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Virial theorem and stability of localized solutions of relativistic classical interacting fields*

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The properties of classical meson fields interacting with the Dirac field are considered in more detail analytically. We obtain a virial relation between the kinetic and potential energy of the interacting Dirac field. The stability of the classical solutions of the coupled fields constrained to the lowest particle solution of the Dirac equation in three space dimensions is investigated. A criterion allowing the determination of stability of Abelian meson fields in interaction with the Dirac field is given.

I. INTRODUCTION

A relation between the kinetic and potential energies for particles in external potentials that follows from the equation of motion is referred to as a virial theorem.[1] A similar relation has been found for classical, relativistically invariant, scalar, self-interacting fields.[2,3] A simple proof is given by Jackiw and Goldstone,[3] who further show that stable solutions in this case exist only in one space dimension. This is a very relevant result in the light of recent discussion of the role of the classical field solutions in quantum theory.[2,4]

The instability of the solutions studied in Ref. 3 is apparently a consequence of the simplicity of the scalar, self-interacting field. Therefore, in this note I consider a wider class of theories associated with a charged spinor (Dirac) field in interaction with neutral meson fields, as well as the case of a self-interacting Dirac field. I find that the physically interesting solutions in these cases must be constrained to the sector of the spectrum of the Dirac equation associated with particle eigensolutions. The only stable solution is built upon the *lowest-energy particle solution* of the Dirac equation.

In order to characterize an eigenstate as a *particle* solution we may simply require that for vanishing strength of the coupling it becomes a positive-frequency free-particle solution. In the particularly interesting case of the interaction with a scalar-meson field this is equivalent to the requirement that we consider the subset of positive-frequency eigensolutions. If vector mesons are present, we must be more careful—even a resonance in the negative-frequency continuum may be a state we wish to consider.[5]

In the next section we will study in detail the implications of the virial theorem that I derive. In particular, I will obtain an expression for the total energy of the interacting fields that depends only on the meson field. Some interesting results concerning the kinetic and potential energies of the meson field will be derived here that point to the unique character of three-dimensional space. In particular, I show that in three space dimensions the total energy can be seen as an explicit function of the meson fields only; no derivative of the meson field enters the explicit expressions that is strictly local. Furthermore, I find that any φ^4-like self-interaction of the meson field is effectively canceled by the contribution of the Dirac field to the total energy. I also consider other interactions of the Dirac field that may not be readily expressed as coupling with other relativistic fields, but can be conveniently described by a two-body potential $V(x-y)$. Particularly interesting here is the case of the contact self-interaction, that is $V\sim\delta^3(x-y)$ also considered in Ref. 6.

I then turn to the discussion of the stability of the solutions. I obtain the second variations of the fields that should be positive for the stable solutions of first-variation equations. These considerations will provide us with a criterion to determine if a solution is stable, provided that the analytic solutions of the field equations are known.

II. THE VIRIAL APPROACH TO CLASSICAL FIELD EQUATIONS

A. Dirac field in an external potential

Basic in our considerations will be a relation describing the kinetic energy of the fermion field ψ in some "external," frozen potential $V_{ex}(x)$. Let H_D be the Dirac operator

$$H_D=\vec{\alpha}\cdot\vec{p}+\beta m+V_{ex}, \tag{2.1}$$

where $\{\vec{\alpha},\beta\}$ are the usual Dirac matrices, and m is the mass of the fermion field. V_{ex} is a matrix in the spinor space and can in principle consist of all possible couplings (S,V,P,A,T). Then we have

$$[\vec{x}\cdot\vec{p},H_D]=i\vec{\alpha}\cdot\vec{p}-i\vec{x}\cdot(\vec{\nabla}V_{ex}). \tag{2.2}$$

Taking the expectation value of Eq. (2.2) between localized eigenfunctions ψ_n of H_D

$$H_D\psi_k=\epsilon_k\psi_k,\quad -m<\epsilon_k<m, \tag{2.3}$$

we find

$$\int d^3x\,\psi_k^\dagger\vec{\alpha}\cdot\vec{p}\psi_k=\int d^3x\,\psi_k^\dagger\vec{x}\cdot(\vec{\nabla}V_{ex})\psi_k. \tag{2.4}$$

This is a very useful relation as we will see in a moment. Beforehand I would like to caution the reader to use Eq. (2.4) only when ψ_k is an eigenfunction of H_D. A superposition of eigenfunctions

$$\psi_t=\sum_k a_k\psi_k \tag{2.5}$$

does not satisfy the virial equation (2.4), since the expectation value of the commutator is

$$\int d^3x\,\psi_t^\dagger[\vec{x}\cdot\vec{p},H_D]\psi_t=\sum_{k\neq m}a_m^*a_k(\epsilon_k+\epsilon_m)\times\int d^3x\,\psi_m^\dagger\vec{x}\cdot\vec{p}\psi_k. \tag{2.6}$$

We record that all of ψ_k considered above are localized discrete eigenstates in configuration space. In general, the last term in Eq. (2.6) will not vanish, since $\vec{x}\cdot\vec{p}$ can induce transitions between the states m and k.

Equation (2.4) has been known for the case of an electron in the external electromagnetic field.[7,8,9]

B. Scalar, time-independent meson field

The Lagrangian of coupled Dirac-scalar-meson fields is

$$\mathcal{L}=\bar\psi(\gamma\cdot p-m)\psi+g_S\varphi_S\bar\psi\psi+\tfrac{1}{2}\partial_\mu\varphi_S\partial^\mu\varphi_S-[U(\varphi_S)-U_0], \tag{2.7a}$$

$$\mathcal{L}=\mathcal{L}_\psi+\mathcal{L}_{\varphi\psi}+\mathcal{L}_\varphi \tag{2.7b}$$

and the equations of motion for the fields are

$$\ddot\varphi_S-\Delta\varphi_S+\frac{\partial U}{\partial\varphi_S}=g_S\bar\psi_k\psi_k, \tag{2.8a}$$

$$[\vec{\alpha}\cdot\vec{p}+\beta(m-g_S\varphi_S)]\psi_k=\epsilon_k\psi_k. \tag{2.8b}$$

