Model for couplings

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We develop a model based on quark conservation which provides a relatively simple description for three-meson vertices, even when the spins of the particles are large. This should lead to a simplified parametrization of the Reggeon-Reggeon-particle vertex needed for multi-Regge phenomenology. Tests of the model involving measured decay rates are encouraging.

I. INTRODUCTION

Phenomenological treatments of reactions in which two particles go to three particles have generally neglected possible dependence on one of the kinematic variables, the Toller angle, unless they relied on a Veneziano-formula parametrization.[1] This is primarily because the general formula[2] describing the reaction of Fig. 1 in the double-Regge limit

$$A_5 \sim \beta_1(t_1)\beta_2(t_2) s_1^{\alpha_1(t_1)} s_2^{\alpha_2(t_2)} \times [\xi_1\xi_{21}\eta_{12}^{\alpha_1} V_1(t_1, t_2; \eta_{12}) + \xi_2\xi_{12}\eta_{12}^{\alpha_2} V_2(t_1, t_2; \eta_{12})] \qquad (1)$$

with

$$\xi_i = e^{-i\pi\alpha_i} + \tau_i, \quad \xi_{ij} = e^{-i\pi(\alpha_i - \alpha_j)} + \tau_i\tau_j,$$

$$\eta_{12} = s_{12}/s_1 s_2$$

has a rather complicated dependence on the functions $\beta(\lambda, t_1, t_2)$ giving the coupling of particle D to the two Reggeons at the central vertex. This coupling β, which depends in an unspecified way on the helicity λ carried off by each of the Reggeons in the X rest frame (X is assumed spinless), appears in the vertex function $V_1(t_1, t_2, \eta_{12})$ as

$$V_1(t_1, t_2, \eta_{12}) = \sum_{j=0}^{\infty} \frac{1}{j!} \Gamma(-\alpha_1 + j)\Gamma(-\alpha_2 + \alpha_1 - j) \times \eta_{12}^{-j} \beta(\alpha_1 - j, t_1, t_2). \qquad (2)$$

A similar formula obtains for V_2. In the absence of a specific model for the β function, it is difficult to parametrize the η_{12} dependence of the V_i functions in a convincing way. The variable η_{12} is related to the Toller angle ω in the double-Regge limit by

$$\eta_{12} = \frac{-t_1 - t_2 + m_X^2 + 2(t_1 t_2)^{1/2}\cos\omega}{t_1^2 + t_2^2 + m_X^4 - 2t_1t_2 - 2t_1 m_X^2 - 2t_2 m_X^2}, \qquad (3)$$

so lack of understanding about $\beta(\lambda, t_1, t_2)$ translates into an inability to parametrize the Toller-angle behavior.

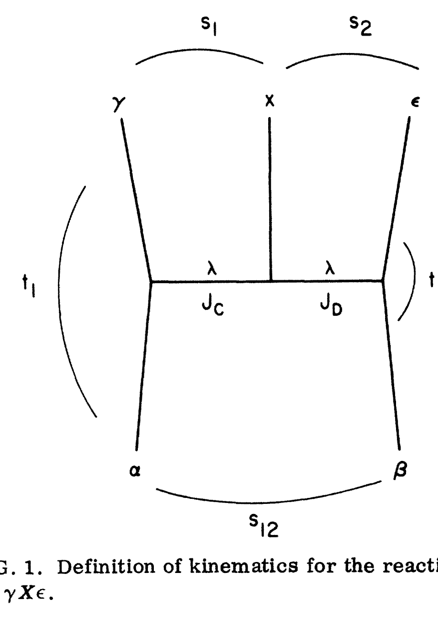

FIG. 1. Definition of kinematics for the reaction $\alpha\beta \rightarrow \gamma X \epsilon$.

Field theory is not much help, for there can be many different couplings of the two particles J_C and J_D on the two Reggeons to particle X, and each such type of coupling should appear with a different coupling constant. The Veneziano model, on the other hand, specifies $\beta(\lambda, t_1, t_2) = 1$; this model has been applied to many $2 \rightarrow 3$ reactions with varying degrees of success. We feel that other models should be investigated; since data for experiments with large numbers of events in these channels is becoming available, such parametrizations can be tested fairly thoroughly. It is almost an axiom of phenomenology that only simple models can, in the long run, be handled with any degree of faith. This paper should be viewed as a first step toward such a simple model.

In Sec. II we explain the basic ideas of our model, which is based on Zweig-rule conservation of quarks at meson vertices. The actual expression for the vertex is derived here and in Appendix A. Section III discusses tests of the formula using measured widths for tensor- and vector-meson decays.

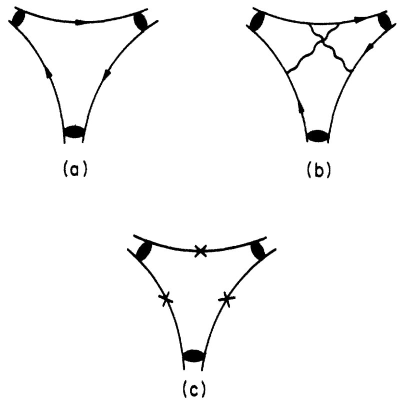

FIG. 2. (a) Triangle diagram illustrating type of contribution envisioned in our model. (b) A contribution we have neglected. (c) In calculations of model, internal quarks are put on the mass shell.

