PHYSICAL REVIEW D VOLUME 16, NUMBER 6 15 SEPTEMBER 1977

SU(2)-flavor Schwinger model on the lattice

Paul J. Steinhardt*
Lyman Laboratory of Physics, Harvard University, Cambridge, Massachusetts 02138
(Received 16 May 1977)

Hamiltonian lattice perturbation methods are used to study the $(1+1)$-dimensional, SU(2)-flavor Abelian gauge model. For a model with coupling constant g and fermion mass m, two distinct vacuum regions characterized by the magnitude of m/g are found. Expansions about the Ising-type vacuum corresponding to large m/g are carried out to order $1/g^8a^8$ (a = lattice spacing) and improved using Padé approximants. The results compare favorably qualitatively and quantitatively with recent studies of the weak-coupling limit of the continuum theory. Expansions to order $1/g^4a^4$ about the Heisenberg antiferromagnetlike vacuum corresponding to small m/g agree qualitatively with known results for the strong-coupling limit of the continuum theory.

I. INTRODUCTION

Recently, Banks, Susskind, and Kogut[1] have used lattice perturbation theory methods to study $(1+1)$-dimensional Abelian gauge theories (massive Schwinger model). Strong-coupling expansions for the theory's mass spectrum were computed to second order in $y=1/g^4a^4$ (a = lattice spacing, g = coupling constant), improved via Padé approximants, and extrapolated to the continuum, $a\to 0$, limit. In general, good agreement between these calculations and exact results for the continuum Schwinger model was obtained. The success of the procedure is not surprising since the continuum theory of free massive (mass $\sim g^2/\pi$) mesons can be described in terms of quark-antiquark pairs bound by gauge strings,[2] and it is precisely this idea that is embodied in the lattice methods.

It is, therefore, an instructive exercise to utilize the same approach to study the $(1+1)$-dimensional, SU(2)-flavor, Abelian gauge theory since certain features of the continuum mass spectrum are not readily comprehensible using the same string picture. Specifically, Coleman[3] has found that the I^{PG} quantum numbers of the lowest-lying bound states above the vacuum are

$$1^{-+}, 0^{--}, 1^{+-}, 0^{++}, \ldots$$

in the weak-coupling ($m \gg g$) limit, but

$$1^{-+}, 0^{++}$$

in the strong-coupling ($g \gg m$) limit. In the latter limit, all other bound states are $O((g/m)^{2/3})$ more massive than the first two.

My intent in this paper is to report the results of Hamiltonian lattice gauge perturbation calculations of the low-lying bound states in both the strong- and weak-coupling limits for the SU(2)-flavor, massive Schwinger model. Except for a few modifications, included in the discussion of Sec. II, the procedure followed is identical to that of Banks *et al.* In Sec. III, the effect of the modifications on the details of the calculations is explained. Two phases with distinct vacuums—one equivalent to the ground state of an Ising model, the other equivalent to the ground state of a Heisenberg antiferromagnet—are shown to exist, characterized by the magnitude of m/g. Expansions about the two vacuums and determination of the respective low-lying bound-state spectra are discussed in Secs. IV and V. It is demonstrated that in the weak-coupling limit the bound-state spectrum calculated through order $1/g^8a^8$ has quantum numbers

$$I^{PG} = 1^{-+}, 0^{++}, 1^{+-}, 0^{++}$$

and the mass splittings are in both qualitative and quantitative agreement with Coleman's weak-coupling predictions for the continuum theory. Expansions about the strong-coupling vacuum through order $1/g^4a^4$ (calculated using known results for the Heisenberg antiferromagnetic chain) imply the low-lying bound-state spectrum is

$$I^{PG} = 1^{-+}, 0^{++}$$

with all other bound states having much greater mass. The results agree qualitatively with Coleman's strong-coupling limit predictions for the continuum theory. Although two vacuum phases exist in the lattice theory, the implications for the continuum model are not clear.

II. SU(2)-FLAVOR MODEL ON THE LATTICE

The first step in the lattice procedure is to formulate the SU(2)-flavor, massive Schwinger Hamiltonian on a spatial lattice. A staggered lattice of spacing a is employed on which, for each type of fermion, a one-component fermion field $\phi_\alpha(n)$ ($\alpha = p, n$ representing the two flavors of fermion

in the theory) is defined at each lattice site n. $\phi_\alpha(n)$ satisfies the anticommutation relations

$$\{\phi_\alpha^\dagger(n), \phi_\alpha(m)\} = \delta_{nm}, \quad \{\phi_\alpha(n), \phi_\alpha(m)\} = 0, \tag{2.1a}$$

$$\{\phi_n^\dagger(n), \phi_\beta(m)\} = 0, \quad \{\phi_n(n), \phi_\beta(m)\} = 0. \tag{2.1b}$$

It is related to the two-component lattice field $\psi_\alpha(n)$ via

$$\psi_\alpha(n) = \begin{pmatrix} \psi_\alpha \text{ even} \\ \psi_\alpha \text{ odd} \end{pmatrix}, \qquad \psi_\alpha\begin{pmatrix}\text{even}\\ \text{odd}\end{pmatrix} = \phi_\alpha(n) \text{ for } n\begin{pmatrix}\text{even}\\ \text{odd}\end{pmatrix}. \tag{2.2}$$