For the time being, I consider only time-independent scalar fields. Then Eq. (2.4) for the kinetic energy of the Dirac field becomes, using Eq. (2.8),

$$\int d^3x\,\psi_k^\dagger\vec{\alpha}\cdot\vec{p}\psi_k=\int d^3x(-\vec{x}\cdot\vec{\nabla}\varphi_S)\left(-\Delta\varphi_S+\frac{\partial U}{\partial\varphi_S}\right). \tag{2.9}$$

We record the useful relations

$$\Delta\varphi_S\vec{x}\cdot\vec{\nabla}\varphi_S=\frac{n-2}{2}(\vec{\nabla}\varphi_S)^2+\vec{\nabla}\cdot[(\vec{x}\cdot\vec{\nabla}\varphi_S)\cdot\vec{\nabla}\varphi_S-\tfrac{1}{2}\vec{x}(\nabla\varphi_S)^2], \tag{2.10a}$$

$$\frac{\partial U}{\partial\varphi_S}\vec{x}\cdot\vec{\nabla}\varphi_S=-n(U-U_0)+\vec{\nabla}[\vec{x}(U-U_0)], \tag{2.10b}$$

where U_0 is an (integration) constant to be fixed by boundary conditions. Furthermore, we have denoted by

$$n=\vec{\nabla}\cdot\vec{x} \tag{2.11}$$

the dimensionality of the space. Normally $n=3$, but to facilitate contact with other work[3] and to dramatize the uniqueness of the three-dimensional, physical space, we use in the remainder of this section the number of dimensions as a parameter. We find using Eq. (2.10) for the kinetic energy of the Dirac field

$$\int d^nx\,\psi_k^\dagger\vec{\alpha}\cdot\vec{p}\psi_k=\int d^nx\left(\frac{n-2}{2}(\vec{\nabla}\varphi_S)^2+n[U(\varphi_S)-U_0]\right)+\int d^nx\,\vec{\nabla}\cdot[\vec{\nabla}\varphi_S(\vec{x}\cdot\vec{\nabla}\varphi_S)-\tfrac{1}{2}\vec{x}(\vec{\nabla}\varphi_S)^2-\vec{x}(U-U_0)]. \tag{2.12}$$

The second part of the integral (2.12) is a surface term which will vanish whenever

$$|x|^n[U-U_0-\tfrac{1}{2}(\nabla\varphi_S)^2]\to 0,\quad |x|\to\infty. \tag{2.13}$$

Let us consider now the special case $n=1$, so far looked at in the literature. Here the situation is simpler, since the vector products collapse in Eq. (2.9) and we obtain directly

$$\int d^1x\,\psi_k^\dagger\alpha i\frac{\partial}{\partial x}\psi_k=\int d^1x\,x\frac{d}{dx}\varphi_S\left(\frac{-d^2\varphi_S}{dx^2}+\frac{\partial U}{\partial\varphi_S}\right)=\int d^1x\,x\frac{d}{dx}\left[-\frac{1}{2}\left(\frac{d\varphi_S}{dx}\right)^2+U-U_0\right]. \tag{2.14}$$

In the case of the σ model for which exact analytical solutions have been studied in Ref. 12, $U-U_0$ is given by ($\chi_S=m/g-\varphi_S$):

$$U-U_0=H(\chi_S{}^2-f^2)^2, \tag{2.15}$$

and the term in the large square brackets in Eq. (2.14) vanishes for the solution

$$\chi_S=f\tanh[\sqrt{2H}f(x-x_0)]. \tag{2.16}$$

That is consistent with the vanishing kinetic energy of the quasi-fermion field. We note that Eq. (2.13) is also satisfied in the one-dimensional example given above; thus the limit $n\to 1$ in Eq. (2.12) can be taken and it yields Eq. (2.14).

C. The effective energy

Using the equation of motion (2.8a) we find for the interaction term of the fields

$$-\int d^nx\, g_S\varphi_S\bar{\psi}_k\psi_k = -\int d^nx\left[(\nabla\varphi_S)^2+\varphi_S\frac{\partial U}{\partial\varphi_S}\right] + \int d^nx\,\vec{\nabla}\cdot(\varphi_S\vec{\nabla}\varphi_S). \qquad (2.17)$$

Again, the second part in the above equation is a surface term that will vanish whenever

$$|\vec{x}|^{n-1}\varphi_S\nabla\varphi_S\to 0, \quad |x|\to\infty. \qquad (2.18)$$

In the same manner we find

$$g_S\int d^nx\,\bar{\psi}_k\psi_k = \int d^nx\frac{\partial U}{\partial\varphi_S} - \int d^nx\,\vec{\nabla}\cdot(\vec{\nabla}\varphi_S), \qquad (2.19)$$

with the most severe restriction in order for the surface term to vanish:

$$|x|^{n-1}|\nabla\varphi_S|\to 0, \quad |x|\to\infty. \qquad (2.20)$$

The Hamiltonian associated with the Lagrangian (2.7) is, with a time-independent φ_S,

$$H^S = \int d^nx[\psi_k^\dagger(\vec{\alpha}\cdot\vec{p}+\beta m)\psi_k - g_S\varphi_S\bar{\psi}_k\psi_k + \tfrac{1}{2}(\nabla\varphi_S)^2 + U(\varphi_S)-U_0]. \qquad (2.21)$$

We may use Eqs. (2.12), (2.15), (2.19) to obtain an expression for the Hamiltonian, provided that Eq. (2.20) is satisfied:

$$E^S_{\text{eff}} = \int d^nx\left[\frac{n-3}{2}(\nabla\varphi_S)^2+(n+1)[U(\varphi_S)-U_0] + \left(\frac{m}{g_S}-\varphi_S\right)\frac{\partial U}{\partial\varphi_S}\right]. \qquad (2.22)$$

As is well known, the above equation, Eq. (2.22), cannot be used as a basis of a variational principle. For known φ_S that minimize Eq. (2.21) it gives the proper value of H^S. Therefore we may view E^S_{eff} as an expression defining the total energy; it gives the proper description of the energy of the fields, but is not the basis of a variational principle.

The first striking feature of Eq. (2.22) is the fact that for $n<3$ (n is the number of space dimensions) the effective kinetic energy of the scalar field becomes negative definite. It just vanishes for $n=3$. Thus the energy content of E^S_{eff} in normal number of space dimensions is independent of the derivatives of the field φ_S. The negative-definite kinetic energy for $n<3$ signals a possible instability of the Hamiltonian; the energy of the solution could be reduced by a small variation of φ_S that involve large gradiants of φ_S.