II. THE MODEL

As a first simplifying feature, we assume that all mesons are made up out of $q\bar{q}$ pairs of spin-$\frac{1}{2}$ quarks, and that the basic meson vertex is somehow related to the Zweig-rule-obeying triangle graph of Fig. 2(a). The blobs at each corner represent meson wave functions of the $q\bar{q}$ system. This is already a very strong assumption; we ignore, for example, all possible complications like the gluon exchanges shown in Fig. 2(b). Models similar to Fig. 2(a) have been considered by several sets of workers,[3] using various models of the meson wave functions. Frequently, their results depend strongly on the behavior of these wave functions for off-mass-shell quarks.

Phenomenologically, however, the great successes of the quark model have occurred when the quarks could be treated as on-mass-shell spin-$\frac{1}{2}$ particles (the parton model and the "concrete quark" model are two such instances). We have a strong bias that off-mass-shell properties of quarks should not be essential to the physics. For this reason, we feel that it should be possible to obtain a reasonable model of the vertex from considering Fig. 2(c), where each quark line has most of its contribution coming from some effective mass μ. We will confine our discussion to the implications of this approximation.

Of course, for general meson masses and quark masses it is impossible to keep all the momenta in the on-shell triangle diagram real and still have momentum conservation at the three vertices. If we insist on conservation of all four components of the momentum, and calculate angles using dot products in the usual way, the cosines of some angles will be greater than one, or even complex, for certain mass configurations. We accept this as the price of having the simplest possible parametrization of the vertex. In other words, we calculate all kinematic quantities in a region of the meson masses and quark effective masses where the triangle diagram is a physical process, and we then continue the expressions in these masses to the region that is actually under consideration.

Although this is a very unorthodox procedure, some such assumption seems necessary to achieve the desired simplification in the results. We will pursue this assumption and see where it leads.

By making this on-mass-shell assumption, we can treat the mesons as poles in the quark-antiquark scattering amplitude. Once complications due to SU(3) are removed, mesons with natural parity and charge parity $[P=(-1)^J,\ C=(-1)^J]$ have only two independent residues in $q\bar{q}$ scattering, and mesons with $P=-(-1)^J$ have only one independent residue, *regardless of the spin of the meson.* Hence, if we use these residues as parameters for our vertex function, we can describe vertices involving particles with very high spin in terms of only a few parameters. This is obviously a great simplification if we are dealing with decays into high-spin particles, or with vertices involving Reggeons. We see that a maximum of nine parameters (three quark effective masses and up to six residue functions) are needed to describe the most general vertex; usually there will be fewer parameters since (a) we expect all quarks of a given flavor to contribute with the same μ, and (b) many of the residues which enter in practical work contain one or more pions, which have only one residue function.

In the rest frame of one of the mesons, the meson pole appears in those of the $q\bar{q}$ helicity amplitudes appropriate to the meson quantum numbers. For example, $P=(-1)^J$, $C=(-1)^J$ meson poles appear in $f_{++;++}$, $f_{++;--}$, $f_{+-;-+}$, and $f_{+-;+-}$ and the amplitudes related to these by parity invariance. The residues of the poles factorize, so we have two independent residue parameters R_{++} and R_{+-}. Hence the *parameters appropriate to each meson are defined in the rest frame of that meson.* To construct the overall vertex in the rest frame of one of the mesons, we must boost the information about the other corners of the triangle from their rest frames into the rest frame where the vertex is being constructed. The quark helicities are rotated under this transformation,[4] so our triangle graph Fig. 2(c) is essentially represented by residue functions for each of the three corners linked by rotation matrices. Each such $d^{1/2}_{\lambda\mu}$ describes the rotation of quark helicities in transforming between the rest frames of the two mesons linked by that quark line.

Finally, we must remove the external quarks in each $q\bar{q}$ scattering; this leaves the meson pole. The different helicity states of the mesons concerned are then present in amounts determined by rotational invariance.

In Appendix A we explain these steps in more detail and demonstrate that our three-meson vertex must be proportional to

$$d^J_{MM_C-M_D}(\theta)d^J_{M_C-M_D5-8}(\alpha)R_{58}(m_X{}^2)d^{1/2}_{5'5}(\psi_5)d^{1/2}_{6'7'}(\psi)d^{1/2}_{8'8}(\psi_8)R_{5'6'}(m_C{}^2)d^{J_C}_{5'-6'M_C}(\theta_C)R_{8'7'}(m_D{}^2)d^{J_D}_{8'-7'M_D}(\theta_D)\,, \tag{4}$$

where the labels of Fig. 3 are used and the various angles are defined as follows:

(a) The vertex is calculated in the rest frame of a meson of mass squared $m_X{}^2$ and spin J; it decays into particles with masses squared and spins $m_C{}^2, J_C$ and $m_D{}^2, J_D$. The spin projections of these mesons are M, M_C, and M_D.

(b) The angle θ is the angle between some z axis and the direction of C, in the s rest frame.

(c) $R_{ij}(m^2)$ is the helicity-labeled residue at the meson pole in the appropriate $q\bar{q}$ scattering. For example, for a pion $R_{++}(m_\pi{}^2) = -R_{--}(m_\pi{}^2)$, $R_{+-} = R_{-+} = 0$.

