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Strong-coupling solutions in an isotopic eikonal problem

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An eikonal representation for small-angle NN scattering is constructed in the strong-coupling limit of the simplest SU(2) massive-vector-meson exchange model. One finds that multiple charged exchanges corresponding to net isospin transfers different from zero tend to be washed out in this limit. The isotopic triplet and singlet amplitudes have the form $T^{(I)}_{\text{eik}}(s,t) = (is/2m^2)\int d^2b\, e^{i\vec{q}\cdot\vec{b}}[1-\exp(i\chi_0^{(S)} + i\langle\vec{\tau}_1\cdot\vec{\tau}_2\rangle\chi_0^{(V)})]\exp[Q^{(I)}(b,\mu,q,s)]$, where $\chi_0^{(S)}(b)$ is the typical eikonal function of isoscalar exchange, $\chi_0^{(V)}(b)$ is the same function with coupling and mass parameters of the isovector exchange, and $\langle\vec{\tau}_1\cdot\vec{\tau}_2\rangle = 2I(I+1)-3$. Very crude estimates, especially for the singlet case, suggest that $Q^{(1)} \simeq -\ln(1+\phi^2)$, and $Q^{(0)} \simeq -(1/2)\ln[1+(4\phi)^2]$, with $\phi \simeq (1/\pi)\ln(\mu b)$, $1 > q/\sqrt{s} > \mu b$, and $\phi \simeq q/\mu b(2\pi s)^{1/2}$, $1 > \mu b > q/\sqrt{s}$. A qualitative discussion is given of the expected small effect of multiple isovector-meson exchange on form-factor damping at large momentum transfers.

I. INTRODUCTION

One of the long-standing problems in high-energy physics has been the construction of an eikonal representation for scattering and production amplitudes when the many quanta exchanged between or emitted from scattering particles can themselves carry isospin, or other quantum numbers. Starting from an underlying field theory, one has been able to give recipes[1] for the construction of amplitudes, assuming that the objects exchanged form the singlet representation of a relevant symmetry group. The methods used cannot apply to the exchange of higher representations, because emission or absorption of such quanta will violate the very fundamental combinatoric demand of a lack of correlation between such quanta. Typically, the only indirect correlations between such "soft" quanta are those of energy-momentum conservation. An exception to this statement was given by Weinberg,[2] who constructed an eikonal representation for soft chiral pions by the special mechanism of including a sufficient number of resonant nuclear states, in a manner chosen to preserve the over-all chiral invariance of the problem. Another, and considerably less ingenious variant, has been the eikonalization of amplitudes constructed from soft-pion-pair emission,[3] where one effectively maintains the idea of isosinglet exchange by permitting the emission of pairs of soft pions, each pair with a net charge zero. One can write phenomenological, and unitary, eikonal amplitudes which include the effects of isospin[4]; but starting from a given field theory, the general problem of constructing an eikonal representation for arbitrary isospin exchange has never been solved.

One need not belabor the importance of such representations; rather, it may suffice to point to just two examples of current interest. The first concerns the recent suggestion of Cornwall and Tiktopoulos[5] that infrared effects can be responsible for color-nonzero hadron confinement. The noncompensating effects (between elastic and inclusive cross sections) described there very much suggest that a certain measure of eikonal solubility may be expected in a massless-gluon, SU(3) Yang-Mills theory. In view of the present virtual certainty that underlying quark fields exist, and the high probability that their interactions may be described within an asymptotically free gauge theory, it would be most useful to have a method of construction of eikonal amplitudes in this and other non-Abelian theories.

A second example is the more phenomenological treatment of large-momentum-transfer processes by the virtual exchange and real emission of massive, low-energy ("soft") vector mesons, soft compared to the incident hadrons of the particular reaction studied. Essentially all large-p_t physics,[6] including same-side correlations in p-p induced jets,[7] can be qualitatively and simply described by imagining that an increasing bremsstrahlung of massive vector mesons (which immediately decay into sprays of pions), from the scattering nucleons, forms the important inelastic production process, as relevant momentum transfers increase.[8] This picture of large-p_t interactions may be thought of as the other side of the coin of the basic, quark-gluon effects, or interchanges,[9] for it deals with the multiple exchange of observed particles, and resonances, rather than with the exchange of their fundamental constituents. In a sensible world, these descriptions should be equivalent; or at least they should coincide below some very large energies, as indeed they seem to do.

The qualitative success of the massive, vector-meson bremsstrahlung model has, however, one serious flaw: It has been formulated only for the exchange of soft, neutral, vector mesons, for example ω's; for reasons alluded to above, it cannot include exchange of the isotriplet $\rho^{\pm}, \rho^0$. Because the ω-nucleon coupling is, experimentally, somewhat larger than the corresponding ρ-nucleon coupling, it may be a reasonable approximation to neglect the ρ's; but as a matter of principle, the argument is incomplete.

The present paper, motivated by the above remarks, provides a partial and very limited solution to the problem of eikonal construction that includes SU(2) isotopic exchanges. There may well be generalizations to the case of higher groups, with complicated gluon self-interactions, but such material is beyond the scope of the present work. What is given here is a construction of an approximate eikonal scattering amplitude, supposedly valid for small momentum transfers in the large-coupling limit of a massive gluon theory. It is obtained by a sequence of approximations which are probably reasonable in the limit of large couplings and/or strong fields. The same construction is then repeated for the soft, vector-meson exchanges which may be expected to provide damping for form factors at large momentum transfers; and by simple extension to the corresponding wide-angle scattering amplitudes (basically simple Born approximations, multiplied by effective form-factor functions in all s, t, u channels).

The results are interesting and reassuring. Although one cannot claim that a precise evaluation of the rather complicated integrals has been performed, qualitative arguments suggest that the present strong-coupling isovector exchange does not significantly modify previous estimates involving isoscalar exchange. The way in which this comes about is quite different in the small-angle and wide-angle cases. In fact, initial estimates in the small-momentum-transfer situation, seem to suggest a strong isovector damping effect; but a more detailed examination, which questions the typical approximations used in eikonal derivations, argues that such damping is not actually present. The reason is simply that there occur functions of the possible lengths $E^{-1}, \mu^{-1}, q^{-1}, b$, which are quite sensitive to small values of the momentum transfer q; and one finds that the conventional $q \to 0$ limit, taken almost everywhere in the usual eikonal derivations, must be postponed until the very end. For form factors, and other processes at large momentum transfers, the evaluation is somewhat different, yielding terms which tend to cancel the isovector dependence between themselves, in this strong-coupling model. More precise evaluations would certainly be welcome; but the qualitative arguments employed suggest that strongly coupled isovector processes lead to eikonal representations not significantly different from those given by multiple isoscalar exchange.

The arrangement of these remarks is such that a very brief review of the basic eikonal formulation, up to the conventional stumbling block of isotopic dependence, is presented in Sec. II. This is followed, in Sec. III, by the approximate evaluation of the relevant nucleon propagators, defined in the presence of an external isotopic source, in the strong-coupling limit. These forms are then used, in Sec. IV, in conjunction with an approximate rendering of the necessary functional integration, one that should be reasonable as a strong-field/semiclassical approximation. In Sec. V, qualitative evaluations of the resulting forms are given, for both small-angle and wide-angle processes. Section VI deals with isotopic projections and integrability conditions, which appear when both isovector and isoscalar exchanges are included; and is followed by a brief summary.

II. BASIC FORMULATION

One begins with the simplest theory, defined by the SU(2)-invariant interaction Lagrangian

$$\mathcal{L}'[\bar\psi, A, \psi] = ig\bar\psi\gamma_\mu A_\mu^\alpha \tau_\alpha \psi\,, \tag{1}$$

where massive meson fields A_μ^α interact with nucleon fields $\psi, \bar\psi$; the τ_α represent the three Pauli matrices. All other interactions are neglected. It is convenient to write the formal solution for the generating functional

$$\mathfrak{z}\{j, \eta, \bar\eta\} = \langle 0 | (\exp[i\int (j_\mu^\alpha A_\mu^\alpha + \bar\eta\psi + \bar\psi\eta)])_+ | 0\rangle$$

in the form[10]

$$\mathfrak{z} = N^{-1} \exp\left[i\int \mathcal{L}'\left(-\frac{1}{i}\frac{\delta}{\delta\eta}, \frac{1}{i}\frac{\delta}{\delta j_\mu^\alpha}, \frac{1}{i}\frac{\delta}{\delta\bar\eta}\right)\right]$$

$$\times \exp\left(i\int \bar\eta S_c \eta + \frac{i}{2}\int j_\mu^\alpha \Delta_{c,\mu\nu}^{\alpha\beta} j_\nu^\beta\right), \tag{2}$$

where S_c and $\Delta_{c,\mu\nu}^{\alpha\beta} = \delta_{\mu\nu}\delta_{\alpha\beta}\Delta_c$ represent the free-nucleon and massive-vector-meson propagators, respectively. It will be useful to rewrite (2) in the somewhat more convenient form

$$\mathfrak{z} = \exp\left(\frac{i}{2}\int j_\mu^\alpha \Delta_c j_\mu^\alpha\right) \exp\left(-\frac{i}{2}\int \frac{\delta}{\delta A_\mu^\alpha}\Delta_c \frac{\delta}{\delta A_\mu^\alpha}\right)$$

$$\times \exp\left(i\int \bar\eta G_c[A]\eta\right) N^{-1} \exp(L[A])\,, \tag{3}$$

where $A_\mu^\alpha(x) = \int \Delta_c(x-y) j_\mu^\alpha(y)$, $G_c(x, y|A)$ represents the nucleon propagator in the presence of an effective external source field $A_\mu^\alpha(z)$, and $L[A]$ denotes the closed-nucleon-loop function that provides, among other things, all radiative corrections to the meson propagator $\Delta_c(x-y)$, $L[A] = +\operatorname{Tr}\ln(G_c[A]S_c^{-1})$. The constant N denotes the vacuum-to-vacuum amplitude constructed by functional operations upon $L[A]$; the familiar approximation of the neglect of all closed-nucleon loops will be made here, $N^{-1}\exp(L[A]) \to 1$, in order to facilitate the comparison between these isotopic forms and the simplest, most familiar case of pure isoscalar exchange.