Another notable feature is that any φ_S^4 proportional term in U cancels out in three space dimensions (in general φ^{n+1} cancels). This is an important feature, since φ_S^4 is commonly held responsible for the stability of a theory with spontaneously broken symmetry (see for example Refs. 10, 3, 4). In the case of a one-dimensional world ($n=1$) where φ_S^2 contribution vanishes, this is obviously a necessary term to stabilize the theory. It would therefore seem that a φ^6 plays the role of the φ^4 term in three space dimensions, as compared with one-dimensional models.

In the context of the above given derivation of E^S_{eff} one may ask the question how an effective potential U_{eff} would look like if the interaction term would be added to U (in one space dimension, where the scalar density vanishes, this is identical to U). Thus, defining

$$U_{\text{eff}} = U(\varphi_S) - g\varphi_S\bar{\psi}_k\psi_k, \qquad (2.23)$$

we find from the equations of motion

$$U_{\text{eff}} = U(\varphi_S) - \varphi_S\left(-\Delta\varphi_S+\frac{\partial U}{\partial\varphi_S}\right), \qquad (2.24)$$

that is,

$$\int d^nx\,U_{\text{eff}} = \int d^nx\left[U(\varphi_S)-\varphi_S\frac{\partial U}{\partial\varphi_S}-(\vec{\nabla}\varphi_S)^2\right] + \int d^nx\,\vec{\nabla}\cdot(\varphi_S\vec{\nabla}\varphi_S). \qquad (2.25)$$

It would appear following the conventional arguments that the solutions which minimize the Hamiltonian are those for which φ_S is a solution of (neglecting the inhomogeneity of the fields φ_S)

$$0=\frac{d}{d\varphi_S}\left[U(\varphi_S)-\varphi_S\frac{\partial U}{\partial\varphi_S}\right] = -\frac{\partial^2 U}{\partial\varphi_S^2}. \qquad (2.26)$$

That this is not the case is shown by Eq. (2.12); the kinetic energy of the Dirac field is a significant contribution and it influences strongly the total energy. However, the above argument is very instructive, *since it proves the dominance of the interaction term over the potential U*; it is not the solution of $\partial U/\partial\varphi_S=0$ that would drive the solution in absence of the Dirac kinetic energy term but rather it is the condition (2.26).

Very often it is the field

$$\chi_S = m/g_S - \varphi_S \qquad (2.27)$$

that is introduced, instead of φ_S. The advantage is that there is no explicit mass term of the fermion field in the Hamiltonian. Our result may be easily adapted and we find for the Hamiltonian

$$H^{S'} = \int d^nx[\psi^\dagger\vec{\alpha}\cdot\vec{p}\psi + g_S\chi_S\bar{\psi}\psi + V(\chi_S)+\tfrac{1}{2}(\vec{\nabla}\chi_S)^2], \qquad (2.28)$$

the result

$$E^{S'}_{\rm eff}=\int d^nx\left[\frac{n-3}{2}(\nabla\chi_S)^2+(n+1)V(\chi_S)-\chi_S\frac{\partial V}{\partial\chi_S}\right], \qquad (2.29)$$

where

$$V(\chi_S)=U(\varphi_S)-U_0. \qquad (2.30)$$

The field χ_S may be considered the effective mass of the Dirac field ψ. The mass of the *free* Dirac field is now

$$m=g_S\chi_S(\vec{x}\to\infty), \qquad (2.31)$$

that is dependent on the specific form of $V(\chi_S)$. Further we note that since the integrability of the Hamiltonian H^S requires $U(\varphi_S=0)=U_0$ we find $V(m/g_S)=U_0$.

In the case of the σ model the function V is considered to be

$$V=H(\chi_S{}^2-f^2)^2. \qquad (2.32)$$

We find in one space dimension

$$E^{\sigma,1}_{\rm eff}=\int dx[-(\nabla\chi_S)^2+2H(f^4-\chi_S{}^4)], \qquad (2.33a)$$

while in three space dimensions we obtain

$$E^{\sigma,3}_{\rm eff}=\int d^3x\, 4Hf^2(f^2-\chi_S{}^2). \qquad (2.33b)$$

Only if χ_S is allowed to vary between $+f$ and $-f$ is the last expression explicitly positive definite. Thus again the form of the solution is essential for the determination of a lower bound on $E_{\rm eff}$. However, an upper limit for $E_{\rm eff}$ may be obtained setting $\chi_S{}^2=0$ in Eq. (2.33) and taking the volume of the solution for the integral:

$$E^{\sigma,3}_{\rm eff}\lesssim(4\pi/3)R^3 4Hf^4\,.$$

D. Other interactions

The Lagrangian of the generalized system of interacting fields involving different mesons

$$\mathcal{L}=\mathcal{L}_\psi+\mathcal{L}_\varphi+\mathcal{L}_{\varphi\psi} \qquad (2.34)$$

may be taken to be

$$\mathcal{L}_\psi=\bar\psi(\gamma\cdot p-m)\psi, \qquad (2.34a)$$

$$\mathcal{L}_\varphi=\sum_i\frac{s(i)}{2}(\partial_\mu\varphi_i)(\partial^\mu\varphi^i)-[U(\varphi_i)-U(0)], \qquad (2.34b)$$

$$\mathcal{L}_{\varphi\psi}=\psi^\dagger I\psi, \qquad (2.34c)$$

$$I=\gamma_0(g_S\varphi_S-g_V\gamma\cdot A+g_Pi\gamma_5\varphi_P+\cdots) \qquad (2.34d)$$

$$=\gamma_0\sum_i\Gamma_i g_i\varphi^i. \qquad (2.34e)$$

The interaction term I is a combination of all possible interactions as demonstrated in Eq. (2.34d). $s(i)$ denotes the character of the field, e.g., it is $+1$ for scalar fields and -1 for the longitudinal component of the vector fields. In general, a similar sign factor will be implicitly built into the potential U. Variation of $\mathcal{L}$ with respect to the different, dynamically independent fields φ_i leads to

$$\frac{\delta\mathcal{L}_\varphi}{\delta\varphi_i}+g_i\bar\psi\Gamma_i\psi=0; \qquad (2.35)$$

