Fate of the false vacuum. II. First quantum corrections*

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It is possible for a classical field theory to have two homogeneous stable equilibrium states with different energy densities. In the quantum version of the theory, the state of higher energy density becomes unstable through barrier penetration. In the first paper in this series, it was argued that the relevant quantity to study was a decay probability per unit time per unit volume, $\Gamma/V = Ae^{-B/\hbar}[1 + O(\hbar)]$, and the theory of the coefficient B was given. This paper gives the theory of the coefficient A.

I. INTRODUCTION

Consider a particle moving in a potential of the form shown in Fig. 1. The classical version of this system possesses a stable equilibrium state in which the particle is at rest at $x=0$; however, quantum corrections render this state unstable; it is a false ground state.

A similar situation can arise in field theory. Consider the theory of a single scalar field with nonderivative self-interactions,

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - U(\phi)\,, \tag{1.1}$$

where U is as shown in Fig. 2. The classical theory possesses two spatially homogeneous stable equilibrium states, $\phi = \phi_+$ and $\phi = \phi_-$; however, quantum corrections render the first of these ununstable; it is a false vacuum.

This is the second of a sequence of papers dealing with the decay of such false vacuums. In the first paper,[1] it was argued that the relevant quantity to compute was a decay probability per unit time per unit volume, Γ/V, and that, in the small-$\hbar$ limit, this quantity was given by an expression of the form

$$\Gamma/V = Ae^{-B/\hbar}[1 + O(\hbar)]\,, \tag{1.2}$$

where A and B are coefficients which depend on the detailed form of U. Reference 1 dealt exclusively with the theory of the coefficient B; this paper deals with the theory of the coefficient A.

Our method of analysis is based on functional integration.[2] In Sec. II, we show how this method can be used to analyze the false ground state of a particle in a potential; in Sec. III, we extend the analysis to field theory. Much of our discussion applies to a general renormalizable field theory; however, at the end we focus on the special case of small energy-density difference between the true and false vacuum. Even in this special case, we are not able to obtain a closed-form expression for A; we are stymied by an obdurate functional determinant. However, we are able to analyze the ultraviolet divergences of A, and to show that they are removed by the usual renormalizations of perturbation theory.

II. A SIMPLE CASE

In this section we restrict ourselves to a particle of unit mass moving in one spatial dimension under the influence of a potential $V(x)$. Our fundamental tool will be the Euclidean (imaginary time) version of Feynman's[3] sum over histories:

$$\langle x_f | e^{-HT/\hbar} | x_i \rangle = N \int [dx] e^{-S/\hbar} \tag{2.1}$$

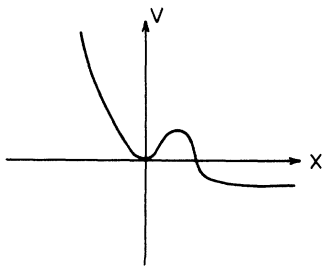

FIG. 1. Potential energy as a function of position for a particle problem with a false (unstable) ground state.

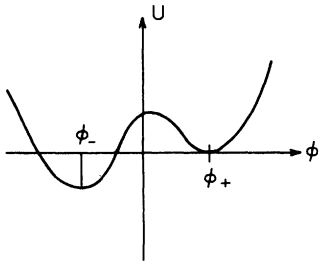

FIG. 2. Potential energy density as a function of field strength for a scalar field theory with a false (unstable) vacuum.

Both sides of this equation require explanation:

On the left-hand side, $|x_i\rangle$ and $|x_f\rangle$ are position eigenstates, H is the Hamiltonian, and T is a positive number. The left-hand side of Eq. (2.1) is of interest because, if we expand in a complete set of energy eigenstates,

$$H|n\rangle = E_n|n\rangle , \tag{2.2}$$

then

$$\langle x_f|e^{-HT/\hbar}|x_i\rangle = \sum_n e^{-E_nT/\hbar}\langle x_f|n\rangle\langle n|x_i\rangle . \tag{2.3}$$

Thus, the leading term in this expression for large T tells us the energy and wave function of the lowest-lying energy eigenstate.

