Field-strength formulation of quantum chromodynamics

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Non-Abelian gauge theories in space-time dimension $D \geq 3$ can be formulated entirely in terms of field strengths. I detail the formulation in four dimensions.

INTRODUCTION

In this paper, I wish to call attention to the fact that $(D \geq 3)$ non-Abelian gauge theories can be reformulated in terms of field strengths, and to work out details in the case of $(D=4)$ quantum chromodynamics (QCD). After completion of this work, I learned that Roskies[1] and Calvo[2] had reached similar conclusions concerning the uniqueness of the Yang-Mills potentials.

The basic idea of the reformulation can be seen in the first-order formalism for $SU(N)_D$ Yang-Mills theory,[3]

$$\partial_\mu G^i_{\mu\nu} = g f^{ijk} V^j_\mu G^k_{\mu\nu}, \tag{1a}$$

$$G^i_{\mu\nu} = \partial_\mu V^i_\nu - \partial_\nu V^i_\mu - g f^{ijk} V^j_\mu V^k_\nu . \tag{1b}$$

The point is that, if the $D(N^2-1)$-dimensional matrix $\mathcal{G}^{ij}_{\mu\nu} \equiv f^{ijk} G^k_{\mu\nu}$ has an inverse, then I can solve Eq. (1a) for the potentials in terms of the field strengths

$$V^i_\mu = -\frac{1}{g} (\mathcal{G}^{-1})^{ij}_{\mu\nu} \partial_\lambda G^j_{\lambda\nu}, \tag{2a}$$

$$\mathcal{G}^{ij}_{\mu\nu} (\mathcal{G}^{-1})^{jl}_{\nu\rho} = \delta^{il} \delta_{\mu\rho} . \tag{2b}$$

Thereafter, reformulation in terms of field strengths is relatively straightforward.

I have concluded that the inversion is generally possible, except when $D=2$. I will discuss below the explicit form of $\det \mathcal{G}$ when $D=4$, but the answer is that $\det\mathcal{G}$ is identically zero only for $D=2$. It may vanish for particular $G^i_{\mu\nu}$ when $D \geq 3$, but this does not appear to be a fundamental problem. Further, $\det\mathcal{G}$ does not vanish around interesting $G^i_{\mu\nu}$, e.g., the pseudoparticle.[4] It is my feeling then that the reformulation may be valuable.

I offer the following understanding of my explicit computations: In fact, the inversion (2a) and (2b) will fail $(\det\mathcal{G}=0)$ if one can make a gauge transformation (T is transpose, O is an element of the regular representation of the group)

$$\begin{aligned} V^i_\mu &\to O^{ia} V^a_\mu + \frac{1}{2g} f^{ijk} O^{ja} \partial_\mu (O^T)^{ak}, \\ G^k_{\mu\nu} &\to O^{kl} G^l_{\mu\nu}, \\ \mathcal{G}^{ij}_{\mu\nu} &\to O^{il} \mathcal{G}^{lm}_{\mu\nu} (O^T)^{mj}, \\ (\mathcal{G}^{-1})^{ij}_{\mu\nu} &\to O^{ib} (\mathcal{G}^{-1})^{ba}_{\mu\nu} (O^T)^{aj} \end{aligned} \tag{3}$$

that alters V^i_μ without rotating $G^k_{\mu\nu}$. When $D=2$, we have $G^k_{\mu\nu} = \epsilon_{\mu\nu} G^k$, and we are dealing with a *single* vector representation. In the case $N=2$, we can gauge transform to $G^k = \delta^{k3} G$. Then, further gauge rotations about the third direction will not change $G^k_{\mu\nu}$, while altering V^i_μ. Hence, $\det\mathcal{G}=0$ identically. This argument is easily extended to all N. Already at $D=3$, however, $G^k_{\mu\nu}$ comprises *three* vector representations, so there cannot be any such damaging gauge transformations for arbitrary $G^k_{\mu\nu}$. In fact, for $N=2$, $D=3$, it is easy to check that $\det\mathcal{G}$ is not identically zero. (Try $G^k_{\mu\nu} = \epsilon_{\mu\nu k}$.)

