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Multiple solutions of the Roy equations: N/D method on a finite interval*

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The Roy equations, combined with unitarity, can be regarded as a system of integral equations for the π-π scattering amplitude in a finite energy region. Even when the partial-wave absorptive parts above this finite range are prescribed, and the two S-wave scattering-length parameters are held fixed, the singular equations have multiple solutions, some of which could be missed in a direct numerical study. We regularize the system by a modified N/D method, in which the full manifold of solutions is parametrized explicitly. If $\delta(s_0)$ is the phase shift of a particular wave at the cutoff point, then that wave carries a number of arbitrary, real parameters equal to the integer part of $2\delta(s_0)/\pi$, provided $\delta(s_0) \geq -\pi/2$. We suggest that the N/D formulation is appropriate for applications of the Roy equations.

I. INTRODUCTION

The Roy equations[1] for π-π scattering may be combined with elastic unitarity to provide a set of nonlinear, singular, integral equations for physical partial-wave amplitudes. The equations are valid in a finite domain of energy, which contains an interval $4 \leq s < s_0$ of the physical region, but they entail also the absorptive parts of partial waves for $s \geq s_0$. In applications the latter are externally assigned parameters, which are usually taken from a Regge-pole model. The S-wave scattering lengths also appear as free parameters.

Calculations by several authors[2] indicate that the Roy equations are effective in constraining the low-energy amplitudes for π-π scattering. That is a pleasing result, since the equations follow from basic principles of field theory; they are derived by taking partial-wave projections of fixed-t dispersion relations. The main theoretical uncertainty, which appears to be not too detrimental, arises from the poorly known high-energy absorptive parts. In practice there has been another source of uncertainty in the circumstance that the equations do not in general have a unique solution, even when the high-energy absorptive parts and the S-wave scattering lengths are fixed. Except for their restriction to a finite range of energy, the Roy equations with unitarity resemble partial-wave dispersion relations. The latter display the Castillejo-Dalitz-Dyson (CDD) ambiguity,[3] which is to say that the multiplicity of solutions increases with the high-energy asymptotic value of the phase shift. As we shall see, the Roy equations have a similar property, but with the value of the phase shift at the cutoff point s_0 determining the multiplicity.

Numerical solutions of the Roy equations have been sought through a procedure of choosing particular unitary formulas for the partial waves.[2] A few parameters in the formulas are adjusted so as to achieve an approximate solution. Although this is certainly a well-justified method of looking for solutions, it has the drawback that one cannot be sure of finding all solutions of interest. The parametrization used may not cover all possibilities, or the search over parameter space may be too costly in computation time. Furthermore, one would like to see a systematic numerical refinement of proposed approximate solutions.[4] A standard method for obtaining a precise solution from an approximate one is to linearize the equation about the approximate solution, i.e., to apply the Newton-Kantorovich method or some similar scheme.[5] The solution of a linear equation gives the first correction to the approximate solution, but in the case of the Roy equation the linear equation has multiple solutions, in general, due to the presence of a Cauchy kernel.[6] Thus, an analysis of the CDD ambiguity cannot be avoided, even in a direct numerical study of the Roy equation.

Our purpose in this paper is to replace the singular Roy equations by equivalent nonsingular N/D equations. The parameters determining the solution multiplicity appear explicitly in the N/D equations, and every solution of the Roy equations corresponds to some choice of those parameters. Furthermore, the N/D equations provide a better framework for numerical computations. The Fréchet derivative of the N/D operator is a Fredholm operator, which ensures that the Newton-Kantorovich method can be applied in a straightforward way, and that solutions can be followed as parameters are varied in small steps.[7]

The finite energy cutoff of the Roy equations introduces a new feature of solution multiplicity not present in the usual CDD problem on an infinite energy interval. As $\delta(s_0)$ is increased, the number of parameters in the N/D equation increases by one whenever $\delta(s_0)$ passes through a positive-integer multiple of $\pi/2$. By contrast, the number of parameters in the usual problem increases by two whenever $\delta(\infty)$ passes through a positive-integer multiple of π. This fact emerged in a recent work of Pomponiu and Wanders,[8] and was implicit in an earlier discussion of the finite-interval N/D equation.[9] Pomponiu and Wanders study solutions of the Roy equation in an infinitesimal neighborhood of a given solution, by making a local linearization in the manner of Refs. 6, 7, and 10. A full justification of this technique would require the methods of bifurcation theory in order to show that solutions of the linearized equation actually correspond to solutions of the nonlinear one. By the N/D method we are able to avoid bifurcation theory, and at the same time obtain a global, rather than a merely local treatment of solution multiplicity. Our results agree with those of Pomponiu and Wanders as far as they are comparable.[11] We find that the number of parameters in the N/D equation for a given partial wave is

$$d=[2\delta(s_0)/\pi], \tag{1.1}$$

provided that $\delta(s_0) \geq -\frac{1}{2}\pi$, where $[x]$ denotes the integer part of x. Barring certain extraordinary cases (such as the kernel of the N/D operator having a unit eigenvalue), the dimension of the manifold of solutions will be equal to the sum over the various partial waves of the numbers d of (1.1).

We find that, if $\delta(s_0) < -\frac{1}{2}\pi$, there are stringent conditions which are necessary for the Roy equations to have a solution at all. It is hard to see how these conditions could be met, and it is also interesting to observe that experimentally determined π-π phase shifts are always greater than $-\frac{1}{2}\pi$.

The unit change in the number of parameters at odd-integral values of $2\delta(s_0)/\pi$ is associated with a logarithmic singularity in the kernel of the N/D equation at $s=s_0$. Because of the singularity, Fredholm theory does not apply. Such marginally singular equations have been regularized by various methods.[9,12-14] We apply the method of Ref. 9, in which a Mehler transform is used to invert the singular part of the integral operator. The equation is thereby reduced to a regular Fredholm equation, but one containing a new free parameter if $\alpha \geq \frac{1}{2}$, where $\alpha=\delta(s_0)/\pi-[\delta(s_0)/\pi]$. As α tends to unity, this parameter drops out, but two new ones enter, namely the position and residue of an extra CDD pole. The number of CDD poles is $[\delta(s_0)/\pi]$, and halfway between the integer values of $\delta(s_0)/\pi$ at which CDD poles enter one acquires "half a CDD pole"; i.e., the single parameter associated with the logarithmic singularity.

Our analysis deals mainly with the unitarity integral over the finite cut $4 \leq s < s_0$, and is not much influenced by the fact that the left-cut terms in the Roy equations are nonlinear functionals of all the partial-wave absorptive parts. Nevertheless, one would like to have an existence proof for solutions of the full nonlinear system, including an assurance that partial waves have the correct threshold behavior.[15] Such a proof has been provided for the original Roy equations with sufficiently small inputs (S-wave subtraction constants and high-energy absorptive parts), and with the restriction that $|\delta(s_0)|$ be sufficiently small.[16] The allowed range of $\delta(s_0)$ is a subinterval of $-\frac{1}{2}\pi \leq \delta(s_0) < \frac{1}{2}\pi$, so that no free parameters would enter the N/D equations. Correspondingly, the existence proof entails a demonstration of uniqueness within a restricted class of functions. Existence proofs for the N/D version of the Roy equation may also be constructed,[17] with the advantage that $\delta(s_0)$ need not be small. The required analysis of CDD terms is similar to that needed for the N/D version of the Low equation,[18,1] and it depends on the smallness of all inhomogeneities, including CDD pole residues. An interesting question arises as to whether all solutions of the N/D equations may be reached by parameter continuation from the case with small inhomogeneities. In any event, an interesting way to look for solutions of the N/D Roy equations is to continue numerically from small-parameter solutions. The latter may be computed by simple iteration of the N/D system.

The plan of this paper is as follows. In Sec. II we give the general form of the Roy equation for a particular angular momentum and isospin. The Cauchy unitarity integral is written separately, and the remainder (an effective driving term) is given in Appendix A in terms of sums and integrals over all partial waves. In Sec. III we set up the N/D representation on a finite interval, and examine its properties. We are then led to a separate study of the singular part of the N/D operator in Sec. IV. The final integral equation is of the Fredholm type, as is proved in Appendix B.

II. ROY EQUATION AS A NONLINEAR SYSTEM

The Roy equations are derived by projecting twice-subtracted, fixed-t dispersion relations onto Legendre polynomials. They have the form

$$F_l^I(s)=\frac{1}{\pi}\int_4^\infty \frac{ds'}{s'-s}\,\frac{s^2}{s'^2}\left(\frac{s-4}{s'-4}\right)^l A_l^I(s') + (s-4)^l C_l^I(s)\,, \tag{2.1}$$

where A_l^I is the absorptive part of the amplitude F_l^I in the state of isospin I and angular momentum

l. The expression for C_l^I, given in Appendix A, involves a sum over absorptive parts for all angular momentum and isospin states, and the two S-wave scattering lengths. Thus the Roy equations provide a means of calculating the real parts of the partial-wave amplitudes, given their imaginary parts and the scattering lengths.

