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Effect of fermions upon tunneling in a one-dimensional system*

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We study the effects of a fermion with Yukawa-type coupling upon the tunneling of the double-well anharmonic oscillator. These effects prove to be nondramatic, despite the zero-frequency bound-state eigenmode one would encounter if one applied the conventional boundary conditions to the path integral.

I. INTRODUCTION

In a previous paper,[1] we evaluated the path integral for the kernel of the double-well anharmonic oscillator,

$$L=\tfrac{1}{2}(\partial_t\varphi)^2-\frac{\lambda}{4}\left(\varphi^2-\frac{\mu^2}{\lambda}\right)^2, \tag{1.1}$$

by expanding about classical solutions of the Euclidean action, and thereby determined the two lowest energy levels. These two levels would be degenerate were it not for quantum tunneling, which shifts one level up and one down by an amount proportional to the probability for barrier penetration. Our analysis led to perfect agreement with the ordinary WKB result.

In this paper we enrich the model by adding to it a fermion with a Yukawa coupling:

$$L=\tfrac{1}{2}(\partial_t\varphi)^2-\frac{\lambda}{4}\left(\varphi^2-\frac{\mu^2}{\lambda}\right)^2+\Psi^\dagger(i\partial_t-g\varphi)\Psi+\frac{\hbar g\varphi}{2}\,. \tag{1.2}$$

Here $\Psi(t)$ is a one-component fermionic degree of freedom. The term linear in φ is included in order to make the fermionic part of the Hamiltonian antisymmetric under $\Psi\to\Psi^\dagger$.

Our interest in introducing fermions is motivated by the dramatic effect which massless fermions have upon the tunneling between the classical vacuums of the pure gauge theory analyzed by Belavin *et al.*[2] It was shown by 't Hooft[3] that in the pseudoparticle sector the fermion determinant possesses a zero-frequency eigenvalue corresponding to a bound-state eigenmode, which leads to no tunneling between classical vacuums of different topology. This effect is interpreted as a consequence of the axial-vector-current anomaly.[3,4] In a recent paper that discusses several models which possess pseudoparticles, Patrascioiu[5] pointed out that in the one-dimensional case described by the Lagrangian (1.2), the fermion determinant again possesses a zero-frequency eigenvalue. (This zero eigenmode was obtained by assuming that a linear combination of the Euclidean eigenfunction and its derivative went to zero at the end points of a box of length T, and then taking the limit $T\to\infty$.[6]) Our attempts to explain the resulting lack of tunneling as a consequence of a symmetry or an anomaly were fruitless.

A key element in our present analysis regards the boundary conditions involved in evaluating the path integral for the fermions. For a boson field, $\varphi(t)$, the kernel $\langle\varphi_2t_2|\varphi_1t_1\rangle$ is evaluated by integrating over all paths such that $\varphi(t_1)=\varphi_1$ and $\varphi(t_2)=\varphi_2$. The fermion field does not admit such a prescription, since fermion states with a definite value of $\Psi(t)$ cannot be constructed. However, we can label fermion states by their occupation number n, where n can equal zero or one. We would therefore be interested in computing $\langle\varphi_2n_2t_2|\varphi_1n_1t_1\rangle$. Unfortunately, we do not know how to translate the information about n_1 and n_2 into boundary conditions obeyed by $\Psi(t)$ and $\Psi^\dagger(t)$. Inspired by the work of Dashen, Hasslacher, and Neveu,[7] we will instead compute $\sum_{n_i}\langle\varphi_2n_it_2|\varphi_1n_it_1\rangle$, with the boundary conditions for the eigenmodes of the fermion determinant chosen to be antiperiodic in time. An important virtue of our model (1.2) is that it can be thoroughly investigated by conventional methods. Thus we can confirm the validity of the above prescription for evaluating the path integral, and we can easily determine the effect of the presence of fermions upon our system. That effect proves to be nondramatic; if the fermion coupling is weak, the boson tunneling is only perturbatively affected.