In general, for $SU(N)$, let the "vectors" and "rotations" be $N \times N$ traceless Hermitian matrices and unitary matrices, respectively. Let G_1 and G_2 be (just) *two* "arbitrary" vectors (not simultaneously block-diagonalizable). If $UG_i U^\dagger = G_i$ $(i=1,2)$, then it is easy to show that $U=1$. A proof goes along the following lines: G_1 may be taken diagonal without loss of generality. The transformations U which leave G_1 invariant form the maximal Abelian subgroup. But then nontrivial elements of this subgroup will not rotate G_2 only if G_2 is block-diagonal. By assumption, it is not, so $U=1$. I am informed by Professor E. Wichmann that this result is well known. Since already for $D=3$ we have *three* vectors, we are safe for all N and $D \geq 3$.

The arguments presented above are classical, but, via functional integrals, I shall show that they apply as well in the quantum case.

FIELD-STRENGTHS AND STRUCTURE OF THE INVERSION

I begin with the action for QCD in the first-order formalism:

$$\begin{aligned} S_1 &= \int d^D x [\tfrac{1}{2} G^i_{\mu\nu} F^i_{\mu\nu} - \tfrac{1}{4} G^i_{\mu\nu} G^i_{\mu\nu} \\ &\qquad + \psi^\dagger (\not\partial + M + ig \not V)\psi], \\ F^i_{\mu\nu} &\equiv \partial_\mu V^i_\nu - \partial_\nu V^i_\mu - g f^{ijk} V^j_\mu V^k_\nu, \\ \not V &\equiv \gamma_\mu V^a_\mu \frac{\lambda_a}{2} . \end{aligned} \tag{4}$$

In the generating functional, we integrate e^{-S_1} over all $G^i_{\mu\nu}, V^i_\mu, \psi, \psi^\dagger$. Performing the (quadratic) integration over V^i_μ, we obtain the *field-strength action*,

$$S=\int d^Dx\left[\frac{1}{2g}(\partial_\rho G^i_{\rho\mu}-gJ^i_\mu)(\mathcal{G}^{-1})^{ij}_{\mu\nu}(\partial_\lambda G^j_{\lambda\nu}-gJ^j_\nu)-\tfrac{1}{4}G^i_{\mu\nu}G^i_{\mu\nu}+\psi^\dagger(\partial\!\!\!/+M)\psi-\chi^i_\mu\frac{g}{2}\mathcal{G}^{ij}_{\mu\nu}\chi^j_\nu\right], \tag{5}$$

$$J^i_\mu\equiv i\psi^\dagger\gamma_\mu\frac{\lambda^i}{2}\psi .$$

χ^i_μ is an auxiliary field, much like a Faddeev-Popov field, which represents $(\det\mathcal{G})^{-1/2}$. As we shall see, χ^i_μ can play a role in quantum corrections.

I will now state the form for $\mathcal{G}^{-1}$ in the case $N=2$, $D=4$. The result, first given by Deser and Teitelboim,[5] is

$$(\mathcal{G}^{-1})^{ij}=\tilde G^j G^i K^{-1} . \tag{6}$$

Here I am using an obvious 4×4 matrix notation, and

$$\tilde G^i_{\mu\nu}\equiv\tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}G^i_{\rho\sigma},\quad \epsilon_{0123}=+1$$
$$K\equiv G^1\tilde G^2G^3-G^3\tilde G^2G^1 . \tag{7}$$

This can be put in the form of Ref. 5 via the identity (Tr is trace)

$$(G^i)^{-1}=\frac{\tilde G^i}{\frac{1}{4}\mathrm{Tr}(G^i\tilde G^i)} , \tag{8}$$

true for any i.

I have also shown explicitly that

$$(\mathcal{G}^{-1})^{ij}=K^{-1}G^j\tilde G^i . \tag{9}$$

This is equivalent to uniqueness of inverse. Hence ($G^T=-G$, $K^T=K$) $\mathcal{G}^{-1}$ is overall symmetric,

$$(\mathcal{G}^{-1})^{ij}_{\mu\nu}=(\mathcal{G}^{-1})^{ji}_{\nu\mu} , \tag{10}$$

as could be anticipated from the symmetry of $\mathcal{G}$. Interest is then focussed on the symmetric matrix K.