Following many previous analyses,[11] the elastic scattering amplitude for a pair of distinguishable nucleons (an approximation used only at small angles) is given in configuration space by

$$M(x_1,y_1;x_2,y_2) = i^2\exp\left(-\frac{i}{2}\int\frac{\delta}{\delta A}\Delta_c\frac{\delta}{\delta A}\right)G_{c1}(y_1,x_1|A) \times G_{c2}(y_2,x_2|A)\Big|_{A\to 0}. \tag{4}$$

Upon mass-shell amputation, it yields the scattering amplitude corresponding to the exchange of all possible mesons between the two nucleons. It also contains self-linkage terms, corresponding to radiative corrections defined along either nucleon line; and these, also, will be dropped

$$M \to i^2\exp\left(-i\int\frac{\delta}{\delta A_1}\Delta_c\frac{\delta}{\delta A_2}\right) \times G_{c1}(y_1,x_1|A_1)G_{c2}(y_2,x_2|A_2)\Big|_{A_{1,2}\to 0}. \tag{5}$$

The construction of this simplest of eikonal representations is made somewhat easier by considering not M, but $\partial M/\partial g^2$, which is given by

$$\frac{\partial M}{\partial g^2} = -i(2\pi)^{-4}\int d^4z_1\int d^4z_2\Delta_c(z_1-z_2)\exp\left(-i\int\frac{\delta}{\delta A_1}\Delta_c\frac{\delta}{\delta A_2}\right) \times\sum_{\alpha,\mu}G_{c1}(y_1,z_1|A_1)\gamma_\mu^{(1)}\tau_\alpha^{(1)}G_{c1}(z_1,x_1|A_1)G_{c2}(y_2,z_2|A_2)\gamma_\mu^{(2)}\tau_\alpha^{(2)}G_{c2}(z_2,x_2|A_2)\Big|_{A_{1,2}\to 0}. \tag{6}$$

Subsequent mass-shell amputation upon (6), and the appropriate projection of initial and final isotopic states, will be necessary in order to extract the actual scattering amplitude $T_{\rm eik}(s,t)$; the normalization used is such that, suppressing isotopic indices,

$$\langle p_1'p_2'|T|p_1p_2\rangle = (2\pi)^{-2}m^2(E_1E_2E_1'E_2')^{-1/2} \times\delta^{(4)}(p_1+p_2-p_1'-p_2')T_{\rm eik}(s,t),$$

with initial $(p_{1,2})$ and final $(p_{1,2}')$ particle momenta.

In the absence of isotopics (all $A_\mu^\alpha\tau_\alpha$ combinations replaced by A_μ), derivation of the conventional eikonal amplitude follows rapidly upon the replacement of each mass-shell amputated $G_c[A]$ by its appropriate Bloch-Nordsieck (BN), or norecoil form. Each $G_c^{\rm BN}[A]$ is given as the exponential of a linear functional of A_μ, so that the functional operations of (6) may be performed immediately to yield an expression for $\partial T_{\rm eik}/\partial g^2$; and this quantity may then be trivially integrated, with respect to g^2, to obtain the customary expression. (The reason for considering $\partial T_{\rm eik}/\partial g^2$ rather than $T_{\rm eik}$ itself, is that the method permits mass-shell amputation to be performed in a simple and direct manner, with no "averaging" required over initial and final momenta; and it leads directly to the integrability conditions of Sec. VI.)

The heart of the problem, then, is the search for an appropriate representation of $G_c^{\rm BN}[A]$ when isotopics are included. Performing all steps of the convention construction, but before isotopic projections to initial and final states are taken, one is left with the expression

$$\frac{\partial T_{\rm eik}}{\partial g^2} = -\left(\frac{s}{2m^2}\right)\int d^4z\,\Delta_c(z)e^{iq\cdot z}\exp\left(-i\int\frac{\delta}{\delta A_1}\Delta_c\frac{\delta}{\delta A_2}\right)\sum_\alpha[\mathcal{F}(\infty;-p_1')\tau_\alpha^{(1)}\mathcal{F}(\infty;p_1)][\mathcal{F}(\infty;-p_2')\tau_\alpha^{(2)}\mathcal{F}(\infty;p_2)], \tag{7}$$

where $q=p_1-p_1'=p_2'-p_2$, $z=z_1-z_2$, and the $z_{1,2}$ dependence inside the $\mathcal{F}$'s is suppressed. The quantity $\mathcal{F}(\xi;p)$ is that solution of the differential equation

$$\frac{\partial\mathcal{F}(\xi;p)}{\partial\xi} = ig\frac{p_\mu}{m}\tau_\alpha A_\mu^\alpha(z-\xi p/m)\mathcal{F}(\xi;p), \tag{8}$$

with the boundary condition $\mathcal{F}(0;p)=1$. For simplicity of notation, in the following (8) shall be written as

$$\frac{\partial\mathcal{F}(\xi)}{\partial\xi} = i\vec{\tau}\cdot\vec{\pi}(\xi)\mathcal{F}(\xi),\quad \mathcal{F}(0)=1, \tag{9}$$

explicitly emphasizing the isotopic coordinates involved.

The solution of (9), of course, may be represented in terms of an ordered exponential (ordered in the dummy variables ξ'),

$$\mathfrak{F}(\xi)=\left(\exp\left[i\int_0^\xi d\xi'\vec{\tau}\cdot\vec{\pi}(\xi')\right]\right)_+ , \tag{10}$$

but for practical computations (10) is worthless. In the absence of isospin, the ordered exponential of (10) becomes an ordinary exponential, and the functional operations of (7) can be trivially obtained; with isotopics, however, one has not been able to progress past this point.

III. $G_c^{\rm BN}[A]$ IN THE STRONG-COUPLING LIMIT

The unitarity property of the formal solution (10), $\mathfrak{F}^\dagger=\mathfrak{F}^{-1}$, suggests a convenient representation for $\mathfrak{F}$ in the form

$$\mathfrak{F}(\xi)=\exp[iG_0(\xi)+i\vec{\tau}\cdot\vec{G}(\xi)],$$

with real functions $\vec{G}(\xi)$ and $G_0(\xi)$ that obey $\vec{G}(0)=0$, $G_0(0)=0$. Substitution into (9), with the aid of the relation

$$\frac{d}{d\xi}e^{Q(\xi)}=\int_0^1 d\lambda e^{\lambda Q(\xi)}\frac{dQ}{d\xi}e^{(1-\lambda)Q(\xi)},$$

immediately shows that $G_0(\xi)=0$ and that $\vec{G}(\xi)$ must satisfy

$$\vec{\tau}\cdot\vec{\pi}(\xi)=\int_0^1 d\lambda e^{i\lambda\vec{\tau}\cdot\vec{G}}\left(\vec{\tau}\cdot\frac{d\vec{G}}{d\xi}\right)e^{-i\lambda\vec{\tau}\cdot\vec{G}}, \tag{11}$$

or

$$\vec{\pi}(\xi)=\frac{d\vec{G}}{d\xi}-\left(\hat{G}\times\frac{d\vec{G}}{d\xi}\right)\frac{1}{2G}[1-\cos(2G)] -\hat{G}\times\left(\frac{d\vec{G}}{d\xi}\times\hat{G}\right)\left(1-\frac{\sin(2G)}{2G}\right), \tag{12}$$

where $\hat{G}=\vec{G}/G$ and $G=+(G^2)^{1/2}$. This exact differential equation is of course most difficult to solve; but it has a weak- and a strong-field limit which are curiously similar.

The weak-field limit, $G\ll 1$, must satisfy the equation

$$\vec{\pi}\simeq d\vec{G}/d\xi,$$

with solution

$$\vec{G}(\xi)\simeq\int_0^\xi d\xi'\vec{\pi}(\xi'). \tag{13}$$

It can be considered a weak-coupling solution in the sense that $\vec{\pi}$ is proportional to $|g|$, which will then enter "weakly" into all expressions built out of (13). The awkwardness of such a model lies here, for one is never sure of how much significance may be attached to the retention of all powers of g^2 in subsequent quantities, when one in fact defined the eikonal by a perturbation expansion in g.

The strong-coupling limit of (12) follows from the assumption $G\gg 1$,

$$\vec{\pi}\simeq\frac{d\vec{G}}{d\xi}-\hat{G}\times\left(\frac{d\vec{G}}{d\xi}\times\hat{G}\right)=\hat{G}\left(\hat{G}\cdot\frac{d\vec{G}}{d\xi}\right)$$

from which one immediately obtains the solution

$$\hat{G}(\xi)=\hat{\pi}(\xi),\quad G(\xi)=\int_0^\xi d\xi'\pi(\xi'). \tag{14}$$

The difference in these two cases is that, effectively, in (14) the direction of the vector $\vec{\pi}$ has been decoupled from its magnitude, as given by the weak-coupling solution (13). Presumably, the exact solution to (12) corresponds to an intermediate situation.

Because $\pi\sim|g|$, (14) may be considered as a strong-coupling limit. As such, there is no question of the validity of perturbation expansions here, if only the necessary functional operations of (7) can be performed upon the functions constructed with (14). [There is, of course, no justification—other than simplicity—for the neglect of closed-nucleon loop structure, which approximation was used in reaching (7).] In fact, the functional operations of (7) will themselves require an approximation, but one that is at least intuitively reasonable for strongly coupled fields.

This strong-field solution has the form

$$\mathfrak{F}(\xi)=\exp\left[i\vec{\tau}\cdot\hat{\pi}(\xi)\int_0^\xi d\xi'\pi(\xi')\right], \tag{15}$$

and one must now decide what value to assign to the quantity

$$\hat{\pi}(\xi)=p\cdot\vec{A}(z-\xi p/m)\{[p\cdot\vec{A}(z-\xi p/m)]^2\}^{-1/2}$$

as $\xi\to\infty$, for this is the form needed in (7). A suggestion for an answer follows from the correlation between the limit $\xi\to\infty$ in the different $\mathfrak{F}(\infty;\pm p)$, and the asymptotic procedure of specifying the mass-shell properties of a particular particle. Imagine, in (7), that the incoming particle p_1 represents a proton. The limit $\xi\to\infty$ in $\mathfrak{F}(\xi;p_1)$ may be looked upon as the statement that, in principle, there was an inifinite amount of time available to measure the 4-momentum p_1 with perfect accuracy. But if that incident particle is prepared as a proton, its Green's function during that time of preparation can only emit or absorb neutral mesons; otherwise, it would be able to change its charge, and no longer be a certified proton. That is, any measurement of that initial state by an external electromagnetic field during the course of its preparation, must yield a charge +1; and hence during that time, as $\xi\to\infty$, only neutral-meson emission or absorption should be permitted. This property is guaranteed if $\hat{\pi}(\infty)$ is chosen to point in the $\pm\hat{e}_3$ direction. (In other words, the preparation and measurement of initial and final states, respectively, breaks the isotopic symmetry.) A similar argument suggests that the corresponding $\hat{\pi}(\infty)$ for a neutron should be $\mp\hat{e}_3$, for this double choice will then satisfy charge independence (of the forces generated by the exchange of neutral mesons between either proton or neutron and an external nucleonic "testing field") during those long asymptotic times of preparation. In fact, as is easily seen upon taking subsequent isotopic projections into states of $I=0,1$, the restriction $\hat{\pi}_p(\infty)=-\hat{\pi}_n(\infty)$ guarantees the necessary degeneracy of the triplet state.