The Dirac Hamiltonian expressed in terms of $\phi_\alpha(n)$ is

$$H = \frac{i}{2a}\sum_n [\phi_\beta^\dagger(n)\phi_\beta(n+1) - \phi_\beta^\dagger(n+1)\phi_\beta(n) + \phi_n^\dagger(n)\phi_n(n+1) - \phi_n^\dagger(n+1)\phi_n(n)] + m\sum_n (-1)^n[\phi_\beta^\dagger(n)\phi_\beta(n) + \phi_n^\dagger(n)\phi_n(n)]. \tag{2.3}$$

From this expression, the one-dimensional fermion problem may be rewritten in terms of a one-dimensional spin chain problem by using a set of transformations derived from a set first found by Jordan and Wigner.[4] The transformations used in Ref. 1 for the U(1) model satisfy the anticommutation relations of Eq. (2.1a), but cannot satisfy the additional relations of Eq. (2.1b) necessary for the SU(2) model. One solution is to modify the Jordan-Wigner transformation for the two fermion case as follows:

$$\phi_\beta(n) = i\left(\prod_{l<n}[i\sigma_{\beta_3}(l)i\sigma_{n_3}(l)]\right)\sigma_\beta^-(n), \tag{2.4a}$$

$$\phi_n(n) = -\left(\prod_{l<n}[i\sigma_{\beta_3}(l)i\sigma_{n_3}(l)]\right)\sigma_{\beta_3}(n)\sigma_n^-(n). \tag{2.4b}$$

The generalization of the transformation for N fermions is ($\alpha = 1, \ldots, N$)

$$\phi_\alpha(n) = i^\alpha\left(\prod_{l<n}[i\sigma_{1_3}(l)\cdots i\sigma_{N_3}(l)]\right) \times \sigma_{1_3}(n)\cdots\sigma_{\alpha-1_3}(n)\sigma_\alpha^-(n). \tag{2.5}$$

The extra $\sigma_{\alpha_3}(n)$'s that appear in the new transformations provide some additional complexity in the formalism. The Dirac Hamiltonian, (2.3), for example, becomes

$$H = \frac{i}{2a}\sum_n [\sigma_\beta^+(n)\sigma_{n_3}(n)\sigma_\beta^-(n+1) - \sigma_\beta^+(n+1)\sigma_{n_3}(n)\sigma_\beta^-(n) + \sigma_n^+(n)\sigma_{\beta_3}(n+1)\sigma_n^-(n+1) - \sigma_n^+(n+1)\sigma_{\beta_3}(n+1)\sigma_n^-(n)] + m\sum_n (-1)^n[\tfrac{1}{2}(1+\sigma_{\beta_3}(n)) + \tfrac{1}{2}(1+\sigma_{n_3}(n))]. \tag{2.6}$$

For the one-flavor model the corresponding expression is a function only of $\sigma^-(n)$ and $\sigma^+(n)$.

The introduction of the gauge field into the theory parallels closely the U(1) case. The gauge field

$$U(n, n+1) = e^{iagA(n)} = e^{i\theta(n)}, \quad A(n) \equiv A^1(n) \tag{2.7}$$

where g is the coupling constant, is defined on the link joining lattice points n and $n+1$. To simplify the Hamiltonian and the space of states, the class of gauges with $A^0 = 0$ is chosen; the electric field is, therefore, $\dot{A}$ and satisfies the canonical commutation relations

$$[A(n), \dot{A}(n)] = i(1/a)\delta_{nm} \tag{2.8}$$

or, rewriting the commutation relations in terms of $L(n) = \dot{A}(n)/g$,

$$[\theta(n), L(m)] = i\delta_{nm}. \tag{2.9}$$

The eigenvalues of $L(m)$ are quantized in integrals steps 0, $\pm 1, \ldots$, and the operator can be represented on a space $\{|l\rangle\}$ such that

$$L|l\rangle = l|l\rangle \tag{2.10a}$$

and

$$e^{\pm i\theta(n)}|l\rangle = |l \pm 1\rangle. \tag{2.10b}$$

The physically relevant states are drawn from the space of gauge-invariant states. The local gauge invariance of the physical sector of the theory is defined by

$$G(n)|\psi\rangle = 0, \tag{2.11a}$$

where

$$G(n) = \frac{a}{g^2}\sum_m [\dot{A}(n+m, -m) - \psi_\beta^\dagger(n)\psi_\beta(n) - \psi_n^\dagger(n)\psi_n(n) + \text{constant}]. \tag{2.11b}$$

This constraint is analogous to the condition $\vec{\nabla}\cdot\vec{E} = \rho_G + \rho_F$, where ρ_G and ρ_F are the color densities of the gauge and Fermi fields, respectively, and the constant represents the vacuum charge density.

Since the gauge contribution is unchanged from Ref. 1, the total Hamiltonian becomes

$$H = \tfrac{1}{2}g^2 a\sum_n L^2(n) + \frac{i}{2a}\sum_n [(\sigma_p^+(n)\sigma_{n_3}(n)\sigma_p^-(n+1)e^{i\theta(n)} - \text{H.c.}) + (\sigma_n^+(n)\sigma_{p_3}(n+1)\sigma_n^-(n+1)e^{i\theta(n)} - \text{H.c.})]$$
$$+ m\sum_n (-1)^n[\tfrac{1}{2}(1+\sigma_{p_3}(n)) + \tfrac{1}{2}(1+\sigma_{n_3}(n))]. \tag{2.12}$$

A renormalized operator,

$$W = \frac{2}{ag^2}H = H_0 + y^{1/2}V, \tag{2.13}$$

where

$$H = \sum_n L^2(n), \tag{2.14a}$$

$$V = i\sum_n [(\sigma_p^+(n)\sigma_{n_3}(n)\sigma_p^-(n+1)e^{i\theta(n)} - \text{H.c.}) + (\sigma_n^+(n)\sigma_{p_3}(n+1)\sigma_n^-(n+1)e^{i\theta(n)} - \text{H.c.})]$$
$$+ \tfrac{1}{2}\mu y^{-1/2}\sum_n (-1)^n(\sigma_{p_3}(n) + \sigma_{n_3}(n)), \tag{2.14b}$$

$$\mu = 2m/g^2 a \text{ and } y = 1/g^4 a^4, \tag{2.14c}$$

is more convenient for the perturbation expansions. For $y \ll 1$, or large $g \cdot a$, one can treat xV as a perturbation on H_0. The magnitude of m/g may be varied independently from this limit.