Consider the generalization of K,

$$K^{ijk}\equiv G^i\tilde G^jG^k-G^k\tilde G^jG^i ,$$
$$K=K^{123} . \tag{11}$$

By examining each matrix element of $K^{ijk}_{\mu\nu}$, I have shown that

$$K^{ijk}=\epsilon^{ijk}K ,\quad K=\tfrac{1}{3}\epsilon^{ijk}G^i\tilde G^jG^k . \tag{12}$$

Hence, K^{ijk}, and, in particular K, is a gauge-invariant color-singlet. In fact, the cyclic properties of K^{ijk} can easily be seen by repeated application of the identities

$$G^i\tilde G^j+G^j\tilde G^i=\tilde G^iG^j+\tilde G^jG^i=\tfrac{1}{2}\mathrm{Tr}(G^i\tilde G^j)1 . \tag{13}$$

I will show that $\det K$ is not identically zero by exhibiting a G for which it is not zero. In fact there are many (most) such G's, so I will show this for an *interesting* case.

There is a spectacular simplification of K in the *self-dual sectors*, $G^i=\pm\tilde G^i$, $G^i_{0l}\equiv E^i_l$. After some algebra, I obtain the result that K is proportional to the unit matrix. Explicitly, for the self-dual sectors,

$$K_{\mu\nu}=\delta_{\mu\nu}\xi ,\quad \xi\equiv\tfrac{1}{3}\epsilon^{\alpha\beta\gamma}\epsilon_{ijk}E^\alpha_iE^\beta_jE^\gamma_k ,$$
$$(\mathcal{G}^{-1})^{ij}=\pm\xi^{-1}G^jG^i . \tag{14}$$

For the pseudoparticle ansatz $E^i_l=\delta^i_l\lambda$, I obtain the even simpler result

$$\xi=2\lambda^3 . \tag{15}$$

Of course, λ is well known[4] and nonvanishing, so $\det K$ cannot vanish identically.

Semiclassical physics in this reformulation must be an expansion around saddle-point configurations with $\det K\neq0$. (Otherwise, as seen below, the saddle-point equations for G are ill-defined. It is problematic whether $\det K=0$ configurations can play a role in more deeply quantum-mechanical approaches.) It is gratifying then that the reformulation is well-defined near the pseudoparticle; it will also be interesting to study more fully these two classes of configurations ($\det K=0$ and $\det K\neq0$). Later, I will illustrate the start of such a semiclassical calculation by solving for λ directly from the classical G equations of motion.

AT THE SADDLE POINT

The classical equations of motion for G, ψ follow by variation of the action S. A helpful identity is

$$\frac{\partial(\mathcal{G}^{-1})^{lm}_{\mu\nu}}{\partial G^i_{\sigma\kappa}}=-(\mathcal{G}^{-1})^{la}_{\mu\sigma}\epsilon^{abi}(\mathcal{G}^{-1})^{bm}_{\kappa\nu}+(\mathcal{G}^{-1})^{la}_{\mu\kappa}\epsilon^{abi}(\mathcal{G}^{-1})^{bm}_{\sigma\nu} \tag{16}$$

which follows directly from $\mathcal{G}\mathcal{G}^{-1}=1$. I obtain then, at the saddle point,

$$F^a_{\mu\nu}(\tilde{\mathcal{J}})+gG^a_{\mu\nu}+g^2\epsilon^{abc}\chi^b_\mu\chi^c_\nu=0 , \tag{17a}$$
$$(i\partial\!\!\!/+iM+\tilde{\mathcal{J}}\!\!\!/)\psi=0 , \tag{17b}$$
$$\mathcal{G}^{ij}_{\mu\nu}\chi^j_\nu=0 , \tag{17c}$$

where

$$F^a_{\mu\nu}(\tilde{\mathcal{J}})\equiv\partial_\mu\tilde{\mathcal{J}}^a_\nu-\partial_\nu\tilde{\mathcal{J}}^a_\mu+\epsilon^{abc}\tilde{\mathcal{J}}^b_\mu\tilde{\mathcal{J}}^c_\nu , \tag{18a}$$
$$\tilde{\mathcal{J}}^a_\mu\equiv\mathcal{J}^a_\mu-g(\mathcal{G}^{-1})^{ab}_{\mu\nu}J^b_\nu , \tag{18b}$$
$$\mathcal{J}^a_\mu\equiv(\mathcal{G}^{-1})^{ab}_{\mu\nu}\partial_\lambda G^b_{\lambda\nu} . \tag{18c}$$