If, for definiteness, one chooses $\hat{\pi}_p(\infty)=-\hat{\pi}_n(\infty)=-\hat{e}_3$, upon appropriate projection of n or p states in (7), one will always find similar (negative) phase factors for both asymptotic proton and neutron Green's functions, simply because the nucleon projection operators have the property $\frac{1}{2}(1\pm\tau_3)\times F\{\pm\tau_3\}=\frac{1}{2}(1\pm\tau_3)F\{+1\}$. The complete phase factors for each nucleon may then be written as $\exp[-i\int d^4u\,\pi(u)f(u)]$, where

$$f(u)=|g|\int_0^\infty d\xi[\delta(u-z+\xi p)+\delta(u-z-\xi p')]\,,$$

or

$$f_{1,2}(u)=|g|\int_{-\infty}^{+\infty} d\xi\,\delta(u-z_{1,2}+\xi p_{1,2})\,, \tag{16}$$

neglecting, as is usual in small-angle eikonal models, the q dependence of all factors other than the phase explicitly exhibited in (7). One then obtains

$$\frac{\partial T_{\text{eik}}}{\partial g^2}=-\left(\frac{s}{2m^2}\right)\int d^4z\,e^{iq\cdot z}\Delta_c(z)\,\langle N_1'N_2'|\vec{\tau}_1\cdot\vec{\tau}_2|N_1N_2\rangle\exp\left(-i\int\frac{\delta}{dA_1}\Delta_c\frac{\delta}{dA_2}\right)\exp\left(-i\int f_1\pi_1-i\int f_2\pi_2\right)\bigg|_{A_{1,2}\to 0}\,, \tag{17}$$

with

$$\pi(u)=\left[\sum_{\alpha=1}^{3}(p_\mu A_\mu^\alpha(u))^2\right]^{1/2}.$$

As obvious notation has been used for the isotopic matrix elements, which here have the form expected in a Born approximation. They shall be suppressed until the isotopic integrability discussion of Sec. VI.

An additional argument can be used to restrict the choice of sign available for $\hat{\pi}_p(\infty)=-\hat{\pi}_n(\infty)$. The explicit negative phase in the source terms of (17) reflects the correct sign of phase with which to represent a particle of positive energy moving in an effective, positive, external field. For example, if $\pi[A]$ is considered as constant, or weakly dependent upon its argument, so that $\int_0^\xi d\xi'\pi\sim\xi\langle\pi\rangle$, the momentum-space propagator for such a (spinless) nucleon would be given by

$$\tilde{G}(p)=i\int_0^\infty d\xi\,\exp[-i\xi(m^2+p^2+\langle\pi\rangle)]$$
$$=(m^2+\vec{p}^2+\langle\pi\rangle-E^2)^{-1}\,,$$

and corresponds to the propagation of a nucleon with averaged potential energy $\langle\pi\rangle$, a positive number for arbitrary field strengths. In the ordinary isoscalar theory, the choice of sign of the corresponding phase is irrelevant; here, however, it should be specified in order to give a description of the scattering of nucleons, rather than that of antinucleons.

IV. APPROXIMATE FUNCTIONAL EVALUATION

The question one must now face is the evaluation of the functional operations of (17), a decidedly nontrivial matter because of the appearance of the magnitudes, rather than the components, of the isotopic fields A_μ^α. Another complication is that it is not clear, using the form (17), just how one may comply with the instructions $A\to 0$, at the end of the computation, since this approximation has been defined for large field strengths, or large couplings.

However, there exists an alternative and formally equivalent procedure, defined by a functional-integration representation for the differential operator of (17),

$$\exp\left(-i\int\frac{\delta}{\delta A_{1\mu}^\alpha}\Delta_c\frac{\delta}{\delta A_{2\mu}^\alpha}\right)=C^{-1}e^{-\operatorname{Tr}\ln\Delta_c}\int d[\varphi]\int d[\psi]\exp\left[-i\sum_{\alpha,\mu}\left(\int\varphi_\mu^\alpha K\psi_\mu^\alpha+\int\varphi_\mu^\alpha\frac{\delta}{\delta A_{1\mu}^\alpha}+\int\psi_\mu^\alpha\frac{\delta}{\delta A_{2\mu}^\alpha}\right)\right]\,, \tag{18}$$

where $K\Delta_c=1$, and $C=(2\pi)^N$ is a typical functional-integration constant for N degrees of freedom (as $N\to\infty$), which will cancel out of the final result. Here, $\int d[\varphi]=\Pi_{l=1}^N\int d\tilde{\varphi}_l$, where $\tilde{\varphi}_l(k_l)$ is the lth Fourier mode of $\varphi(x)$, with isotopic and 4-momentum coordinates suppressed.

With the representation (18), one may write in place of (17),

$$\frac{\partial T_{\text{eik}}}{\partial g^2} = -\left(\frac{s}{2m^2}\right) \int d^4z e^{iq\cdot z}\, C^{-1} e^{-\operatorname{Tr}\ln\Delta_c} \times \int d[\varphi] \int d[\psi] \exp\left(-i \sum_{\alpha,\mu} \int \varphi^\alpha_\mu K \psi^\alpha_\mu - i \int f_1 \pi_1[\varphi] - i \int f_2 \pi_2[\psi]\right), \tag{19}$$

where $\pi_{1,2}[\chi] = \{\sum_\alpha [p^\mu_{1,2}\chi^\alpha_\mu(u)]^2\}^{1/2}$; and one may now look for alternative methods of evaluation which maintain large values of $\pi_1[\varphi]$ and $\pi_2[\psi]$. The method that comes to mind immediately, in this strong-field approximation, is that of stationary phase. Here, one imagines that the coupling is so strong, and the fields in question so large, that virtual processes must contain very large numbers of quanta; and that a simpler way of picturing the effects of so many quanta is obtained by replacing them by a "semiclassical" field, here defined in some self-consistent, nonlinear way. It is understood that, by the imposition of such an "averaging" approximation, some of the fine details of the theory may be lost; but the intuitive hope remains that the important qualitative features of the strong-field limit will be preserved.

The stationary-phase approximation now adopted to evaluate (19) treats the φ and ψ coordinates as independent variables; that is, one considers the functional integral

$$\int d[\varphi] \int d[\psi] \exp(f[\varphi,\psi]), \tag{20}$$

and expands $f[\varphi,\psi]$ about nonzero (and large) φ_0, ψ_0 values, which are determined by the simultaneous conditions $\delta f/\delta\varphi_0 = \delta f/\delta\psi_0 = 0$. One retains in $f[\varphi,\psi]$ only quadratic dependence in the variables $\varphi - \varphi_0$ and $\psi - \psi_0$, so that (20) is replaced by

$$\exp(f[\varphi_0,\psi_0]) \int d[\varphi] \int d[\psi] \exp\left(\tfrac{1}{2} \int (\varphi-\varphi_0) \frac{\delta^2 f}{\delta\varphi_0\delta\varphi_0} (\varphi-\varphi_0) + \tfrac{1}{2} \int (\psi-\psi_0) \frac{\delta^2 f}{\delta\psi_0\delta\psi_0} (\psi-\psi_0) + \int (\varphi-\varphi_0) \frac{\delta^2 f}{\delta\varphi_0\delta\psi_0} (\psi-\psi_0)\right). \tag{21}$$

In the variables $\bar\varphi = \varphi - \varphi_0$, $\bar\psi = \psi - \psi_0$, (21) now corresponds to a pair of Gaussian functional integrals, and can be immediately evaluated in terms of the functions φ_0, ψ_0, or more properly $\varphi^\alpha_{0\mu}, \psi^\alpha_{0\mu}$. For notational ease, the subscript 0 is henceforth omitted.

The Euler equations for $\varphi^\alpha_\mu, \psi^\alpha_\mu$ obtained in this way are

$$K\psi^\alpha_\mu(u) = -f_1(u) p_{1\mu} \frac{(p_{1\nu}\varphi^\alpha_\nu)}{\pi_1[\varphi]}, \tag{22}$$

and

$$K\varphi^\alpha_\mu(u) = -f_2(u) p_{2\mu} \frac{(p_{2\nu}\psi^\alpha_\nu)}{\pi_2[\psi]}, \tag{23}$$

while the necessary quadratic derivatives are given by

$$\frac{\delta^2 f}{\delta\varphi^\alpha_\mu(u)\delta\varphi^\beta_\nu(v)} = -if_1\delta(u-v)\frac{p_{1\mu}p_{1\nu}}{\pi_1[\varphi]} \times \left(\delta_{\alpha\beta} - \frac{(p_1\cdot\varphi^\alpha)(p_1\cdot\varphi^\beta)}{\pi_1{}^2[\varphi]}\right), \tag{24}$$

$$\frac{\delta^2 f}{\delta\psi^\alpha_\mu(u)\delta\psi^\beta_\nu(v)} = -if_2\delta(u-v)\frac{p_{2\mu}p_{2\nu}}{\pi_2[\psi]} \times \left(\delta_{\alpha\beta} - \frac{(p_2\cdot\psi^\alpha)(p_2\cdot\psi^\beta)}{\pi_2{}^2[\psi]}\right), \tag{25}$$

and

$$\frac{\delta^2 f}{\delta\varphi^\alpha_\mu(u)\delta\psi^\beta_\nu(v)} = -i\delta_{\alpha\beta}\delta_{\mu\nu}\delta(u-v)K_u, \tag{26}$$

with $K_u = \mu^2 - \partial_u{}^2$.

To look for solutions it is simplest to extract all $p_{1,2}$ dependence by making the ansatz (physically, the most important, although not the most general) $\varphi^\alpha_\mu(u) = p_{2\mu}\varphi^\alpha(u)$, $\psi^\alpha_\mu(u) = p_{1\mu}\psi^\alpha(u)$, so that this pair of Euler equations becomes

$$K\varphi^\alpha = f_2 \frac{\psi^\alpha}{(\vec\psi^2)^{1/2}}, \quad K\psi^\alpha = f_1 \frac{\varphi^\alpha}{(\vec\varphi^2)^{1/2}}.$$

The overall sign change, compared to (22) and (23), occurs because $(p_1\cdot p_2) = m^2 - s/2$ is negative. The unit vectors $\hat p \equiv \vec\psi/(\vec\psi^2)^{1/2}$ and $\hat k \equiv \vec\varphi/(\vec\varphi^2)^{1/2}$ are, in general, position-dependent. Thus, the sources of these "averaged" meson fields associated with each nucleon line are given by the kinematics of the other nucleon and the isotopic direction of its meson field. One may write this dependence in terms of integral equations, involving causal propagation,

$$\varphi^\alpha(u) = \int \Delta_c(u-v) f_2(v) \hat p_\alpha(v), \tag{27}$$

$$\psi^\alpha(u) = \int \Delta_c(u-v) f_1(v) \hat k_\alpha(v), \tag{28}$$

but the general solution of this pair of equations is not immediately obvious.

