2)(z + \tfrac{1}{2}\rho)^{-2}(z - \tfrac{1}{2}\rho)^{-2} \times \bar{\lambda}_L \gamma_\lambda \mathfrak{N}_L. \tag{10}$$

The function $D(\rho, m_W^2)$ falls off exponentially at $\rho > m_W^{-1}$ and the integration over ρ in Eq. (10) just reduces to the substitution of ρ by m_W^{-1}. In this way we get in the logarithmic approximation

$$\Gamma_\lambda \sim \frac{m_c^2}{m_W^2} \lambda \gamma_\lambda \frac{1+\gamma_5}{2} \mathfrak{N} \int_{z > m_W^{-1}}^{z < m_c^{-1}} d^4z/z^4.$$

From this expression we see that the whole region of $m_W^{-1} < z < m_c^{-1}$ is essential. Thus far we have considered free quarks. However, the gluon corrections are logarithmic in nature and, therefore, they do not change the convergence properties of the integrals and the characteristic values of z.

After this preliminary exposition of the differences in various approaches let us describe in more detail our procedure of calculating the gluon corrections. To this end let us return to the consideration of the T product of three operators [see Eq. (8)]. As the first step in evaluating the gluon corrections we use the Wilson expansion for two currents

$$T\{j_\mu^+(x), j_\mu^-(y)\} \xrightarrow[x \to y]{} \beta_+(\rho) O_+(z) + \beta_-(\rho) O_-(z), \tag{11}$$

where

$$O_\pm = \tfrac{1}{2}(\bar{\lambda}_L \gamma_\mu \mathcal{P}_L \bar{\mathcal{P}}_L \gamma_\mu \mathfrak{N}_L \pm \bar{\lambda}_L \gamma_\mu \mathfrak{N}_L \bar{\mathcal{P}}_L \gamma_\mu \mathcal{P}_L) - (\mathcal{P} \to c) \tag{12}$$

and

$$\beta_\pm(\rho) = [g^2(m^2)/g^2(1/\rho^2)]^{\alpha_\pm}, \tag{13}$$

where $g^2(1/x^2)$ is the effective quark-gluon coupling constant squared, and in the model considered g^2 is equal to

$$g^2(1/x^2) = \frac{g^2(m^2)}{1 + \frac{25}{3}[g^2(m^2)/16\pi^2] \ln(1/m^2x^2)}, \tag{14}$$

and $\alpha_+ = -\frac{6}{25}$, $\alpha_- = \frac{12}{25}$. Mass scale m stands in Eq. (14) for the normalization point, so that

$$g^2(m^2)/4\pi = 1.$$

Let us also notice that Eqs. (12) and (13) were first obtained in fact in Refs. 10 and correspond to the calculation of the effective four-fermion Hamiltonian of weak interactions provided that $1/\rho^2 = m_W^2$.

To find the z dependence of matrix element (8a) we use the Wilson expansion of the product of the operators $O_\pm$ and $\bar{c}(0)\gamma_5 c(0)$. This expansion looks as follows

$$m_c O_\pm(z) \bar{c}(0) \gamma_5 c(0) \xrightarrow[z \to 0]{} m_c^2 A^\pm(z) \frac{z_6}{z^6} J_6(0),$$
$$J_6(0) = \bar{\lambda}_L \gamma_6 \mathfrak{N}_L. \tag{15}$$

The z dependence of the coefficients $A^\pm(z)$ introduced in this way is determined by the anomalous dimensions of the operators appearing in Eq. (15), namely, operators $O_\pm$, c-quark mass, and current J_6. The anomalous dimension of the current is equal to zero, while the anomalous dimensions of the other operators can be easily read off from Eqs. (13) and (6). Combining all the factors we get for $A^\pm(z)$

$$A^\pm(z) = a^\pm \left[\frac{g^2(m^2)}{g^2(1/z^2)}\right]^{-\alpha_\pm} \left[\frac{g^2(m_c^2)}{g^2(1/z^2)}\right]^{2\delta_m}, \tag{16}$$

where

$$\delta_m = -\tfrac{12}{25},$$

and the coefficients $a^\pm$ can be found from an explicit calculation in the free-quark limit since we require that for the strong interactions switched off the result coincides with the bare-quark calculation.