Although the general form (2.1) is valid for all s, the infinite series for $C_l^I(s)$ converges only for limited values of s; in the physical region the domain of convergence is $4 \leq s \leq s_0$. If one has a fully crossing symmetric amplitude at hand, then one may use Bose symmetry to halve the interval on which the partial-wave projection is made, and then it may be shown that $s_0 = 60$. However, in practice one needs to combine (2.1) with unitarity in order to obtain a nonlinear system of equations for the amplitudes. In seeking solutions of this system only two-channel crossing symmetry is enforced exactly, and this means that the projection interval may not be halved. It can be shown that one is then limited to $s_0 = 32$ (Ref. 15).

We rewrite the Roy equation as follows:

$$f_l^I(s) = \frac{1}{\pi}\int_4^{s_0} \frac{ds'}{s'-s}\, a_l^I(s') + B_l^I(s)\,, \tag{2.2}$$

where

$$f_l^I(s) = (s-4)^{-l} F_l^I(s)\,, \tag{2.3}$$

$$a_l^I(s) = (s-4)^{-l} A_l^I(s)\,, \tag{2.4}$$

and

$$B_l^I(s) = C_l^I(s) - \frac{1}{\pi}\int_4^{s_0} ds'\, \frac{s'+s}{s'^2}\, a_l^I(s') + \frac{1}{\pi}\int_{s_0}^{\infty} \frac{ds'}{s'-s}\, \frac{s^2}{s'^2}\, a_l^I(s')\,. \tag{2.5}$$

Suppose now that the scattering lengths, and $a_l^I(s')$ for $s' \geq s_0$, are prescribed. We may then use elastic unitarity,

$$a_l^I(s) = (s-4)^l \left(\frac{s-4}{s}\right)^{1/2} |f_l^I(s_+)|^2\,, \tag{2.6}$$

to convert (2.2) into a nonlinear equation for f_l^I. Strictly speaking, elastic unitarity is exact only for $s < 16$; but one knows from experiment that there is essentially no inelasticity below the $K\bar{K}$ threshold, so that one may take (2.6) to be correct for $s \leq s_0 = 32$.

The cutoff at $s_0 = 32$ may perhaps be smaller than one would like. It should be mentioned, however, that an improved set of equations, due to Mahoux, Roy, and Wanders,[19] to which our treatment is equally applicable, allows a value $s_0 \approx 125$.

In the following it will be convenient to take s_0 to be slightly less than 32, so that the series defining $C_l^I(s)$ actually converges for $s \leq s_0$. In fact, with such a choice, $C_l^I(s)$ is analytic in a complex neighborhood of $[4, s_0]$. The third term in $B_l^I(s)$ has a logarithmic singularity at $s = s_0$, which we separate as follows:

$$\frac{1}{\pi}\int_{s_0}^{\infty} \frac{ds'}{s'-s}\, \frac{s^2}{s'^2}\, a_l^I(s') = -\frac{1}{\pi} a_l^I(s_0)\left[\ln\left(\frac{s_0-s}{s_0}\right) + \frac{s}{s_0}\right] + \frac{1}{\pi}\int_{s_0}^{\infty} \frac{ds'}{s'-s}\, [a_l^I(s') - a_l^I(s_0)]\, \frac{s^2}{s'^2}\,. \tag{2.7}$$

We may then represent $B_l^I(s)$ as

$$B_l^I(s) = -\frac{1}{\pi}\, a_l^I(s_0) \ln(s_0 - s) + \hat{B}_l^I(s)\,, \tag{2.8}$$

where $\hat{B}_l^I(s)$ is analytic in a domain Ω, which consists of some complex neighborhood of $[4, s_0]$, minus a cut along the real axis to the right of $s = s_0$. The exact form of Ω may be computed from the theory of the Roy equations, but it is of no importance in our discussion. We assume that the input function $a_l^I(s)$ is Hölder-continuous for $s_0 \leq s \leq s_0 + \epsilon$. It follows that $\hat{B}_l^I(s)$ is Hölder-continuous, and hence bounded, for $4 \leq s \leq s_0$.

In this paper we shall be exclusively interested in studying the manifold of solutions of the system (2.2), (2.6), for a given B_l^I. Solution of the system for given B_l^I is just one step in the solution of the complete problem, since B_l^I actually depends in a nonlinear way on the $a_l^I(s)$ for $4 \leq s \leq s_0$. After replacing the equation with given B_l^I by an N/D system, we can express a_l^I, and hence B_l^I, in terms of the N functions, $N_l^I(s)$, $4 \leq s \leq s_0$. We thereby obtain a complete nonlinear system for the N_l^I, which we regard as the appropriate system of equations for a thorough study of the Roy problem. The multiplicity of solutions arising from the Cauchy unitarity integral [the first term in (2.2)] is accounted for by explicit parameters in the N/D system, but we cannot rule out additional multiplicities arising from the nonlinear character of B_l^I, i.e., from possible bifurcation phenomena. An investigation of the latter would require numerical computation.

III. N/D METHOD ON A FINITE INTERVAL

In this section we suppress the indices (l, I) and invoke the unitarity equation (2.6) to write the Roy equation (2.2) for a particular channel as

$$f(s) = B(s) + \frac{1}{\pi}\int_4^{s_0} \frac{q(s') f(s'_+) f(s'_-) ds'}{s'-s}\,, \tag{3.1}$$

where

$$q(s) = \left(\frac{s-4}{s}\right)^{1/2} (s-4)^l\,, \tag{3.2}$$

$$B(s)=-\frac{1}{\pi}\frac{\sin^2\delta(s_0)}{q(s_0)}\ln(s_0-s)+\hat{B}(s)\,. \qquad (3.3)$$

The function $\hat{B}$ is analytic in the domain Ω described in the preceding section, is Hölder-continuous for $4\leq s\leq s_0$, and satisfies the reality condition $\hat{B}(s)=\hat{B}(s^*)^*$. The parameter $\sin^2\delta(s_0)$ comes from the externally assigned absorptive part, and may be regarded as given.

A solution of the Roy equation (3.1) is understood as being a function f having the representation (3.1), with Hölder-continuous boundary values $f(s_\pm)$ on the closed interval $[4,s_0]$, such that $f(s)=f(s^*)^*$. Notice that $s=s_0$ is required to be a point of continuity of $f(s_+)$, but that $B(s)$ is logarithmically infinite at $s=s_0$. The two terms on the right-hand side of (3.1) must cancel at $s=s_0$. We shall find that the required cancellation occurs automatically in an amplitude constructed by the N/D method.

We show first that any solution of the Roy equation has a certain N/D representation, and that the N function satisfies an integral equation, the so-called N equation. Second, we prove that each solution of the N equation leads to a solution of the Roy equation, provided that the corresponding D function has no zero on the physical sheet. Finally, we show how to reduce the problem of solving the N equation to that of inverting a regular Fredholm equation.

Suppose that f is any solution of (3.1). The corresponding phase shift δ may be defined so that it is Hölder-continuous and zero at threshold. We construct from δ the function

$$\mathfrak{D}(s)=\exp\left[-\frac{1}{\pi}\int_4^{s_0}\frac{\delta(s')ds'}{s'-s}\right], \qquad (3.4)$$

which has a phase opposite to that of $f(s_+)$:

$$\mathfrak{D}(s_+)=e^{-i\delta(s)}\left|\mathfrak{D}(s_+)\right| =e^{-i\delta(s)}\exp\left[-\mathrm{P}\,\frac{1}{\pi}\int_4^{s_0}\frac{\delta(s')ds'}{s'-s}\right]. \qquad (3.5)$$

As in the usual N/D method, one sees that $\mathfrak{N}=f\mathfrak{D}$ has no cut between 4 and s_0; this follows from $\mathfrak{N}(s)=\mathfrak{N}(s^*)^*$, and the fact that $\mathfrak{N}(s_+)$ is real. Thus, the numerator function $\mathfrak{N}$ is analytic in Ω. The function $\mathfrak{D}$ is not always an appropriate denominator function for the N/D method, because of a possible strong divergence at $s=s_0$.