III. VACUUM TRANSITION

The first lattice perturbation calculations concern the determination of the zero-energy or vacuum state. The vacuum of the unperturbed Hamiltonian H_0 is enormously degenerate since fluxless states have unperturbed energy eigenvalue $H_0 = 0$ independent of the fermion configuration. To determine the vacuum, degenerate perturbation theory is applied and the perturbation is diagonalized in the degenerate subspace. When the fermion bare mass is set equal to zero, the lowest-order nonzero contribution is

$$\Delta E = y\left\langle L=0 \middle| V\frac{1}{-H_0}V \middle| L=0\right\rangle, \tag{3.1}$$

where $|L=0\rangle$ is a fluxless state with an arbitrary spin configuration. We choose the vacuum to be the fluxless spin state that minimizes ΔE.

With two spins (p and n type) defined for each lattice site, four possible spin configurations—$\langle \text{in}\rangle$, $\langle \text{out}\rangle$, $\langle \text{up}\rangle$, and $\langle \text{down}\rangle$ (see Fig. 1)—are possible at each site. A fluxless spin state may be written as a linear combination of "basis" states with one of the four spin configurations set for each lattice site, e.g.,

$$|(\text{site } 1;\langle \text{in}\rangle), (\text{site } 2,\langle \text{up}\rangle), \ldots, (\text{site } N-1;\langle \text{in}\rangle), (\text{site } N;\langle \text{out}\rangle)\rangle$$

for a periodic spin chain of N links.

Using the translation and isospin symmetries that the vacuum is expected to possess, it is possible to prove that all fluxless spin states having these symmetries that also lie in the gauge-invariant (physical) sector can be written as either a linear combination of vacuum basis states with only $\langle \text{in}\rangle$ or $\langle \text{out}\rangle$ configurations set for each lattice site or as a single basis state with only $\langle \text{up}\rangle$ or $\langle \text{down}\rangle$ configurations set for each site (e.g., the example basis state above cannot be a vacuum basis state because it contains both $\langle \text{up}\rangle$ and $\langle \text{out}\rangle$ configurations):

Translational invariance in the continuum theory is equivalent to invariance under translations by two spacings on the lattice, just as in Ref. 1. The current density for the third component of isospin, $j_3^0(x)$, has the form on the lattice

$$j_3^0(n) = \tfrac{1}{4}(\sigma_{p_3}(n) - \sigma_{n_3}(n)); \tag{3.2a}$$

this operator has eigenvalues $+\frac{1}{2}$ $(-\frac{1}{2})$ for $\langle \text{out}\rangle$ ($\langle \text{in}\rangle$) configurations and zero for $\langle \text{up}\rangle$ and $\langle \text{down}\rangle$ configurations. The charge current density, on the other hand, has the form

$$j^0(n) = \tfrac{1}{4}(\sigma_{p_3}(n) + \sigma_{n_3}(n)), \tag{3.2b}$$

which has eigenvalue $+\frac{1}{2}$ $(-\frac{1}{2})$ for $\langle \text{up}\rangle$ ($\langle \text{down}\rangle$) con-

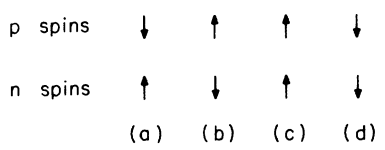

FIG. 1. For the SU(2)-flavor model, there are four spin configurations possible at each site: (a) $\langle \text{in}\rangle$, (b) $\langle \text{out}\rangle$, (c) $\langle \text{up}\rangle$, and (d) $\langle \text{down}\rangle$.

figurations and zero for ⟨in⟩ and ⟨out⟩ configurations. Since the vacuum of the lattice theory has total charge zero and total isospin zero, (3.2a) and (3.2b) imply that each vacuum basis state must have an equal number of ⟨in⟩ and ⟨out⟩ sites and an equal number of ⟨up⟩ and ⟨down⟩ sites. Since the vacuum of the lattice theory has to be translation invariant, for every vacuum basis state in the linear combination of basis states that form the vacuum that has a specific configuration at site k, there must be another vacuum basis state in the linear combination (possibly the same basis state) which has the same configuration at sites $k \pm 2, \pm 4, \ldots, \pm(N-2)$. For the vacuum to be locally gauge invariant, all vacuum basis states must have the same charge density at each lattice site in order that a single choice of the constant in (2.11b) is sufficient to guarantee that $G(n)|\text{vacuum}\rangle = 0$. Therefore, for a vacuum to have basis states with any sites of nonzero charge density (⟨up⟩ or ⟨down⟩ configurations) and to be translation invariant, total charge zero, and locally gauge invariant, it must be one of two (trivial) linear combinations of basis states:

$$|(\text{site } 1;\langle\text{up}\rangle), (\text{site } 2;\langle\text{down}\rangle), (\text{site } 3;\langle\text{up}\rangle), \ldots, (\text{site } N;\langle\text{down}\rangle)\rangle$$

or

$$|(\text{site } 1;\langle\text{down}\rangle), (\text{site } 2;\langle\text{up}\rangle), (\text{site } 3;\langle\text{down}\rangle), \ldots, (\text{site } N;\langle\text{up}\rangle)\rangle.$$