(d) The angles of rotation of the quarks are defined by

$$\cos\psi_5 = \left(\frac{m_C{}^2+\mu_5{}^2-\mu_6{}^2}{2m_C\mu_5}\;\frac{m_X{}^2+\mu_5{}^2-\mu_8{}^2}{2m_X\mu_5} - \frac{m_X{}^2+m_C{}^2-m_D{}^2}{2m_Xm_C}\right)\Big/\frac{[\Lambda(m_C{}^2,\mu_5{}^2,\mu_6{}^2)]^{1/2}}{2m_C\mu_5}\;\frac{[\Lambda(m_X{}^2,\mu_5{}^2,\mu_8{}^2)]^{1/2}}{2m_X\mu_5}\,,$$

$$\cos\psi = \left(\frac{m_C{}^2+\mu_6{}^2-\mu_5{}^2}{2\mu_6m_C}\;\frac{m_D{}^2+\mu_6{}^2-\mu_8{}^2}{2m_D\mu_6} - \frac{m_C{}^2+m_D{}^2-m_X{}^2}{2m_Cm_D}\right)\Big/\frac{[\Lambda(m_C{}^2,\mu_6{}^2,\mu_5{}^2)]^{1/2}}{2m_C\mu_6}\;\frac{[\Lambda(m_D{}^2,\mu_6{}^2,\mu_8{}^2)]^{1/2}}{2m_D\mu_6}\,, \tag{5}$$

$$\cos\psi_8 = \left(\frac{m_D{}^2+\mu_8{}^2-\mu_6{}^2}{2m_D\mu_8}\;\frac{m_X{}^2+\mu_8{}^2-\mu_5{}^2}{2m_X\mu_8} - \frac{m_X{}^2+m_D{}^2-m_C{}^2}{2m_Dm_X}\right)\Big/\frac{[\Lambda(m_D{}^2,\mu_8{}^2,\mu_6{}^2)]^{1/2}}{2\mu_8m_D}\;\frac{[\Lambda(m_X{}^2,\mu_8{}^2,\mu_5{}^2)]^{1/2}}{2\mu_8m_X}$$

with $\Lambda(A,B,C) = A^2+B^2+C^2-2AB-2AC-2BC$.

(e) The angles at the meson vertices are given by

$$\cos\alpha = \frac{2m_X{}^2(\mu_6{}^2-\mu_5{}^2-m_C{}^2)+(m_X{}^2+m_C{}^2-m_D{}^2)(m_X{}^2+\mu_5{}^2-\mu_8{}^2)}{[\Lambda(m_X{}^2,m_C{}^2,m_D{}^2)]^{1/2}[\Lambda(m_X{}^2,\mu_5{}^2,\mu_8{}^2)]^{1/2}}\,,$$

$$\cos\theta_C = \frac{2m_C{}^2(m_X{}^2+\mu_5{}^2-\mu_8{}^2)+(m_X{}^2+m_C{}^2-m_D{}^2)(\mu_6{}^2-\mu_5{}^2-m_C{}^2)}{[\Lambda(m_X{}^2,m_C{}^2,m_D{}^2)]^{1/2}[\Lambda(m_C{}^2,\mu_5{}^2,\mu_6{}^2)]^{1/2}}\,, \tag{6}$$

$$\cos\theta_D = \frac{2m_D{}^2(m_X{}^2+\mu_8{}^2-\mu_5{}^2)+(m_X{}^2+m_D{}^2-m_C{}^2)(\mu_6{}^2-\mu_8{}^2-m_D{}^2)}{[\Lambda(m_X{}^2,m_C{}^2,m_D{}^2)]^{1/2}[\Lambda(m_D{}^2,\mu_8{}^2,\mu_6{}^2)]^{1/2}}\,.$$

This vertex may be multiplied by factors of $\Lambda(m_X{}^2,m_C{}^2,m_D{}^2)$ sufficient to give the proper threshold behavior. The residue functions $R_{ij}(m^2)$ may depend on quark mass; it is possible that this dependence is smooth except for threshold factors. Otherwise all functional dependences are explicit in Eq. (4).

Notice that all dependence of the vertex functions on the helicities M_C and M_D is displayed. The partial waves of a reaction such as the one in Fig. 1 can then be constructed by assembling an appropriate number of vertices (3) and propagators (2);

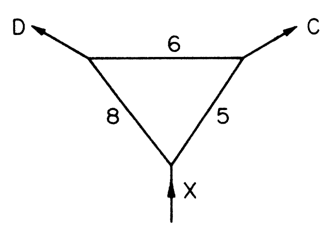

FIG. 3. Meson of mass m_X is composed of quarks of mass μ_5 and μ_8. Meson of mass m_C is composed of quarks of mass μ_5 and μ_6.

high-energy behavior is extracted from the sums over the partial waves. If the residue parameters and quark-mass parameters can be determined from some other reaction, we then have an explicit representation for the double-Regge exchange. Even if the parameters are not known very well, Eq. (4) provides a relatively economical description of this vertex with only a few parameters to be varied.

In the next section we use this model to study meson decay widths, with an eye to extracting the effective-mass parameters of the strange and nonstrange quarks.

III. APPLICATION TO MESON DECAYS

The model can be applied most easily to complicated vertices if the quark mass parameters are already known. These can be derived from simple decays of vector and tensor mesons into two pseudoscalar mesons, provided we are willing to make some additional assumptions about the residue functions R_{ij}, and about the factor multiplying Eq. (4). These assumptions are the following:

(a) The only strong dependence of $R_{ij}(m^2)$ on the quark masses lies in threshold factors $\{[\Lambda(m^2, \mu_i^2, \mu_j^2)]^{1/2}\}^{L_{\min}}$, where $L_{\min}$ is the lowest orbital angular momentum state the $q\bar{q}$ pair can have in coupling to the meson considered.