To determine the behavior of $\mathfrak{D}$ at s_0 we note the following decomposition of the integral in (3.5):

$$-\mathrm{P}\,\frac{1}{\pi}\int_4^{s_0}\frac{\delta(s')ds'}{s'-s} =-\frac{1}{\pi}\int_4^{s_0}\frac{\delta(s')-\delta(s_0)}{s'-s}ds'-\frac{\delta(s_0)}{\pi}\ln\left(\frac{s_0-s}{s-4}\right) =-\frac{\delta(s_0)}{\pi}\ln(s_0-s)+O(1),\quad s\to s_0\,. \qquad (3.6)$$

This result implies the asymptote

$$\left|\mathfrak{D}(s)\right|\sim(s_0-s)^{-\delta(s_0)/\pi},\quad s\to s_0\,. \qquad (3.7)$$

We first assume that $\delta(s_0)\geq 0$, and later discuss the case of negative $\delta(s_0)$. Let us define the numbers

$$n=[\delta(s_0)/\pi]\,, \qquad (3.8)$$

$$\alpha=\delta(s_0)/\pi-n\,, \qquad (3.9)$$

where $[x]$ means the integral part of x. The denominator function of interest is

$$D(s)=\mathfrak{D}(s)(s-s_0)^n\prod_{i=1}^{n}(s-s_i)^{-1}\,, \qquad (3.10)$$

where it is understood that $D=\mathfrak{D}$ if $n=0$. The s_i are any distinct points in the interval $[4,s_0]$ such that $\sin\delta(s_i)=0$. There are always at least n such points, but since there may be more than n in general, D is not always uniquely defined. The poles of D at $s=s_i$ are mathematically analogous to the usual Castillejo-Dalitz-Dyson (CDD) poles, but their physical interpretation is not necessarily the same, owing to the fact that D is defined with a finite cut. We shall call them *finite-interval CDD poles*, or, less exactly, just *CDD poles*. The behavior of D as s tends to s_0 is

$$D(s)\sim\kappa(s_0-s)^{-\alpha},\quad s\to s_0$$
$$0\leq\alpha<1,\quad \kappa\neq 0\,. \qquad (3.11)$$

We define N as the numerator function corresponding to the denominator function (3.10):

$$N(s)=f(s)D(s)\,. \qquad (3.12)$$

Since f is unitary, one knows that $\mathrm{Im}f^{-1}=-q$, and therefore

$$\mathrm{Im}D(s_+)=-q(s)N(s)\,. \qquad (3.13)$$

The N function has no singularities at the points $s=s_i<s_0$. One can see this by noting that $N(s_+)-N(s_-)=0$, $4\leq s<s_0$, $s\neq s_i$, and that

$$N(s)=O(|s-s_i|^{-1+\mu}),\quad s\to s_i,\quad 0<\mu\leq 1\,. \qquad (3.14)$$

The bound (3.14) follows from (3.12), (3.5), and the requirement $\sin\delta(s_i)=0$, if μ is identified as the exponent of Hölder continuity of δ. Since the bound (3.14) rules out a pole at $s=s_i<s_0$, and since N has zero discontinuity over the real exis, we conclude that N is in fact analytic in Ω. An application of Cauchy's integral theorem then shows that D has the representation

$$D(s)=1+\sum_{i=1}^{n}\frac{c_i}{s-s_i}-\frac{1}{\pi}\int_4^{s_0}\frac{q(s')N(s')ds'}{s'-s}\,,\quad s\neq s_0\,. \qquad (3.15)$$

From (3.12), (3.15), and the analyticity of N we see that $f(s_\pm)$ has derivatives of all orders for $4<s<s_0$. Unitarity was an essential assumption in deducing this property of f from analyticity in Ω and mere Hölder continuity of boundary values.

We are now ready to derive the N equation, which we do by examining the auxiliary function

$$\Lambda(s)=[f(s)-B(s)]D(s)-\frac{1}{\pi}\int_4^{s_0}\frac{B(s')q(s')N(s')ds'}{s'-s}$$
$$=N(s)-B(s)\left(1+\sum_{i=1}^{n}\frac{c_i}{s-s_i}\right)-\frac{1}{\pi}\int_4^{s_0}\frac{B(s)-B(s')}{s-s'}q(s')N(s')ds'. \qquad (3.16)$$

It is clear from the first line of (3.16) that Λ is analytic in the s plane, except for possible poles arising from D, and a possible cut $[4, s_0]$, if we recall that $f-B$ is just the unitarity integral that appears in (3.1). Since N and B are analytic in Ω, the second line of (3.16) shows that there is in fact no discontinuity of Λ over $[4, s_0]$. Since Λ vanishes at infinity, it must be simply a sum of poles (if $n>0$) or zero (if $n=0$):

$$\Lambda(s)=-\sum_{i=1}^{n}\frac{c_iB(s_i)}{s-s_i}. \qquad (3.17)$$

It should be noticed that the separate terms that appear in the second line of (3.16) can be infinite at $s=s_0$. Each term is bounded by $\kappa(s-s_0)^{-\beta}$, $\beta<1$, so that the absence of a discontinuity of Λ implies that it is not singular at $s=s_0$ (except in the case in which the largest s_i coincides with s_0). It is seen from (3.16) and (3.17) that a necessary condition on the N function corresponding to a solution of the Roy equation with $\delta(s_0)\geq 0$ is

$$N(s)=B(s)+\sum_{i=1}^{n}c_i\frac{B(s)-B(s_i)}{s-s_i}+\frac{1}{\pi}\int_4^{s_0}\frac{B(s)-B(s')}{s-s'}q(s')N(s')ds', \quad s\neq s_0. \qquad (3.18)$$

This is the N equation of the N/D method.

If $\delta(s_0)<0$, one can put $D=\mathfrak{D}$ and show that $N=fD$ satisfies the same equation (3.18) (of course without the CDD terms). This case is extraordinary, however, in that the homogeneous version of Eq. (3.18) has at least one nontrivial solution. Suppose $\delta(s_0)<0$ and define

$$m=[-\delta(s_0)/\pi], \qquad (3.19)$$

$$\beta=-\delta(s_0)/\pi-m. \qquad (3.20)$$

Consider the following sequence of denominator functions:

$$D^{(j)}(s)=(s-s_0)^{-j}\mathfrak{D}(s), \quad j=1,2,\ldots,m+1. \qquad (3.21)$$

Because of (3.7), the first m of the $D^{(j)}$ are zero at $s=s_0$, and $D^{(m+1)}$ has a bound $\kappa(s_0-s)^{-1+\beta}$, $0\leq\beta<1$. Since the $D^{(j)}$ vanish at infinity, they satisfy Cauchy representations of the form

$$D^{(j)}(s)=-\frac{1}{\pi}\int_4^{s_0}\frac{q(s')N^{(j)}(s')ds'}{s'-s}, \qquad (3.22)$$

$$j=1,2,\ldots,p;\quad p=\begin{cases}m+1, & \beta>0\\ m, & \beta=0.\end{cases}$$

In particular, if $m=0$ we have $\beta>0$, and (3.21) holds for $D^{(1)}$. By the arguments given above one proves that

$$N^{(j)}(s)=\frac{1}{\pi}\int_4^{s_0}\frac{B(s)-B(s')}{s-s'}q(s')N^{(j)}(s')ds', \quad j=1,2,\ldots,p. \qquad (3.23)$$

Thus, if $\delta(s_0)<0$ we have p linearly independent nontrivial solutions of the homogeneous version of (3.18), with $p\geq 1$. This suggests that solutions of the Roy equations with $\delta(s_0)<0$ (if any) have a singular character. It will turn out presently that the case $-\frac{1}{2}\pi\leq\delta(s_0)<0$ is actually not singular in general, because of peculiarities associated with the non-Fredholm character of the kernel in (3.18). The region of actual singularity is $\delta(s_0)<-\frac{1}{2}\pi$, as our later discussion will show.

We next seek a converse to the statement that any solution of the Roy equation provides a solution of the N equation. Suppose that we have a solution of (3.18) for $4\leq s<s_0$, with some choice of the real CDD parameters (c_i, s_i). We require that the s_i be distinct points in $(4, s_0]$. A solution is understood as being a function N which satisfies (3.18) for $4\leq s<s_0$, which is continuous in that region, and which has the behavior $N(s)\sim\kappa(s_0-s)^{-\gamma}$, $0\leq\gamma<1$, $s\to s_0$. The right-hand side of (3.18) then provides an extension of N to a function analytic in Ω. Given the solution N we may construct an amplitude

$$f(s)=N(s)/D(s)$$
$$=B(s)+\frac{1}{D(s)}\left[\sum_{i=1}^{n}c_i\frac{B(s_i)}{s_i-s}+\frac{1}{\pi}\int_4^{s_0}\frac{B(s')q(s')N(s')ds'}{s'-s}\right], \qquad (3.24)$$

where D is computed from (3.15). Suppose that D has no zeros in the cut plane (including points on the cut). Then $f-B$ is clearly analytic except for the cut $[4, s_0]$. Now f is unitary, because of (3.13), so that the discontinuity of $f-B$ is $q(s)f(s_+)f(s_-)$. Further, the singularity of $f-B$ at $s=s_0$ is at most logarithmic, because of the assumed behavior of N at $s=s_0$, and the behavior (3.3) of B. [In general, the numerator in $f-B$ behaves as $\ln(s_0-s)(s_0-s)^{-\gamma}$, and the denominator D as $(s_0-s)^{-\gamma}$.] Since $f-B$ clearly vanishes at infinity, it follows that it must be equal to the unitarity integral, so that f solves the Roy equation (3.1). Whatever the choice of the parameters (c_i, s_i), a solution of the N equation gives a solution of the Roy equation, provided that D has no zero (no so-called "ghost"). For general B, c_i, and s_i, we should not expect D to be free of ghosts. There are probably many choices of the CDD parameters (c_i, s_i) which do imply ghosts, but every solution of the Roy equation corresponds to at least one ghost-free choice of the parameters.