Combining Eqs. (11)–(13) and Eqs. (15) and (16) we get

$$T_{\mu\mu}(x,y)=\left[\frac{g^2(m_c^2)}{g^2(1/z^2)}\right]^{2\delta_m} \times\left[a^+\left(\frac{g^2(1/z^2)}{g^2(1/\rho^2)}\right)^{\alpha_+}+a^-\left(\frac{g^2(1/z^2)}{g^2(1/\rho^2)}\right)^{\alpha_-}\right] \times(z_6/z^6)\bar{\lambda}_L(0)\gamma_6\mathfrak{N}_L(0)\,. \qquad (17)$$

It is worth noting that if we put $x\sim y\sim\rho$ in this equation, then we reproduce the results of Refs. 7, 9. However, as is explained above, the distances $z\gg\rho$ are also essential and Eq. (17) differs from the corresponding results of Refs. 7, 9.

To get the final answer for the triangle graph we substitute Eq. (17) into Eq. (8b) and come in this way to the integrals of the type

$$\frac{m_c^2}{m_W^2}J_6(0)\int_{m_W^{-1}}^{m_c^{-1}}\frac{d^4z}{z^4}\left[\frac{g^2(m_c^2)}{g^2(1/z^2)}\right]^{2\delta_m}\left[\frac{g^2(1/z^2)}{g^2(m_W^2)}\right]^{\alpha_\pm}, \qquad (18)$$

which can be easily evaluated to be

$$\frac{m_c^2}{m_W^2}\pi^2J_6(0)\frac{1}{\frac{25}{3}g^2(m_c^2)/16\pi^2}\frac{1}{2\delta_m-\alpha_\pm+1} \times\left\{\left[\frac{g^2(m_c^2)}{g^2(m_W^2)}\right]^{2\delta_m+1}-\left[\frac{g^2(m_c^2)}{g^2(m_W^2)}\right]^{\alpha_\pm}\right\}. \qquad (19)$$

Thus, the difference between our calculations and those of Refs. 7, 9 in the case of the triangle graph can be summarized as follows. We evaluate the integrals of the form

$$\int_{m_W^{-1}}^{m_c^{-1}}\frac{dx}{x}\left(\frac{1-C\ln m_cx}{1-C\ln(m_c/m_W)}\right)^{-\alpha_\pm}(1-C\ln m_cx)^{2\delta_m}=\frac{1}{C}\frac{1}{2\delta_m-\alpha_\pm+1}\{[1+C\ln(m_W/m_c)]^{2\delta_m+1}-[1+C\ln(m_W/m_c)]^{\alpha_\pm}\} \qquad (20)$$

explicitly, while the authors of Refs. 7, 9 use a certain procedure which amounts in fact to an assumption that all the integrals can be found by replacing the integrand by its value on the lower limit of integration. In the case considered this assumption corresponds to the replacement

$$\int_{m_W^{-1}}^{m_c^{-1}}\frac{dx}{x}\left(\frac{1-C\ln m_cx}{1-C\ln(m_c/m_W)}\right)^{-\alpha_\pm}(1-C\ln m_cx)^{2\delta_m} \to[1+C\ln(m_W/m_c)]^{2\delta_m}\int_{m_W^{-1}}^{m_c^{-1}}dx/x\,,$$

which differs from Eq. (20) both in numerical coefficient and in the power of the logarithmic factor.

IV. $K_L\to 2\mu$ DECAY. RESULTS

In the two previous sections we have exemplified all the principal problems which are encountered in the course of calculation of the gluon corrections to the matrix element of the $K_L\to 2\mu$ decay. For the sake of complete[illegible] we will present in this section the final result for the gluon corrections to the amplitude of the $K_L\to 2\mu$ decay derived first in our paper.[8]