The classical field equations (17a) and (17b) are ill-defined unless $\det K\neq0$. If our approach is to be semiclassical, we must begin with only this class of configurations. But then $\mathcal{G}^{ij}_{\mu\nu}$ has no zero eigenvalues and we must take $\chi^i_\mu = 0$, by (17c). In quantum corrections to such configurations, the χ^i_μ will in general play a role. In other, more deeply quantum-mechanical approaches (strong-coupling?), the χ^i_μ may play a more immediate role. This is closely related to the statement that in such "other" approaches, configurations with $\det K = 0$ may play a role. Many interesting questions arise; e.g., Do all nontrivial self-dual solutions in the *usual* formulation satisfy $\det K \neq 0$?

The equations of motion (17a) and (17b) (with $\chi = 0$) are precisely what one expects from solving the first-order equation for V^i_μ, as discussed above. The identification is

$$V^i_\mu \longleftrightarrow -\frac{1}{g}\tilde{\mathcal{J}}^i_\mu = -\frac{1}{g}(\tilde{G}^j G^i K^{-1})_{\mu\nu}(\partial_\lambda G^j_{\lambda\nu} - gJ^j_\nu). \quad (19)$$

Notice also the induced "four-Fermi interaction (times $\mathcal{G}^{-1}$)" apparent in the action and in the field equations. In what follows, I will drop the quark terms for simplicity, stating full results when relevant.

EXPLICITLY GAUGE-INVARIANT ACTION

Under gauge transformation [Eq. (3)], all quantities transform as expected from the first-order formalism. The field equations are invariant. The action

$$S = \int d^4x \left[\frac{1}{2g}(\partial G \mathcal{G}^{-1}\partial G) - \tfrac{1}{4}G^2 - \chi\frac{g}{2}\mathcal{G}\chi\right] \quad (20)$$

is not, however, explicitly gauge-invariant. (It is if one drops surface terms after the transformation.) This is because, in obtaining S from Eq. (4), we had to do an integration by parts to put the V^i_μ integration in standard form. I can regain an explicitly gauge-invariant action via the useful identities

$$\begin{aligned}\partial_\lambda G^i_{\lambda\mu}(\mathcal{G}^{-1})^{ij}_{\mu\nu}\partial_\rho G^j_{\rho\nu} &= G^i_{\mu\nu}\epsilon^{ijk}\mathcal{J}^j_\mu \mathcal{J}^k_\nu \\ &= \partial_\lambda G^i_{\lambda\mu}\mathcal{J}^i_\mu \\ &= -G^i_{\lambda\mu}\partial_\lambda \mathcal{J}^i_\mu + \partial_\lambda(G^i_{\lambda\mu}\mathcal{J}^i_\mu). \quad (21)\end{aligned}$$

To obtain these, I have repeatedly used the *definition* of $\mathcal{J}^a_\mu$ [Eq. (18c)], and the implied fact that

$$\partial_\mu G^i_{\mu\nu} + \epsilon^{ijk}\mathcal{J}^j_\mu G^k_{\mu\nu} = 0 \quad (22)$$

even *off* the saddle point. This gives us a choice of *many* forms for the action. Taking appropriate combinations of the second and fourth forms in Eq. (21), and dropping the surface term, I obtain the explicitly gauge-invariant form

$$S = \int d^4x \left[-\frac{1}{2g}G\mathcal{F}(\mathcal{J}(G)) - \tfrac{1}{4}G^2 - \chi\frac{g}{2}\mathcal{G}\chi\right]. \quad (23)$$

This form is what might be expected from the first-order formalism. A similar manipulation, including quarks, yields the expected result: Equation (23) with $\mathcal{J} \to \tilde{\mathcal{J}}$, an an additional $\Delta S = \int d^4x\, \psi^\dagger \times (\not{\partial} + M - \tilde{\not{\mathcal{J}}})\psi$.