In a practical calculation, it is a simple matter to check for ghosts. Suppose, for instance, that $\delta(s_0)\geq 0$, and define $r=[\delta(s_0)/\pi]$. To check for ghosts we have only to compare r with the number n of CDD poles of the D function. The D function, as computed through (3.15) from a solution N of (3.18), may be represented also as

$$D(s)=(s-s_0)^r\mathfrak{D}(s)R(s)\,, \tag{3.25}$$

where R is a rational function with the asymptotic behavior

$$R(s)\sim s^{-r}, \quad s\to\infty\,. \tag{3.26}$$

Since $(s-s_0)^r\mathfrak{D}(s)$ has no zeros, ghosts can appear only as zeros of R. Since R has n poles, (3.26) shows that there are no ghosts if and only if $n=r$.

Our next concern is the problem of solving (3.18) for given B and (c_i, s_i). This is the practical problem that arises when the Roy equation is solved by the N/D method. We first seek solutions of (3.18) which lead to solutions of the Roy equation with $\delta(s_0)\geq 0$. According to (3.11), such solutions should have the behavior

$$N(s)\sim\kappa f(s_0)(s_0-s)^{-\alpha}, \quad s\to s_0$$
$$0\leq\alpha<1, \quad \kappa\neq 0, \tag{3.27}$$

where α is given by (3.9). In view of (3.9), $\sin^2\pi\alpha$ is to be identified with the input parameter $\sin^2\delta(s_0)$ that appears in (3.3). In fact, we shall henceforth write

$$B(s)=\frac{-\sin^2\pi\alpha}{\pi q(s_0)}\ln(s_0-s)+\hat{B}(s)\,, \tag{3.28}$$

and study solutions of the N equation as functions of the input parameter $\sin^2\pi\alpha$. The input function B is invariant under the transformation $\alpha\to 1-\alpha$, whereas the asymptotic behavior (3.27) is not. We must then explain how the solutions in the range $0<\alpha<\frac{1}{2}$ can be different from those in the range $\frac{1}{2}<\alpha<1$. The explanation is that solutions with $\alpha\geq\frac{1}{2}$ in (3.27) are not uniquely determined by the input parameters $(B, \sin^2\pi\alpha, c_i, s_i)$. We shall find that a new parameter enters for $\alpha\geq\frac{1}{2}$, because of the non-Fredholm nature of the N equation.

The behavior (3.27) is generated, so to speak, by the logarithmic singularity in $B(s)$. To study this matter, it is convenient to isolate the singular parts in the kernel of the N equation. Taking account of (3.28), we write

$$\frac{1}{\pi}\int_4^{s_0}\frac{B(s)-B(s')}{s-s'}q(s')N(s')ds'=-\left(\frac{\sin\pi\alpha}{\pi}\right)^2\int_4^{s_0}\frac{\ln(s_0-s)-\ln(s_0-s')}{s-s'}N(s')ds'+\frac{1}{\pi}\int_4^{s_0}M(s,s')N(s')ds'\,, \tag{3.29}$$

$$M(s,s')=\frac{\hat{B}(s)-\hat{B}(s')}{s-s'}q(s')-\frac{\sin^2\pi\alpha}{\pi q(s_0)}\,\frac{\ln(s_0-s)-\ln(s_0-s')}{s-s'}[q(s')-q(s_0)]\,. \tag{3.30}$$

Next, let us change variables as follows:

$$\frac{x+1}{2}=\frac{s_0-4}{s_0-s}, \quad n(x)=\frac{N(s)}{x+1}\,. \tag{3.31}$$

This yields

$$n(x)=a(x)+\left(\frac{\sin\pi\alpha}{\pi}\right)^2\int_1^\infty\frac{\ln[(x'+1)/(x+1)]n(x')dx'}{x'-x}+\int_1^\infty k(x,x')n(x')dx'\,, \tag{3.32}$$

where

$$a(x)=\frac{1}{x+1}\left[B(s)+\sum_{i=1}^{n}c_i\,\frac{B(s)-B(s_i)}{s-s_i}\right]\,, \tag{3.33}$$

$$\int_1^\infty k(x,x')n(x')dx'=\frac{1}{x+1}\,\frac{1}{\pi}\int_4^{s_0}M(s,s')N(s')ds'\,. \tag{3.34}$$

In view of (3.28), a is continuous on $[1, \infty)$ and as $x \to \infty$,

$$a(x) = \begin{cases} O(x^{-1}\ln x), & \alpha \neq 0 \\ O(x^{-1}), & \alpha = 0 . \end{cases} \tag{3.35}$$

According to (3.27), we are interested in solutions of (3.32) such that n is continuous on $[1, \infty)$ and

$$n(x) = 0\left(\frac{1}{x^{1-\alpha}}\right), \quad x \to \infty, \; 0 \leq \alpha < 1 . \tag{3.36}$$

Consequently, it is natural to look for solutions of (3.32) in a Banach space S_α consisting of all real continuous functions $\phi(x)$ on $[1, \infty)$ such that

$$\|\phi\| = \sup |x^{1-\alpha}\phi(x)| < \infty, \quad 0 \leq \alpha < 1 . \tag{3.37}$$

The kernel $k(x, x')$ in (3.32) defines a compact integral operator K on S_α, for any $\alpha \in [0, 1)$, as we demonstrate in Appendix B. The other kernel in (3.32) gives a noncompact operator, because of the logarithmic singularity. It is not amenable to Fredholm theory, but it may be analyzed completely by the special method of the following section.

In the next section, we analyze the equation

$$\phi(x) = g(x) + \left(\frac{\sin\pi\alpha}{\pi}\right)^2 \int_1^\infty \frac{\ln[(x'+1)/(x+1)]\phi(x')dx'}{x'-x}, \tag{3.38}$$

where $g(x)$ is a given continuous function with the bound

$$g(x) = \begin{cases} O(x^{-1+\gamma}), & 0 < \gamma < \frac{1}{2}, \; \gamma < \alpha \\ O(x^{-1}), & \alpha = 0 . \end{cases} \tag{3.39}$$

We find that the general solution of (3.38) in the space S_α has the form

$$\phi(x) = c\,\theta(\alpha - \tfrac{1}{2})P_{-1+\alpha}(x) + g(x) + \int_1^\infty l(x, x')g(x')dx' , \tag{3.40}$$

where θ is the step function $[\theta(\xi) = 1, \; \xi \geq 0, \; \theta(\xi) = 0, \; \xi < 0]$, c is an arbitrary real constant, and P_ν is the Legendre function of degree ν. The kernel l is defined in Eq. (4.26); the corresponding integral operator is called L. If $n \in S_\alpha$, then $g = a + Kn$ in (3.32) obeys (3.39) with $\gamma = \alpha - \mu$, where μ is the exponent of Hölder continuity of B, assumed to satisfy $\mu > \alpha - \frac{1}{2}$. This property of $g = a + Kn$ follows from (3.35) and the analysis of K in Appendix B. Now if $n \in S_\alpha$ is a solution of the singular equation (3.32), we see from (3.32), (3.38), and (3.40) that n must also satisfy

$$n(x) = c\,\theta(\alpha - \tfrac{1}{2})P_{-1+\alpha}(x) + a(x) + \int_1^\infty k(x, x')n(x')dx' + \int_1^\infty l(x, x')\left[a(x') + \int_1^\infty k(x', x'')n(x'')dx''\right]dx' . \tag{3.41}$$

For $\alpha \geq \frac{1}{2}$ a new arbitrary constant c enters the inhomogeneous term. In Appendix B we show that the operator $(1+L)K$ appearing in (3.41) is compact on S_β, where $\beta > \alpha$ and $0 < \beta < 1$. Since S_α is a subspace of S_β, the solutions of interest may be found by studying the equation on S_β by means of standard Fredholm theory. Let us suppose henceforth that $(1+L)K$, regarded as an operator on S_β, does not have 1 as an eigenvalue. One expects that unit eigenvalues will not normally occur in applications of the theory. Then (3.41) will have a unique solution in S_β when the parameters (c, c_i, s_i) are specified. Furthermore, the argument of Sec. IV shows that this solution actually belongs to S_α (if $\beta - \alpha$ is sufficiently small) and in general[20] has the asymptotic behavior $n(x) \sim \kappa x^{-1+\alpha}$, $\kappa \neq 0$. If there are no ghosts, this latter behavior implies that Eqs. (3.8) and (3.9) hold, where δ is the phase shift computed from the N/D method, n is the number of CDD poles, and α is such that $\sin^2\pi\alpha$ is the input parameter in (3.28). The proof of this assertion follows from the representation (3.10) of the D function, which is valid for the D function computed from a solution of (3.18), provided that there are no ghosts. We see that a solution in S_β of the N equation (3.18) generally leads to a solution of the Roy equation (for small $\beta - \alpha$), and the latter belongs to a manifold of solutions depending on d real parameters, where

$$d = 2n + \theta(\alpha - \tfrac{1}{2}) = [2\delta(s_0)/\pi] , \qquad n = [\delta(s_0)/\pi], \quad \alpha = \delta(s_0)/\pi - n . \tag{3.42}$$