(b) Apart from the dependence on meson mass contained in the threshold factors extracted in (a) there is no mass dependence of $R_{ij}(m^2)$ for members of a given SU(3) multiplet. The ratios of residues between members of a multiplet are then determined by assumption of SU(3) invariance (after this threshold factor has been removed).

(c) The entire vertex is determined by multiplying Eq. (4) by the minimum power of $[\Lambda(m_C^2, m_D^2, m_X^2)]^{1/2}$ necessary to assure proper threshold behavior. This factor is (see Appendix B)

$$\{[\Lambda(m_C^2, m_D^2, m_X^2)]^{1/2}\}^{\mathcal{J}},$$

where

$$\mathcal{J} = J + J_C + J_D + \min\{|J - J_C - J_D|, |J_C - J - J_D|, |J_D - J - J_C|\}.$$

(We may also have an overall numerical constant multiplier; however, this will not affect the considerations of this section.)

If we make these assumptions, the four common tensor-meson decays into pseudoscalars $K^* \to K\pi$, $f' \to K\bar{K}$, $f \to \pi\pi$, and $A_2 \to K\bar{K}$ are all determined (up to an overall constant) by three parameters: the ratio of R_{+-}/R_{++} for the tensor octet, and the masses of the strange and nonstrange quarks. The four decay rates then give us three ratios; we can vary our three parameters in an attempt to fit the decays.

We have carried out this study under the assumption that the mesons were "ideal" composites of the appropriate quarks (i.e., that f' contained only strange quarks and that f and A_2 contained no strange quarks). This is a simplification, of course, but we feel the main worth of the model should not be dependent on small mixing angles. Similarly, we did not perform a least-squares fit; we just searched through sets of values for the three parameters until we found a range which fit the decays. Clearly a more thorough study could be performed; our results at this stage are encouraging and we believe that they support such further study.

The tensor-meson decay ratios could be fit with nonstrange quarks with masses ranging from 179 MeV to 214 MeV. The acceptable range of the other two parameters depended on the nonstrange mass. If the nonstrange mass was low (~179 MeV), allowable values of the strange-quark mass ranged from 557 MeV to 566 MeV, and allowable values of the residue ratio R_{+-}/R_{++} ranged from −0.94 to −0.90. At the larger nonstrange-quark masses (~214 MeV), the strange-quark mass was within 1 MeV of 573.5 MeV and the range of R_{+-}/R_{++} was between −0.86 and −0.84.

It is encouraging that a set of three parameters can be found which fits the three data points, given the highly nonlinear dependence of our expressions on some of these parameters. The fact that the quark "masses" determined this way are similar to those determined in various other models is nothing less than astonishing. (We had hoped that the squares of the effective masses would come out negative, accounting for nonobservable quarks. However, we were not able to find any solution with μ^2 negative.)

To see whether our solution was accidental, we next examined the vector decays into two pseudoscalars $\rho \to \pi\pi$, $K^* \to K\pi$, and $\phi \to K\bar{K}$ using this same range of quark masses. The $(K^* \to K\pi)/(\rho \to \pi\pi)$ ratio can be fitted within this same range of masses for the strange and nonstrange quarks if a residue ratio near −1 is used (values of the ratio which fitted again vary with the particular quark masses used, over a range of 0.1 or so to each side of −1). However, these same values gave widths for $\phi \to \bar{K}K$ which were too *small* by factors of 10 to 20.

Since we expect that R_{+-}/R_{++} should probably change slowly with mass along an (exchange-degenerate) Regge trajectory, we believe the fit to $(K^* \to K\pi)/(\rho \to \pi\pi)$ for a ratio near −1 is in approximate agreement with our fit of the tensor decays (where this ratio was near −0.9). This is not conclusive, however, because we were also able to obtain fits in the same mass range if very large positive values (6, 10, 12, or 16) were used for R_{+-}/R_{++}. In no case could we obtain a simultaneous fit to the two ratios of the three decays $\rho \to \pi\pi$, $K^* \to K\pi$, and $\phi \to K\bar{K}$. All values of the parameters within this "interesting" range gave small widths for $\phi \to K\bar{K}$.

Since the fundamental experimental peculiarity about ϕ decay is the narrow width, we feel that a model which underestimates this width is better than one which overestimates it. It is also possible that the ϕ decay rate may be much more sensitive to our extra assumptions than the other rates because of the peculiar closeness of the ϕ mass to the $q_s\bar{q}_s$ and $K\bar{K}$ thresholds. Tentatively, therefore, we conclude that our model (with the additional simplifying assumptions mentioned here) is in substantial agreement with data on tensor and vector decays into pseudoscalars. The effective-mass parameters deduced from this data for the strange and nonstrange quarks seem reasonable. Further tests of the model in more complex situations are therefore warranted.