then the analog of Eq. (2.9) may be written

$$\int d^nx\,\psi_k^\dagger\vec\alpha\cdot\vec{p}\psi_k=-\int d^nx\left[\sum_i(\vec{x}\cdot\vec\nabla\varphi_i)\frac{\delta\mathcal{L}_\varphi}{\delta\varphi_i}\right], \qquad (2.36)$$

then we find using Eq. (2.34b) the analog of Eq. (2.12)

$$\int d^nx\,\psi_k^\dagger\vec\alpha\cdot\vec{p}\psi_k=\int d^nx\left[\sum_i s(i)\frac{n-2}{2}(\nabla\varphi_i)^2+n(U-U_0)\right], \qquad (2.37)$$

and for the interaction Lagrangian, Eq. (2.34), we find in a similar way as for Eq. (2.17)

$$-\int d^nx\sum_i g_i\bar\psi_k\Gamma_i\psi_k\varphi^i=-\int d^nx\left[\sum_i s(i)(\nabla\varphi_i)^2+\varphi_i\frac{\partial U}{\partial\varphi_i}\right]. \qquad (2.38)$$

The scalar density is given as by Eq. (2.19). Then for the Hamiltonian (with time-independent fields)

$$H=\int d^nx\left[\psi_k^\dagger(\vec\alpha\cdot\vec{p}+\beta m)\psi_k-\sum_i g_i\varphi_i\bar\psi_k\Gamma_i\psi_k+\sum_i s(i)\tfrac{1}{2}(\nabla\varphi_i)^2+U-U_0\right], \qquad (2.39)$$

we find

$$E_{\rm eff}=\int d^nx\left[\sum_i s(i)\frac{n-3}{2}(\nabla\varphi_i)^2+(n+1)(U-U_0)+\left(\frac{m}{g_S}-\varphi_S\right)\frac{\partial U}{\partial\varphi_S}-\sum_{i\neq S}\varphi_i\frac{\partial U}{\partial\varphi_i}\right]. \qquad (2.40)$$

Whenever there is no explicit coupling between the different meson fields, that is when U is a sum of the contributions from the different fields

$$U=\sum_i s(i)U_i(\varphi_i), \qquad (2.41)$$

then we have

$$E_{\rm eff}=\sum_i s(i)\int d^nx\left[\frac{n-3}{2}(\nabla\varphi_i)^2+(n+1)(U_i-U_{i0})\right.$$
$$\left.-\varphi_i\frac{\partial U}{\partial\varphi_i}\right], \qquad (2.42)$$

where we have included the fermion mass in the scalar field, in the manner discussed in Eqs. (2.27)–(2.33). Naturally, U need not be a sum of the individual meson contributions. Then we will find the more general Eq. (2.40) very useful.

In addition to the comments made previously, we note that $E_{\rm eff}$ may be negative definite for some interactions. Since the virial approach assumes existence of a valid solution to the Dirac equation, that may not exist in each case, care must be exercised when such result is obtained. We consider such a case now.

E. Vector-type interaction

We specialize our general results to the particular case of a Lorentz vector field A_μ of mass μ_V in interaction with the Dirac field.[5] The Lagrangian takes the form

$$\mathcal{L}_A=-\tfrac{1}{2}\partial_\mu A^\nu\partial^\mu A_\nu+\tfrac{1}{2}\mu_V{}^2A_\mu A^\mu$$
$$-g_VA_\mu\overline{\psi}\gamma^\mu\psi+\overline{\psi}(\gamma\cdot p-m)\psi, \qquad (2.43)$$

and we find

$$E^A_{\rm eff}=-\int d^nx\left[\frac{n-3}{2}(\nabla A_\mu)^2+\frac{n-1}{2}\mu_V{}^2A^2\right]$$
$$+m\int d^nx\,\overline{\psi}\psi, \qquad (2.44)$$

which is negative definite, considering longitudinal component A_0 only. This is in agreement with the generally known facts that the longitudinal part of the vector-type interaction is repulsive in the particle-particle channel and attractive only in the particle-antiparticle channel.

The situation changes when the sign in the part of the Lagrangian corresponding to the free vector field is changed, that is when we consider

$$\mathcal{L}_{A'}=\tfrac{1}{2}\partial_\mu A^\nu\partial_\mu A_\nu-\tfrac{1}{2}\mu_V{}^2A^2$$
$$-g_VA_\mu\overline{\psi}\gamma^\mu\psi+\overline{\psi}(\gamma\cdot p-m)\psi. \qquad (2.45)$$

Then we find

$$E^{A'}_{\rm eff}=\int d^nx\left[\frac{n-3}{2}(\nabla A_\mu)^2+(n-1)\tfrac{1}{2}\mu_V{}^2A^2\right]$$
$$+m\int d^nx\,\overline{\psi}_k\psi_k, \qquad (2.46)$$

which is positive definite, constrained to the longitudinal part of A_μ, provided that the scalar integral $\int d^nx\,\overline{\psi}_k\psi_k>0$. The above-described change in sign accomplishes at the same time a change in the polarity of the vector interaction; the longitudinal part is now attractive in the particle-particle channel, while the particle-antiparticle channel becomes repulsive. The quantum field theory, if based on Eq. (2.45) would suffer from the well-known difficulties associated with the possible need for negative metric particles (ghosts) to guarantee a bounded below spectrum. Therefore, such modifications are usually not considered seriously. Such an example considered in the frame of classical field theory may, however, serve as an educational example in order to gain experience with "attractive" vector-type fields encountered in non-Abelian meson theories. Returning for a moment to Eq. (2.44) we wish to mention again that it is in principle possible to find a solution in which the space vector part of A_μ dominates thus allowing a stable solution, even with the conventional choice for the sign of the vector-field action.