However, there is at least one solution which can be readily obtained in the (absurd) limit of very large μ^2, that is, $K=\mu^2-\partial^2\to\mu^2$, for these equations then become algebraic. For bounded functions $f_{1,2}$ (not the case here), one has $\mu^2\varphi^\alpha = f_2\hat{p}_\alpha$ and $\mu^2\psi^\alpha = f_1\hat{k}_\alpha$, so that $(\vec{\varphi}^2)^{1/2}=f_2/\mu^2$ and $(\vec{\psi}^2)^{1/2}=f_1/\mu^2$. Thus $f_1\varphi^\alpha=f_2\psi^\alpha$, and since f_1 and f_2 are positive, $\hat{k}=\hat{p}$. The $\exp(f[\varphi_0,\psi_0])$ factor of (21) then becomes $+i(p_1\cdot p_2)\int\int f_1\mu^{-2}f_2$, which has the same form as the "limit" of an isoscalar eikonal term, $\exp[i(p_1\cdot p_2)\int f_1\Delta_c f_2]$. Of course, this particular solution is not to be taken seriously, but is only written to exhibit an example of a solution in which $\hat{k}=\hat{p}$. Since the analysis here has been algebraic, there is no specification as to whether these unit vectors are constants. If they were constants, the total isospin carried by this total "averaged" meson-field ($\vec{T}\sim\int d^3x\,\vec{\phi}_\mu\times\partial_0\vec{\phi}_\mu$, $\phi_\mu^\alpha=p_{2\mu}\varphi^\alpha+p_{1\mu}\psi^\alpha$) would vanish locally; and it would be a reasonable feature to build into any such fields which must lead to an overall isotopic-spin conservation. Incidentally, solutions of the homogeneous differental equations added to (27) and (28), which would correspond to (mass-shell) meson dependence independent of the "other nucleon" source, would suggest spontaneous violations of isospin invariance, and are not considered.

In order to solve (27) and (28) in an explicit way, it is now assumed that physically reasonable solutions can be obtained with constant unit vectors, so that solutions to these equations now become

$$\varphi^\alpha(u)=\hat{p}_\alpha\int\Delta_c(u-v)f_2(v)\equiv\hat{p}_\alpha\varphi(u)\,, \tag{29}$$

$$\psi^\alpha(u)=\hat{k}_\alpha\int\Delta_c(u-v)f_1(v)\equiv\hat{k}_\alpha\psi(u)\,, \tag{30}$$

where the isotopic magnitudes φ and ψ are (in the $q\to 0$ limit) real and positive functions. Hence, from the definition of $\hat{p}$ and $\hat{k}$, $\hat{k}=\hat{p}$. The isospin in these averaged meson fields is then zero, by this simplest construction; and (21) then becomes

$$\exp\Big[i(\hat{k}\cdot\hat{p})(p_1\cdot p_2)\int f_1\Delta_c f_2\Big]\int d[\varphi]\int d[\psi]\exp\Big(\frac{i}{2}\int\overline{\varphi}_\mu^\alpha S_{\mu\nu}^{\alpha\beta}\overline{\varphi}_\nu^\beta+\frac{i}{2}\int\overline{\psi}_\mu^\alpha T_{\mu\nu}^{\alpha\beta}\overline{\psi}_\nu^\beta-i\int\overline{\varphi}_\mu^\alpha K\overline{\psi}_\mu^\alpha\Big)\,, \tag{31}$$

with

$$S_{\mu\nu}^{\alpha\beta}=\frac{p_{1\mu}p_{1\nu}}{(p_1\cdot p_2)}\frac{f_1(u)}{\varphi(u)}(\delta_{\alpha\beta}-\hat{k}_\alpha\hat{k}_\beta) \tag{32}$$

and

$$T_{\mu\nu}^{\alpha\beta}=\frac{p_{2\mu}p_{2\nu}}{(p_1\cdot p_2)}\frac{f_2(u)}{\psi(u)}(\delta_{\alpha\beta}-\hat{k}_\alpha\hat{k}_\beta)\,. \tag{33}$$

In a one-dimensional isospace, that is, with isoscalar exchange only, (32) and (33) vanish, and the entire functional integral [including the normalization factors of (18)] reduces to the coefficient multiplying the integrals of (31), which is just the familiar isoscalar result. This coefficient may be thought of, in this "averaged" isovector model, as equivalent to the contribution of $I_3=0$ isovector exchanges only, and will subsequently be referred to as "equivalent isoscalar exchange." It will be seen, in Sec. VI, that this solution is possible only for the isotriplet amplitude. There, independent and equivalent constructions will be performed for both triplet and singlet amplitudes.

The Gaussian functional integrals of (31) now generate for (19) the quantity

$$\frac{\partial T_{\rm eik}}{\partial g^2}=-\Big(\frac{s}{2m^2}\Big)\int d^4z\,e^{iq\cdot z}\Delta_c(z)\times\exp\Big[i(p_1\cdot p_2)\int f_1\Delta_c f_2+Q\Big]\,, \tag{34}$$

where

$$Q=-\tfrac{1}{2}\,\mathrm{Tr}\ln(1-S\Delta_c T\Delta_c)\,. \tag{35}$$

The physical interpretation of Q, in this stationary-phase approximation, is that it takes into account—in terms of the averaged isotopic fields φ,ψ—the possibility of virtual emission of multiple pairs of zero-net-charge fields, within a strong-coupling framework. Together with the effectively isoscalar term, it corresponds to the sum of all possible "averaged" isovector exchanges.

The trace of (35) includes isotopic, 4-momentum, and configuration-space coordinates. In particular, the factor $(\delta_{\alpha\beta}-\hat{k}_\alpha\hat{k}_\beta)$ acts as an isotopic projection operator, and effectively factors through every term of the expansion of (35) to multiply the over-all expression by a factor $\mathrm{tr}(\delta_{\alpha\beta}-\hat{k}_\alpha\hat{k}_\beta)=+2$. The effect of tracing all 4-momentum coordinates in every iterate of (35) is, even more simply, the removal of all $(p_1\cdot p_2)$ dependence. In this way, (35) may be replaced by

$$Q=-\mathrm{Tr}\ln(1-S'\Delta_c T'\Delta_c)\,, \tag{36}$$

where $S'(u)=f_1(u)/\varphi(u)$, $T'(u)=f_2(u)/\psi(u)$, and the trace operation runs over spatial (or momentum) coordinates only. Finally, since S' and T' are independent of g, (34) may be integrated to generate the desired modification of the old Lévy-Sucher result,[12]

$$T_{\text{eik}}(s,t)=\frac{is}{2m^2}\int d^2b e^{i\vec{q}\cdot\vec{b}}\left[1-\exp\left(-i\frac{g^2}{2\pi}K_0(\mu b)\right)\right]\times e^{Q(b,s)}\,. \qquad (37)$$

In obtaining (31), it has been assumed that $s/m^2 \gg 1$; and that in this asymptotic region Q will depend only upon s and $b=z_T$, but not upon q or $z^{(\pm)}$. In fact, this is not exactly the case, for the q dependence of Q will subsequently play an important role.

It has also been assumed that T_{eik} must vanish if g^2 is set equal to zero, even though the derivation has been performed assuming that the fields are large; this point is discussed more fully in Sec. VII. As a result, e^Q multiplies both terms in the bracket of (31); Q is not just added on to the eikonal function $i\chi_0=-i(g^2/2\pi)K_0(\mu b)$, as one might expect in a general isoscalar theory, or weak-coupling approximation. Because Q is independent of g, it factors out of any expression which depends upon the counting of coupling-constant insertions; this property is a legacy of the stationary-phase approximation used above, and is quite compatible with the idea of strong coupling (one may imagine that a summation over all powers of g has been performed to obtain the φ and ψ denominators of S' and T', respectively). It may also be noted that Q is independent of any overt spin dependence of the exchanged mesons [all $(p_1\cdot p_2)$ factors have cancelled]; the same set of approximations in a theory of $J=0$, $I=1$ boson exchanges will yield just the Q of (36), although the χ_0 function would then decrease as s^{-1}.

V. APPROXIMATE INTEGRAL EVALUATIONS

In this section the evaluation of the integrals representing Q is (a) first carried out in the present $q=0$ limit, and the difficulty and impropriety of this limit is made clear. This is followed by a second calculation (b) for large momentum transfers; and the forms found there are used to suggest an alternative calculation (c) of the small-momentum-transfer region, in which the integrals are properly behaved if the limit $q/\sqrt{s}\to 0$ in Q is postponed until the end of the calculation.

A. Small-angle scattering (first attempt)

All the complexity of the problem now resides in Q, which may be written in the form

$$Q=\int_0^1 d\lambda\,\text{Tr}[S'\Delta_c T'\Delta_c(1-\lambda S'\Delta_c T'\Delta_c)^{-1}]\,, \qquad (38)$$

and it is natural to expand the integrand in powers of λ, and so compute all the iterates of (38). In fact, were it not for the denominator factors of φ and ψ, in the definition of S' and T', (38) would correspond to the sum of virtual pairs of soft charged (or neutral) mesons; that is, to the small-angle counterpart of the wide-angle damping factor previously discussed[3] in pseudoscalar-pion/nucleon theory. Those denominator factors of φ and ψ, and the corresponding absence of dependence on the coupling, suggest that these strong-coupling isovector forms may be thought of as virtual-meson pair emission modified by a nonperturbative normalization of each such meson propagator. Integration over these factors will require care.