A mixture of the two vacuums is not even an allowed vacuum because it is not gauge invariant. Otherwise the vacuum must be a linear combination of vacuum basis states with ⟨out⟩ or ⟨in⟩ configurations at each lattice site. Local gauge invariance places no constraint on this vacuum since each site has zero charge density. Thus, the vacuum is either in a subspace with ⟨up⟩ or ⟨down⟩ configurations at each lattice site or in a subspace with ⟨in⟩ or ⟨out⟩ configurations at each site of the vacuum basis states. Q.E.D.

Since the two subspaces cannot mix to form the vacuum, the perturbation can be diagonalized separately in the ⟨in⟩-⟨out⟩ and ⟨up⟩-⟨down⟩ subspaces and the state of minimum energy determined. Inserting the spin expression for the kinetic energy, (2.14b), into the perturbation Hamiltonian, (3.1), one finds

$$\Delta E = -y\Big\langle L=0\Big|\Big\{-i\sum_n [(\sigma_p^+(n)\sigma_{n_3}(n)\sigma_p^-(n+1)e^{i\theta(n)} - \text{H.c.}) + (\sigma_n^+(n)\sigma_{p_3}(n+1)\sigma_n^-(n+1)e^{i\theta(n)} - \text{H.c.})]\Big\}\{\text{same operator}\}\Big| L=0\Big\rangle \tag{3.3a}$$

$$= y\sum_n \langle L=0|[-1 + \tfrac{1}{2}\sigma_{p_3}(n)\sigma_{p_3}(n+1) + \tfrac{1}{2}\sigma_{n_3}(n)\sigma_{n_3}(n+1) + 2\sigma_p^+(n)\sigma_n^-(n)\sigma_p^-(n+1)\sigma_n^+(n+1) + 2\sigma_p^-(n)\sigma_n^+(n)\sigma_p^+(n+1)\sigma_n^-(n+1)]\,|\,L=0\rangle. \tag{3.3b}$$

If $|L=0\rangle$ is restricted to the ⟨up⟩-⟨down⟩ subspace, the last two terms of (3.3b) contribute zero. Equation (3.3b) may then be reexpressed in terms of a new spin operator,

$$\sigma_3'(n) = \tfrac{1}{4}(\sigma_{p_3}(n) + \sigma_{n_3}(n)), \tag{3.4}$$

which has eigenvalue $\sigma_3'(n) = \frac{1}{2}$ $(-\frac{1}{2})$ for ⟨up⟩ (⟨down⟩) configurations at site n and which has the properties of a z-component Pauli spin operator. In terms of σ_3', (3.3b) has the form

$$\Delta E = y\Big\langle L=0\Big|\sum_n [-1 + 4\sigma_3'(n)\sigma_3'(n+1)]\Big| L=0\Big\rangle. \tag{3.5}$$

The problem of minimizing ΔE in the ⟨up⟩-⟨down⟩ subspace corresponds, therefore, to finding the ground state of an Ising spin chain (for this reason, the ⟨up⟩-⟨down⟩ subspace will henceforth be designated as the Ising subspace). The solution is well known; there exist two degenerate vacuums (related by translations of one lattice spacing) with alternating $\sigma' = +1$ and $\sigma' = -1$ [see for example Fig. 2(a)]. The two solutions have exactly the two ⟨up⟩-⟨down⟩ configurations shown above to be gauge invariant, total isospin zero, and total charge zero. The existence of two solutions corresponds to spontaneous breaking of discrete chiral symmetry on the lattice, as was the case for the one-flavor Abelian gauge model.[1] The energy density for the Ising ground state is $E_0/N = -2y$, where N is the number of lattice links.

If $|L=0\rangle$ is in the ⟨in⟩-⟨out⟩ subspace, the last two terms in relation (3.3b) are a nonzero contribution. Equation (3.3b) may then be rewritten in terms of the spin operators

$$\sigma_3''(n) = \tfrac{1}{4}(\sigma_{p_3}(n) - \sigma_{n_3}(n)), \tag{3.6a}$$

$$\sigma_-''^{+}(n) = \sigma_+''(n) = \sigma_p^+(n)\sigma_n^-(n), \tag{3.6b}$$

where these operators have the properties of Pauli spin-$\frac{1}{2}$ matrices and $\sigma_3''(n) = \frac{1}{2}$ $(-\frac{1}{2})$ if the configuration is ⟨in⟩ (⟨out⟩) on site n. In terms of σ_i'',

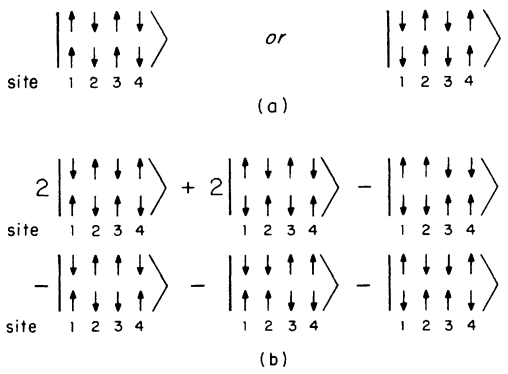

FIG. 2. Vacuum spin configuration for $N=4$ link chain for (a) Ising and (b) Heisenberg subspaces. The latter is a linear combination of $N!/[(-N)!]$ vacuum basis states.