F. Absence of an explicit meson field

We consider the Hamiltonian

$$H^D=\int d^nx\,\psi_k^\dagger(\vec{\alpha}\cdot\vec{p}+\beta m)\psi_k$$
$$-\sum_i\tfrac{1}{2}G_i\int d^nx\,d^ny\,\rho_{ik}(\vec{x})K(\vec{x}-\vec{y})\rho_{ik}(\vec{y}) \qquad (2.47)$$

with

$$\rho_{ik}=\overline{\psi}_k\Gamma_i\psi_k.$$

The equation of motion for the Dirac field is similar to that given in Eq. (2.8b). With the potential

$$V(\vec{x})=-\sum_i G_i\gamma_0\Gamma_i\int d^ny[K(\vec{x}-\vec{y})\rho_{ik}(\vec{y})] \qquad (2.48)$$

we have

$$(\vec{\alpha}\cdot\vec{p}+\beta m+V)\psi_k=\epsilon_k\psi_k. \qquad (2.49)$$

Application of Eq. (2.4) leads in a straightforward manner to

$$\int d^nx\,\psi_k^\dagger\vec{\alpha}\cdot\vec{p}\psi_k=-\sum_i G_i\int d^ny\,d^nx[\rho_{ik}(\vec{x})\tfrac{1}{2}(\vec{x}-\vec{y})\cdot\vec{\nabla}_{x-y}K(\vec{x}-\vec{y})\rho_{ik}(\vec{y})], \qquad (2.50)$$

where we have made use of the relation

$$\int d^nx\, d^ny[F(\vec{x},\vec{y})\vec{x}\cdot\vec{\nabla}_x K(\vec{x}-\vec{y})] = \int d^nx\, d^ny\{F(x,y)[-\vec{y}\cdot\vec{\nabla}_y K(\vec{x}-\vec{y})]\},$$

that holds for functions F satisfying $F(x,y) = F(y,x)$.

As a particular example for the interaction function K we consider

$$K_l(x-y) = |\vec{x}-\vec{y}|^l \tag{2.51}$$

For some l this ansatz is equivalent to our previous discussion. However, in most cases it is quite difficult to find a theory based on interacting meson-fermion fields to describe some of the choices for K.

In the example (2.51) we find

$$\int d^nx\, \psi_k^\dagger \vec{\alpha}\cdot\vec{p}\psi_k = lV_l, \tag{2.52}$$

where V is

$$V_l = \sum_i \frac{G_i}{2} \int d^nx\, d^ny[\rho_{ik}(\vec{x})|\vec{x}-\vec{y}|^l \rho_{ik}(\vec{y})]. \tag{2.53}$$

Therefore we find in that case

$$E^l_{\text{eff}} = (l+1)V_l + mS. \tag{2.54}$$

We note that for $l=-1$ (Coulomb-type interaction) we find

$$E^{(-1)}_{\text{eff}} = mS \tag{2.55}$$

while for $l=1$ and $m=0$ the kinetic and potential energies contribute equally to the energy of the solution. We find for the Hamiltonian with a scalar-type coupling

$$H^{S_1} = \int d^nx\, \psi_k^\dagger \vec{\alpha}\cdot\vec{p}\psi_k + \frac{G}{2}\int d^nx\, d^ny\, \bar{\psi}_k(\vec{x})\psi_k(\vec{x})|\vec{x}-\vec{y}|\bar{\psi}_k(\vec{y})\psi_k(\vec{y}) \tag{2.56}$$

the effective energy to be

$$E^{S_1}_{\text{eff}} = G\int d^nx\, d^ny \bar{\psi}_k\psi_k(\vec{x})|\vec{x}-\vec{y}|\bar{\psi}_k\psi_k(\vec{y}) \tag{2.57}$$

$$= 2\int d^nx \psi^\dagger \vec{\alpha}\cdot\vec{p}\psi. \tag{2.58}$$

We note that the energy, Eq. (2.57), rises linearly with the size of the solutions.

We now consider the special case of the self-interacting Dirac field

$$K_p(\vec{x}-\vec{y}) = \delta^3(\vec{x}-\vec{y}). \tag{2.59}$$

Then H assumes the form, for scalar-type self-interaction

$$H^p = \int d^nx\left[\psi_k^\dagger(\vec{\alpha}\cdot\vec{p}+\beta m)\psi_k - \frac{G}{2}(\bar{\psi}_k\psi_k)^2\right]. \tag{2.60}$$

Numerical solutions associated with the Eq. (2.60) have been obtained previously,[6] we only remark here that in view of the equations of motion

$$[\vec{\alpha}\cdot\vec{p}+\beta(m-G\bar{\psi}_k\psi_k)]\psi_k = \epsilon_k\psi_k \tag{2.61}$$

the virial relation reads

$$T_k = \int d^nx\psi_k^\dagger\vec{\alpha}\cdot\vec{p}\psi_k = \int d^nx(\vec{x}\cdot\vec{\nabla})\left[-\frac{G}{2}(\bar{\psi}_k\psi_k)^2\right]. \tag{2.62}$$

Upon partial integration of the right-hand side, we obtain, up to a vanishing surface term,

$$T_k = -nV_k, \tag{2.63}$$

where

$$V_k = -\int d^nx\, \frac{G}{2}(\bar{\psi}_k\psi_k)^2. \tag{2.64}$$

The energy for the self-interacting Dirac field can be written now as

$$E^p_{\text{eff}} = (n-1)(-V_k) + mS_k. \tag{2.65}$$

S is the scalar integral

$$S_k = \int d^nx(\bar{\psi}_k\psi_k). \tag{2.66}$$

This result, Eq. (2.65), corrects a superficial impression that H is unbound, following from Eq. (2.60) in which the self-interaction is attractive. We find that the kinetic energy of the Dirac field more than offsets the attractive self-interaction. Since $-V$ is always positive, we find that for all dimensions the positivity of the solution depends on the sign of the scalar integral. Further we note that since the eigenfrequency can be written as

$$\epsilon_k = (n-2)(-V_k) + mS_k, \tag{2.67}$$

a solution with finite, positive ϵ_k may be found for $n \geq 2$.

III. STABILITY OF THE CLASSICAL SOLUTIONS

A. Dirac field in an external potential

We have so far used a relation derived from the equations of motion in order to obtain several properties of the interacting fields. We have not yet considered the stability of the eventually existing solutions. In principle, the solutions of the coupled nonlinear equations of motion need not be actual stable minima of the action; they let vanish the first variation of the action but no information is available about the second variations. Let us consider here as an example the case of an externally prescribed potential V_{ex}. Then the Hamiltonian is

$$H=\int d^3x\,\psi^\dagger(\vec{\alpha}\cdot\vec{p}+\beta m-V_{ex})\psi \tag{3.1}$$

and the equation of motion follows from equating to zero the first variation of $(H-\epsilon N)$ with respect to ψ. Here N is the norm of the field ψ while ϵ is the Lagrange multiplier that ensures the normalizability of the solution. We find, as usual, the eigenvalue equation

$$(\vec{\alpha}\cdot\vec{p}+\beta m-V_{ex})\psi_k=\epsilon_k\psi_k. \tag{3.2}$$

As is well known, the set of eigensolutions $\{\psi_k\}$ is complete. Suppose we take a trial function

$$\psi_t=\sum a_k^t\psi_k \tag{3.3}$$

in an attempt to minimize the Hamiltonian, Eq. (3.1). Using the expansion (3.3) into the complete basis set $\{\psi_k\}$ generated by Eq. (3.2) we find

$$(H-\epsilon_tN)=\sum_k |a_k^t|^2(\epsilon_k-\epsilon_t). \tag{3.4}$$

The spectrum of the Dirac equation is unbounded below and therefore the expression (3.4) does not reflect in any way on the extrema described by solutions of Eq. (3.2).