With

$$Q=\sum_{n=1}^{\infty} Q_n\,,$$

and using the representations of (16) and (36), the nth iterate of (38) may be written as

$$Q_n=\frac{1}{n}\int d^4u_1\cdots\int d^4u_n\int d^4v_1\cdots\int d^4v_n\frac{f_1(u_1)}{\varphi(u_1)}\Delta_c(u_1-v_1)\frac{f_2(v_1)}{\psi(v_1)}\Delta_c(v_1-u_2)\frac{f_1(u_2)}{\varphi(u_2)}\cdots\frac{f_2(v_n)}{\psi(v_n)}\Delta_c(v_n-u_1)\,,$$

or

$$Q_n=\frac{1}{n}\int_{-\infty}^{+\infty}d\xi_1\int_{-\infty}^{+\infty}d\eta_1\cdots\int_{-\infty}^{+\infty}d\xi_n\int_{-\infty}^{+\infty}d\eta_n\,\frac{\Delta_c(z-\xi_1p_1+\eta_1p_2)\Delta_c(z-\xi_2p_1+\eta_1p_2)\cdots\Delta_c(z-\xi_1p_1+\eta_np_2)}{\mathfrak{D}_1(z_1-\xi p_1)\mathfrak{D}_2(z+\eta_1p_2)\mathfrak{D}_1(z-\xi_2p_1)\cdots\mathfrak{D}_2(z+\eta_np_2)}\,, \qquad (39)$$

where

$$\mathfrak{D}_1(z-\xi p_1)=\int_{-\infty}^{+\infty}d\eta'\Delta_c(z-\xi p_1+\eta' p_2)\,, \qquad (40)$$

and

$$\mathfrak{D}_2(z+\eta p_2)=\int_{-\infty}^{+\infty}d\xi'\Delta_c(z-\xi' p_1+\eta p_2)\,. \qquad (41)$$

Evaluation of the $\mathfrak{D}_{1,2}$ is simple; but one immediately sees that the typical high-energy limit, wherein one neglects the mass difference between E and p, is not appropriate. With that approximation, there would result

$$\mathfrak{D}_1(Q)=\frac{1}{2\pi E}K_0(\mu b)\delta(Q^{(+)})\,,$$

$$\mathfrak{D}_2(Q)=\frac{1}{2\pi E}K_0(\mu b)\delta(Q^{(-)})\,,$$

and the presence of the $\delta(Q^{(\pm)})$ functions in the denominators of (41) would be troublesome. However, (40) and (41) may be evaluated exactly, and give (in the c.m. frame)

$$\mathfrak{D}_1(Q) = \frac{1}{4\pi}[(mb)^2 + R_1{}^2E^2]^{-1/2} \times \exp\left(-\frac{\mu}{m}[(mb)^2 + R_1{}^2E^2]^{1/2}\right),$$

$$R_1 = Q_3 + Q_0\frac{p}{E}, \tag{42}$$

and

$$\mathfrak{D}_2(Q) = \frac{1}{4\pi}[(mb)^2 + R_2{}^2E^2]^{-1/2} \times \exp\left(-\frac{\mu}{m}[(mb)^2 + R_2{}^2E^2]^{1/2}\right),$$

$$R_2 = Q_3 - Q_0\frac{p}{E}, \tag{43}$$

where p denotes the longitudinal component of p_1. Now one may take the high-energy limit, so that $R_1 \to z^{(+)} - 2\xi E$, $R_2 \to z^{(-)} + 2\eta E$. Further, every numerator propagator is a function of the square of its argument, as written in (39); that is, the functional dependence here is on quantities of form $(z - \xi p_1 + \eta p_2)^2$. In the high-energy limit, these become simply $b^2 - (z^{(+)} - 2\xi E)(z^{(-)} + 2\eta E)$, and involve just the $R_{1,2}$ variables found in $\mathfrak{D}_{1,2}$. Hence a redefinition of every ξ_i, η_i of (39), in terms of the new variables $\bar{\xi}_i = \xi_i - z^{(+)}/2E$, $\bar{\eta}_i = \eta_i + z^{(-)}/2E$, removes all dependence on $z^{(\pm)}$, as expected. This replacement of $(z - \xi p_1 + \eta p_2)^2$ by its asymptotic form, which neglects quadratic ξ, η dependence, is justified as long as each of the ξ, η intergration variables subsequently cuts off at values $\ll 1/m^2$, such as $1/m\sqrt{s}$ or $b/\sqrt{s}$.

Although the denominator factors are now properly free of singular parts, the integrals of (39) are still not without ambiguity. This is because there exists a denominator factor $\mathfrak{D}_1$ or $\mathfrak{D}_2$ for each ξ and η integration, and for large values of ξ and η it is not clear that the integrals converge: if each $\Delta_c \sim H_1^{(2)}(\mu[b^2 + s\eta\xi]^{1/2}) \sim \exp[-i\mu(\eta\xi s)^{1/2}]$, in the region where ξ and η have the same sign, this will not be sufficient to produce convergence because of the

$$\exp\left(+\frac{\mu}{m}\frac{s}{2}\xi\right)$$

arising from the appropriate $\mathfrak{D}_1{}^{-1}(Q)$, and a factor of

$$\exp\left(+\frac{\mu}{m}\frac{s}{2}\eta\right)$$

coming from the corresponding $\mathfrak{D}_2{}^{-1}(Q)$. When ξ and η have opposite signs, and the Hankel functions of Δ_c are replaced by $K_1 \sim \exp[-\mu(s|\xi\eta|)^{1/2}]$, one will not obtain convergence. Large values of ξ and η correspond to small values of the virtual momenta exchanged in all the relevant Feynman graphs; one is familiar with the need to avoid divergences at small ξ, η,[3] but never before, in a massive theory, at the infrared end of the spectrum.

This difficulty arises, clearly, because of the special nature of the previous approximations, wherein integrals over propagators appear in the denominator. One might try to avoid these troubles by imagining that all Feynman integrations are calculated before the infinite summations over graphs (which yield the denominator factors) have been performed. Perhaps the simplest way of arranging this would be to suppose that each $\mathfrak{D}_{1,2}{}^{-1}(Q)$ is replaced by $(1/\epsilon)\ln[1 + \epsilon/\mathfrak{D}_{1,2}(Q)]$, and the limit $\epsilon \to 0$ is taken after the ξ, η integrations are performed; here, ϵ may be thought of as $\sim 1/g^2$, in a version of a strong-coupling limit. Convergence of every integral of (39), for large ξ, η values now follows, in this example, because of the Bessel functions of the numerator propagators. However, it is not at all clear that the results of such an ϵ-limiting procedure have anything to do with this strong-coupling approximation. Rather, this ϵ procedure acts to define what is meant by the $\mathfrak{D}_{1,2}{}^{-1}$ factors; and the real difficulty is that the results, while quite finite, must be rejected on physical grounds.

To see this, imagine that such an interchange of limits supplies effective cutoffs into every ξ, η integral of $r_{\max}$. Then, as a simple estimate, one may replace each $\int\int_{-\infty}^{+\infty} d\eta d\xi$ by $\int\int_{-r_{\max}}^{+r_{\max}} d\xi d\eta$, and accordingly approximate all ξ, η dependence in the remainder of the integrands. If $s|\xi\eta| > b^2$, each $\Delta_c \sim \exp(i\mu r_{\max}\sqrt{s})$ or $e^{-\mu\sqrt{s}\, r_{\max}}$, and we expect $r_{\max} \sim 1/\mu\sqrt{s}$. (Incidentally, this supposes that $sr_{\max}{}^2 > b^2$ or that $\mu b < 1$, which will be the important region of impact parameter in all of these estimates.) To simplify the evaluation, we replace the numerator Δ_c by their massless forms, but cut off all the ξ, η integrals at $\pm 1/\mu\sqrt{s}$. It follows that the different ξ, η integrands decouple from each other, with integration over each pair generating an amount

$$\left[\frac{2i}{\pi}\int_0^{1/\mu\sqrt{s}} d\eta\, \frac{1}{\bar{\eta}\sqrt{s}}\left(\frac{s\bar{\eta}}{2}\right)\exp\left(\frac{\mu}{2m}s\bar{\eta}\right)\right]^2. \tag{44}$$

The most important contributions to (44) will come from large values of the variable, and yield, approximately,

$$\left(\frac{i}{\pi}\frac{2m}{\sqrt{s}}e^{\sqrt{s}/2m}\right)^2, \tag{45}$$

which quantity is to be raised to the nth power, and divided by n, to compute Q_n. One obtains

$$e^{Q} \to \left[1+\left(\frac{2m}{\pi\sqrt{s}}\ e^{\sqrt{s}/2m}\right)^2\right]^{-1}, \qquad (46)$$

which damps the entire scattering amplitude almost completely, effectively multiplying a factor $(\pi^2 s/4m^2)e^{-\sqrt{s}/m}$ into the over-all impact-parameter representation. This evaluation is crude, but reasonable. The only trouble is that the results are physically wrong; e.g., such damping would be independent of all other (heretofore neglected) isoscalar processes. Thus this ϵ-limiting procedure must be rejected on physical grounds; and the same difficulty will haunt any similar *ad hoc* definition of $\mathfrak{D}_{1,2}{}^{-1}$.

B. Form factors at large momentum transfer

All the formal results obtained in the previous sections are directly applicable to form factors, or vertex functions, with q^2-dependent damping of a simple large-q^2 vertex expected to arise from the exchange of soft vector mesons. The quantity

$$ie\gamma_\mu \exp[Q(t)-Q(0)] \qquad (47)$$

may be interpreted as one part of the normalized vertex function of a nucleon, where $Q(t)$ is again given by (38) with the replacements: $p_1 \to p$, $p_2 \to -p'$, $z\to 0$, and all ξ,η integrals ranging over the semiinfinite limits 0 to ∞; here, $t=-(p_1-p')^2$. The "effectively isoscalar" term of (34),

$$\exp\left[i(p\cdot p')\int f_1\Delta_c f_2\right], \qquad (48)$$

will, after renormalization, again generate damping that can lead to form-factor falloff with increasing $|t|$; and one now asks what effects will result from $Q(t)$, when $|t|$ is large.

The denominator integral $\int_0^\infty d\xi'\Delta_c(\xi' p+\eta p')$ may be evaluated (neglecting $m^2/|t|$ effects) approximately as

$$-\frac{i}{(2\pi)^2}\ \frac{L}{\eta|t|}\ ,\quad L\simeq \ln(1+\mu_c{}^2\mu^2)\ , \qquad (49)$$

where a factor $e^{-i\alpha k^2}$ has been inserted into the momentum-space integrals to limit the magnitude of the 4-momentum included in this soft approximation, with the replacement $\alpha\to -i\mu_c{}^2$ subsequently performed; this simplest cutoff method has been used several times previously.[13] (One can always return to the more conventional, but more complicated, representations of the soft form-factor integrals by choosing the cutoff $\mu_c \sim |t|^{1/2}$, for large $|t|$.) The important thing to notice is the factor $1/\eta$ of (49), which then enters as a factor η in the numerator of the Q_n integrands. This, together with similar factors of ξ arising fron denominator integrals of form $\int_0^\infty d\eta'\Delta_c(\xi p_1+\eta' p_2)$, corresponds to polynomial enhancement of the numerator integrals, rather than to the exponential enhancement previously found in the small-angle case.

One can readily see that such η^{-1} dependence of (49) would be removed were the dummy ξ' integration that produced (49) to cover values from $-\infty$ to $+\infty$, for then one would obtain a result proportional to $\delta(\eta)$, similar to that noted in Sec. IIA. Thus, the extra coherence of small-angle processes ($\sim\int_{-\infty}^{+\infty} d\xi'$) compared to those of wide angles ($\sim\int_0^\infty d\xi'$) is the physical reason for the previous difficulties.