(3.3b) has the form

$$\Delta E = \left\langle L=0 \left| \sum_n [\,-1 + 4\vec{\sigma}''(n+1)\cdot\vec{\sigma}''(n)] \right| L=0 \right\rangle . \quad (3.7)$$

The problem of minimizing ΔE in the $\langle \text{in}\rangle$-$\langle \text{out}\rangle$ subspace corresponds, therefore, to finding the ground state of a Heisenberg antiferromagnetic spin chain (the $\langle \text{in}\rangle$-$\langle \text{out}\rangle$ subspace will therefore be designated as the Heisenberg subspace). The determination of the vacuum for a Heisenberg antiferromagnet is considerably more complicated than the comparable problem for the Ising chain. Bethe and, later, Hulthén[5-7] first solved the problem and found the vacuum to be a linear combination of all vacuum basis states containing an equal number of $\sigma_3''=+1$ and $\sigma_3''=-1$ sites [see for example, Fig. 2(b)]. The vacuum is nondegenerate, and so is symmetric under discrete chiral transformations. The energy density is a function of N, the number of lattice sites, and as $N\to\infty$ the limit is $E_0/N=-(4\ln 2)\,y=-2.77y$.[7]

When $m\neq 0$, the mass contribution to the perturbation Hamiltonian, V, is of the form

$$M=\left[\frac{(m/g)}{y^{3/4}}\right] y\sum_n (-1)^n \tfrac{1}{2}\left(\sigma_{p_3}(n)+\sigma_{n_3}(n)\right), \quad (3.8)$$

where the factor of y has been factored so that, along the lines of $\alpha=[(m/g)/y^{3/4}]=\text{constant}$, the lowest-order contribution is of the same order as the kinetic energy contribution. Since (3.8) is of the form $2\alpha\sum_n(-1)^n\sigma_3'(n)$, M contributes $E_0/N=\pm\alpha$ for the Ising vacuums (depending on whether $\langle$up$\rangle$ or $\langle$down$\rangle$ configurations are on even-numbered sites) and contributes $E_0/N=0$ for the Heisenberg vacuum.

The sum of the mass and kinetic energy contributions to order y is

$$E_0/N=(\pm\alpha-2)y \quad (3.9a)$$

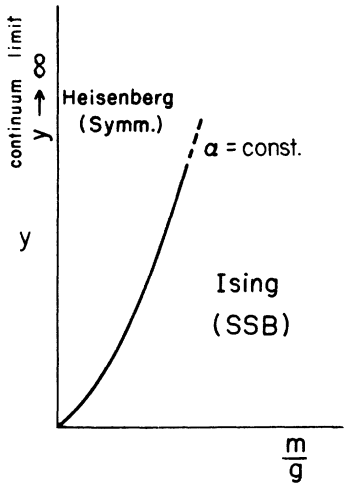

FIG. 3. Phase transition as a function of $y=1/g^4a^4$ and m/g is presented. $\alpha=(m/g)/(y^{3/4})=0.77$ defines the transition line (to first order in y) between the Ising and Heisenberg subspaces.

for the Ising vacuums, and

$$E_0/N=(-2.77)y \quad (3.9b)$$

for the Heisenberg vacuum. The real vacuum is the state with minimum $E_{0_{\text{total}}}$ which is determined by the magnitude of α. Specifically, for $\alpha>0.77$, or large m/g, the real vacuum is the spontaneous symmetry-breaking Ising vacuum, while for $\alpha<0.77$, or small m/g, the real vacuum is the Heisenberg vacuum. Figure 3 is a graphical representation of the transition. These results have been confirmed for spin chains of 4, 6, and 8 links by explicitly computing $E_{0_{\text{total}}}$ for all possible spin configurations as a function of α.

Results of the strong-coupling expansions in each of the vacuum regions are presented in Secs IV and V.

IV. WEAK-COUPLING ($m\gg g$) LIMIT

For the Ising vacuum, each lattice site has a configuration with p- and n-type spins parallel [Figs. 1(c) and 1(d)]. Since a fermion-antifermion pair cannot be created on a single site of this kind, a gauge-invariant excited state above the vacuum must contain at least one flux link. The simplest gauge-invariant excited state is formed by creating a fermion on one site, an antifermion on a neighboring site, and a flux link connecting the two.

More precisely, the low-lying states (at zero momentum) are generated by invariant operators with definite isospin and parity. For example, the lowest-lying bound-state multiplet has isospin 1 and odd total parity (even spatial parity), so the most local lattice construction on the Ising vacuum is

$$|I_3=1, I=1, PG=-+\rangle = -i\sum_n (\sigma_p^+(n)\sigma_{n_3}(n)\sigma_{p_3}(n+1)e^{i\theta(n)}\sigma_n^-(n+1)+\sigma_p^+(n+1)\sigma_n^-(n)e^{-i\theta(n)})\,|\text{Ising vac}\rangle . \quad (4.1)$$

The other low-lying bound-state multiplets may be guessed similarly; they are listed in Table I.

With the bound states determined, the calculation proceeds exactly as in Ref. 1 with three exceptions. Firstly, the σ_{α_3}'s that appear throughout the formalism introduce extra negative signs into the calculation. Secondly, several new types of diagrams appear in the expansion. Thirdly, although the theory is manifestly SU(2) isospin symmetric, the diagrams that contribute to the $I=1$ elements of a given multiplet are significantly different from those that contribute to the $I=0$ element.