To discuss the case $\delta(s_0) < 0$, we first seek solutions such that $-1 < \delta(s_0)/\pi < 0$. That is, $m = 0$ and $0 < \beta < 1$ in (3.19) and (3.20). The function $\mathfrak{D}$ vanishes at $s = s_0$, which suggests that it is not a convenient denominator function; the constraint $\mathfrak{D}(s_0) = 0$ would have to be imposed as a separate condition. We choose instead $D^{(1)}$ of Eq. (3.21):

$$D(s) = D^{(1)}(s) = (s - s_0)^{-1}\mathfrak{D}(s) . \tag{3.43}$$

Thus, we look for solutions such that

$$D(s) \sim \kappa(s_0 - s)^{-1+\beta}, \quad N(s) \sim \kappa f(s_0)(s_0 - s)^{-1+\beta}, \quad s \to s_0, \; 0 < \beta < 1 . \tag{3.44}$$

Since D vanishes at infinity it has a Cauchy representation of the type (3.22), and the corresponding N function satisfies

$$N(s) = \frac{1}{\pi}\int_4^{s_0} \frac{B(s) - B(s')}{s - s'} q(s')N(s')ds' . \tag{3.45}$$

In terms of the function $n(x)$ this equation takes the form (3.32) with $a = 0$. Since the input parameter $\sin^2\pi\alpha$ in (3.32) is to be identified with $\sin^2\pi\beta$, the analysis of Sec. IV shows that n is obtained by solving the Fredholm equation

$$n(x)=cP_{-\beta}(x)+\int_1^\infty k(x,x')n(x')dx' +\int_1^\infty l(x,x')dx'\int_1^\infty k(x',x'')n(x'')dx'' . \tag{3.46}$$

According to (3.44), the desired behavior of n at large x is

$$n(x)\sim\kappa x^{-\beta},\quad 0<\beta<1 . \tag{3.47}$$

The term $cP_{-\beta}$ in (3.46) is then allowed only if $0<\beta\leq\frac{1}{2}$, since $P_{-\beta}$ never decreases faster than $x^{-1/2}$. For $0<\beta\leq\frac{1}{2}$ and $c\neq0$ one has a unique solution of (3.46) [we maintain our assumption that $(1+L)$ does not have a unit eigenvalue], and that solution is proportional to c. According to (3.22), D is also proportional to c, so that $f=N/D$ is independent of c. Thus there are no free parameters associated with solutions for which $-\frac{1}{2}\pi\leq\delta(s_0)<0$. Combining this with our previous results we see that there are no free parameters associated with solutions for which

$$-\tfrac{1}{2}\pi\leq\delta(s_0)<\tfrac{1}{2}\pi . \tag{3.48}$$

The situation changes drastically if $\delta(s_0)<-\frac{1}{2}\pi$. Then we must have $c=0$ in (3.46), and there must be a unit eigenvalue of $(1+L)$ for a solution to exist at all. There is no reason to expect such eigenvalues to occur, and in general one may doubt the existence of solutions to the Roy equation for $\delta(s_0)<-\frac{1}{2}\pi$. It is interesting that experimental phase shifts in π-π scattering are either positive or negative but small; they do not fall below $-\frac{1}{2}\pi$.

To look for a solution with $m\geq1$ (and $\beta>0$, say) one would consider N equations for the $m+1$ functions $N^{(j)}$ corresponding to the $D^{(j)}$ of (3.22). It is easy to see that these equations become more and more restrictive as m increases. In general the existence of solutions demands that $(1+L)$ have multiple eigenvectors with eigenvalue 1.

To check for ghosts when the amplitude is computed by means of (3.46), one argues as in (3.25) ff. As long as the computed phase shift satisfies $-\frac{1}{2}\pi\leq\delta(s_0)<0$, there will be no ghost.

To summarize the work of this and the following section, we may say that solutions of the Roy equation with $\delta(s_0)\geq-\frac{1}{2}\pi$ are members of solution manifolds depending on d real parameters, where d is given by (3.42). Furthermore, these solutions may be computed through the integral equations (3.41) and (3.46), in which the d parameters appear explicitly. An exception to these statements could occur if the Fredholm operator $(1+L)$ had a unit eigenvalue, but there is no reason to expect that to happen.

IV. SINGULAR PART OF THE N/D OPERATOR

We wish to find the general solution of Eq. (3.38) in the space S_α. Our method is based on the observation that the Legendre function $P_\nu(x)$ has the following Cauchy representation, for $-1<\mathrm{Re}\,\nu<0$ (Ref. 21):

$$\begin{aligned}P_\nu(x)&=-\frac{\sin\pi\nu}{\pi}\int_1^\infty\frac{dt\,P_\nu(t)}{t+x}\\&=\left(\frac{\sin\pi\nu}{\pi}\right)^2\int_1^\infty\frac{dt}{t+x}\int_1^\infty\frac{du}{u+t}P_\nu(u)\\&=\left(\frac{\sin\pi\nu}{\pi}\right)^2\int_1^\infty du\,P_\nu(u)\,\frac{\ln[(x+1)/(u+1)]}{x-u} .\end{aligned}\tag{4.1}$$

Recall also that

$$P_\nu(x)=P_{-1-\nu}(x) \tag{4.2}$$

and that the asymptotic behavior of P_ν is

$$P_\nu(x)\sim\kappa x^\nu,\quad x\to\infty,\ \kappa\neq0,\ \nu\geq-\tfrac{1}{2} . \tag{4.3}$$

By (4.2) and (4.3) together we have

$$P_\nu(x)\sim\kappa x^{-1/2+|\nu+1/2|},\quad x\to\infty , \tag{4.4}$$

for all real ν.

We see from (4.1) that $P_{-1+\alpha}(x)$ is a solution in S_α of the homogeneous form of (3.38) [i.e., of (3.38) with $g(x)=0$], provided that $\frac{1}{2}\leq\alpha<1$. According to (4.4), no $P_\nu(x)$ decreases faster than $x^{-1/2}$, which means that we do not get a solution in S_α with $\alpha<\frac{1}{2}$ by means of the identity (4.1). In fact, it is possible to show that in S_α the homogeneous form of (3.38) has only the trivial solution for $0\leq\alpha<\frac{1}{2}$, and only the solution $P_{-1+\alpha}(x)$ for $\frac{1}{2}\leq\alpha<1$. This assertion is proved by an elementary method in Appendix C, except for the case $\alpha=\frac{1}{2}$. The case $\alpha=\frac{1}{2}$ requires a more involved proof.

If ϕ is any solution of (3.38), and ψ a particular solution, then $\phi-\psi$ solves the homogeneous equation. The general solution in S_α is then

$$\phi(x)=c\,\theta(\alpha-\tfrac{1}{2})P_{-1+\alpha}(x)+\psi(x) , \tag{4.5}$$

where ψ is any fixed solution in S_α, and $-\infty<c<\infty$. To find a particular solution ψ we make use of the Mehler transform,

$$\hat{\psi}(y)=\int_1^\infty dx\,P_{-1/2+iy}(x)\psi(x) , \tag{4.6}$$

which is defined for $-\infty<y<\infty$ provided

$$\psi(x)=O(x^{-\beta}),\quad x\to\infty,\ \beta>\tfrac{1}{2} . \tag{4.7}$$

Let us try to find a continuous solution ψ obeying the bound (4.7). Proceeding formally, we take the Mehler transform of (3.38) and make use of (4.1).

This yields

$$\hat{\psi}(y)=\hat{g}(y)+\left(\frac{\sin\pi\alpha}{\cosh\pi y}\right)^2\hat{\psi}(y), \quad (4.8)$$

or

$$\hat{\psi}(y)=\hat{g}(y)+\frac{\sin^2\pi\alpha\,\hat{g}(y)}{\cosh^2\pi y-\sin^2\pi\alpha}. \quad (4.9)$$

We hope to find a solution by taking the inverse Mehler transform of the right-hand side of (4.9). Mehler transforms obey the following inversion theorem[22]: Suppose that for all $a>1$,

$$f(t)(t-1)^{-1/4}\ln(t-1)\in L(1,a),$$

$$f(t)t^{-1/2}\in L(a,\infty),$$

where $L(\alpha,\beta)$ is the class of functions Lebesgue-integrable on (α,β). Suppose also that $f(t)$ is of bounded variation in a neighborhood of $t=x$. Then

$$\tfrac{1}{2}[f(x+0)+f(x-0)] = \tfrac{1}{2}\int_{-\infty}^{\infty} y\,dy\tanh\pi y\,P_{-1/2+iy}(x)\hat{f}(y). \quad (4.10)$$

Since g obeys the conditions of the theorem, the inverse transform of (4.9) yields

$$\psi(x)=g(x)+\frac{\sin^2\pi\alpha}{2}\int_{-\infty}^{\infty}\frac{y\,dy\tanh\pi y}{\cosh^2\pi y-\sin^2\pi\alpha}\times P_{-1/2+iy}(x)\hat{g}(y). \quad (4.11)$$