On the right-hand side, N is a normalization factor, S is the Euclidean action[4]

$$S = \int_{-T/2}^{T/2} dt\left[\tfrac{1}{2}\left(\frac{dx}{dt}\right)^2 + V\right] , \tag{2.4}$$

and $[dx]$ denotes integration over all functions $x(t)$, obeying the boundary conditions, $x(-T/2) = x_i$ and $x(T/2) = x_f$. To be more specific, if $\bar{x}$ is any function obeying the boundary conditions, then a general function obeying the boundary conditions can be written as

$$x(t) = \bar{x}(t) + \sum_n c_n x_n(t) , \tag{2.5}$$

where the x_n's are a complete set of real orthonormal functions vanishing at the boundaries,

$$\int_{-T/2}^{T/2} dt\, x_n(t)x_m(t) = \delta_{nm} , \tag{2.6a}$$

$$x_n(\pm T/2) = 0 . \tag{2.6b}$$

Then the measure $[dx]$ is defined by

$$[dx] = \prod_n (2\pi\hbar)^{-1/2} dc_n . \tag{2.7}$$

(This measure differs in normalization from the measure defined by Feynman;[3] this is why we need the normalization constant N. However, as we shall see, we shall never need an explicit formula for N.)

The right-hand side of Eq. (2.1) is of interest because it can readily be evaluated in the semiclassical (small-$\hbar$) limit. In this case the functional integral is dominated by the stationary points of S. For simplicity, let us assume for the moment that there is only one such stationary point, which we denote by $\bar{x}$,

$$\frac{\delta S}{\delta \bar{x}} = -\frac{d^2\bar{x}}{dt^2} + V'(\bar{x}) = 0 , \tag{2.8}$$

where the prime denotes differentiation with respect to x. Further, let us choose the x_n's to be eigenfunctions of the second variational derivative of S at $\bar{x}$,

$$\frac{d^2x_n}{dt^2} + V''(\bar{x})x_n = \lambda_n x_n . \tag{2.9}$$

Then, in the small-$\hbar$ limit, the integral becomes a product of Gaussians, and we find

$$\begin{aligned}\langle x_f|e^{-HT/\hbar}|x_i\rangle &= Ne^{-S(\bar{x})/\hbar}\prod_n \lambda_n^{-1/2}[1+O(\hbar)] \\ &= Ne^{-S(\bar{x})/\hbar}\{\det[-\partial_t^2 + V''(\bar{x})]\}^{-1/2} \\ &\quad\times[1+O(\hbar)] .\end{aligned} \tag{2.10}$$

(Of course, we are tacitly assuming here that all the eigenvalues are positive. We shall shortly see what to do when this is not the case.) If there are several stationary points, in general one has to sum over all of them.

Equation (2.8) is the equation of motion for a particle of unit mass moving in a potential *minus* V. Thus,

$$E = \tfrac{1}{2}\left(\frac{d\bar{x}}{dt}\right)^2 - V(\bar{x}) \tag{2.11}$$

is a constant of the motion. This can be used to determine the qualitative features of the solutions of Eq. (2.8) by inspection.

As a simple example, let us consider the potential shown in Fig. 3(a), and let us choose $x_i = x_f = 0$. It is obvious from Fig. 3(b) that the only solution obeying the boundary conditions is $\bar{x} = 0$. For this solution, $S = 0$. If we define ω^2 to be $V''(0)$, then a standard computation[3] shows that for large T

$$N[\det(-\partial_t^2 + \omega^2)]^{-1/2} = \left(\frac{\omega}{\pi\hbar}\right)^{1/2} e^{-\omega T/2} . \tag{2.12}$$

Of course, this gives the correct semiclassical shift in the ground-state energy, $\hbar\omega/2$.