The last two points are illustrated by Figs. 4 and 5, where the diagrammatic conventions used are the same as those in Ref. 1. Because p- and n-type spins exist on the same lattice site, diagrams such as 4(b), 4(d), 4(e), 4(f), 5(c), and 5(h) are possible for the SU(2) problem. Note particularly that in diagrams such as 5(c) and 5(h) the intermediate state has a gauge field link $L=2$ so that the H_0 eigenvalue is $4+4\mu$. Comparing diagrams in Figs. 4 and 5, one observes that both the disconnected bubble diagrams and the interaction diagrams differ significantly for members of the same isomultiplet. Nevertheless, it would be a disaster if the results were not isospin symmetric; this, therefore, offers an important consistency check on the calculated results.

Otherwise, to complete the calculations as in Ref. 1, the energy of the vacuum and four lowest-lying bound states can be computed and the expansion for the masses can be found by subtracting the vacuum contribution and multiplying by $g/2y^{1/4}$.

The results of calculations of the mass of the four lowest-lying bound states for $m \gg g$ are shown in Table II. In column I, the ratio $2y^{1/4}m_{\text{bound}}/g$ is shown to second order in the expansion parameter $y=1/g^4a^4$ (where the mass term has been added to the unperturbed Hamiltonian since $m \gg g$). Although not listed separately in the table, expansions for the $I_3=1$ and $I_3=0$ members of $I=1$

TABLE I. Spin creation operators for the four lowest-lying isospin multiplets in the weak-coupling (large m/g) limit.

State	State creation operators
$I^{PG}=1^{-+}$	
$I_3=1$	$-i\sum_n [\sigma_p^+(n)\sigma_{n_3}(n)\sigma_{p_3}(n+1)e^{i\theta(n)}\sigma_n^-(n+1)+\sigma_p^+(n+1)\sigma_n^-(n)e^{-i\theta(n)}]$
$I_3=0$	$\frac{1}{\sqrt{2}}\sum_n [(\sigma_p^+(n)\sigma_{n_3}(n)\sigma_p^-(n+1)e^{i\theta(n)}+\sigma_p^+(n+1)\sigma_{n_3}(n)\sigma_p^-(n)e^{-i\theta(n)})$ $-(\sigma_n^+(n)\sigma_{p_3}(n+1)\sigma_n^-(n+1)e^{i\theta(n)}+\sigma_n^+(n+1)\sigma_{p_3}(n+1)\sigma_n^-(n)e^{-i\theta(n)})]$
$I_3=-1$	$-i\sum_n [\sigma_n^+(n)\sigma_p^-(n+1)e^{i\theta(n)}+\sigma_n^+(n+1)\sigma_{n_3}(n)\sigma_{p_3}(n+1)e^{-i\theta(n)}\sigma_p^-(n)]$
$I^{PG}=1^{+-}$	
$I_3=1$	$-i\sum [\sigma_p^+(n)\sigma_{n_3}(n)\sigma_{p_3}(n+1)e^{i\theta(n)}\sigma_n^-(n+1)-\sigma_p^+(n+1)\sigma_n^-(n)e^{-i\theta(n)}]$
$I_3=0$	$\frac{1}{\sqrt{2}}\sum_n [(\sigma_p^+(n)\sigma_{n_3}(n)\sigma_p^-(n+1)e^{i\theta(n)}-\sigma_p^+(n+1)\sigma_{n_3}(n)\sigma_p^-(n)e^{-i\theta(n)})$ $-(\sigma_n^+(n)\sigma_{p_3}(n+1)\sigma_n^-(n+1)e^{i\theta(n)}-\sigma_n^+(n+1)\sigma_{p_3}(n+1)\sigma_n^-(n)e^{-i\theta(n)})]$
$I_3=-1$	$-i\sum_n [\sigma_n^+(n)\sigma_p^-(n+1)e^{i\theta(n)}-\sigma_n^+(n+1)\sigma_{n_3}(n)\sigma_{p_3}(n+1)e^{-i\theta(n)}\sigma_p^-(n)]$
$I^{PG}=0^{--}$	$\frac{1}{\sqrt{2}}\sum [(\sigma_p^+(n)\sigma_{n_3}(n)\sigma_p^-(n+1)e^{i\theta(n)}+\sigma_p^+(n+1)\sigma_{n_3}(n)\sigma_p^-(n)e^{-i\theta(n)})$ $+(\sigma_n^+(n)\sigma_{p_3}(n+1)\sigma_n^-(n+1)e^{i\theta(n)}+\sigma_n^+(n+1)\sigma_{p_3}(n+1)\sigma_n^-(n)e^{-i\theta(n)})]$
$I^{PG}=0^{++}$	$\frac{1}{\sqrt{2}}\sum_n [(\sigma_p^+(n)\sigma_{n_3}(n)\sigma_p^-(n+1)e^{i\theta(n)}-\sigma_p^+(n+1)\sigma_{n_3}(n)\sigma_p^-(n)e^{-i\theta(n)})$ $+(\sigma_n^+(n)\sigma_{p_3}(n+1)\sigma_n^-(n+1)e^{i\theta(n)}-\sigma_n^+(n+1)\sigma_{p_3}(n+1)\sigma_n^-(n)e^{-i\theta(n)})]$

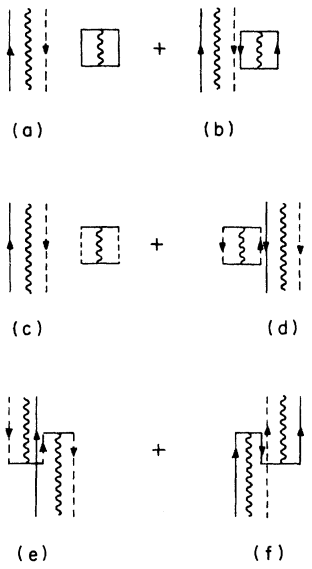

FIG. 4. The diagrams that contribute to order y to the mass spectrum for the $I^{PG}=1^{-+}$, $I_3=1$ state. Notation similar to Ref. 1 is used; solid lines represent p spins, dotted lines represent n spins, and wiggly lines represent gauge links. Arrows distinguish particles (↑) from antiparticles (↓).