From this example we recognize the need to constrain the number of allowed degrees of freedom of the Dirac field. We must exclude from our considerations the possibility of a transition that a classical Dirac particle can undergo into a (classically) unoccupied state of arbitrary large negative frequency that in quantum field theory corresponds to antiparticle state of positive energy.

B. The scalar coupling

The solutions under investigation here thus must be obtained by a method that allows *a priori* rejection of all unwanted modes of the Dirac equation. In a particular application this means that the Dirac equation must always be solved exactly for some prescribed potential, and an eigenstate ψ_k obtained.

In the case of the interacting Dirac-scalar fields, we do actually minimize the action constrained by this consideration, that is given by

$$H^S=\epsilon_k[\varphi]+\int d^3x\left[\tfrac{1}{2}(\vec{\nabla}\varphi_S)+U(\varphi_S)-U_0\right], \tag{3.5}$$

where ϵ_k follows from

$$[\vec{\alpha}\cdot\vec{p}+\beta(m-g_S\varphi_S)]\psi_k=\epsilon_k\psi_k. \tag{3.6}$$

The above equation must be always solved exactly to obtain the Dirac eigenvalue ϵ_k as a functional depending on φ_S. Using the obvious relation

$$\frac{\delta\epsilon_k}{\delta\varphi_S(x)}=-g_S\bar{\psi}_k(x)\psi_k(x) \tag{3.7}$$

we recover the conventional equation of motion for the field $\varphi_S(\vec{x})$ that we now write in the form

$$-\Delta\varphi_S+\frac{\partial U}{\partial\varphi_S}=-\frac{\delta\epsilon_k}{\delta\varphi_S(x)}. \tag{3.8}$$

The question which we now will address ourselves to is: Are the solutions, constrained to the lowest positive-frequency (particle) solutions of the Dirac field, stable? To wit, let us continue for a moment with the above example; writing

$$\varphi_S=\varphi_0+\delta\varphi \tag{3.9}$$

we find

$$H=H_0+H_1+H_2, \tag{3.10}$$

where

$$H_0=\epsilon_k[\varphi_0]+\int d^3x\left[\tfrac{1}{2}(\nabla\varphi_0)+U(\varphi_0)-U_0\right], \tag{3.11}$$

$$H_1=\int d^3x\left[\left(\frac{\delta\epsilon_k}{\delta\varphi(x)}\right)_{\varphi_0}-\Delta\varphi_0+\left(\frac{\partial U}{\partial\varphi}\right)_{\varphi_0}\right]\delta\varphi(x), \tag{3.12}$$

$$H_2=\int d^3x\,d^3y\,\tfrac{1}{2}\delta\varphi(x)\times\left[\left(\frac{\delta^2\epsilon_k}{\delta\varphi(x)\delta\varphi(y)}\right)_{\varphi_0}+\delta^3(x-y)\left(\frac{\partial U}{\partial\varphi^2}\right)_{\varphi_0}\right]\delta\varphi(y)+\int d^3x\left[\tfrac{1}{2}(\nabla\delta\varphi)^2+\tfrac{1}{2}\pi_\varphi{}^2\right]. \tag{3.13}$$

We have included the conjugate momentum of the meson field

$$\pi_\varphi=-\delta\dot{\varphi} \tag{3.14}$$

into our considerations under the assumption that it is of the same order as the variation $\delta\varphi$.

The first term H_0 is the term studied previously; H_1 vanishes in consequence of the equations of motion [provided that φ_0 is actually a solution of Eq. (3.8)] while H_2 is the correction quadratic in the small variations. Our aim is to show that this term is positive definite for arbitrarily chosen $\delta\varphi$.

Let us denote by

$$T(\vec{x},\vec{y})=\left(\frac{\delta\epsilon_0}{\delta\varphi(\vec{x})\delta\varphi(\vec{y})}\right)_{\varphi_0}, \tag{3.15}$$

then varying H_2 with respect to the $\delta\varphi$ we find

$$\delta\ddot{\varphi} - \Delta\delta\varphi + \left(\frac{\partial^2 U}{\partial\varphi^2}\right)_{\varphi_0}\delta\varphi = -\int d^3y\, T(\vec{x},\vec{y})\delta\varphi(\vec{y}), \tag{3.16}$$

Now, taking the usual form of the time dependence of a neutral field

$$\delta\varphi = \chi(\vec{x})\sqrt{2}\cos\omega t, \tag{3.17}$$

we find the nonlocal eigenvalue equation

$$-\Delta\chi(\vec{x}) + \left(\frac{\partial^2 U}{\partial\varphi^2}\right)_{\varphi_0(x)}\chi(\vec{x}) + \int d^3y[T(\vec{x},\vec{y})\chi(\vec{y})] = \omega^2\chi(\vec{x}). \tag{3.18}$$

Inserting this equation of motion into H_2 we find (for spatially localized χ)

$$H_2 = \int d^3x\, \omega^2|\chi(\vec{x})|^2. \tag{3.19}$$

We first study the case $\omega = 0$. It is well known that such solutions correspond to spurious center-of-mass motion.[3] This is easily recognized to be the case here. Differentiation of the Eq. (3.8) leads to

$$-\Delta(\vec{\nabla}\varphi_0) + \left(\frac{\partial^2 U}{\partial\varphi^2}\right)_{\varphi_0}(\vec{\nabla}\varphi_0) = -\vec{\nabla}\left(\frac{\delta\epsilon_0}{\delta\varphi(\vec{x})}\right)_{\varphi_0} \tag{3.20a}$$

and

$$\vec{\nabla}\left(\frac{\delta\epsilon_0}{\delta\varphi(\vec{x})}\right)_{\varphi_0} = \int d^3y\left(\frac{d^2\epsilon_0}{\delta\varphi(\vec{x})\delta\varphi(\vec{y})}\right)_{\varphi_0}\vec{\nabla}\varphi_0. \tag{3.20b}$$