APPENDIX A: DERIVATION OF THE VERTEX FORMULA

The form of the decay vertex will be calculated in the rest frame of particle J, m_X^2. However, meson J_C, m_C^2 with particular quantum numbers is represented by a pole in certain well-known combinations of helicity amplitudes *in its own rest frame*. Hence we need to be able to transform particle helicities between the rest frames of J and J_C, J_C and J_D, J and J_D. This transformation is well known[4]; for spin-$\frac{1}{2}$ quarks it consists of the matrix $d^{1/2}_{\lambda_1\lambda_2}(\theta)$, where

$$\cos\theta = \frac{(E_1/\mu)(E_2/\mu) - \cosh\xi}{(p_1/\mu)(p_2/\mu)}$$

for E_1, E_2, the energies of the quark in the two frames considered, and $\cosh\xi$, the parameter of the Lorentz transformation between them. We will therefore have three such d functions in our vertex: one transforming quark 5 from the rest frame of J to that of J_C, one transforming quark 6 from the rest frame of J_C to that of J_D, and one transforming quark 8 from the rest frame of J to that of J_D.

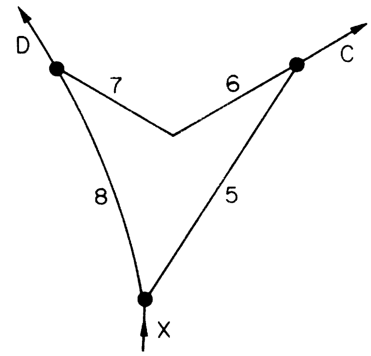

FIG. 4. We can equally well envision the top line in Fig. 3 to be a creation of a $q\bar{q}$ pair from vacuum. This labeling is used in the kinematics in Appendix A.

To set up the kinematics, it is easiest to break line 6 as shown in Fig. 4 with $p_7 = -p_6$. The energies and magnitudes of the momenta of quarks 5 and 8 are determined by two-body kinematics for a system of mass-squared m_X^2, and those of quarks 6 and 7 are determined by the known momenta of C, D, 5, and 8 and momentum conservation at the vertices:

$$\begin{aligned} P_C &= (E_C, p\sin\theta, 0, p\cos\theta)\,, \\ P_D &= (E_D, -p\sin\theta, 0, -p\cos\theta) \end{aligned} \tag{A1}$$

with

$$E_C = \frac{m_X^2 + m_C^2 - m_D^2}{2m_X}\,, \quad E_D = \frac{m_X^2 + m_D^2 - m_C^2}{2m_X}\,,$$

$$p = \frac{1}{2m_X}\left[\Lambda(m_X^2, m_C^2, m_D^2)\right]^{1/2},$$

$$\Lambda(x, y, z) = x^2 - 2x(y+z) + (y-z)^2\,.$$

We now use the direction of the outgoing mesons in the overall center-of-mass system to set up an axis system for the quarks. Note that we have arbitrarily chosen the final-state mesons to be in the xz plane, but there is no reason for the internal quarks to be confined to this plane. Hence

$$p_5 = \left(\frac{m_X^2 + \mu_5^2 - \mu_8^2}{2m_X}, p_q\cos\alpha\sin\theta + p_q\sin\alpha\cos\beta\cos\theta, p_q\sin\alpha\sin\beta, p_q\cos\alpha\cos\theta - p_q\sin\alpha\cos\beta\sin\theta\right) \tag{A2}$$

with $p_q = (1/2m_X)[\Lambda(m_X^2, \mu_5^2, \mu_8^2)]^{1/2}$. Thus momentum conservation yields (in the overall c.m. system)

$$\begin{aligned} p_6 = p_C - p_5 = \Big(&\frac{m_C^2 - m_D^2 - \mu_5^2 + \mu_8^2}{2m_X}, (p - p_q\cos\alpha)\sin\theta - p_q\sin\alpha\cos\beta\cos\theta, -p_q\sin\alpha\sin\beta, \\ &(p - p_q\cos\alpha)\cos\theta + p_q\sin\alpha\cos\beta\sin\theta\Big)\,. \end{aligned} \tag{A3}$$

Note that with this construction $p_6{}^2=\mu_6{}^2$ if

$$\cos\alpha=\frac{\mu_6{}^2-\mu_5{}^2-m_C{}^2+2E_CE_5}{2pp_q}\ . \tag{A4}$$

This is compatible with $p_7=p_D-p_8$, $p_7{}^2=\mu_6{}^2$ at the other vertex.

The quark-antiquark scattering distribution, in its own center-of-mass frame, has angular dependence $d^{J_C}_{\lambda_1-\lambda_2\lambda_3-\lambda_4}(\theta)$ for a partial wave of angular momentum J_C. We can decompose this angle θ between the incoming and outgoing quarks as $\theta=\theta_1+\theta_2$, where θ_1 is the angle between the incident quark and some arbitrarily defined z axis. Then $d^{J_C}_{\lambda_1-\lambda_2\lambda_3-\lambda_4}(\theta)=d^{J_C}_{\lambda_1-\lambda_2 M_C}(\theta_1)d^{J_C}_{M_C\lambda_3-\lambda_4}(\theta_2)$. Let this z axis for particle C be defined by the direction C takes in the rest frame of J. Then the decay amplitude $d^{J_C}_{M_C\lambda_3-\lambda_4}(\theta_2)$ can be removed, leaving us with an amplitude for production of a meson with helicity M_C proportional to $d^{J_C}_{\lambda_1-\lambda_2 M_C}(\theta_1)$. We therefore need to calculate the angle θ_1 for particles C and D, the angle between the incoming quarks and the direction of the outgoing meson, in the meson rest frame.