One can now show by direct substitution that ψ as given by (4.11) indeed satisfies (3.38). Let I denote the second term on the right-hand side of (4.11). We must show that

$$I=I_1+I_2, \quad (4.12)$$

$$I_1=\left(\frac{\sin\pi\alpha}{\pi}\right)^2\int_1^{\infty}\frac{dx'\ln[(1+x')/(1+x)]g(x')}{x'-x}, \quad (4.13)$$

$$I_2=\left(\frac{\sin\pi\alpha}{\pi}\right)^2\int_1^{\infty}\frac{dx'\ln[(1+x')/(1+x)]}{x'-x}\frac{\sin^2\pi\alpha}{2}\times\int_{-\infty}^{\infty}\frac{y\,dy\tanh\pi y\,P_{-1/2+iy}(x)\hat{g}(y)}{\cosh^2\pi y-\sin^2\pi\alpha}. \quad (4.14)$$

In I_2 we may reverse the order of integrations (by virtue of absolute convergence) and then apply (4.1) to eliminate the logarithm. It is then seen that

$$I_2=I-I_3, \quad (4.15)$$

$$I_3=\frac{\sin^2\pi\alpha}{2}\int_{-\infty}^{\infty}\frac{y\,dy\tanh\pi y\,P_{-1/2+iy}(x)\hat{g}(y)}{\cosh^2\pi y}. \quad (4.16)$$

Now it is easy to verify that $I_1=I_3$: Express $g(x')$ in terms of $\hat{g}(y)$, and then apply (4.1). We see that (4.11) satisfies (3.38) for $0\leq\alpha<1$. Since (3.38) and (4.11) involve α only through $\sin^2\pi\alpha$, there is no distinction between the cases $0<\alpha<\frac{1}{2}$ and $\frac{1}{2}<\alpha<1$. A distinction enters only in the solution of the homogeneous equation in the space S_α.

The asymptotic behavior of $\psi(x)$ may be extracted by noticing that $\hat{g}(y)$ is analytic in a strip in the y plane. Since

$$P_{-1/2+iy}(x)=O(x^{-1/2+|\mathrm{Im}y|}),\quad x\to\infty, \quad (4.17)$$

and $P_\nu(x)$ is entire in ν, it follows from (3.39) and (4.6) that $\hat{g}(y)$ is analytic in the strip

$$-\tfrac{1}{2}+\gamma<\mathrm{Im}y<\tfrac{1}{2}-\gamma. \quad (4.18)$$

It is possible to move the y integration contour in (4.11) so as to display the large -x behavior, as in Regge theory. We must first decompose $P_{-1/2+iy}$ according to the well-known formula[21]

$$P_{-1/2+iy}(x)=\frac{i}{\pi}\coth\pi y[Q_{-1/2+iy}(x)-Q_{-1/2-iy}(x)], \quad (4.19)$$

where Q_ν is the Legendre function of the second kind, which is analytic in ν for $\mathrm{Re}\nu>-1$. With $y=u+iv$, the second-kind Legendre function has the uniform bound

$$|Q_{-1/2\pm iy}(x)|\leq\kappa(1+|u|)^{-1/2}x^{-1/2\pm v},\quad x\geq1+\epsilon>1. \quad (4.20)$$

To get the best bound on $\psi(x)$ at large x, one should move the y contour downward for the term in $Q_{-1/2+iy}$, and upward for the term in $Q_{-1/2-iy}$. The zeros of the denominator in (4.11) closest to the real axis are at

$$y=\pm i(\alpha-\tfrac{1}{2}). \quad (4.21)$$

If $0<\alpha<\frac{1}{2}$ we move each contour a small distance δ beyond the corresponding pole; i.e., the contours are to follow the lines

$$\mathrm{Im}y=\pm(|\alpha-\tfrac{1}{2}|+\delta). \quad (4.22)$$

The pole contributions dominate the large-x behavior and

$$\psi(x)=\frac{2(\alpha-\frac{1}{2})\sin^2\pi\alpha}{\pi\sin2\pi(\alpha-\frac{1}{2})}\hat{g}(i(\alpha-\tfrac{1}{2}))Q_{-1/2+|\alpha-1/2|}(x)+O(x^{-1/2-|\alpha-1/2|-\delta}),\quad x\to\infty. \quad (4.23)$$

Since $Q_\nu(x)$ is actually asymptotic to $\kappa x^{-\nu-1}$ for real ν, we have

$$\psi(x)\sim\kappa x^{-1/2-|\alpha-1/2|},\quad x\to\infty, \quad (4.24)$$

if $\hat{g}(i(\alpha-\frac{1}{2}))\neq0$ and $0<\alpha<\frac{1}{2}$. Of the remaining cases $\alpha=0,\leq\frac{1}{2}$, the case $\alpha=0$ is trivial since it gives $\psi(x)=g(x)\sim\kappa x^{-1}$. For $\alpha=\frac{1}{2}$ the upper bound $\psi(x)=O(x^{-1/2})$ is evident from (4.11) without any contour displacements. To get an asymptote $\psi(x) \sim \kappa x^{-1/2}$, rather than a mere upper bound, we note that (4.11) is continuous at $\alpha = \frac{1}{2}$, and that the limit $\alpha \to \frac{1}{2}$ is uniform in x; there is no pole of the integrand at $y = 0$ in the limit, because of the factor $y \tanh \pi y$. We may then simply take the limit $\alpha \to \frac{1}{2}$ of the formula (4.23) to obtain the required asymptote:

$$\psi(x) \sim \frac{\hat{g}(0)}{\pi^2} Q_{-1/2}(x), \quad \alpha = \tfrac{1}{2}. \tag{4.25}$$

For $\alpha > \frac{1}{2}$ we see without moving the contour that $\psi(x) = O(x^{-1/2})$, which is to say that the first term in (4.5) dominates at large x. We may then conclude that $\psi \in S_\alpha$ for $0 \le \alpha < 1$. For the purposes of the argument following Eq. (3.41), we remark that if $n \in S_\beta$, then $g = a + Kn$ satisfies (3.39) with $\gamma = \beta - \mu$, provided $\beta - \alpha > 0$ is chosen to be sufficiently small. Thus, if n belongs to S_β in (3.41), it also belongs to S_α, if β is close enough to α. We also note that ψ satisfies (3.38) if $\alpha = \frac{1}{2}$, even though the integral (4.6) defining its Mehler transform does not converge absolutely in that case.

From (4.11) and (4.6) one sees that the kernel appearing in (3.40) is

$$l(x, x') = \frac{\sin^2 \pi\alpha}{2} \int_{-\infty}^{\infty} \frac{y\, dy \tanh \pi y}{\cosh^2 \pi y - \sin^2 \pi \alpha} \times P_{-1/2+iy}(x) P_{-1/2+iy}(x'). \tag{4.26}$$

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APPENDIX A

In this appendix we write the fixed-t dispersion relations of Roy[1] in the following form:

$$F(s,t) = \tfrac{1}{4}(s + tC_{st} + uC_{su})\alpha + \frac{1}{\pi}\int_4^\infty \frac{ds'}{s'^2} g(s,t,s') A(s',0) + \frac{1}{\pi}\int_4^\infty \frac{ds'}{s'^2}\left[\frac{s^2}{s'-s} + h(s,t,s')\right] A(s',t), \tag{A1}$$

where

$$g(s,t,s') = C_{st}\left(\frac{1+C_{tu}}{2} + \frac{s-u}{t-4}\,\frac{1-C_{tu}}{2}\right)\left[\frac{t^2}{s'-t} + \frac{(4-t)^2 C_{su}}{s'-4+t} - \frac{4t + 4(4-t)C_{su}}{s'-4}\right], \tag{A2}$$

and

$$h(s,t,s') = \frac{u^2}{s'-u} C_{su} - \frac{(4-t)^2}{s'-4+t}\left(\frac{1+C_{su}}{2} - \frac{s-u}{t-4}\,\frac{1-C_{su}}{2}\right). \tag{A3}$$

Here C_{st}, C_{su}, and C_{tu} are the well-known 3×3 isospin crossing matrices, and α is a three-component vector, the first and third components of which are the isospin-0 and isospin-2 S-wave scattering lengths, and the second component of which is zero.

Following the discussion of Ref. 15, we project (A1) onto Legendre polynomials, using the full interval $-1 \le z_s \le 1$. The result is

$$F_l(s) = \frac{1}{\pi}\int_4^\infty \frac{s^2}{s'^2}\,\frac{ds'}{s'-s}\left(\frac{s-4}{s'-4}\right)^l A_l(s') + (s-4)^l C_l(s), \tag{A4}$$

where

$$\begin{aligned} C_l(s) = (s-4)^{-l-1}\int_{4-s}^{0} dt\, P_l\left(1+\frac{2t}{s-4}\right) & \times \left\{\frac{s + tC_{st} + uC_{su}}{4}\alpha + \frac{1}{\pi}\int_4^\infty \frac{ds'}{s'^2} g(s,t,s') \sum_{l'=0}^\infty (2l'+1) A_{l'}(s') \right. \\ & + \frac{1}{\pi}\int_4^\infty \frac{ds'}{s'^2}\,\frac{s^2}{s'-s} \sum_{l'=l+2}^\infty (2l'+1) A_{l'}(s')\left[P_{l'}\left(1+\frac{2t}{s'-4}\right) - P_{l'}\left(1+\frac{2t}{s-4}\right)\right] \\ & \left. + \frac{1}{\pi}\int_4^\infty \frac{ds'}{s'^2} h(s,t,s') \sum_{l'=0}^\infty (2l'+1) A_{l'}(s') P_{l'}\left(1+\frac{2t}{s'-4}\right)\right\}. \end{aligned} \tag{A5}$$

It may be shown that, if $s < 32$, the infinite series converge for all $s' \in [4, \infty)$ and all $t \in [4-s, 0]$. More-over, $C_l(s)$ is an analytic function of s in some neighborhood of the segment $[4, s_0]$ for any $s_0 < 32$. These are the only general properties of $C_l(s)$ that we need in this paper.