As shown in Refs. 1–5, 9 the amplitude considered, $M(K_L\to 2\mu)$, is determined by the graphs of the three types represented in Figs. 1, 3, and 4. Using the Ward identities for the $Z\lambda\mathfrak{N}$ vertex the general expression for the amplitude of the $\lambda\mathfrak{N}\to\mu^+\mu^-$ transition can be transformed into the following form:

$$M=M_1+M_2+M_3\,,$$

$$M_1=8G^2i\int\frac{d^4k}{(2\pi)^4}\frac{m_W^4}{(k^2-m_W^2)^2}\bar{\mu}_L\gamma_\mu\frac{1}{\hat{k}}\gamma_\nu\mu_LM_{\mu\nu}\,, \qquad (21)$$

$$M_2=-4G^2i\bar{\mu}\gamma_\lambda\gamma_5\mu\times\int\frac{d^4k}{(2\pi)^4}\frac{m_W^2}{(k^2-m_W^2)^2}(k_\mu M_{\mu\lambda}+k_\nu M_{\lambda\nu})\,,$$

$$M_3=2G^2i\bar{\mu}\gamma_\lambda\gamma_5\mu\int\frac{d^4k}{(2\pi)^4}\frac{m_W^2}{k^2-m_W^2}\frac{\partial}{\partial q_\lambda}T_{\mu\mu}\,,$$

where

$$M_{\mu\nu}=i\int d^4xe^{-ik\cdot x}\langle 0|T\{j_\mu^+(x),j_\nu^-(0)\}|K_L\rangle\,,$$

$$T_{\mu\nu}=-\int d^4xd^4ye^{iq\cdot x-ik\cdot y}\times\langle 0|T\{m_c\bar{c}(x)\gamma_5c(x)j_\mu^+(y)j_\nu^-(0)\}|K_L\rangle\,, \qquad (22)$$

$$j_\mu^+=\bar{\mathcal{P}}_L\gamma_\mu(\mathfrak{N}_L\cos\theta_C+\lambda_L\sin\theta_C)+\bar{c}_L\gamma_\mu(-\mathfrak{N}_L\sin\theta_C+\lambda_L\cos\theta_C)\,.$$

In the free-quark approximation $M_{\mu\nu}$, $T_{\mu\nu}$ are equal to

$$M_{\mu\nu}^{(0)}=\sin\theta_C\cos\theta_C\times\left(\bar{\lambda}_L\gamma_\nu\frac{1}{\hat{k}-m_c}\gamma_\mu\mathfrak{N}_L-\bar{\lambda}_L\gamma_\nu\frac{1}{\hat{k}-m_{\mathcal{P}}}\gamma_\mu\mathfrak{N}_L\right),$$

$$T^{(0)}_{\mu\nu} = -m_c \sin\theta_C \cos\theta_C \left(\bar{\lambda}_L \gamma_\nu \frac{1}{\hat{k} - \hat{q} - m_c} \gamma_5 \frac{1}{\hat{k} - m_c} \gamma_\mu \mathfrak{N}_L\right),$$

and we come to the result obtained first in Ref. 1

$$\begin{aligned} M^{(0)} &= M_1^{(0)} + M_2^{(0)} + M_3^{(0)} \\ &= (-G^2 m_c^2 \sin\theta_C \cos\theta_C / 4\pi^2) \bar{\mu}\gamma_\lambda\gamma_5\mu \\ &\quad \times \langle 0 | \bar{\lambda}\gamma_\lambda\gamma_5 \mathfrak{N} | K^0 \rangle . \end{aligned}$$

If we include now the strong-interaction corrections, then the mass of the c quark in the matrix elements M_1, M_2 must be replaced by

$$m_c \to m_c \left(1 + \frac{25}{3} \frac{g^2(m_c^2)}{16\pi^2} \ln \frac{k^2}{m_c^2}\right)^{-12/25} \tag{23}$$

(see Sec. II). As for the matrix element M_3 the general expression for it can be transformed in the following way:

$$M_3 = 2G^2 \mu\gamma_\lambda\gamma_5\mu \int d^4x x_\lambda \langle 0 | T\{m_c \bar{c}(x)\gamma_5 c(x) \int d^4y m_W^2 D(y, m_W^2) j_\mu^+(y) j_\mu^-(0)\} | K_L \rangle .$$