To calculate this angle, we must transform the momentum of quark 5 to the C rest frame. This gives $p_5\cos\alpha\rightarrow p_5\cos\alpha\cosh\xi_C-E_5\sinh\xi_C$ ($\cosh\xi_C=E_C/m_C$, $\sinh\xi_C=p_C/m_C$), so

$$\cos\theta_C=\frac{p_5\cos\alpha\cosh\xi_C-E_5\sinh\xi_C}{p_5'}$$

with

$$p_5'=\frac{1}{2m_C}\,[\Lambda(m_C{}^2,m_5{}^2,m_6{}^2)]^{1/2}.$$

A little algebra yields

$$\begin{aligned}\cos\theta_C&=\frac{(m_X{}^2+m_C{}^2-m_D{}^2)(\mu_6{}^2-\mu_5{}^2-m_C{}^2)+2m_C{}^2(m_X{}^2+\mu_5{}^2-\mu_8{}^2)}{[\Lambda(m_X{}^2,m_C{}^2,m_D{}^2)]^{1/2}[\Lambda(m_C{}^2,\mu_5{}^2,\mu_6{}^2)]^{1/2}}\ ,\\ \cos\theta_D&=\frac{(m_X{}^2+m_D{}^2-m_C{}^2)(\mu_6{}^2-\mu_8{}^2-m_D{}^2)+2m_D{}^2(m_X{}^2+\mu_8{}^2-\mu_5{}^2)}{[\Lambda(m_X{}^2,m_C{}^2,m_D{}^2)]^{1/2}[\Lambda(m_D{}^2,\mu_8{}^2,\mu_6{}^2)]^{1/2}}\ .\end{aligned} \tag{A5}$$

The transformation angles for the spins of the quarks are

$$\begin{aligned}\cos\psi_5&=\left(\frac{m_C{}^2+\mu_5{}^2-\mu_6{}^2}{2m_C\mu_5}\ \frac{m_X{}^2+\mu_5{}^2-\mu_8{}^2}{2m_X\mu_5}-\cosh\xi_C\right)\Big/\frac{[\Lambda(m_C{}^2,\mu_5{}^2,\mu_6{}^2)]^{1/2}}{2m_C\mu_5}\ \frac{[\Lambda(m_X{}^2,\mu_5{}^2,\mu_8{}^2)]^{1/2}}{2m_X\mu_5}\ ,\\ \cos\psi_8&=\left(\frac{m_D{}^2+\mu_8{}^2-\mu_6{}^2}{2m_D\mu_8}\ \frac{m_X{}^2+\mu_8{}^2-\mu_5{}^2}{2m_X\mu_8}-\cosh\xi_D\right)\Big/\frac{[\Lambda(m_D{}^2,\mu_8{}^2,\mu_6{}^2)]^{1/2}}{2\mu_8m_D}\ \frac{[\Lambda(m_X{}^2,\mu_8{}^2,\mu_5{}^2)]^{1/2}}{2\mu_8m_X}\ ,\end{aligned} \tag{A6}$$

and

$$\cos\psi=\left(\frac{m_C{}^2+\mu_6{}^2-\mu_5{}^2}{2\mu_6m_C}\ \frac{m_D{}^2+\mu_6{}^2-\mu_8{}^2}{2m_D\mu_6}-\frac{m_C{}^2+m_D{}^2-m_X{}^2}{2m_Cm_D}\right)\Big/\frac{[\Lambda(m_C{}^2,\mu_6{}^2,\mu_5{}^2)]^{1/2}}{2m_C\mu_6}\ \frac{[\Lambda(m_D{}^2,\mu_6{}^2,\mu_8{}^2)]^{1/2}}{2m_D\mu_6}\ .$$

We now have all the ingredients to assemble the amplitude. First, consider the vertex at particle J. To get from the z axis to p_5 in this rest frame, we must first rotate through α about the y axis, then rotate through $+\beta$ about the z axis, and finally rotate through θ about the y axis. The quark state thus rotated is

$$e^{-i\theta J_y}e^{-i\beta J_z}e^{-i\alpha J_y}|85\rangle\ ,$$

so its overlap with our initial meson state of angular momentum J and helicity M is

$$\langle J58|e^{i\alpha J_y}e^{+i\beta J_z}e^{i\theta J_y}|JM\rangle=\sum_\epsilon d^J_{\epsilon\,5-8}(\alpha)e^{+i\beta\epsilon}d^J_{M\epsilon}(\theta)\ .$$

The quark pair 6, 7 is created without contact with any gluons, so the helicities of 6 and 7 must be the same in the $J, m_X{}^2$ system. We thus have a factor which expresses the change of helicity under transformation to the rest frame of C or D:

$$d^{1/2}_{5'5}(\psi_5)d^{1/2}_{6'6}(\psi_6)d^{1/2}_{7'7}(\psi_7)d^{1/2}_{8'8}(\psi_8)|_{\text{with}\,6=7}=d^{1/2}_{5'5}(\psi_5)d^{1/2}_{6'7'}(\psi)d^{1/2}_{8'8}(\psi_8)\ .$$

Finally, within the rest frame of C (or D) we must calculate the overlap of our quark state $e^{-i\beta J_z}e^{-i\theta_C J_y}|5'6'\rangle$ with the meson wave function $|J_CM_C\rangle$. We therefore obtain the vertex factors

$$e^{-i\beta M_C}d^{J_C}_{5'-6'M_C}(\theta_C)\quad\text{and}\quad e^{i\beta M_D}d^{J_D}_{8'-7'M_D}(\theta_D)\ .$$