Let us now turn to the potential of Fig. 1. We still choose $x_i = x_f = 0$. We can see from Fig. 4 that there are now nontrivial solutions of Eq. (2.8): The particle can begin at the top of the hill, bounce off the potential wall on the right, and return to the top of the hill. Since we will ultimately be inter-

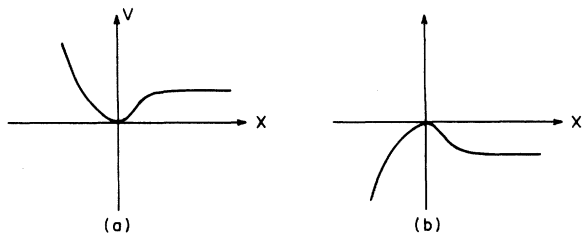

FIG. 3. (a) Potential energy as a function of position for a particle problem with a true ground state. (b) The same thing, upside down.

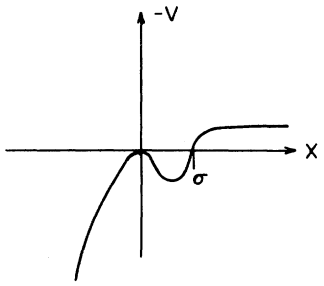

FIG. 4. The graph in Fig. 1 turned upside down.

ested only in the form of the functional integral for very large T, we will focus our attention on the limiting form of such motions for infinite T. We call this limiting form "the bounce."

The bounce has $E=0$. Thus, if we denote S for the bounce by B, then

$$B=\int_{-\infty}^{\infty} dt(d\bar{x}/dt)^2 \tag{2.13a}$$

$$=\int_0^{\sigma} dx[2V(x)]^{1/2}, \tag{2.13b}$$

where σ is the second zero of V. (See Fig. 4.)

We define "the center of the bounce" as the point where $dx/dt=0$. By time translation invariance, the center of the bounce can be anywhere along the t axis. For very large T, a bounce centered anywhere in the interval of integration is an approximate stationary point of the functional integrand. So also are n widely separated bounces, with centers at $t_1, \cdots, t_n$, where $T/2>t_1>t_2>\cdots>t_n>-T/2$. We propose to evaluate the functional integral by summing over all these configurations.

First we will assemble the factors that go into this expression; then we will do the summation.

(1) For n bounces, S is nB. This takes care of the exponential of the action.

(2) Now for the determinant: The bounces are separated by vast regions in which $x=0$. Thus we can evaluate the determinant as a product of contributions from large time intervals surrounding each bounce and even larger time intervals containing the vacant regions between the bounces. In this way we obtain

$$\left(\frac{\omega}{\pi\hbar}\right)^{1/2} e^{-\omega T/2} K^n, \tag{2.14}$$

where K is defined by demanding that this expression give the right answer for one bounce. We will evaluate K shortly.

(3) Finally, we must integrate over the locations of the centers:

$$\int_{-T/2}^{T/2} dt \int_{-T/2}^{t_1} dt_2 \cdots \int_{-T/2}^{t_{n-1}} dt_n = T^n/n! \,. \tag{2.15}$$

We can now do the summation:

$$\sum_{n=0}^{\infty}\left(\frac{\omega}{\pi\hbar}\right)^{1/2} e^{-\omega T/2}\frac{(Ke^{-B/\hbar}T)^n}{n!}$$

$$=\left(\frac{\omega}{\pi\hbar}\right)^{1/2}\exp(-\omega T/2+Ke^{-B/\hbar}T)\,. \tag{2.16}$$

Thus, the inclusion of multibounce configurations in the functional integral has modified our estimate of the ground-state energy:

$$E_0=(\hbar\omega/2-\hbar Ke^{-B/\hbar})[1+O(\hbar)]\,. \tag{2.17}$$

(The added term is very small in magnitude for small $\hbar$; it would be completely uninteresting were it not that K has an imaginary part, as we shall shortly see.)

Now let us turn to the evaluation of K. We must study Eq. (2.9) with $\bar{x}$ a single bounce. Because of time translation invariance, this equation necessarily possesses an eigenfunction of eigenvalue zero,

$$x_1=B^{-1/2}d\bar{x}/dt\,. \tag{2.18}$$

[The normalization factor comes from Eq. (2.13a).] Were we to integrate over the corresponding expansion coefficient, c_1, this would lead to a disaster in the determinant. Fortunately, we have already done this integration, in the guise of evaluating the integrals over time translations of the bounce in Eq. (2.15):

$$dx=(dx/dt)dt=x_1dc_1\,. \tag{2.19}$$

Thus

$$(2\pi\hbar)^{-1/2}dc_1=(B/2\pi\hbar)^{1/2}dt\,. \tag{2.20}$$

To summarize[5]: In evaluating the determinant, we should not include the zero eigenvalue, but we should include in K a factor of $(B/2\pi\hbar)^{1/2}$.