II. DERIVATION OF THE DESIRED RESULT

The Euclidean version of our Lagrangian (1.2) is

$$L_E=-\tfrac{1}{2}(\partial_t\varphi)^2-\frac{\lambda}{4}\left(\varphi^2-\frac{\mu^2}{\lambda}\right)^2+\Psi^\dagger(\partial_t-g\varphi)\Psi+\frac{\hbar g}{2}\,\varphi\,, \tag{2.1}$$

where Ψ and $\Psi^\dagger$ are now independent degrees of freedom. These fermion fields obey the equations of motion

$$(\partial_t-g\varphi)\Psi=0 \tag{2.2a}$$

and

$$(-\partial_t-g\varphi)\Psi^\dagger=0\,. \tag{2.2b}$$

They yield

$$\Psi(t) = T\exp\left[\int_0^t d\tau\, g\varphi(\tau)\right]\Psi(0) \qquad (2.3a)$$

and

$$\Psi^\dagger(t) = T\exp\left[-\int_0^t d\tau\, g\varphi(\tau)\right]\Psi^\dagger(0)\,, \qquad (2.3b)$$

where T stands for the time-ordered product. The fermion number operator,

$$n(t) \equiv \frac{1}{\hbar}\Psi^\dagger(t)\Psi(t)\,,$$

is thus conserved, so that

$$\langle\varphi_2 n_2 t_2|\varphi_1 n_1 t_1\rangle \propto \delta_{n_1,n_2}\,. \qquad (2.4)$$

The eigenvalues of n are 0 and 1.

The Hamiltonian of the full system can therefore be written as

$$H = -\tfrac{1}{2}(\partial_t\varphi)^2 + \frac{\lambda}{4}\left(\varphi^2 - \frac{\mu^2}{\lambda}\right)^2 + \hbar g\varphi(n-\tfrac{1}{2})\,. \qquad (2.5)$$

Its effect on the fermion label is trivial, so that the problem reduces to studying a boson system with the Hamiltonian

$$H_{b\pm} = -\tfrac{1}{2}(\partial_t\varphi)^2 + \frac{\lambda}{4}\left(\varphi^2 - \frac{\mu^2}{\lambda}\right)^2 \pm \frac{\hbar g}{2}\varphi\,, \qquad (2.6)$$

where + and − refer to $n=0$ and $n=1$, respectively. Thus the presence of the fermion can be described in terms of replacing the original boson potential,

$$V(\varphi) = \frac{\lambda}{4}\left(\varphi^2 - \frac{\mu^2}{\lambda}\right)^2\,, \qquad (2.7)$$

by two new potentials,

$$V_\pm(\varphi) = \frac{\lambda}{4}\left(\varphi^2 - \frac{\mu^2}{\lambda}\right)^2 \pm \frac{\hbar g}{2}\varphi\,. \qquad (2.8)$$

Since it is obvious that H_{b+} and H_{b-} have the same spectrum, one effect of the fermion is to introduce a twofold degeneracy in the energy levels. In the weak-coupling limit, the system can be analyzed perturbatively. We show in Appendix A that in fact the first-order correction in g to the kernel $\langle\varphi_2 n_2 t_2|\varphi_1 n_1 t_1\rangle$ is zero. There is no dramatic effect due to fermions.

III. A PATH-INTEGRAL ANALYSIS

In principle one would like to compute the Euclidean kernel, $\langle\varphi_2 n_2 t_2|\varphi_1 n_1 t_1\rangle$, via a path integral, but for the reasons stated in the Introduction, we shall settle for the trace of the kernel over the fermion label,

$$\sum_{n_i}\langle\varphi_2 n_2 t_2|\varphi_1 n_1 t_1\rangle = \int \mathfrak{D}\varphi(t)\mathfrak{D}\Psi(t)\mathfrak{D}\Psi^\dagger(t)\exp\left\{-\frac{1}{\hbar}\int_{t_1}^{t_2} dt\left[\tfrac{1}{2}(\partial_t\varphi)^2 + \frac{\lambda}{4}\left(\varphi^2 - \frac{\mu^2}{\lambda}\right)^2 - \Psi^\dagger(\partial_t - g\varphi)\Psi - \frac{\hbar g}{2}\varphi\right]\right\}. \qquad (3.1)$$