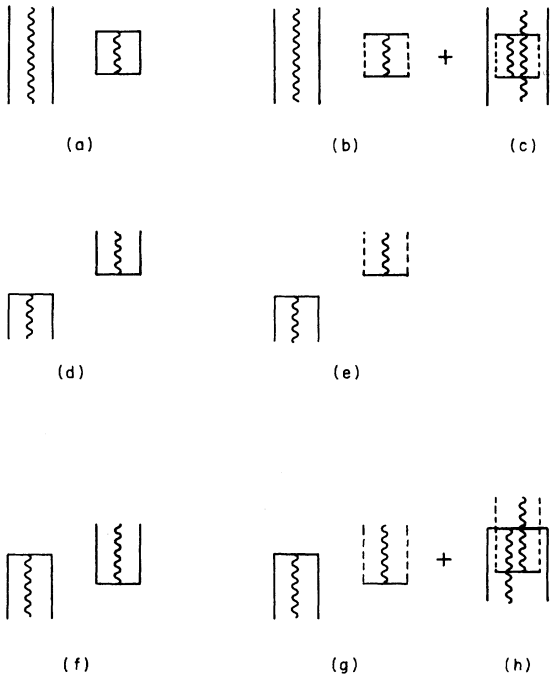

FIG. 5. The diagrams that contribute to order y to the mass spectrum of the $I^{PG}=1^{-+}$, $I_3=0$ state.

multiplets were calculated independently and found to give identical results. In column II, the (1, 1) Padé approximants[8] to these expansions are provided.

The results of the Padé approximants for several values of m/g are graphed in Fig. 6 as a function of y. The conclusion to be drawn from the figure is that, throughout the range of m/g, the mass spectrum for a wide range of y is

$$I^{PG} = \text{vacuum}, 1^{-+}, 0^{--}, 1^{+-}, 0^{++} \cdots .$$

The splitting between the pairs of states with opposite parity is much greater than the splitting between states of the same parity. The only diagrams to lowest order that split two states of the

TABLE II. Results of lattice calculations in the weak-coupling (large m/g) limit for masses of low-lying multiplets. Column I contains expansion to second order and column II shows (1, 1) Padé approximations, where $y=x^2=1/g^4a^4$.

I^{PG}	$2y^{1/4}m_{\text{bound}}/g$ I	II
1^{-+}	$1+2\mu+\left(\frac{2}{1+2\mu}\right)x^2+\left[\frac{-4\mu-16}{(1+2\mu)^3}+\frac{12}{(4+4\mu)(1+2\mu)^2}\right]x^4$	$\left[\frac{(1+2\mu)^2(4+4\mu)+(8\mu^2+36\mu+34)y}{(1+2\mu)^2(4+4\mu)+(8\mu^2+28\mu+26)y}\right](1+2\mu)$
1^{+-}	$1+2\mu+\left(\frac{6}{1+2\mu}\right)x+\left[\frac{-4\mu-32}{(1+2\mu)^3}+\frac{12}{(4+4\mu)(1+2\mu)^2}\right]x^4$	$\left[\frac{3(1+2\mu)^2(4+4\mu)+(8\mu^2+132\mu+130)y}{3(1+2\mu)^2(4+4\mu)+(8\mu^2+60\mu+58)y}\right](1+2\mu)$
0^{--}	$1+2\mu+\left[\frac{-2}{3+2\mu}+\frac{4}{(1+2\mu)}\right]x^2+\left[\frac{-4\mu-32}{(1+2\mu)^3}+\frac{(4+16\mu+16\mu^2)}{(3+2\mu)^2(4+4\mu)(1+2\mu)^2}\right]x^4$	$\left[\frac{(3+2\mu)(4+4\mu)(1+2\mu)^2(4\mu+10)+y(64\mu^4+800\mu^3+2720\mu^2+3528\mu+1548)}{(3+2\mu)(4+4\mu)(1+2\mu)^2(4\mu+10)+y(64\mu^4+736\mu^3+2336\mu^2+2808\mu+1148)}\right]\times(1+2\mu)$
0^{++}	$1+2\mu+\left(\frac{-2}{3+2\mu}+\frac{8}{1+2\mu}\right)x^2+\left[\frac{-4\mu-52}{(1+2\mu)^3}+\frac{(352+496\mu+176\mu^2)}{(3+2\mu)^2(4+4\mu)(1+2\mu)^2}\right]x^4$	$\left[\frac{(3+2\mu)(4+4\mu)(1+2\mu)^2(22+12\mu)+y(64\mu^4+1312\mu^3+5184\mu^2+7400\mu+3476)}{(3+2\mu)(4+4\mu)(1+2\mu)^2(22+12\mu)+y(64\mu^4+736\mu^3+2496\mu^2+3352\mu+1540)}\right]\times(1+2\mu)$