We now come to a sticky point: $d\bar{x}/dt$ has a zero; thus x_1 has a node and is not the eigenfunction of lowest eigenvalue. The nodeless eigenfunction, x_0, must have a lower eigenvalue, that is to say, a negative eigenvalue. Thus the bounce is not a minimum of the action but a saddle point, and the Gaussian integral over the expansion coefficient c_0 diverges.

We are in trouble. We should be, for we have been foolish: We have tried to compute an eigenvalue that is not in the spectrum of the Hamiltonian, the energy of an unstable state. This is a quantity that can be defined only by analytic continuation. We shall now perform such a continuation and save our computation.

To keep things as simple as possible, let us consider not an integral over all function space, but an integral over some path in function space parametrized by a real variable, z,

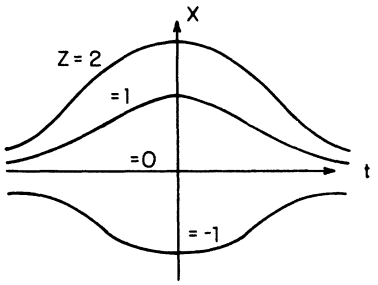

FIG. 5. Some functions along a path in function space parametrized by a real variable z.

$$J = \int dz (2\pi\hbar)^{-1/2} e^{-S(z)/\hbar}, \tag{2.21}$$

where $S(z)$ is the action along the path. The functions along this path are sketched in Fig. 5. We have chosen the path to include the two important functions that occur in the real problem: $\bar{x} = 0$, at $z = 0$, and the bounce, at $z = 1$. Furthermore, we have chosen our path such that the tangent vector to the path at $z = 1$ is x_0. Thus the bounce is a maximum of $S(z)$, as is shown in Fig. 6(a). (S goes to minus infinity as z goes to infinity because the functions spend more and more time in the region beyond the turning point, where V is negative.)

If $x = 0$ were the absolute minimum of V, that is to say, if V were as in Fig. 3(a), we would have, for the same path, the situation shown in Fig. 6(b). In this case, we would have no divergence in the integral (2.21). Now let us suppose we analytically change V in some way such that we go from this situation back to the one of interest. To keep the integral convergent, we must distort the right-hand portion of the contour of integration into the complex plane. How we distort it depends on the details of the analytic passage from one potential to the other. In Fig. 7, we have assumed that it is distorted into the upper half plane. Following the standard procedure of the method of steepest descents, we have led the contour along the real axis to $z = 1$ (the saddle point) and then out along

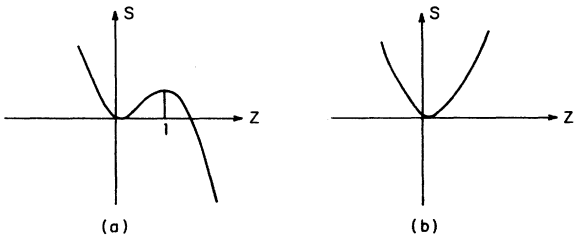

FIG. 6. (a) Euclidean action as a function of the path parameter for the path shown in Fig. 5 and the potential shown in Fig. 1. (b) The same for the potential shown in Fig. 3.

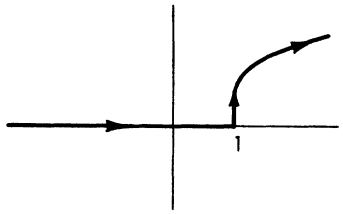

FIG. 7. Distortion of the contour of integration in the complex z plane needed as one passes analytically from the situation shown in Fig. 6(b) to that shown in Fig. 6(a).

a line of constant imaginary part of S. The integral thus acquires an imaginary part; in the steepest-descent approximation,

$$\begin{aligned} \mathrm{Im}J &= \mathrm{Im}\int_1^{1+i\infty} dz (2\pi\hbar)^{-1/2} e^{-S(1)/\hbar} \\ &\quad \times e^{-(1/2)S''(z-1)^2/\hbar} \\ &= \tfrac{1}{2} e^{-S(1)/\hbar} |S''(1)|^{-1/2}. \end{aligned} \tag{2.22}$$

Note the factor of $\frac{1}{2}$; this arises because we are integrating over only half of the Gaussian peak.