We will first perform the $\Psi, \Psi^\dagger$ integrations, treating the boson field as an arbitrary external source. Again, as discussed in the Introduction, we evaluate

$$\sum_{n_i}\langle n_i t_2|n_i t_1\rangle = \int \mathfrak{D}\Psi(t)\mathfrak{D}\Psi^\dagger(t)\exp\left\{\frac{1}{\hbar}\int_{t_1}^{t_2} dt\left[\Psi^\dagger(\partial_t - g\varphi)\Psi + \frac{\hbar g\varphi}{2}\right]\right\} \qquad (3.2)$$

by setting it equal to

$$\sum_{n_i}\langle n_i t_2|n_i t_1\rangle = 2\,\frac{\mathrm{Det}[\partial_t - g\varphi]}{\mathrm{Det}[\partial_t]} = 2\prod_m \frac{E_m(\varphi)}{E_m(0)}\,, \qquad (3.3)$$

where the $E_n(\varphi)$ satisfy

$$(\partial_t - g\varphi)\Psi_m = E_m(\varphi)\Psi_m\,,\qquad (-\partial_t - g\varphi)\Psi_m^\dagger = E_m(\varphi)\Psi_m^\dagger\,, \qquad (3.4)$$

with

$$\Psi_m(t_2) = -\Psi_m(t_1)\,,\qquad \Psi_m^\dagger(t_2) = -\Psi_m(t_1)\,. \qquad (3.5)$$

Note that Ψ and $\Psi^\dagger$ are independent degrees of freedom satisfying different differential equations and that the differential operator is not Hermitian. Yet we do obtain the following eigenfunctions (normalized to $\hbar$):

$$\Psi_m(t) = \left(\frac{\hbar}{t_2 - t_1}\right)^{1/2}\exp\left[\int_0^t d\tau\, g\varphi - tg\hat{\varphi} - i\,\frac{(2m+1)\pi t}{t_2 - t_1}\right],$$

$$\Psi_m^\dagger(t) = \left(\frac{\hbar}{t_2 - t_1}\right)^{1/2}\exp\left[-\int_0^t d\tau\, g\varphi + tg\hat{\varphi} + i\,\frac{(2m+1)\pi t}{t_2 - t_1}\right], \qquad (3.6)$$

where

$$\hat{\varphi} \equiv \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} dt\, \varphi(t)\,. \tag{3.7}$$

The eigenvalues are

$$E_m(\varphi) = -g\hat{\varphi} - i\,\frac{(2m+1)\pi}{t_2 - t_1}\,, \quad m = 0, \pm 1, \pm 2, \ldots\,. \tag{3.8}$$

It is easy to check that both Ψ_m and $\Psi_m^\dagger$ form a complete orthonormal set, where the inner product is defined as $\int_{t_1}^{t_2} dt\,\Psi_m^\dagger(t)\Psi_n(t)$. The fact that Ψ_m and $\Psi_n^\dagger$ obey Eqs. (3.4) and obey the completeness relation make it especially simple to expand the propagator of our system in terms of these functions. Since such an expansion can sometimes be used to extract interesting information about the theory, we include a derivation of it in Appendix B.

Using our result to evaluate (3.3), we obtain

$$\sum_{n_i} \langle n_i t_2 | n_i t_1\rangle = 2 \prod_{m=-\infty}^{\infty} \left[1 - \frac{ig\hat{\varphi}(t_2 - t_1)}{(2m+1)\pi}\right] = 2\cosh[\tfrac{1}{2} g\hat{\varphi}(t_2 - t_1)]\,. \tag{3.9}$$