APPENDIX B

The Banach space S_α was defined in Sec. III as the set of all real continuous functions $\phi(x)$ on $[1, \infty)$ such that $\|\phi\| = \sup|x^{1-\alpha}\phi(x)|$, the norm on the space, is finite. We show that the operator $(1+L)K$ appearing in (3.41) is compact on S_β, for some $\beta > \alpha$, $0 < \beta < 1$. Here α is the parameter that appears in the kernel k as defined in (3.34) and (3.30), and in the kernel l defined in (4.26). The function $\hat{B}(s)$ in (3.30) is assumed to be Hölder-continuous with exponent μ on the closed interval $[4, s_0]$.

We first show that K is compact on S_β, for any $\beta \in (0,1)$. Let $\{\phi_n\}$ be any bounded sequence in S_β; i.e., $\|\phi_n\| < a$. We must show that $\{K\phi_n\}$ has a convergent subsequence. Consider the sequence $\{\psi_n\}$ of functions of s,

$$\psi_n(s) = x^{1-\beta} K\phi_n(x), \quad 4 \le s \le s_0, \tag{B1}$$

where s and x are related as in (3.31). If the $\psi_n(s)$ are uniformly bounded and equicontinous, then there is a subsequence $\{\psi_{n_k}\}$ which converges uniformly to a continuous limit $\psi(s)$, by the Ascoli-Arzelâ theorem. Then $\{K\phi_{n_k}\}$ also converges to a limit with respect to the metric induced by the norm on S_β.

To show that the $\psi_n(s)$ are uniformly bounded, we first note that

$$|\psi_n(s)| = \left|\frac{x^{1-\beta}}{x+1}\int_4^{s_0} M(s,s')(x'+1)\phi_n(x')ds'\right| \le \kappa\|\phi_n\|(s_0-s)^\beta \int_4^{s_0} |M(s,s')|(s_0-s')^{-\beta}ds'. \tag{B2}$$

The two terms of M, defined in (3.30) may be majorized as follows:

$$|M_1(s,s')| = \left|\frac{\hat{B}(s)-\hat{B}(s')}{s-s'}q(s')\right| \le \kappa|s-s'|^{-1+\mu}, \tag{B3}$$

$$|M_2(s,s')| = \kappa\left|\frac{\ln(s_0-s)-\ln(s_0-s')}{s-s'}[q(s')-q(s_0)]\right|$$

$$\le \begin{cases} \kappa, & s \le s' \\ \kappa\left(\dfrac{1}{s_0-s}\right)^\delta \left|\dfrac{\ln(s_0-s)}{s-s'}\right|^{1-\delta}|s_0-s'|, & s > s',\ 0<\delta<1. \end{cases} \tag{B4}$$

The contribution of M_1 to the right-hand side of (B2), call it $m_1(s)$, has the bound

$$m_1(s) = \kappa(s_0-s)^\beta \int_4^{s_0} |s'-s|^{-1+\mu}(s_0-s')^{-\beta}ds' = \kappa(s_0-s)^\mu \int_0^{(s_0-4)/(s_0-s)} |1-v|^{-1+\mu}v^{-\beta}dv \le \kappa(s_0-s)^\mu, \tag{B5}$$

where the last inequality is true if $\beta > \mu$. For convenience in notation we suppose that $\beta > \mu$. This entails no loss of generality, since a function Hölder-continuous with exponent μ is also Hölder-continuous with any positive exponent $\mu' < \mu$. By (B5) the contribution of M_1 to ψ_n is uniformly bounded, and it is easy to see from (B4) that the contribution of M_2 has the same property.

To prove equicontinuity of the ψ_n, we show that

$$|\psi_n(s_1) - \psi_n(s_2)| \le \kappa|s_1-s_2|^\delta, \tag{B6}$$

for some $\delta > 0$, with κ independent of n, s_1, s_2. A calculation using the relation of x to s shows that

$$|\psi_n(s_1)-\psi_n(s_2)| \le \kappa|s_1-s_2|^\beta(s_0-s_1)^\beta \int_4^{s_0}|M(s_1,s')|(s_0-s')^{-\beta}ds'$$
$$+\kappa(s_0-s_2)^\beta\int_4^{s_0}|M(s_1,s')-M(s_2,s')|(s_0-s')^{-\beta}ds'. \tag{B7}$$

By the analysis given above, the first term on the right-hand side has a bound like (B6). To handle the second term, we split the exponent. With a small η, $0<\eta<1$, we write

$$|M(s_1,s')-M(s_2,s')|^{\eta+(1-\eta)} \le [|M(s_1,s')|^{1-\eta}+|M(s_2,s')|^{1-\eta}]|M(s_1,s')-M(s_2,s')|^\eta. \tag{B8}$$

For the $1-\eta$ powers we introduce the majorizations (B3), (B4), while for the η power we use (B3) and the mean-value theorem to bound differences as follows:

$$|M_1(s_1,s')-M_1(s_2,s')| \le \frac{\kappa}{|s_1-s'|^\eta}\left(|s_1-s_2|^{\mu\eta}+\frac{|s_1-s_2|^\eta}{|s_2-s'|^{(1-\mu)\eta}}\right), \tag{B9}$$

$$|M_2(s_1,s')-M_2(s_2,s')|\le\kappa\left|\frac{s_2-s_1}{s'-s}\left[\frac{1}{s_0-s}+\frac{\ln(s_0-s)-\ln(s_0-s')}{s-s'}\right]\right|,\quad s_1<s<s_2\,. \tag{B10}$$

When these results are introduced in (B7), one finds that all singularities are integrable for sufficiently small η, and the desired bound (B6) comes out. The proof of compactness of K is finished, and it remains to show that LK is compact.

We must show that the $\chi_n(s)$ are uniformly bounded and equicontinuous, where

$$\chi_n(s)=x^{1-\beta}LK\phi_n(x)=\tfrac{1}{2}x^{1-\beta}\sin^2\pi\alpha\int_{-\infty}^{\infty}\frac{y\,dy\,\tanh\pi y}{\cosh^2\pi y-\sin^2\pi\alpha}P_{-1/2+iy}(x)f_n(y)\,, \tag{B11}$$

$$f_n(y)=\int_1^\infty dx\,P_{-1/2+iy}(x)K\phi_n(x)\,. \tag{B12}$$

If $\alpha\ge\frac{1}{2}$, the required estimates are obtained immediately if $\beta>\alpha$; the y integral converges exponentially, and one has only to invoke the known behavior of $P_\nu(x)$ and $P'_\nu(x)$ at large x and large $\mathrm{Im}\nu$. If $\alpha<\frac{1}{2}$, the y-integration contour must be displaced, as in (4.17) ff. By (B5) and a similar bound for the contribution of M_2 we find that

$$|P_{-1/2+iy}(x)K\phi_n(x)|\le\kappa x^{-3/2+\beta-\mu+|\mathrm{Im}y|}\,, \tag{B13}$$

from which it follows that $f_n(y)$ is analytic in a strip

$$|\mathrm{Im}y|<\tfrac{1}{2}-\beta+\mu\,. \tag{B14}$$

The nearest poles in the integrand of (B11) are at $y=\pm i(\frac{1}{2}-\alpha)$, so that for $\alpha<\frac{1}{2}$ we may move the integration path beyond the poles if we make an appropriate choice of β, namely, $\alpha<\beta<\alpha+\mu$. The behavior of the integral (B11) at large x is then obtained from (4.11) and (4.23). We see that $\chi_n(s)$ behaves as $(s_0-s)^{\beta-\alpha}$ at $s=s_0$. One easily establishes uniform boundedness and equicontinuity of the χ_n, through simple estimates of Legendre functions. The restriction $\beta>\alpha$ is needed for equicontinuity at at $s=s_0$ in the case $\alpha\le\frac{1}{2}$, whereas for $\alpha>\frac{1}{2}$ one may take $\alpha=\beta$.

APPENDIX C

Here we wish to study the homogeneous integral equation

$$\phi(x)=\frac{\lambda}{\pi}\int_1^\infty\frac{dt}{t+x}\,\phi(t) \tag{C1}$$

and the first iteration of it, namely

$$\phi(x)=\left(\frac{\lambda}{\pi}\right)^2\int_1^\infty du\,\frac{\ln[(x+1)/(u+1)]}{x-u}\,\phi(u)\,. \tag{C2}$$

We limit our search for solutions to the Banach space, S_β, of continuous functions with bounded norm of the type

$$\|\phi\|=\sup_{1\le x<\infty}|x^{1-\beta}\phi(x)|\,, \tag{C3}$$

for some $\beta\in(0,1)$.