Thus we find the classic result

$$\chi_i(\vec{x}, \omega = 0) = \nabla_i\varphi_0, \quad i = 1, 2, 3. \tag{3.21}$$

When studying the eigenvalue spectrum of Eq. (3.18) we have to allow both positive and negative values of ω^2. The eventual stability of the solutions φ_0 is then a consequence of the absence of negative ω^2 modes. In the absence of an analytical expression for $\epsilon[\varphi]$ the simplest numerical method to test the stability of the solutions involves a procedure that has been described in the paragraph preceding Eq. (3.5); find for each given φ_S the Dirac eigenvalue $\epsilon_k[\varphi_S]$, compute H^S, Eq. (3.5), and find the minimum of H_S as a function of φ_S. I have carried out such calculations and have found that the numerical solutions given previously[5,6,11,12] are stable.

C. Other interactions

Taking the Hamiltonian (2.39) constrained to the particular solution of the Dirac field

$$H = \epsilon_k[\varphi_i] + \sum_i \frac{s(i)}{2}[\pi_i{}^2 + (\nabla\varphi_i{}^2)] + U(\varphi_i) - U_0, \tag{3.22}$$

where ϵ_k is an eigensolution of the equation

$$[\vec{\alpha}\cdot\vec{p} + \beta(m - I)]\psi_k = \epsilon_k\psi_k, \tag{3.23}$$

with

$$I = \gamma_0\sum_i \Gamma_i g_i \varphi_i. \tag{3.24}$$

The conjugate momenta are

$$\pi_i = -s(i)\varphi_i. \tag{3.25}$$

We record the relation

$$\frac{\delta\epsilon_0}{\partial\varphi_i(\vec{x})} = -g_i\bar{\psi}(\vec{x})\Gamma_i\psi(\vec{x}). \tag{3.26}$$

Let us write

$$\varphi_i(\vec{x}) = \varphi_i^0(\vec{x}) + \delta\varphi_i(\vec{x}), \tag{3.27}$$

where $\delta\varphi_i$ must transfer under Lorentz transformations in the same manner as the field φ_i. We can carry the expansion of Eq. (3.22) out in the same way as previously,

$$H = H_0 + H_1 + H_2 \tag{3.28}$$

with

$$H_0 = H[\varphi_i^0] \tag{3.29}$$

and

$$H_1 = \int d^3x\left[\left(\frac{\delta H}{\delta\varphi_i}\right)_{\varphi_i^0}\delta\varphi_i\right]. \tag{3.30}$$

H_1 vanishes in view of the equations of motion. For H_2 we find

$$H_2 = \int d^3x\, d^3y\, \tfrac{1}{2}\sum_{i,j}\left(\frac{\delta^2 H}{\delta\varphi_i(\vec{x})\delta\varphi_j(\vec{y})}\right)_{\varphi^0}\delta\varphi_i(\vec{x})\delta\varphi_j(\vec{y}) + \int d^3x\sum_i\frac{s(i)}{2}\pi_i\pi^i. \tag{3.31}$$

Denoting by

$$T_{ij}(\vec{x},\vec{y}) = \left(\frac{\delta\epsilon_0}{\delta\varphi_i(\vec{x})\delta\varphi_j(\vec{y})}\right)_{\varphi^0(\vec{x})} = T_{ji}(\vec{y},\vec{x}), \tag{3.32}$$

we have explicitly

$$H_2 = \int d^3x\left\{\sum_i\frac{s(i)}{2}[\pi_i\pi^i + (\vec{\nabla}\delta\varphi_i)(\vec{\nabla}\delta\varphi^i)] + \tfrac{1}{2}\sum_{i,j}\left(\frac{\partial^2 U}{\partial\varphi_i\partial\varphi_j}\right)_{\varphi^0}\right\} + \int d^3x\, d^3y\, \tfrac{1}{2}\sum_{i,j}\delta\varphi_i(\vec{x})T_{ij}(\vec{x},\vec{y})\delta\varphi_j(\vec{y}). \tag{3.33}$$

The equation of motion that follows under variation of H_2 with respect to $\delta\varphi_i$ is

$$s(i)(\delta\ddot{\varphi}_i - \Delta\delta\varphi_i) + \int d^3y\left[\sum_j T_{ij}(\vec{x},\vec{y})\delta\varphi_j(\vec{y})\right] + \sum_j\left(\frac{\partial^2 U}{\partial\varphi_i\partial\varphi_j}\right)_{\varphi^0}\delta\varphi_j = 0. \tag{3.34}$$

Then, for localized eigenfunctions φ_i we find, reinserting into the Eq. (3.33),

$$H_2 = \int d^3x \sum_i \frac{s(i)}{2}(\pi_i\pi^i - \delta\varphi_i\delta\ddot{\varphi}_i). \tag{3.35}$$

With the ansatz

$$\delta\varphi_i = \chi_i\sqrt{2}\cos(\omega_i t)$$

we find

$$H_2 = \int d^3x \sum_i s(i)\omega_i^2|\chi_i|^2. \tag{3.36}$$

However, this ansatz for the field $\delta\varphi_i$ becomes compatible with the equations of motion only if all ω_i are equal. Even in that particular case we find that the stability of the field solutions depends on the balance between the different interaction involved. Thus we obtain the result that, for example under the simultaneous presence of the attractive scalar interaction and the repulsive vector fields, there will be a region of the coupling strengths for which no stable solutions will be found. I have found in explicit calculations that at such a point the numerical algorithms fail to give a solution of equations of motion.