Returning now to the original system, where the boson field is also quantized, our result implies that

$$\sum_{n_i} \langle \varphi_2 n_i t_2 | \varphi_1 n_i t_1\rangle = \int \mathfrak{D}\varphi(t) \left(\exp\left\{ -\frac{1}{\hbar} \int_{t_1}^{t_2} dt \left[(\partial_t \varphi)^2 + \frac{\lambda}{4}\left(\varphi^2 - \frac{\mu^2}{\lambda}\right)^2 + \frac{\hbar g \varphi}{2}\right]\right\} + \exp\left\{ -\frac{1}{\hbar} \int_{t_1}^{t_2} dt \left[(\partial_t \varphi)^2 + \frac{\lambda}{4}\left(\varphi^2 - \frac{\mu^2}{\lambda}\right)^2 - \frac{\hbar g \varphi}{2}\right]\right\} \right)\,. \tag{3.10}$$

But clearly this is exactly what one would expect from the boson system of (2.6), i.e., we have reproduced the "desired" result of Sec. II.

Note that if $\hat{\varphi} = 0$, $E_0 \xrightarrow[t_1 - t_2 \to \infty]{} 0$, but Ψ_0 is not a bound-state eigenfunction, but rather it has plane-wave normalization [i.e., $\int_{-T}^{T} dt\,\Psi^\dagger(t)\Psi(t) \xrightarrow[T\to\infty]{} \infty$]. Thus the antiperiodic boundary conditions do not lead to a zero-frequency bound-state solution.

We are currently applying these methods to the four-dimensional problem, and will report on our results in a forthcoming paper.

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APPENDIX A

We show here that first-order correction in g to the kernel $\langle \varphi_2 n_2 t_2 | \varphi_1 n_1 t_1\rangle$ is zero. To solve the boson problem we need to assume weak coupling, i.e., $\lambda\hbar/\mu^3 \ll 1$. Weak coupling of the fermion occurs if

$$\frac{\hbar}{2}\sqrt{2}\,\mu \gg \frac{\hbar}{2}\, g(\mu^2/\lambda)^{1/2}, \text{ i.e., } g \ll \sqrt{\lambda}\,. \tag{A1}$$

We recall[1] that for $g = 0$ the lowest-lying energy eigenstates occur at

$$E_{g\mp} = \frac{\hbar}{2}\sqrt{2}\,\mu\left[1 \mp \left(\frac{16\sqrt{2}\,\mu^3}{\pi\lambda\hbar}\right)^{1/2} e^{-2\sqrt{2}\,\mu^3/3\lambda\hbar}\right]. \tag{A2}$$

Since for the unperturbed-potential $V(\varphi) = V(-\varphi)$, the first-order effect of the perturbation $\pm(\hbar g/2)\varphi$ is zero. However, the wave functions are changed:

$$\begin{aligned} |g_-\rangle &= \frac{|L_0\rangle + |R_0\rangle}{\sqrt{2}} \\ &\to |g_-'\rangle_\pm = |g_-\rangle \pm \frac{\hbar g}{2}\,|g_+\rangle \frac{2(\mu^2/\lambda)^{1/2}}{E_{g+} - E_{g-}}\,, \\ |g_+\rangle &= \frac{|L_0\rangle - |R_0\rangle}{\sqrt{2}} \\ &\to |g_+'\rangle_\pm = |g_+\rangle \pm \frac{\hbar g}{2}\,|g_-\rangle \frac{2(\mu^2/\lambda)^{1/2}}{E_{g+} - E_{g-}}\,, \end{aligned} \tag{A3}$$

where $|L_0\rangle$ and $|R_0\rangle$ are the ground states of the left and right well, respectively, when one ignores the presence of the other well.

One can now compute the kernel, $\langle \varphi_2 n_2 t_2 | \varphi_1 n_1 t_1\rangle$, for $-\varphi_1 = \varphi_2 = (\mu^2/\lambda)^{1/2}$, with a large Euclidean time interval, $t_2 - t_1$. It is saturated by the two lowest-energy eigenstates, so that we obtain