(If we had passed from one potential to the other in the conjugate manner, the contour would have been distorted into the lower half plane, and we would have obtained the opposite sign for the imaginary part of the integral. This is just a reflection of the well-known fact that what sign you get for the imaginary part of the energy of an unstable state depends on how you do your analytic continuation.)

Boldly extending this analysis from a one-dimensional integral to an integral over a function space, we find that the one-bounce contribution to the functional integral is given by

$$\begin{aligned} &\mathrm{Im}\left(N\int [dx] e^{-S/\hbar}\right)_{\text{one bounce}} \\ &\quad = \tfrac{1}{2} N e^{-B/\hbar} (B/2\pi\hbar)^{1/2} T \left|\det'[-\partial_t^2 + V''(\bar{x})]\right|^{-1/2}, \end{aligned} \tag{2.23}$$

where det′ indicates that the zero eigenvalue is to be omitted when computing the determinant.

Comparing this to the definition of K, we find

$$\mathrm{Im}K = \tfrac{1}{2}(B/2\pi\hbar)^{1/2} \left|\frac{\det'[-\partial_t^2 + V''(\bar{x})]}{\det(-\partial_t^2 + \omega^2)}\right|^{-1/2}. \tag{2.24}$$

Hence, the decay probability per unit time of the unstable state is given by

$$\begin{aligned} \Gamma &= -2\,\mathrm{Im}E_0/\hbar \\ &= (B/2\pi\hbar)^{1/2} e^{-B/\hbar} \left|\frac{\det'[-\partial_t^2 + V''(\bar{x})]}{\det(-\partial_t^2 + \omega^2)}\right|^{-1/2} \\ &\quad \times [1 + O(\hbar)]. \end{aligned} \tag{2.25}$$

For one-dimensional quantum mechanics, it is possible to evaluate functional determinants such as those occurring above in closed form. We have done this, to check that Eq. (2.25) agrees with the prediction obtained (so very much more easily) by the standard methods of wave mechanics. However, for our purposes here, we prefer to stop with Eq. (2.24), because this is the expression that generalizes immediately to quantum field theory.

III. FALSE VACUUMS

A. Preliminary remarks

We wish to consider the theory of a single scalar field, ϕ, in four-dimensional space-time. The Euclidean action is

$$S=\int d^4x[\tfrac{1}{2}\partial_\mu\phi\partial_\mu\phi+U(\phi)]. \tag{3.1}$$

Here μ ranges from 1 to 4, the summation over repeated indices is implied, and U is some function of ϕ. (For renormalizable theories, U is a quartic polynomial.) Furthermore, we assume that U has two relative minima, ϕ_+ and ϕ_-, of which only the second is an absolute minimum, and we add a constant to U such that $U(\phi_+)=0$. (See Fig. 2.) $\phi=\phi_+$ is a false vacuum, analogous to the false ground state, $x=0$, of Sec. II.

For a theory of this kind, the bounce, $\bar{\phi}(x)$, was found in the first paper in this series. $\bar{\phi}$ is a spherically symmetric [O(4)-invariant] function of x, going monotonically to ϕ_+ as x goes to infinity. $\bar{\phi}$ is a stationary point of the action and is unique, except, of course, that its center can be anywhere in Euclidean four-space.

Thus, the situation is much like that of Sec. II, with three exceptions: (1) In particle mechanics, we had only one infinitesimal translation, and thus one zero eigenvalue, to worry about; here we have four. (2) Whenever we study a relativistic field theory, we must deal with ultraviolet divergences. (3) It was critical in the analysis of Sec. II that the second variational derivative of the action at the bounce had one and only one negative eigenvalue. Is the same true here? We will deal with these three problems in the order in which we have stated them.