In Sec. III we have shown that the gluon corrections to this matrix element are not reduced to the mass renormalization, but include the modification by the strong interactions of the weak $\bar{\lambda}\mathfrak{N} \to \bar{c}c$ transition as well. Proceeding with the calculation outlined in the previous section, we get the following answer:

$$M = M^{(0)}\eta ,$$

$$\begin{aligned} \eta = \frac{1}{2} \frac{16\pi^2}{g^2(m_c^2)} \Big[&\tfrac{3}{11} (\kappa_1^{12/25} - \kappa_1^{1/25}) + 3(\kappa_1^{1/25} - 1) \\ &+ \tfrac{6}{7}(\kappa_1^{-6/25} - \kappa_1^{1/25}) \Big] \\ &+ \tfrac{1}{2}(-\kappa_1^{12/25} + 2\kappa_1^{-6/25} + \kappa_1^{-24/25}) , \end{aligned} \tag{24}$$

where

$$\kappa_1 = g^2(m_c^2)/g^2(m_W^2) .$$

It is worth emphasizing that if $[g^2(m_c^2)/4\pi] \times \ln(m_W^2/m_c^2) \sim 1$, then the second term in the right-hand side of Eq. (24) is, in principle, of the same order of magnitude as neglected "corrections" to the first term. We have kept it for two reasons. Firstly, it interpolates smoothly the calculation of the gluon corrections to the free-quark case in the limit of $g^2 \to 0$. Secondly, the leading term is canceled out in this limit, and just because of this cancellation the leading term is not so large numerically as it might be expected to be. Thus, it seems reasonable to present the result in the form (24). Let us notice that Eq. (24) reflects the fact that the strong-interaction corrections do not just reduce to multiplicative renormalization of the result of the free-quark approximation as is asserted in Refs. 7, 9. The reasons for this were in fact discussed above as well as in our previous paper.[8] The point was criticized in Ref. 9 and we turn to the consideration of this criticism in the next section.

V. GAUGE INDEPENDENCE OF THE RESULTS OBTAINED

In this section we will consider the gluon corrections to the contributions depending on the choice of the gauge of the W-boson field. The problem was brought up by Gaillard *et al.*,[9] who argue that the strong-interaction corrections are the same both to the gauge-dependent and gauge-independent pieces. On the basis of this argument the authors of Ref. 9 conclude that the cancellation of the $\ln(m_W^2/m_c^2)$ terms which takes place in the free-quark approximation maintains in the presence of the strong interactions as well, and this conclusion is in variance with the results of the explicit calculation presented in the preceding section.

We will show here that in the case of the T product of three currents the gauge-dependent and gauge-independent pieces are in fact modified by strong interactions in different ways, so that the assertion of Ref. 9 is not valid, to our mind.

Let us introduce the gauge-dependent term in the Green's function of the W-boson field in the following way:

$$D^W_{\mu\nu}(x) = g_{\mu\nu} D(x, m_W^2) + \frac{1}{m_W^2} \partial_\mu \partial_\nu \left[D(x, m_W^2) - D\left(x, \frac{m_W^2}{\xi}\right)\right], \tag{25}$$

where parameter ξ corresponds to the so called R_ξ gauge.

Furthermore, in the case of the free-quark approximation we find for the contribution of the ξ-dependent part of $D^W_{\mu\nu}(x)$ into the vertex Γ_λ the following expression:

$$\Gamma_\lambda^{(\xi)} \sim m_c^2 \int d^4z d^4\rho \bar{\lambda}_L \gamma_\mu\gamma_\lambda\gamma_\nu \mathfrak{N}_L \frac{1}{(z+\frac{1}{2}\rho)^2 (z-\frac{1}{2}\rho)^2} \times \frac{\partial}{\partial\rho_\mu} \frac{\partial}{\partial\rho_\nu} [D(\rho, m_W^2) - D(\rho, m_W^2/\xi)] , \tag{26}$$

where we used Eqs. (9), (25).

After integrating over ρ by parts we see that the integral over z converges at $z \sim \rho$, unlike the case considered in Sec. III, where we used the Feynman gauge for the W-boson Green's function. On the other hand, the gluon corrections depend crucially on the value of ρ/z [see Eq. (17)]. Therefore, the contribution of the ξ-dependent part of $D^W_{\mu\nu}$ is affected by the strong interactions in a different way than that of $D(x, m_W^2)g_{\mu\nu}$.