$$\begin{aligned} \langle \varphi_2 n_2 t_2 | \varphi_1 n_1 t_1\rangle = \delta_{n_1, n_2} \tfrac{1}{2} |\langle \varphi = (\mu^2/\lambda)^{1/2} | R_0\rangle|^2 \Bigg\{ &\left[1 \pm \frac{\hbar g(\mu^2/\lambda)^{1/2}}{E_{g+} - E_{g-}}\right]\left[1 \mp \frac{\hbar g(\mu^2/\lambda)^{1/2}}{E_{g+} - E_{g-}}\right] e^{-(1/\hbar)E_{g-}(t_2 - t_1)} \\ &- \left[1 \pm \frac{\hbar g(\mu^2/\lambda)^{1/2}}{E_{g+} - E_{g-}}\right]\left[1 \mp \frac{\hbar g(\mu^2/\lambda)^{1/2}}{E_{g+} - E_{g-}}\right] e^{-(1/\hbar)E_{g+}(t_2 - t_1)} \Bigg\}\,, \end{aligned} \tag{A4}$$

where $\pm$ refers to $n = 1$ or 0, respectively. Notice that the $O(g)$ terms cancel. Q.E.D.

APPENDIX B

In this section we present the expansion of the Euclidean propagator in the presence of an external source φ in terms of Ψ_m and $\Psi_n^\dagger$. Let $S(t_2', t_1')$ be the propagator; then

$$S(t_2', t_1') \equiv \langle 0|T\Psi(t_2')\Psi^\dagger(t_1')|0\rangle = \Theta(t_2' - t_1')\hbar \exp\left[\int_{t_1'}^{t_2'} d\tau g\varphi(\tau)\right]. \tag{B1}$$

S obeys the equations

$$(-\partial_{t_1'} - g\varphi)S(t_2', t_1') = \hbar\,\delta(t_2' - t_1')$$

and

$$(\partial_{t_2'} - g\,\varphi)S(t_2', t_1') = \hbar\,\delta(t_2' - t_1'). \tag{B2}$$

It is easy to check that

$$\sum_{m=-\infty}^{\infty} \Psi_m(t_2')\Psi_m^\dagger(t_1') = \hbar\,\delta(t_2' - t_1'). \tag{B3}$$

Then the expansion of S [on the interval (t_2, t_1)] in terms of Ψ_m and $\Psi_n^\dagger$ is simple:

$$S(t_2', t_1') = \sum \frac{\Psi_m(t_2')\Psi_m^\dagger(t_1')}{E_m} + C\hbar \exp\left[\int_{t_1'}^{t_2'} d\tau g\varphi(\tau)\right], \tag{B4}$$

where all that remains to be determined is C, the coefficient of the solution to the homogeneous equation. To evaluate C we explicitly plug in for the eigenmodes,

$$S(t_2', t_1') = \sum_{m=-\infty}^{\infty} \frac{\hbar}{t_2 - t_1} \frac{\exp\left[\int_{t_1'}^{t_2'} d\tau g\varphi + (t_1' - t_2')g\hat{\varphi} + i\dfrac{(t_1' - t_2')(2n+1)\pi}{t_2 - t_1}\right]}{-g\hat{\varphi} - i\dfrac{(2n+1)\pi}{t_2 - t_1}} + C\hbar \exp\left(\int_{t_1'}^{t_2'} d\tau g\varphi\right), \tag{B5}$$

and consider the limit $t_2 \to \infty$ and $t_1 \to -\infty$,

$$\begin{aligned} S(t_2', t_1') &\to -\frac{\hbar}{2\pi i}\int_{-\infty}^{\infty} dE \frac{\exp\left[\int_{t_1'}^{t_2'} d\tau g\varphi + (t_1' - t_2')g\hat{\varphi} + i(t_1' - t_2')E\right]}{-ig\hat{\varphi} + E} + C\hbar \exp\left(\int_{t_1'}^{t_2'} d\tau g\varphi\right) \\ &= [-\theta(t_1' - t_2')\theta(g\hat{\varphi}) + \theta(t_2' - t_1')\theta(-g\hat{\varphi}) + C]\hbar \exp\left(\int_{t_1'}^{t_2'} d\tau g\varphi\right). \end{aligned} \tag{B6}$$

Comparing with (B1), we conclude that, in the limit which we considered,

$$C = \theta(g\hat{\varphi}). \tag{B7}$$

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