We shall show shortly that (C1) has only the trivial solution if the real part of λ is negative. If λ has a positive real part, then there is only one independent solution in S_β, namely

$$\phi(x)=P_\nu(x)\Theta(\beta-1-\mathrm{Re}\nu)\,, \tag{C4}$$

where ν is defined uniquely by

$$\lambda=-\sin\pi\nu,\quad 0>\mathrm{Re}\nu\ge-\tfrac{1}{2},\ \mathrm{Im}\nu\ge0\,. \tag{C5}$$

It is trivial that (C4) is also a solution of (C2). We shall now show that it is the most general independent solution of (C2). For suppose that $\phi(x)$ is any solution of (C2), which we may write in the form

$$\phi=\left(\frac{\lambda}{\pi}F\right)^2\phi\,. \tag{C6}$$

Since the same equation is obtained if we replace λ by $-\lambda$, we may limit our attention for convenience to the right half plane, $\mathrm{Re}\lambda>0$. Define

$$\xi=\phi-\frac{\lambda}{\pi}F\phi\,, \tag{C7}$$

then (C6) implies

$$\xi=-\frac{\lambda}{\pi}F\,\xi\,. \tag{C8}$$

Now when $\mathrm{Re}\,\lambda>0$, (C8) has only the trivial solution; but this means that the left-hand side of (C7) is necessarily zero, so that ϕ, an arbitrary solution of (C2), also satisfies (C1).

We see thus that, although we are primarily interested in the iterated equation (C2), we may limit our attention to (C1). We shall give the proof now that (C1) has no nontrivial solution if $\mathrm{Re}\,\lambda<0$, and at most one independent solution in S_β if $\mathrm{Re}\,\lambda>0$. Suppose that $\phi(x)$ belongs to S_β, and that it satisfies (C1). Then the number

$$\phi(0)=\frac{\lambda}{\pi}\int_1^\infty\frac{dt}{t}\,\phi(t) \tag{C9}$$

is certainly finite, and the number

$$I=\int_1^\infty\frac{|\phi(x)|^2}{x}\,dx \tag{C10}$$

is finite, and positive. We may use (C1) to rewrite this in the form

$$I=\frac{\lambda}{\pi}\int_1^\infty dx\int_1^\infty dt\,\frac{\phi^*(x)\phi(t)}{x(t+x)}. \tag{C11}$$

Since I is real, we may freely complex-conjugate the right-hand side of (C11), so that also

$$\begin{aligned}I&=\frac{\lambda^*}{\pi}\int_1^\infty dx\int_1^\infty dt\,\frac{\phi(x)\phi^*(t)}{x(t+x)}\\&=\frac{\lambda^*}{\pi}\int_1^\infty dx\int_1^\infty dt\,\phi^*(t)\phi(x)\left(\frac{1}{x}-\frac{1}{t+x}\right)\frac{1}{t}\\&=\frac{\pi}{\lambda}\,|\phi(0)|^2-\frac{\lambda^*}{\pi}\int_1^\infty dx\int_1^\infty dt\,\frac{\phi^*(x)\phi(t)}{x(t+x)},\end{aligned} \tag{C12}$$

where we have interchanged the dummy integration variables in the last step. On comparing (C11) with (C12), we see that

$$(\lambda+\lambda^*)I=\pi|\phi(0)|^2, \tag{C13}$$

and from this it is clear that the real part of λ must be non-negative unless $\phi(x)$ vanishes identically.

In the text we are exclusively interested in real values of λ in the interval $[0,1]$. We shall now give an elementary proof that if $|\lambda|<1$ there cannot be more than one independent solution of (C1). This covers all cases of interest, except the point $\lambda=1$. In fact the uniqueness theorem holds in the whole complex λ plane, but more powerful methods are needed to treat the case $|\lambda|\geq 1$. Suppose then that $|\lambda|<1$, $\mathrm{Re}\lambda>0$, and that $\phi(x)$ is a nontrivial solution of (C1). Then

$$\frac{\phi(x)}{x}=\frac{\phi(0)}{x}-\frac{\lambda}{\pi}\int_1^\infty\frac{dt}{t+x}\,\frac{\phi(t)}{t}, \tag{C14}$$

where we have used the definition (C9). Now $\phi(0)$ cannot vanish, since if it did $\phi(x)/x$ would satisfy the homogeneous equation with $-\lambda$ in place of λ, and we know that that equation has only the trivial solution. Hence we can define

$$\psi(x)=\frac{1}{x}\,\frac{\phi(x)}{\phi(0)}, \tag{C15}$$

so that

$$\psi(x)=\frac{1}{x}-\frac{\lambda}{\pi}\int_1^\infty\frac{dt}{t+x}\,\psi(t). \tag{C16}$$

We shall show that (C16) has a unique solution, $\psi\in S_{1/2}$, and it then follows that there cannot be more than one independent solution ϕ of (C1) in S_β, for if there were more independent solutions, these would yield independent solutions, $\psi\in S_{1/2}$, of (C16).

The uniqueness proof for (C16) is a simple application of the Banach-Cacciopoli contraction mapping theorem.[5,7] Set

$$P[\psi;x]=\frac{1}{x}-\frac{\lambda}{\pi}\int_1^\infty\frac{dt}{t+x}\,\psi(t). \tag{C17}$$

Clearly if ψ belongs to $S_{1/2}$, so does $P[\psi]$. If ψ_1 and ψ_2 belong to $S_{1/2}$, then

$$\|P[\psi_1]-P[\psi_2]\|_{1/2}\leq|\lambda|\,\|\psi_1-\psi_2\|_{1/2}, \tag{C18}$$

so we have contraction if $|\lambda|<1$.

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[4]A frequent viewpoint in numerical work with S-matrix equations is that it is sufficient to find a solution good to a few percent, since the theoretical uncertainty about inputs is at least that great. It seems to us that this view needlessly compounds mathematical and physical uncertainties. For given inputs one should try to establish that an apparent approximate solution is close to a true solution. For this question, a study of a local linearization of the equation at the proposed approximate solution is usually quite informative.

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[11]In making a comparison one should keep in mind that we regard the absorptive part for $s\geq s_0$ as fixed when we state the dimension of the solution manifold, where-

as Pomponiu and Wanders regard it as variable. Thus, for the $I=1$ P wave with $\delta(s_0)=160°$ they have a two-parameter manifold, but one of the parameters corresponds to variation of the input value of $\delta(s_0)$ itself. We prefer to call this a one-parameter manifold, since there is actually an infinite-dimensional parameter (an arbitrary function) associated with variations of high-energy absorptive parts. It is only because of an approximation that this arbitrary function is replaced by a single number in Ref. 8. Pomponiu and Wanders compare their two parameters of the $I=1$ P wave to the two parameters of a CDD pole, but this seems inappropriate. In our discussion we have the Pomponiu-Wanders parameters, and also CDD pole parameters, and they are distinct.

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[20]According to the discussion in Sec. IV, the asymptotic behavior $n(x)\sim\kappa x^{-1+\alpha}$, $\kappa\neq 0$, will not occur if the coefficient $\hat{g}(i(\alpha-\frac{1}{2}))$ in (4.23) happens to be zero. In that case, we cannot find an exact asymptote for $n(x)$, but we can prove the upper bound $n(x)=O(x^{-1+\delta})$, all $\delta>0$; i.e., $N(s)=O((s_0-s)^{-\delta})$. This case can occur if the phase shift oscillates about a multiple of π, so that the number of points s_i such that $\sin\delta(s_i)=0$ is larger than the minimum number implied by the value of $\delta(s_0)$. Suppose, for instance, that $\delta(s_1)=\delta(s_2)=\pi$, $s_1, s_2<s_0$, and $[\delta(s_0)/\pi]=1$, $\alpha>0$. Then we may form the denominator function $\tilde{D}(s)=(s-s_0)^2(s-s_1)^{-1}(s-s_2)^{-1}\mathfrak{D}(s)$ which is $O((s_0-s)^{1-\alpha})$ at $s=s_0$. The corresponding $\tilde{N}(s)$ is $O((s_0-s)^{1-\alpha})$, which is compatible with the bound we obtain when $\hat{g}(i(\alpha-\frac{1}{2}))=0$. Of course, there are also D functions which are $O((s_0-s)^{-\alpha})$, for instance $D(s)=(s-s_0)(s-s_1)^{-1}\mathfrak{D}(s)$, but the corresponding N functions satisfy integral equations having inhomogeneities different from that of the equation for $\hat{N}$. Thus, if $\hat{g}(i(\alpha-\frac{1}{2}))=0$ in a particular application of our method, one should check the phase-shift behavior to see if there is another equivalent N equation for which $\hat{g}(i(\alpha-\frac{1}{2}))\neq 0$.

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