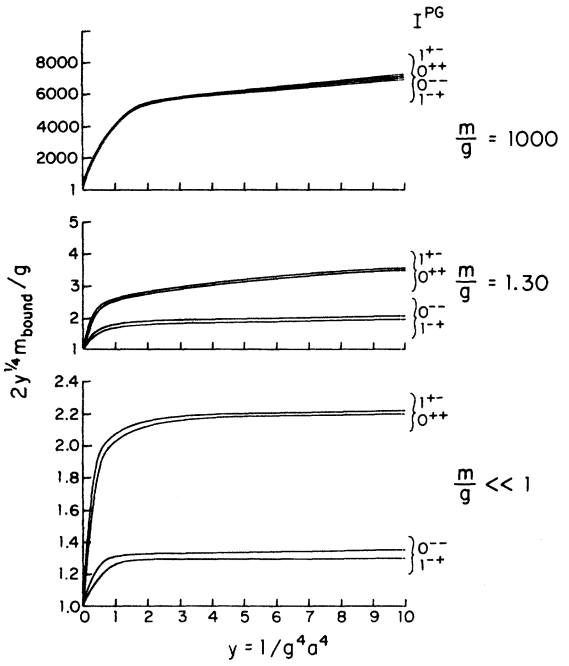

FIG. 6. Plots of $2y^{1/4}m_{\text{bound}}/g$ as a function of y for various values of m/g. The weak-coupling limit corresponds to the highest set of curves. Note changes in scale of the ordinate. Basic structure and splittings of the mass spectrum are the same for all values of m/g but are difficult to determine graphically (although not numerically) for large m/g.

same parity are of the form of Fig. 5(d) or 5(e). These diagrams are the lattice analogy of the Feynman annihilation diagrams of the continuum theory. Furthermore, the annihilation diagrams on the lattice lead to a smaller splitting in the Padé approximants for the total parity $P = +$ pair than for the total parity $P = -$ pair. These results are in complete accord with all of Coleman's results in Ref. 3 for the weak-coupling limit of the continuum theory.

V. STRONG-COUPLING ($m \ll g$) LIMIT

In the Heisenberg vacuum, each lattice site has a configuration with antiparallel p- and n-type spins [see Figs. 1(a) and (b)]. Gauge-invariant configurations of fermion-antifermion pairs which contain no flux links ($H_0 = 0$) can be made on single lattice sites of this type. Gauge-invariant states consisting of a fermion-antifermion pair on neighboring sites connected by a flux link are also possible but such states have $H_0 = 1$ rather than $H_0 = 0$.

The calculation of the bound-state spectrum for the strong-coupling limit is considerably more difficult than that for the weak-coupling limit owing to the complexity of the Heisenberg vacuum. To make matters worse, to calculate the bound-state spectrum in the strong-coupling limit to order y, one must understand the excitations of the spin chain as well as the vacuum. Fortunately, pioneering work on the lowest-lying excited states of the Heisenberg antiferromagnetic chain has already been done by des Cloizeaux and Pearson.[9] Using an extension of the methods of Hulthén and Bethe, they found the lowest-lying spin wave states to to be a spin-1 triplet corresponding to an isotriplet multiplet for the SU(2)-flavor Schwinger model. The $I_3 = 1$ member of the multiplet, for example, corresponds to a p-$\bar{n}$ pair on the same site created by flipping $\sigma' = -1$ sites to $\sigma' = +1$. The triplet's energy (above the vacuum level) was found to be proportional to $(1/a)|\sin pa|$, where p is the total momentum of the states. As $a \to 0$, the energy to lowest order is proportional to p and the lattice states are massless. Since the states correspond to the fermion and antifermion on the same site, the state can only have even spatial parity or odd total parity. Thus, as Coleman finds in Ref. 2, the lattice theory in the strong-coupling limit ($m \sim 0$) has only one (nearly) massless isotriplet state with $I^{PG} = 1^{-+}$.

Coleman also finds a low-lying state with $I^{PG} = 0^{++}$ in his strong-coupling analysis. In Bethe's analysis of the Heisenberg spin chain, he proposes that in addition to a spin-1 triplet there exists bound states of the spin-1 excitations.[5] Recent calculations by Endo and Ishikawa[10] imply that only one bound state with spin 0 exists below the two spin-1 wave spectrum. Such a state, which is a parity-even bound state of two $I^{PG} = 1^{-+}$ waves must be $I^{PG} = 0^{++}$. All other bound states created from the Heisenberg vacuum have at least one flux link and much greater mass.

VI. DISCUSSION

These calculations demonstrate that only small technical alterations from the methods used for the U(1) Schwinger model are necessary to carry out high-order strong-coupling calculations for the (1+1)-dimensional SU(2)-flavor Abelian gauge model. However, examination of the vacuum of the lattice theory reveals the existence of two vacuum phases characterized by the magnitude of m/g. Expansion about each vacuum results in excellent qualitative agreement with conclusions for the continuum theory in both the weak- and strong-coupling limits. Quantitative agreement also exists in the weak-coupling limit, but higher-

order perturbation calculations will be necessary in the strong-coupling limit to quantitatively compare lattice results to continuum theory. Whether or not present agreement implies that the continuum theory has a phase transition from a spontaneous chiral-symmetry-breaking sector to a chiral-symmetry-preserving sector is not clear. These and related questions are currently being studied.

ACKNOWLEDGMENTS

I would like to thank L. Susskind for many useful conversations and his kind hospitality on my visit to Yeshiva University. I am especially indebted to S. Coleman for his advice and guidance throughout the course of this work. I have also had useful conversations with P. C. Martin, B. I. Halperin, A. Carroll, and M. Peskin.

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*Research supported in part by the National Science Foundation under Grant No. PHY75-20427.

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