The gauge-invariant action Eq. (23) can also be cast in the form

$$S = \int d^4x \left[-\tfrac{1}{4}\left(G + \frac{\mathcal{F}}{g}\right)^2 + \frac{1}{8g^2}(\mathcal{F} \pm \tilde{\mathcal{F}})^2 \mp \frac{1}{4g^2}\mathcal{F}\tilde{\mathcal{F}}\right]. \quad (24)$$

The first term vanishes at the saddle point. Then, at fixed $\int d^4x\, \mathcal{F}\tilde{\mathcal{F}}$, S is a minimum only if $\mathcal{F} \pm \tilde{\mathcal{F}} = 0$ $(G = \pm\tilde{G})$.

THE PSEUDOPARTICLE

As a simple application of the field-strength reformulation, I will solve for the $G = \tilde{G}$ pseudoparticle directly from the (no-quark) G equations of motion. In this sector, we have, using Eqs. (18c) and (14),

$$\mathcal{J}^i_\mu = \xi^{-1}(G^j G^i)_{\mu\nu}\partial_\lambda G^i_{\lambda\nu}. \quad (25)$$

With the ansatz $E^i_l = B^i_l = \delta^i_l\lambda$, I obtain, using Eq. (15),

$$\begin{aligned}\mathcal{J}^i_0 &= -\tfrac{1}{2}\partial_i \ln\lambda, \\ \mathcal{J}^i_l &= \tfrac{1}{2}(\delta_{il}\partial_0 - \epsilon_{ilm}\partial_m)\ln\lambda.\end{aligned} \quad (26)$$

The field equations, $\mathcal{F}(\mathcal{J}(G)) + gG = 0$, are conveniently grouped as

$$\mathcal{F} + \tilde{\mathcal{F}} + 2gG = 0, \quad (27a)$$

$$\mathcal{F} = \tilde{\mathcal{F}}. \quad (27b)$$

Both are second-order differential equations. With the assumption[6] that $\lambda = \lambda(R)$ $(R^2 \equiv x_\mu x_\mu)$, these become respectively ($f \equiv \ln\lambda$, prime means d/dR)

$$\frac{1}{4}f'' + \frac{3}{4}\frac{f'}{R} + \frac{1}{8}(f')^2 + g\lambda = 0, \quad (28a)$$

$$\frac{1}{4}f'' - \frac{(f')^2}{8} - \frac{f'}{4R} = 0. \quad (28b)$$

In general, two second-order differential equations for the same function would be cause for concern. This system, however, is easily solved: Eliminating the second derivatives, we have

$$\frac{(f')^2}{4} + \frac{f'}{R} + g\lambda = 0, \quad (29)$$

the most general solution to which is

$$\lambda(R) = \frac{4b}{g}\frac{1}{(R^2+b)^2}, \quad (30)$$

b arbitrary. Further, this λ *also* solves Eqs. (28a) and (28b). This is the usual pseudoparticle. I have spent some time trying to see a similar structure in the full Eq. (27) (with no ansatz), but, thus far, to no avail.

REMARKS

Finally, I would like to make a few remarks about directions. In the first place, *the inversion, Eq. (19), is singular at $g=0$.* As a result, I have *not* been able to expand the field-strength formulation in the usual perturbation series; I strongly suspect that the field-strength formulation is fundamentally nonperturbative.

Two other weak-coupling schemes present themselves, however. (1) The usual semiclassical expansion is easily available [scale $G \equiv (1/g)\overline{G}$]. (2) An expansion directly in the form Eq. (23) yields the gauge-invariant (but useless) $\mathfrak{F}(\mathcal{G})=0$ in the leading approximation.

A similar *gauge-invariant strong-coupling scheme* is also suggested. The point is that *each* of the three terms in Eq. (23) is *separately* gauge-invariant. We might call $-(1/2g)\int G\mathfrak{F}(\mathcal{G}(G))d^4x$ the gauge-invariant "kinetic energy," and $-\frac{1}{4}\int G^2 d^4x$ the gauge-invariant "interaction." The strong-coupling expansion is then in powers of the kinetic energy. This has the usual strong-coupling problems in lowest order, but, for non-Abelian gauge theories, it is the *only* gauge-invariant strong-coupling expansion I know. It also has the usual *advantage* (of strong-coupling schemes) that such an expansion is expected to *converge*. This is easy to see via the methods of Lipatov[7]: Under a scaling $G \to \sqrt{N}G$, the interaction $G^2 \to NG^2$, while the kinetic energy $G\mathfrak{F} \to \sqrt{N}G\mathfrak{F}$ ($\mathcal{G}$ is invariant under G-scaling). Thus, the strong-coupling expansion is expected to be of the convergent form