B. Zero eigenvalues

In the one-dimensional problem of Sec. II, the second variational derivative of the action at the bounce had one eigenfunction of eigenvalue zero, x_1, proportional to $d\bar{x}/dt$. Now, in four dimensions, there are four such eigenfunctions, ϕ_μ, proportional to $\partial_\mu\bar{\phi}$. We must determine the constant of proportionality, that is to say, the normalization of the eigenfunctions.

By the spherical symmetry of the bounce

$$\int d^4x\,\partial_\mu\bar{\phi}\partial_\nu\bar{\phi}=\tfrac{1}{4}\delta_{\mu\nu}\int d^4x\,\partial_\lambda\bar{\phi}\partial_\lambda\bar{\phi}. \tag{3.2}$$

The bounce is a stationary point of S under general variations. Thus, in particular, S is stationary under infinitesimal scale transformations,

$$\delta\phi=x_\lambda\partial_\lambda\bar{\phi}. \tag{3.3}$$

Hence,

$$0=\delta S=-\int d^4x[\partial_\lambda\bar{\phi}\partial_\lambda\bar{\phi}+4U(\bar{\phi})]. \tag{3.4}$$

Thus, if we denote $S(\bar{\phi})$ by B,

$$B=\tfrac{1}{4}\int d^4x\,\partial_\lambda\bar{\phi}\partial_\lambda\bar{\phi}. \tag{3.5}$$

Thus, for each zero eigenvalue we have the same normalization as in the one-dimensional case [Eq. (2.13a)].

Let us assume for the moment that no qualitatively new problems are introduced by renormalization or by negative eigenvalues. If this is so, then we can obtain the formula for the decay width of the false vacuum by a straightforward transcription of the analysis of Sec. II. The only difference is that we have four factors of $(B/2\pi\hbar)^{1/2}$ instead of one, and that, in integrating over the center of the bounce, we pick up a factor of VT, where V is the volume of three-space, instead of just a factor of T. Thus,

$$\Gamma/V=\frac{B^2}{4\pi^2\hbar^2}e^{-B/\hbar}\left|\frac{\det'[-\partial^2+U''(\phi)]}{\det[-\partial^2+U''(\phi_+)]}\right|^{-1/2}\times[1+O(\hbar)]. \tag{3.6}$$

where, as before, det′ denotes the determinant computed with the zero eigenvalues omitted, and $\partial^2=\partial_\mu\partial_\mu$.

As a mild consistency check, let us verify that this expression has the right dimensions. Both B and $\hbar$ have the dimensions of action. The differential operators have the dimensions of 1/length^2, as do their eigenvalues. Since four eigenvalues are omitted from det′, the ratio det′/det has the dimensions of (length)^8. Thus the total expression has the dimensions of 1/length^4, just right for a decay probability per unit time per unit volume.

C. Renormalization

Until now we have been working implicitly with the action expressed in terms of unrenormalized fields, masses and coupling constants. We must now recast all of our formulas in terms of the renormalized versions of these quantities. We begin

with S itself:

$$S = S_R + \sum_{n+1}^{\infty} \hbar^n S^{(n)} . \tag{3.7}$$

Here S_R is the renormalized action, a functional of exactly the same form as S, but with all unrenormalized quantities replaced by their renormalized counterparts. $S^{(n)}$ is the action induced by the standard renormalization counterterms computed from the sum of all n-loop graphs. In order to avoid excessive clutter in our equations, we will redefine ϕ to be the renormalized field, U to be the polynomial that occurs in S_R, $\bar{\phi}$ to be the bounce as computed from S_R, and B to be $S_R(\bar{\phi})$.

The renormalization counterterms serve to remove all ultraviolet divergences from all one-particle-irreducible Green's functions. Equivalently, they serve to remove all ultraviolet divergences from the effective action, $\gamma(\phi)$, the generating functional of these Green's functions. To one-loop order, the effective action is given by[6]

$$e^{\gamma(\phi)} = \exp[S_R(\phi)/\hbar + S^{(1)}(\phi)] \times (\det[-\partial^2 + U''(\phi)])^{1/2} . \tag{3.8}$$

It will be important to us shortly that the right-hand side of this equation is free of ultraviolet divergences for arbitrary ϕ.