Using this method of the μ_c cutoff, the integrals correspondint to Q_1 can be evaluated without too much difficultiy, and generate for large $|t|$ the dependence

$$Q_1(t)\sim a_1 \ln\left|\frac{t}{m^2}\right|\ , \qquad (50)$$

with $a_1 = L-1/L^2$. Similar evaluation of all the remaining Q_n suggests that each of them is also proportional to a single factor of $\ln|t|$, $Q_n(t) \sim a_n \ln|t/m^2|$, $a_n>0$; but the calculations become exceedingly tedious for $n>1$, and hence the present discussion will center only on a_1. [In view of the basic logarithmic nature of the expansion for Q, one might conjecture that $a_n\sim -\ln(1-a_1)\sim 1/L$.] Equations (50) and (47) correspond to a coupling-independent enhancement of the form factor, which acts to counterbalance the damping of the effective isoscalar term of (48). Using the same μ_c cutoff method, the latter was long ago shown to given form-factor damping contributions of form

$$\exp\left(-\frac{g^2}{8\pi^2}\ L\ln\left|\frac{t}{m^2}\right|\ \right), \qquad (51)$$

where g denotes the appropriate meson-nucleon coupling.

Suppose now that the strong-coupling analysis presented here is relevant for coupling strengths of ρ to nucleon, say $g^2/4\pi\sim 3$, to pick a rough, average, experimental value. Assume also that soft ω exchanges occur. Since L is not sensitive to the ρ-ω mass difference, one would require the relation

$$\frac{1-L}{L^2}+\frac{1}{2\pi}\left(\frac{g_\rho{}^2}{4\pi}+\frac{g_\omega{}^2}{4\pi}\right)L\simeq 2.2\ , \qquad (52)$$

which corresponds to the $|t|^{-2.2}$ falloff of the proton's electromagnetic form factor. To satisfy (52), using the rough values $g_\rho{}^2/4\pi\sim 3$ and $g_\omega{}^2/4\pi \sim 10$, one should adopt the reasonable value $L\gtrsim 1$. Thus the isovector contributions tend to cancel, assuming that $\sum_n a_n$ does not greatly exceed a_1, leaving the most important terms, as before, coming from virtual isoscalar exchange. Thus, as long as this strong-coupling model is relevant to ρ exchange, the phenomenological picture of multiple vector-meson exchange generating form-factor damping can be retained. In a similar fashion, all the other multiple soft-vector-meson descriptions of large-p_t processes remain essentially unchanged by the inclusion of isovector ρ exchange.

C. Small-angle scattering (second attempt)

Having seen how the wide-angle analysis produces polynomial ξ,η dependence in every Q_n integrand, and thus automatically generates finite integrals, it is natural to ask if a similar effect can occur if the $q\to 0$ limit is postponed in the $\mathfrak{D}_{1,2}^{-1}$ terms. If this is the case, then one will also have to imagine that the $z^{(\pm)}$ integrations, leading to the conventional eikonal forms, must be postponed until the last stages of the calculation. Since $q^{(\pm)}\sim \pm\vec{q}^{\,2}/2E$, each $z^{(\pm)}$ may be thought of as of order $\sqrt{s}/q^2$, and the way in which they enter the remaining ξ,η integrals could conceivably be of importance.

"Undoing" the $q=p_1-p_1'\to 0$ limit used in the previous evaluation of $\mathfrak{D}_2$, one considers in place of (41)

$$\mathfrak{D}_2=\int_0^\infty d\xi[\Delta_c(z-\xi p_1+\eta p_2)+\Delta_c(z+\xi p_1'+\eta p_2)]\ . \tag{53}$$

A distinction should also be made for the ηp_2 variables, using p_2 when $\eta>0$ and $p_2'=p_2+q$ when $\eta<0$; but this last qualification merely complicates the arithmetic and is therefore omitted, so that (53) will be used for either sign of η. The previous μ_c cutoff will also not be needed.

In the high-energy limit, $s\gg m^2$, (53) may be written in the form

$$\mathfrak{D}_2\to \frac{1}{16\pi^2}\int_0^\infty \frac{da}{a^2}\,e^{-ia\mu^2}\exp\left(+\frac{i}{4a}\,(b^2-2E\bar{\eta}z^{(+)})\right)\times\int_{-\infty}^{+\infty}d\xi\,\exp\left(\frac{i}{4a}\left[\frac{\xi^2q^2}{4}-|\xi|(\vec{b}\cdot\vec{q})+s\bar{\eta}\xi\right]\right)\ . \tag{54}$$

Because of the $|\xi|$ phase terms, the ξ integral of (54) is an incomplete Gaussian, whose value may be estimated easily in the region $b\mu<1$. Hence, the subsequent analysis is restricted to small impact parameter; for $b\mu>1$, the final b integration is cut off and effectively removed by the χ_0 function, in the standard way.

When the effective ξ cutoff is given by the quadratic ξ dependence of the phase terms of (54), $|\xi_{\max}|\sim 1/q\mu$ (since the variable a scales with $1/\mu^2$). Hence the awkward $|\xi|$ dependence is of size $O(bq\mu^2/q\mu)\sim O(b\mu)$; and, when $b\mu<1$, it may be neglected, and the ξ integral performed immediately,

$$\mathfrak{D}_2\sim\frac{1}{16\pi^2}\int_0^\infty\frac{da}{a^{3/2}}\left(\frac{16\pi}{-iq^2}\right)^{1/2}\times\exp\left\{-ia\mu^2+\frac{i}{4a}\left[b^2-2E\bar{\eta}z^{(+)}-\left(\frac{\bar{\eta}s}{q}\right)^2\right]\right\}. \tag{55}$$

Now consider the contribution of various terms in the phase coefficient of $i/4a$, in (55). We are interested in large $|\bar{\eta}|$ values and in the way such quantities enter into the numerator Q_n integrals. As mentioned above, assume that $|z^{(+)}|$ may be considered to be of order E/q^2. If the important large $|\bar{\eta}|$ values are such that

$$|\bar{\eta}|s>1\ , \tag{56}$$

the the linear $\bar{\eta}$ term in the phase of (55) may be neglected in comparison with the quadratic $\bar{\eta}$ dependence there. Suppose that this is true; and that we are interested only in those large values of $|\bar{\eta}|$ for which

$$b^2<\left(\frac{\bar{\eta}s}{q}\right)^2\ . \tag{57}$$

Then the phase of (55) will have a definite negative sign, and the resulting integral has a guaranteed phase dependence

$$\mathfrak{D}_2\sim-\frac{1}{2\pi}\,\frac{e^{-i|\bar{\eta}|s\mu/q}}{s|\bar{\eta}|}\ . \tag{58}$$

Exactly the same arguments may be applied to the $\mathfrak{D}_1$ denominator function, "undoing" the $q=p_2'-p_2\to 0$ limit by using

$$\mathfrak{D}_1=\int_0^\infty d\eta[\Delta_c(z-\xi p_1+\eta p_2)+\Delta_c(z-\xi p_1-\eta p_2')]\ , \tag{59}$$

to yield

$$\mathfrak{D}_1\sim-\frac{1}{2\pi}\,\frac{e^{-i|\xi|s\mu/q}}{s|\bar{\xi}|}\ . \tag{60}$$

Note that the sign of the phase of each of these forms for $\mathfrak{D}_1$ and $\mathfrak{D}_2$ is the same, and hence there is no possibility of multiplicative cancellations. These forms may be considered as appropriate replacements for the $\delta(s\bar{\xi})$ and $\delta(s\bar{\eta})$ obtained earlier, after taking the $q=0$ limit.

When the phase factors of (58) and (60) enter the Q_n numerators as factors $\sim s^2|\bar{\xi}\bar{\eta}|\exp[+i(|\bar{\eta}|+|\bar{\xi}|)\times s\mu/q]$, they must be considered in comparison to the existing numerator Bessel-function dependence, $H_1^{(2)}(\mu[b^2+s\,|\bar{\xi}\bar{\eta}\,|]^{1/2})$ or $K_1(\mu(s\,|\bar{\xi}\bar{\eta}\,|-b^2)^{1/2})$; these latter forms are correct if $s/m^2\gg 1$, so that the correct high-energy argument of these functions,

$$\mu\left(b^2+s\bar{\xi}\bar{\eta}-\frac{m^2\bar{\xi}z^{(+)}}{2E}-\frac{m^2\bar{\eta}z^{(-)}}{2E}-\frac{m^4z^{(+)}z^{(-)}}{s^2}\right)^{1/2} \tag{61}$$

may be approximated by

$$\mu(b^2+s\bar{\xi}\bar{\eta})^{1/2}\,. \tag{62}$$

This replacement also assumes that the remaining linear $\bar{\xi}$ and $\bar{\eta}$ dependence of (61) can be neglected, which will be verified later.

Were the denominator terms to bring simple polynomial dependence into the numerators, as in the case of the wide-angle analysis, the $\bar{\xi},\bar{\eta}$ cutoffs would again be given by $r_{\max}\sim 1/\mu\sqrt{s}$. Here, however, the denominator phases cut off the integrals at far lower values,

$$r_{\max}\sim q/s\mu\,. \tag{63}$$

In order to satisfy (56), it follows that this analysis requires the kinematical restriction $q>\mu$. (This is compatible with $b\mu<1$ in the sense that for $q>\mu$ only the specification of small-b dependence is necessary.) Similarly, the replacement of (61) by (62) requires $q^3>m^2\mu$, again compatible with a fixed q^2 and large s.

In order to evaluate the Q_n, we shall again perform the simplest possible estimation, and treat each numerator Δ_c as a massless propagator, while restricting the integration over the largest $|\xi|,|\eta|$ values which do not exceed $r_{\max}$. The $|$minimum$|$ value of these variables included here is such that (57) holds; and similarly for a similar relation bounding the ξ dependence; that is, $r_{\min}\sim bq/s$. Thus, for the case $sr_{\max}>b^2$, corresponding to $1>q/\sqrt{s}>\mu b$, one may roughly estimate

$$Q_1\sim-\frac{1}{\pi^2}\int\!\!\int_{r_{\min}}^{r_{\max}}d\bar{\xi}d\bar{\eta}\;\frac{|\bar{\xi}||\bar{\eta}|s^2}{(\bar{\xi}\bar{\eta}s)^2}=-\left(\frac{1}{\pi}\ln(\mu b)\right)^2; \tag{64}$$

and, in general,

$$Q_\kappa\sim\frac{(-1)^\kappa}{\kappa}\left(\frac{1}{\pi}\ln(\mu b)\right)^{2\kappa}. \tag{65}$$

Of course, only the largest $|\bar{\xi}|,|\bar{\eta}|$ contributions have been considered here; but it is interesting to see that the result is finite and is independent of explicit q dependence, which suggests the possibility that the restriction $q>\mu$ is not essential. One thus obtains

$$e^Q\simeq\left[1+\left(\frac{1}{\pi}\ln(\mu b)\right)^2\right]^{-1},\quad 1>\frac{q}{\sqrt{s}}>\mu b\,, \tag{66}$$

a form free of the previous overdamped behavior of Sec. V A.