Our expression for the vertex thus takes the form (inserting the residues R_{ij} of the meson helicity amplitudes at the poles)

$$\sum_{\substack{5,6,7,8\\5',6',7',8'}} \int d\beta \sum_{\epsilon} d^{J}_{\epsilon\,5-8}(\alpha)e^{i\beta\epsilon} d^{J}_{M\epsilon}(\theta)R_{58}(m_X^2)d^{1/2}_{5'5}(\psi_5)d^{1/2}_{6'7'}(\psi)d^{1/2}_{8'8}(\psi_8)R_{5'6'}(m_C^2)$$
$$\times e^{-i\beta M_C}d^{J_C}_{5'-6'\,M_C}(\theta_C)R_{8'7'}(m_D^2)e^{i\beta M_D}d^{J_D}_{8'-7'\,M_D}(\theta_D)\,.$$

The β integral is trivial; we find $\epsilon = M_C - M_D$ so that the vertex becomes

$$\sum_{\substack{5678\\5'6'7'8'}} d^{J}_{M M_C - M_D}(\theta)d^{J}_{M_C - M_D\,5-8}(\alpha)R_{58}(m_X^2)d^{1/2}_{5'5}(\psi_5)d^{1/2}_{6'7'}(\psi_6-\psi_7)$$
$$\times d^{1/2}_{8'8}(\psi_8)R_{5'6'}(m_C^2)d^{J_C}_{5'-6'\,M_C}(\theta_C)R_{8'7'}(m_D^2)d^{J_D}_{8'-7'\,M_D}(\theta_D)\,. \quad \text{(A7)}$$

The sums over quark helicities can now be performed for any particular mesons of interest. One should bear in mind the relations $R_{\lambda_1\lambda_2}=\eta R_{-\lambda_1-\lambda_2}$ for quark-antiquark residues, where η is the naturality of the meson, $P=\eta(-1)^J$. The other simplification of importance is the fact that unnatural-parity trajectories of natural charge conjugation (like the π, B, etc.) couple only to R_{++} and not to R_{+-}, whereas those of unnatural charge conjugation (like the A_1, if it exists) couple only to R_{+-}.

APPENDIX B: SOME PROPERTIES OF THE VERTEX

1. Fundamentals: Parity and threshold behavior

Our expression should exhibit all the helicity dependence of the vertex. We note that the actual vertex coupling [after removing $d^J_{MM_C-M_D}(\theta)$] obeys

$$V_{M_C M_D}=\eta\eta_C\eta_D V_{-M_C-M_D}\,,$$

as required by parity invariance. Because of our somewhat cavalier treatment of the triangle integral, we allow this expression to be multiplied by any nonsingular function of the masses m_X^2, m_C^2 and m_D^2; this is in fact necessary in order to obtain the correct threshold behavior at the meson threshold.

The overall vertex should vanish like $\{[\Lambda(m_X^2, m_C^2, m_D^2)]^{1/2}\}^{L_{\min}}$ at $\Lambda=0$, where $L_{\min}$ is the lowest allowed orbital angular momentum state between the outgoing mesons. In general, if $\eta\eta_C\eta_D=+1$,

$$L^{\min}=\min\{|J-J_C-J_D|,|J_C-J-J_D|,|J_D-J-J_C|\}\,,$$

whereas if $\eta\eta_C\eta_D=-1$,

$$L^{\min}=\min\{|J-J_C-J_D|,|J_C-J-J_D|,\ |J_D-J-J_C|\}+1\,.$$

Since each of the displayed angles $\cos\alpha$, $\cos\theta_1$, and $\cos\theta_2$ behaves like $1/\sqrt{\Lambda}$, Eq. (4) itself will behave like $(1/\sqrt{\Lambda})^{J+J_C+J_D}$ unless there is some cancellation between terms which might possibly yield a behavior like $(1/\sqrt{\Lambda})^{J+J_C+J_D-1}$. We therefore multiply the expression [Eq. (4)] by $(\sqrt{\Lambda})^{\mathcal{J}}$, where

$$\mathcal{J}=J+J_C+J_D+\min\{|J-J_C-J_D|,|J_C-J-J_D|,|J_D-J-J_C|\}\,.$$

Using the relation $R_{\lambda_1\lambda_2}=\eta R_{-\lambda_1-\lambda_2}$, we find that the terms in our expression will look like

$$\frac{R_{58}R_{5'6'}R_{8'7'}}{2}\,[d^J_{M_C-M_D\,5-8}(\alpha)d^{1/2}_{5'5}(\psi_5)d^{1/2}_{6'7'}(\psi)d^{1/2}_{8'8}(\psi_8)d^{J_C}_{5'-6'\,M_C}(\theta_C)d^{J_D}_{8'-7'\,M_D}(\theta_D)$$
$$+\eta\eta_C\eta_D d^J_{M_C-M_D-5+8}(\alpha)d^{1/2}_{-5'\,-5}(\psi_5)d^{1/2}_{-6'\,-7'}(\psi)d^{1/2}_{-8'\,-8}(\psi_8)d^{J_C}_{-5'+6'\,M_C}(\theta_C)d^{J_D}_{-8'+7'\,M_D}(\theta_D)]$$

or (B1)

$$\frac{R_{58}R_{5'6'}R_{8'7'}}{2}\,d^{1/2}_{5'5}(\psi_5)d^{1/2}_{6'7'}(\psi)d^{1/2}_{8'8}(\psi_8)[d^J_{M_C-M_D\,5-8}(\alpha)d^{J_C}_{5'-6'\,M_C}(\theta_C)d^{J_D}_{8'-7'\,M_D}(\theta_D)$$
$$+\eta\eta_C\eta_D d^J_{-M_C+M_D\,5-8}(\alpha)d^{J_C}_{5'-6'\,-M_C}(\theta_C)d^{J_D}_{8'-7'\,-M_D}(\theta_D)]\,.$$