D. Linear elongations

The method, introduced into the study of classical field theories in Ref. 3 employs the linear transformation

$$\vec{x}' = a\vec{x}. \tag{3.37}$$

The virial theorem is then obtain from

$$\left.\frac{\partial[H(a\vec{x})/N(a\vec{x})]}{\partial a}\right|_{a=1} = 0, \tag{3.38}$$

while the stability of the solutions should be tested by

$$H'' = \left.\frac{\partial^2[H(\vec{x}a)/N(a\vec{x})]}{\partial a^2}\right|_{a=1}, \tag{3.39}$$

where N is the norm of the Dirac field. This approach, in view of our constrain to exact solutions of the Dirac equation is, *in principle inapplicable to our particular case*. Nonetheless, we find that Eq. (3.38) works since in the case of the Dirac equation at $a=1$ the exact solution must be a local extremum of the Hamiltonian. Thus the virial theorem follows in a straightforward manner when norm is conserved from the variation of

$$H[a\vec{x}]/N[a\vec{x}] = \frac{\int d^3x\, a^3\psi^\dagger[(1/a)\vec{\alpha}\cdot\vec{p} + \beta(m - V(a\vec{x}))]\psi}{\int d^3x\, a^3\psi^\dagger\psi}, \tag{3.40}$$

and takes the form

$$\int d^3x\, \psi^\dagger\vec{\alpha}\cdot\vec{p}\psi = -\int d^3x\, \bar{\psi}\psi\vec{x}\cdot\vec{\nabla}V(\vec{x}) \tag{3.41}$$

in agreement with Eq. (2.4) (note that in this comparison $V_{ex} = -V$).

However, the calculation of H'' does not help us further. Should we obtain a negative value for H'', it may only indicate that a number of negative-frequency modes have been included into the trial wave function. We now consider this point in some more detail. Since the solutions ψ_k of the Dirac equation for a fixed "potential" V form a complete set, we may write

$$\psi_0(a\vec{x}) = \sum_k b_k(a)\psi_k(\vec{x}). \tag{3.42}$$

That leads in a straightforward manner to the expression, using Eq. (3.40),

$$H^S[a\vec{x}] = a^3\sum_k |b_k(a)|^2\epsilon_k[\varphi(a\vec{x})] + E_\varphi(a\vec{x}) \tag{3.43}$$

for the energy of the Hamiltonian (2.21), where E_φ is the energy content of the scalar field. The particular term that destroys the conventional line of arguments in the cases that involve the Dirac field is

$$\left.\frac{\partial^2 H^S[a\vec{x}]}{\partial a^2}\right|_{a=1} = \sum_k \left|\frac{\partial b_k(a=1)}{\partial a}\right|^2 \epsilon_k[\varphi(\vec{x})] + \cdots. \tag{3.44}$$

Since the spectrum is not positive definite, there is no way to determine the sign of the expression (3.44). We note in passing that in view of Eq. (3.42) we have

$$b_k(a) = \int \psi_k^\dagger(\vec{x})\psi_0(a\vec{x})d^3x, \tag{3.45}$$

and therefore,

$$\left.\frac{\partial b_k}{\partial a}\right|_{a=1} = \int \psi_k^\dagger(\vec{x})\vec{x}\cdot\vec{\nabla}\psi_0(\vec{x})d^3x. \tag{3.46}$$

The angular momentum selection rules allow coupling of the lowest positive-frequency S state ψ_0 in Eq. (3.46) to other S states only. The nonvanishing of the expression (3.46) for negative frequencies ϵ_k is easily documented [see also Eq. (2.6)].

An explicit example of this inconclusiveness of the elongation method is given by the self-interacting Dirac fields considered previously. Beginning with the Hamiltonian, Eq. (2.60), we find using Eq. (3.40)

$$H(a\vec{x})/N(a\vec{x}) = \int d^3x\left[\psi^\dagger\left(\frac{1}{a}\vec{\alpha}\cdot\vec{p} + \beta m\right)\psi - \frac{1}{2a^3}(\bar{\psi}\psi)^2\right] \Big/ \int d^3x\,\psi^\dagger\psi, \tag{3.47}$$

wherefrom we obtain again the relation (2.63) using Eq. (3.38). However, we find for the second derivative

$$\frac{\partial^2[H(a\vec{x})/N(a\vec{x})]}{\partial a^2}\bigg|_{a=1} = -\frac{6G}{2}\int d^3x(\bar{\psi}\psi)^2, \qquad (3.48)$$

a result that is explicitly negative definite. This would make us worry about the stability of the self-interacting field had we not convinced ourselves with the help of Eq. (3.44) that such a result can arise as a consequence of the indefinite character of the Dirac equation spectrum and has nothing to do with the stability of the solutions.

IV. CONCLUSIONS

We have shown that a consistent classical field theory in three space dimensions involving the Dirac field is possible. Considering both the general case and some particularly interesting examples involving Abelian meson fields we have found that stable solutions to the field equations may arise. We have been able to derive several useful relations between the integrals of the fields, such that significant physical insights into the nature and structure of the solutions have been revealed.

In particular, relations that we derive for the self-interacting Dirac field are very useful in the understanding of the basic mechanisms involved. Let us, for example, return to the case of the scalar self-interaction of the massless fermions with linearly rising potential [see Eq. (2.56)]. In this case the interaction energy between the fermions rises linearly with the distance between the particles. We then find that half of the energy of the interacting particles resides in their kinetic energy, while the other half is absorbed in the interaction term.

In the self-consistent bag approximation of quark confinement[11] the solutions of the Dirac field studied here are associated with the quark wave functions in the quarkic bag that is described by the self-consistent meson fields. In the case of the linearly rising fermion-fermion interaction we thus find that half of the energy (in the rest frame of the bag) would reside in the "free" quark fields, while the other half would be associated with the bag.

We have also considered other types of interactions than the scalar electrodynamics, and have found that qualitative features found in the simplest case persist. Some additional complications arise from the possible instabilities associated with the repulsive character of some of the other interactions. The detailed consideration of the stability of the solutions leads us to a universal condition, which in the absence of an exact solution to the complicated problem of the interacting Dirac-meson fields has not been at present used in actual calculations. However, a related numerical approach is described that has been used to establish the stability of the numerical solutions available in the literature.[5,6,11,12]

Aside from the many useful and practical relations that have been derived in this paper we have been able to obtain interesting qualitative results concerning the nature of the solutions of the coupled fields. We have shown that in the case of three space dimensions it is the interaction term between the Dirac and meson fields that determines largely the form of the solution. This is opposite to the case of one space dimension; there the self-interaction term of the meson field is the most essential part of the Lagrangian. We further find that the use of φ^4 self-interaction in three space dimensions is not as essential as in the case of one space dimension, also in view of the cancellation of its immediate contribution to the total energy of the solution.

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