For somewhat larger values of μb, $1>\mu b>q/\sqrt{s}$, the Δ_c denominators $(b^2+s\bar{\xi}\bar{\eta})^{-1}$ should be replaced by $(b^2)^{-1}$; and one obtains

$$Q_1\sim\frac{-1}{\pi^2}\int\!\!\int_{r_{\min}}^{r_{\max}}d\bar{\xi}d\bar{\eta}\;\frac{|\bar{\xi}||\bar{\eta}|}{[b^2]^2}=-\left(\frac{q^2}{2\pi\mu^2b^2s}\right)^2 \tag{67}$$

and

$$e^Q\simeq\left[1+\frac{q^2}{2\pi\mu^2b^2s}\right]^{-1},\quad 1>\mu b>q/\sqrt{s}\,. \tag{68}$$

Clearly the difference of (68) from unity vanishes in the limit $q/\sqrt{s}\to 0$. Further, in this limit, (66) is appropriate only for vanishing impact parameter, and in effect gives no contribution to the integral over $\int d^2b$. Thus, according to these crude estimates, Q may be thought of as small, real, and negative.

The sign and reality property of Q is not surprising, in view of the possibility of inelastic processes, in which pairs of charged mesons are emitted by the two nucleons acting coherently, and which must tend to damp out the elastic amplitude; rather, what is surprising is that the effect is so small. One can express this in another way by writing a conventional eikonal parametrization of the scattering amplitude, in terms of a complex eikonal function χ',

$$1-e^{i\chi'}=(1-e^{i\chi})e^Q\,,$$

where, for small Q, $i\chi'\simeq i\chi+Q(1-e^{i\chi})$. Thus, for χ and Q real,

$$\mathrm{Re}(i\chi')\sim Q(1-\cos\chi)\,,$$

and is always negative.

VI. ISOTOPIC PROJECTIONS AND INTEGRABILITY

The final step of this analysis is the necessary projection and construction of isotopic $I=1,0$ amplitudes. It will be seen that the form given in (37) is appropriate only for the triplet amplitude; and a somewhat different construction will be used in order to obtain both singlet and triplet projections.

Suppose that there exist both isotopic scalar and vector exchanges, in terms of the propagators $\Delta_c^{(V)}$ and $\Delta_c^{(S)}$, and the coupling constants g_V and g_S. This follows from the addition to (1) of the term $\mathcal{L}''=ig_S\bar{\psi}\gamma_\mu A_\mu\psi$, where for simplicity the restriction to vector mesons is maintained. All of the above analysis has then, in effect, been carried through for the case $g_S=0$. For $g_S\neq 0$, however, the only difference would be that the $\mathfrak{F}(\xi)$ of (10) is multiplied by the familiar

$$\exp\left[-ig_S\int_0^\xi d\xi'\frac{p_\mu}{m}A_\mu\left(z-\xi'\frac{p}{m}\right)\right]$$

factor. But the scalar functional linkages proceed independently, with no regard for the complexities of the isovector interactions. Thus (19) is multiplied by a factor

$$\exp\left[i(p_1\cdot p_2)\int f_2^{(S)}\Delta_c^{(S)}f_1^{(S)}\right]\equiv\exp(ig_S{}^2\Omega_S)\,, \tag{69}$$

where $f_{1,2}^{(S)}$ contains the factor g_S, rather than the $|g_V|$ of (16); this dependence on coupling is made explicit by the second form of (69), involving

$$\begin{aligned}ig_S{}^2\Omega_S &= ig_S{}^2(p_1\cdot p_2)\iint_{-\infty}^{+\infty}d\xi d\eta\Delta_c^{(S)}(z-\xi p_1+\eta p_2)\\ &= i\chi_0^{(S)}(b)\to -i\frac{g_S{}^2}{2\pi}K_0(\mu_S b)\,,\end{aligned}$$

for $s\gg m^2$. (In all of this section, the conventional $q=0$ limit is assumed.) Thus, the result of performing all functional operations is to replace (34) by

$$\frac{\partial T}{\partial g_V{}^2}=\int\cdots\Omega_V\exp(ig_V{}^2\Omega_V+ig_S{}^2\Omega_S)\,, \tag{70}$$

where

$$\begin{aligned}ig_V\Omega_V &= ig_V{}^2(p_1\cdot p_2)\iint_{-\infty}^{+\infty}d\xi d\eta\Delta_c^{(V)}(z-\xi p_1+\eta p_2)\\ &= i\chi_0^{(V)}(b)\,,\end{aligned} \tag{71}$$

and all irrelevant multiplicative factors and integrations have been implicitly indicated by the symbols $\int\cdots$.

Were the identical operations to be performed for the scalar interaction, one would find

$$\frac{\partial T}{\partial g_S{}^2}=\int\cdots\Omega_S\exp(ig_S{}^2\Omega_S+ig_V{}^2\Omega_V)\,. \tag{72}$$

Equations (70) and (72) are compatible, in the sense that they satisfy the integrability condition $\partial^2T/\partial g_S{}^2\partial g_V{}^2=\partial^2T/\partial g_V{}^2\partial g_S{}^2$. But a glance at the original isotopic factors of (17) shows that this is only true for the triplet amplitude, where $(\vec\tau_1\cdot\vec\tau_2)$ has the eigenvalue $+1$; that is, a factor of $+1$ multiplying (70) corresponds to the projection of this operator in the triplet state. The pair of Eqs. (70) and (72) may be integrated immediately and yield (now supplying all factors)

$$T_{\text{eik}}^{(I=1)}(s,t)=\frac{is}{2m^2}\int d^2b\,e^{i\vec q\cdot\vec b}\,e^Q(1-e^{i\chi_0^{(V)}+i\chi_0^{(S)}})\,, \tag{73}$$

as the correct eikonal representation of the triplet amplitude.

On the other hand, suppose that the same analysis is performed for the singlet amplitude. Here, $(\vec\tau_1\cdot\vec\tau_2)$ has the eigenvalue -3; and one would obtain (72), still valid in the singlet state, together with

$$\frac{\partial T^{(0)}}{\partial g_V{}^2}=(-3)\int\cdots\Omega_V\exp(ig_V{}^2\Omega_V+ig_S{}^2\Omega_S)\,. \tag{74}$$

Equations (72) and (74) are not compatible; the integrability condition is not satisfied, and one cannot construct $T_{\text{eik}}^{(0)}$ in this way. Incidentally, the $\int\cdots$ symbols can differ in the singlet and triplet cases, owing to their dependence on e^Q, which factors are independent of coupling.

The power of the present method, which calculated $\partial T/\partial g^2$ rather than T, can now be seen, for in this strong-coupling limit it generates precise integrability conditions. For example, it is clear that the isosinglet amplitude will necessarily have an exponential dependence of form $\exp(-3ig_V{}^2\Omega_V+ig_S{}^2\Omega_S)$. This does not at all specify the corresponding isosinglet Q term; and one must search for solutions to the Euler equations, other than those of (29)–(31), which will generate for the isosinglet $f[\varphi_0,\psi_0]$ of (21) the quantity $-3ig_V{}^2\Omega_V$.

For the previous choice of constant, numerical unit vectors, it is not difficult to see that self-consistency requires the condition $(\hat\kappa\circ\hat\rho)^2=1$; and hence that the possible values of $\hat\kappa\circ\hat\rho$ are restricted to ±1. What is needed for the $\hat\kappa\circ\hat\rho$ term of (31) is the possibility of having the value -3, when computing the singlet amplitude. This conclusion is unchanged if the fields and unit vectors are allowed to become complex, a property recently noted[14] as useful when constructing phenomenological representations of complete, coherent isotopic pion states. The use of variable unit vectors is always a possibility, but one that is too complicated to be transparent.

The possibility of a successful construction is suggested by the physical observation that, in the previous pair of equations

$$K\varphi^\alpha=f_2\psi^\alpha(\vec\psi^2)^{1/2}\equiv f_2p_\alpha\,, \tag{75}$$

and

$$K\psi^\alpha=f_1\varphi^\alpha/(\vec\varphi^2)^{1/2}\equiv f_1q_\alpha\,, \tag{76}$$

the $f_{1,2}$ terms, representing classical nucleon currents, act as the sources of these averaged fields. Previously, the unit vectors (now called q_α and p_α) were taken as arbitrary constants; but there is no reason to exclude the possibility that the nucleon isotopic coordinates could not be used to represent the isotopic direction of these sources. One is then led to consider the possibility that the fields φ^α, ψ^α are themselves matrices in the isospace of the two nucleons; and accordingly, one may set $p_\alpha=a\tau_1^\alpha+b\tau_2^\alpha$, $q_\alpha=c\tau_1^\alpha+d\tau_2^\alpha$, where the subscripts 1, 2 refer to the distinguished nucleons, and the parameters a,b,c,d are numerical constants.

In terms of such a representation, φ^α, $(\vec\varphi^2)^{1/2}$,

ψ^α, $(\vec{\psi}^2)^{1/2}$,q_α, p_α all become matrices, satisfying (75) and (76). The manipulations which lead to these equations, as well as their matrix solutions, should be quite independent of the particular isotopic projections to follow; and it should be possible to build in the requirements of the integrability condition while retaining the independence of the matrix solutions to the subsequent triplet and singlet projections. (One can approach the problem differently, by looking for different solutions for the two amplitudes.) Such independence is the only measure of uniqueness in this problem; and it is the only reason for believing in the relevance of the singlet solution given below.

There is one qualification to this procedure, however, which demands critical attention. One must now allow for the possibility of one essential change, in comparison with the previous situation of constant numerical matrices, for the quantity $\sum_\alpha(\varphi^\alpha)^2$, constructed from the sum of the squares of the matrix φ^α need not be equal to the square of the quantity $(\vec{\varphi}^2)^{1/2}$; here, the sum-over-squares relation between φ^α and $(\vec{\varphi}^2)^{1/2}$ may no longer act as a constraint on the possible matrix solutions of φ^α if the integrability conditions are to be satisfied, for q^2 will have two eigenvalues, of which only one is unity. (A constant numerical vector has not been included in the definition of q_α, since q^2 would then have nondiagonal terms in both singlet and triplet states.) Similar remarks hold for ψ^α, $(\vec{\psi}^2)^{1/2}$, and p_α. The only value of this construction is that it proceeds from a single variational principle, and builds in the integrability condition in a way which permits the solutions φ_0, ψ_0 to be written for arbitrary isospin. In the triplet state, one finds just the results of the previous sections; and one is thus led to associate the singlet projection of these forms as the appropriate $f[\varphi_0,\psi_0]$ and Q. Of course, from the integrability condition one knows exactly what must be the proper form for the singlet $f[\varphi_0,\psi_0]$; and the entire construction is essentially to discover the singlet Q.