As $\cos\theta \to \infty$,

$$d^J_{MM'}(\theta) \to (-1)^M d^J_{-MM'}(\theta),$$
$$d^J_{MM'}(\theta) \to (-1)^{M'} d^J_{M-M'}(\theta).$$

Hence as $\Lambda \to 0$, the term like $(1/\sqrt{\Lambda})^{J+J_C+J_D}$ is multiplied by $(1+\eta\eta_C\eta_D)$, and the contributions for $\eta\eta_C\eta_D = -1$ are less singular, by one power of $\sqrt{\Lambda}$. Hence we need multiply by only one power of Λ for any choice of J, J_C, and J_D and all the possible parity combinations will automatically be taken care of. Notice that in fact $(\sqrt{\Lambda})^{\mathscr{J}}$ is a power of Λ, not $\sqrt{\Lambda}$. The various choices of $L_{\min}$ here correspond to the different contributions to the triple-Regge vertex.[2]

The vertex thus has all desired kinematic properties.

2. Regge couplings in two-body phenomenology

Because the vertex has all proper parity properties, various selection rules due only to parity conservation are automatically fulfilled. For example, consider the production of particle J_D off a pion ($J_C = 0$) in the t-channel center-of-mass frame. From (B1) we easily see that the $M_D = 0$ amplitude will be populated only when $\eta\eta_C\eta_D = +1$.

This leads us to consider the common situation of production of an $\eta = +$particle (such as f or ρ) from an incident π via exchange of an off-shell π. We then find that (B1) reduces to

$$R_{++}(t)R_{++}(m_\pi{}^2)R_{8'7'}(m_D{}^2)$$
$$\times d^{1/2}_{++}(\psi_5)d^{1/2}_{+7'}(\psi)d^{1/2}_{8'+}(\psi_8)d^J_{-M_D0}(\alpha)$$
$$\times [d^{J_D}_{8'-7'M_D}(\theta_D) + (-1)^{M_D} d^{J_D}_{8'-7'-M_D}(\theta_D)].$$

Once the two specific couplings R_{++} and R_{+-} for the external particle have been supplied, this provides a model for the relative amount of helicities greater than 0 found away from the $t = m_\pi{}^2$ poles [note particularly that the unknown function $R_{++}(t)$ cancels out when taking the ratios of different helicity couplings]. The higher the spin of the produced particle, the more useful this parametrization becomes in calculating density-matrix elements.

3. The case when all quark masses are zero

If the quark masses are zero, the quark helicities do not rotate in going from one frame to another. Our general formula then simplifies considerably to

$$R_{58}(m_X{}^2)R_{56}(m_C{}^2)R_{86}(m_D{}^2)$$
$$\times [d^J_{M_C-M_D5-8}(\alpha)d^{J_C}_{5-6M_C}(\theta_C)d^{J_D}_{8-6M_D}(\theta_D)$$
$$+\eta\eta_C\eta_D d^J_{-M_C+M_D5-8}(\alpha)d^{J_C}_{5-6-M_C}(\theta_C)d^{J_D}_{8-6-M_D}(\theta_D)].$$

Unfortunately it is fairly easy to demonstrate that this is in conflict with the data. The decay $B \to \omega\pi$ is well known to produce predominantly transversely polarized ω's.[5] In the zero-quark-mass approximation, the ratio of longitudinal to transverse ω's is (using the constraints $5=8$, $5=6$ required to obtain the proper quantum numbers of the B and the π)

$$\left|\frac{\text{Long}}{\text{Trans}}\right| = \left|\frac{d^1_{00}(\alpha)d^1_{00}(\theta_D)}{d^1_{01}(\alpha)d^1_{01}(\theta_D)}\right| = \left|\frac{2\cos\alpha\cos\theta_D}{\sin\alpha\sin\theta_D}\right|$$
$$= \left|\frac{2(m_B{}^2 - m_\pi{}^2 - m_\omega{}^2)(m_B{}^2 - m_\omega{}^2 + m_\pi{}^2)}{2im_\pi m_\omega 2im_\pi m_B}\right|$$
$$= +\frac{(m_B{}^2 - m_\omega{}^2)^2 - m_\pi{}^4}{2m_\pi{}^2 m_\omega m_B} \gg 1,$$

a serious conflict with the data.

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[1]We can list only a few of the many studies in the literature. An example of a successful model which neglects Toller-angle dependence is the usual Reggeized Deck calculation, such as the work of E. L. Berger, Phys. Rev. 166, 1525 (1968); and G. Ascoli *et al.*, Phys. Rev. D 9, 1963 (1974). The Veneziano model was applied to $\pi N \to \pi\omega N$ by P. Aurenche, *ibid.* 9, 1514 (1974) and by many other workers in the early 1970's to other reactions. One example of a study which included Toller-angle dependence in a way not derived from the Veneziano model is the work by E. Berger and J. Vergeest on $Kp \to K^*\pi N$ [Nucl. Phys. B116, 317 (1976)].

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