This does not imply of course that the final result is gauge-dependent. Moreover, the cancellation of the gauge-dependent terms can be followed in a rather general way. The central point here is that the contribution of the ξ-dependent part of $D^W_{\mu\nu}$ reduces to the integral from the T product of two currents. Indeed, substituting $D^W_{\mu\nu}$ by its longitudinal part [see Eq. (8b)] and integrating by parts over x, y we come to the integral from

$$\frac{\partial}{\partial x_\mu}\,\frac{\partial}{\partial y_\nu}\,T\{j^+_\mu(x)j^-_\nu(y)\bar{c}(0)\gamma_5 c(0)\}\,.$$

Performing the differentiation explicitly we get both the terms containing equal-time commutators of currents and the T product of divergences of the corresponding currents. The latter terms give a contribution to Γ_λ which is proportional to an extra factor of m_c^2/m_W^2 and can be neglected for this reason.

The gluon corrections to the commutator terms which are in fact T products of two currents reduce to the replacement of the mass of the c quark by its effective mass [see Eq. (23)] in all the equations obtained in the free-quark approximation. Thus, all the gauge-dependent terms in the general case can be obtained by this simple substitution and the cancellation of the gauge-dependent terms proceeds just in the same way as for free quarks.

Thus, we have shown that the gauge-dependent and gauge-independent terms are modified by gluon exchanges in different ways, contrary to the assertion of Gaillard *et al.* This circumstance does not affect the gauge independence of the final result, which can be easily checked.

VI. K_L-K_S MASS DIFFERENCE

The calculation of the gluon corrections to the K_L-K_S mass difference proceeds mostly in the same way as in the case of the amplitude of the $K_L \to 2\mu$ decay considered above. We will give the final result here for the sake of completeness. We will also use the opportunity to rectify the numerical error contained in our paper.[8a] Because of this error the final result presented in this paper is not correct, although the approach to the calculation seems to be adequate to the problem.

The K_L-K_S mass difference is determined by the T product of four currents

$$\int d^4x\,d^4y\,d^4z\,D(x-y, m_W^2)D(z, m_W^2) \times \langle \bar{K}^0 \,|\, T\{j^+_\mu(x)j^-_\mu(y)j^+_\nu(z)j^-_\nu(0)\} \,|\, K^0\rangle\,.$$

As follows from the properties of the function $D(x, m_W^2)$ the characteristic values of $(x-y)$ and z are of the order m_W^{-1}. As for the values of x they are of the order m_c^{-1} as is readily seen from the consideration of the graphs in the perturbation theory.

Using the Wilson expansion for the pairs of currents $j^+_\mu(x)j^-_\mu(y)$ and $j^+_\nu(z)j^-_\nu(0)$ [see Eq. (12)], we come to the T product of two effective Hamiltonians of weak interactions. Furthermore, applying the Wilson expansion to this product we come to the operator

$$(\bar{\lambda}_L\gamma_\alpha\mathfrak{N}_L)(\bar{\lambda}_L\gamma_\alpha\mathfrak{N}_L)\,.$$

Accounting to the anomalous dimension of this operator, we finally find that the effect of gluon exchanges in the case of the K_L-K_S mass difference reduces to the following factor:

$$\tilde{\eta} = (\tfrac{1}{2}\,\kappa_1^{24/25} - \kappa_1^{6/25} + \tfrac{3}{2}\,\kappa_1^{-12/25})\kappa_2^{-2/9}\,, \qquad (27)$$
$$\kappa_1 = g^2(m_c^2)/g^2(m_W^2); \quad \kappa_2 = g^2(m^2)/g^2(m_c^2)\,.$$

Two remarks concerning this result are now in order. Firstly, we have kept not only the highest power of the ratio $g^2(m_c^2)/g^2(m_W^2)$, but the other terms as well. In other words we sum up all the leading log terms. This is justified for the case of $[g^2(m_c^2)/4\pi]\ln(m_W^2/m_c^2) \sim 1$ and $g^2(m_c^2)/4\pi \ll 1$. In the rather unrealistic case of $[g^2(m_c^2)/4\pi] \times\ln(m_W^2/m_c^2) \gg 1$ only the first term in the right-hand side of Eq. (27) can be retained. Secondly, there is no logarithmic term in the effective mass of the c quark at the distances $m^{-1} > x > m_c^{-1}$. For this reason the power of κ_2 in Eq. (27) does not include the anomalous dimension of the mass operator.

In conclusion let us notice that numerically the factor $\tilde{\eta}$ is close to unity for all the reasonable choices of the parameters m, m_c, m_W entering Eq. (27). In other words, the result of the free-quark approximation is only slightly modified by the gluon corrections.

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