With this choice of constant matrix forms for p_α and q_α, one can see from (75) and (76) that there is still a measure of self-consistency which must be respected by these solutions,

$$p_\alpha = q_\alpha \frac{1}{q^2}(q\cdot p) \text{ and } q_\alpha = p_\alpha \frac{1}{p^2}(p\cdot q). \tag{77}$$

The explicit solutions are

$$\varphi^\alpha = p_\alpha \int \Delta_c f_2, \quad (\vec{\varphi}^2)^{1/2} = \frac{1}{q^2}(q\cdot p)\int \Delta_c f_2, \tag{78}$$

and

$$\psi^\alpha = q_\alpha \int \Delta_c f_1, \quad (\vec{\psi}^2)^{1/2} = \frac{1}{p^2}(p\cdot q)\int \Delta_c f_1, \tag{79}$$

so that the $f[\varphi_0,\psi_0]$ of (21) then becomes

$$i(p_1\cdot p_2)\left(-1+\frac{1}{q^2}+\frac{1}{p^2}\right)(q\cdot p)\int f_1\Delta_c f_2. \tag{80}$$

This will provide the desired results if the constants a, b, c, d are chosen to satisfy

$$(\vec{\tau}_1\cdot\vec{\tau}_2) = \left(\frac{1}{q^2}+\frac{1}{p^2}-1\right)(q\cdot p),$$

along with the restrictions of (77). The latter can most simply be satisfied by the symmetric choice $a=c$, $b=d$, so that $p_\alpha = q_\alpha$. Since, then, $q\cdot p = q^2 = p^2$, the requirements

$$2ab=-1 \text{ and } 2-3(a^2+b^2)=0 \tag{81}$$

are sufficient to produce the desired form for (80),

$$i(p_1\cdot p_2)(\vec{\tau}_1\cdot\vec{\tau}_2)\int f_1\Delta_c f_2, \tag{82}$$

in which the $(\vec{\tau}_1\cdot\vec{\tau}_2)$ factor may be assigned its isotopic projection value of +1 or −3. The solutions to (81) exist only for complex a,b,

$$a^2 = \tfrac{1}{3}(1\pm i\sqrt{5}/6),$$

but may be used for both singlet and triplet cases. Thus φ^α and ψ^α have been constructed independently of the isotopic states, although their a, b values are determined by the integrability condition.

With such matrix solutions, the Gaussian functional integration of (31) goes through as before, except that now the $S^{\mu\nu}_{\alpha\beta}$, $T^{\mu\nu}_{\alpha\beta}$ tensors contain nucleon isotopic matrices as well. For the triplet case, where q^2 has the expectation value +1, it is easy to see that the isospin operators effectively reproduce the net factor of +2, previously used to convert (35) to (36); and hence the amplitude as written in (37) is again found to be the triplet solution. For the singlet case, the Q_n iterates are slightly more involved; but one can see that every iterate is multiplied by a factor: $2+(1-q^2)^{2n}$. Thus, for the singlet solution with $q^2=5$, the crude estimates of the previous section suggest that $e^Q \simeq [1+(4\phi)^2]^{-1/2}$, where $\phi\simeq(1/\pi)\ln(\mu b)$, $1>q/\sqrt{s}>\mu b$, and $\phi\simeq q/\mu b(2\pi s)^{1/2}$, $1>\mu b>q/\sqrt{s}$.

Having found this matrix solution to (75) and (76), and taken its projections in the different isospin states, one might observe that, really, one is interested in *any* solution of these equations, whether self-consistent in the sense of $\sum_\alpha {\varphi_\alpha}^2 = [(\vec{\varphi}^2)^{1/2}]^2$ or not; for any such solution will permit the evaluation of the functional integrals by the stationary-phase approximation. Hence, it was not really necessary to use matrix representations for q_α and p_α, for, in the $\vec{\kappa}\cdot\vec{\rho}$ solutions of the previous section, one could have merely chosen $(\vec{\kappa}\cdot\vec{\rho})^2 = \vec{\kappa}^2 = \vec{\rho}^2 = 1$ for the triplet amplitude (as was done), and $(\vec{\kappa}\cdot\vec{\rho})^2=\vec{\kappa}^2=\vec{\rho}^2=5$ for the singlet. The latter's $\vec{\kappa},\vec{\rho}$ vectors are now no longer of unit magnitude; but one still has a solution of (75) and (76) if only $(\vec{\varphi}^2)^{1/2}$ is defined in terms of φ_α and p_α. A "justification" for that procedure is suggested by the present method, which uses the same variational principle and matrix solutions for both states, while allowing the integrability condition to dominate the normalization requirements (which are not requirements for stationary phase). Of course, it is not known if the results of this construction are unique. But it is reassuring that the triplet projection does agree with its form previously found; and it is hoped that the singlet projection is the correct one. (It would be a pleasure to be able to discard the latter, in favor of a self-consistent singlet solution.) Practically speaking, both Q terms are probably small, and may be neglected at sufficiently high energies; while the coupling-dependent eikonal functions are unambiguously determined by the integrability conditions.

VII. SUMMARY

Putting all terms together, the result of these sections is an eikonal representation for NN scattering by the exchange of multiple, massive SU(2) vector mesons, in the form

$$T_{\text{eik}}^{(I)}(s,t)=\frac{is}{2m^2}\int d^2b\,e^{i\vec{q}\cdot\vec{b}}\times\left[1-\exp(i\chi_0^{(S)}+i\langle\vec{\tau}_1\cdot\vec{\tau}_2\rangle\chi_0^{(V)})\right]\times e^{Q^{(I)}(b,\mu,q,s)}, \tag{83}$$

where $\chi_0^{(S)}$ is the typical eikonal function of isoscalar exchange, $\chi_0^{(V)}$ is the *same* function with coupling and mass parameters of the isovector exchange, and $\langle\vec{\tau}_1\cdot\vec{\tau}_2\rangle=2I(I+1)-3$. Crude estimates suggest that $Q^{(1)}\simeq-\ln(1+\phi^2)$, $Q^{(0)}\simeq-\frac{1}{2}\ln[1+(4\phi)^2]$, where $\phi\simeq(1/\pi)\ln(\mu b)$, $1>q/\sqrt{s}>\mu b$, and $\phi\simeq q/\mu b(2\pi s)^{1/2}$, $1>\mu b>q/\sqrt{s}$. One may expect better estimates of the Q to show some fine details missed by the qualitative arguments above.

In all of the above, the requirement $T(g_V=0,\ g_S=0)=0$ has been made, thereby eliminating an arbitrary function of impact parameter independent of either coupling constant; and this point deserves special attention. Although derived under the assumption that $G\sim|g_V|\times 0$ (field magnitudes) >1, one may ask if the limit of the strong-field solution (15) for $\mathfrak{F}(\xi)$, as $g_V\to 0$, is not simply unity. The answer is not necessarily, because the field magnitudes could conceivably become infinite as the coupling vanishes. But in this calculation, as in any calculation (e.g., of T rather than $\partial T/\partial g_V{}^2$) which uses stationary phase, the field magnitudes are bounded in the sense that they are replaced by "average" values, proportional to $|g_V|$, to which are added fluctuations that are themselves damped by Gaussian weight factors independent of g_V. In effect, the field magnitudes in any stationary-phase calculation of T are going to be bounded, as $g_V{}^2\to 0$; and hence the limit $|g_V|\to 0$ in the strong-coupling $\mathfrak{F}(\xi)$ produces $\mathfrak{F}(\xi)\to 1$, and therefore $G_c^{\text{BN}}\to S_c$, the free-particle propagator. That is, the scattering amplitude must vanish in this limit, even though the entire derivation has been carried through assuming strong coupling or strong fields. The first few terms in the $g_V{}^2$ expansion of (83) or (37) need have nothing to do with the proper small-coupling expansion of T; but the amplitude should vanish when $g_S,\ g_V\to 0$. (Example: suppose the exact form of $\pi(\xi)$ in (10) were given by $g\,dA(\xi)/d\xi+g^2dB(\xi)/d\xi$, a form which, while absurd, illustrates the point. The exact solution for $\mathfrak{F}(\xi)$ would then be $\exp[gA(\xi)+g^2B(\xi)]$, with strong-coupling limit $\exp[g^2B(\xi)]$; and the latter $\to 1$ when $g^2\to 0$, even though its weak-coupling expansion is incorrect.) The result (83) is consistent with this idea, for it is the $Q^{(I)}$ that generate the difference between this strong-coupling form and the simple Born approximation, in the small-coupling limit.

For wide-angle processes, the evaluations here are clearly incomplete; but their indications are that, in part because of relative cancellations, and in part because ρ's couple less strongly to nucleons than do ω's, form-factor damping due to $\rho^\pm$, ρ^0 exchange, as well as the contributions to large-p_t scattering and production amplitudes, may be neglected as a first approximation in the construction of massive-gluon bremsstrahlung models. One assumes, of course, that the strong-coupling limits and approximations used are actually applicable to ρ exchange.

Finally, there are some interesting speculations that follow from this analysis:

(i) It would be interesting to calculate the effects of isospin on the "tower-graph" contributions to the eikonal neglected in all of the above. Would a phenomenological procedure similar to that used to generate Pomerons from isoscalar ladder graphs be sufficient to produce unitarized and Reggeized ρ exchange?

(ii) As μ decreases, at fixed q and s, the damping of the e^Q factor can become appreciable. In the limit as $\mu\to 0$, one must consider suitably inclusive cross sections, combining scattering with soft-meson emission; and because of existing theorems,[15] one expects that all logarithmic mass dependence in every order of perturbation theory will cancel. In the present case of strongly coupled fields, where the limit of large coupling is taken before μ vanishes, it would be interesting to see if one can have an SU(2) version of the Yang-Mills behavior suggested in Ref. 5. The graphical origins of that effect, if it exists, would be quite different in each case; but it would be interesting to find an explicit case of damping by logarithmic mass dependence in a strongly coupled, non-Abelian theory.

(iii) If the strong-coupling analysis of Sec. III could be extended to SU(3), and the self-interactions of Yang-Mills gluons included along with gluon exchange in a generalization of the stationary-phase procedure of Sec. IV, one would have a rough model for quark-quark scattering. It would then be most interesting to see if the Cornwall-Tiktopoulos damping occurs, as $\mu \to 0$.

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