$$Z \sim \sum_N \frac{1}{g^N}\frac{K^N}{(N!)^{1/2}} . \tag{31}$$

I have also concluded that gauge-fixing is not a problem; e.g., ghost-free gauges are easy to translate into the field-strength formulation. (Ghost gauges are, however, problematic.) What would be interesting would be an investigation of allowed "G gauges" (gauges directly in terms of G). This would be easiest to study in phase space, where our formulation is close to the usual (but with "coordinates" and "momenta" reversed).

Finally, I remark that the field-strength formulation allows the introduction of certain *gauge-invariant quantities as dynamical variables*. E.g., in a representation such as

$$\{G^i_{\mu\nu}\} = r_{\mu\nu}(\sin\theta_{\mu\nu}\cos\phi_{\mu\nu},\ \sin\theta_{\mu\nu}\sin\phi_{\mu\nu}, \cos\theta_{\mu\nu}) , \tag{32}$$

the variables $r_{\mu\nu} = (\sum_i G^i_{\mu\nu}G^i_{\mu\nu})^{1/2}$ are gauge-invariant. Such directions are interesting, and may lead to a gauge-invariant formulation of non-Abelian gauge theories.

Note added in proof. There is a subtle point about first-order formalisms (in general) which shows up most clearly in Euclidean space. The reader may want to check for himself that the same thing happens in first-order formulation of, say, $\lambda\phi^4$ theory. *For convergence* of the Euclidean functional integrals, the variables $G^a_{\mu\nu}$ must be integrated over purely imaginary contours, i.e., $\mathfrak{D}G = i\mathfrak{D}G'$, G' real. (Alternately, replace $G \to iG$ and integrate over real contours.) By symmetry properties ($G \to -G$), it is easy to show that the functional integrals are still real. Of course, the saddle-point equations

$$\mathfrak{F}^a_{\mu\nu}(\mathcal{G}) + gG^a_{\mu\nu} = 0$$

show that the saddle points are at real G (because $\mathcal{G}$, and hence $\mathfrak{F}$, is invariant under G scaling). The saddle points must be approached by contour distortions.

To go beyond semiclassical expansions about nonsingular ($\det\mathcal{G} \neq 0$) configurations, one must find a consistent prescription in the neighborhood of the singular configurations.

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[1]R. Roskies, Phys. Rev. D 15, 1731 (1977).

[2]M. Calvo, Phys. Rev. D 15, 1733 (1977).

[3]I use Euclidean variables throughout this paper. My Minkowski space notation is that of J. D. Bjorken and S. D. Drell, *Relativistic Quantum Fields* (McGraw-Hill, New York, 1965). Attaching labels M (Minkowski) and E (Euclidean), my translation is $x_{0M} = -ix_{0E}$, $x_{iM} = x_{iE}$, $\partial_{0M} = i\partial_{E}$, $\partial_{iM} = \partial_{iE}$, $V_{0M} = iV_{0E}$, $V_{iM} = V_{iE}$, $\gamma_{0M} = \gamma_{0E}$, $\gamma_{iM} = -i\gamma_{iE}$, $(\gamma_{\mu E}, \gamma_{\nu E})_+ = 2\delta_{\mu\nu}$, $\gamma^\dagger_{\mu E} = \gamma_{\mu E}$. In the (implied) Euclidean functional integrals, I am calling $\overline{\psi}$ by the name $\psi^\dagger$.

[4]A. Belavin, A. Polyakov, A. Schwartz, and Y. Tyupkin, Phys. Lett. 59B, 85 (1975).

[5]S. Deser and C. Teitelboim, Phys. Rev. D 13, 1592 (1976).

[6]I have also sought all nonspherical solutions $E^a_i = B^a_i = \delta^a_i\lambda(x_\mu)$. Aside from the translated pseudoparticle, there are no further solutions of this form.

[7]L. N. Lipatov, Leningrad Nuclear Physics Institute report, 1976 (unpublished).