Now let us imagine computing Γ/V iteratively, first treating S_R as if it were the total action, and then taking account of the effect of the renormalization counterterms perturbatively. To the order to which we are working, the only counterterm we need consider is $S^{(1)}$.

The first thing we must realize is that the renormalization counterterms may destroy our convention that $S(\phi_+)$ vanishes. We can take care of this problem trivially by replacing $S(\phi)$ in Eq. (3.6) by the difference $S(\phi) - S(\phi_+)$.

Secondly, adding new terms to S_R will change the stationary points of S. In particular, it will change the bounce. Let us write

$$\bar{\phi} \to \bar{\phi} + \hbar\phi^{(1)} + O(\hbar^2) . \tag{3.9}$$

Then

$$S(\bar{\phi}) \to S_R(\bar{\phi}) + \hbar \int d^4x \frac{\delta S_R}{\delta \bar{\phi}} \phi^{(1)} + \hbar S^{(1)}(\phi) + O(\hbar^2)$$

$$= B + \hbar S^{(1)}(\bar{\phi}) + O(\hbar^2) \tag{3.10}$$

because the bounce is a stationary point of S_R. Similar reasoning applies to ϕ_+.

Thus, we arrive at the renormalized version of Eq. (3.6),

$$\Gamma/V = \frac{B^2}{4\pi^2\hbar^2} \exp[-B/\hbar - S^{(1)}(\bar{\phi}) + S^{(1)}(\phi_+)] \times \left| \frac{\det'[-\partial^2 + U''(\bar{\phi})]}{\det[-\partial^2 + U''(\phi_+)]} \right|^{-1/2} [1 + O(\hbar)] . \tag{3.11}$$

As a good renormalized expression should be, this is free of ultraviolet divergences; each determinant is paired with an exponential of $S^{(1)}$, just as in Eq. (3.8). (That one of our factors is a primed determinant is irrelevant; omitting any finite number of eigenvalues has no effect on the ultraviolet divergence.)

D. Negative eigenvalues

We now investigate the operator

$$-\partial^2 + U''(\bar{\phi}) . \tag{3.12}$$

This operator is rotationally invariant, and thus its eigenfunctions are, in their angular dependence, four-dimensional scalar spherical harmonics. We denote these functions by $Y_{jmm'}$, where j is 0, $\frac{1}{2}$, 1, ..., and m and m' independently range from $-j$ to j by integer steps. We remind the reader that these functions transform according to the representation $D^{(j,j)}$ of SO(4) = SO(3) × SO(3); thus $j=0$ is a scalar, $j=\frac{1}{2}$ is a vector, etc. If we write the eigenfunction as

$$\chi_{nj}(\rho) Y_{jmm'}/\rho^{3/2} , \tag{3.13}$$

where ρ is the four-dimension radial variable, then the eigenvalue equation becomes

$$\left[-\frac{d^2}{d\rho^2} + \frac{8j(j+1)+3}{4\rho^2} + U''(\bar{\phi}) \right] \chi_{nj} = \lambda_{nj}\chi_{nj} . \tag{3.14}$$

Aside from the coefficient of the centrifugal potential, this is identical in form to the familiar three-dimensional radial Schrödinger equation and can be analyzed by familiar methods. The eigenfunctions of zero eigenvalue transform according to $j=\frac{1}{2}$. Furthermore, since $\bar{\phi}$ is a monotone increasing function of ρ, the radial part of these eigenfunctions is free of nodes. Thus, there are no negative eigenvalues with $j \neq 0$, and there is at least one negative eigenvalue with $j=0$.

The question is: Are there more than one? For, if there are, we will have to modify our formulas drastically; the whole analysis of Sec. II was based on the existence of one and only one negative eigenvalue. Regrettably, we have been able to settle this point only in the limit of small energy-density difference ϵ, between the true and false vacuum. In this case, there is indeed only one negative eigenvalue, as we shall now show.

In the first paper in this series, it was shown that, in the limit of small ϵ,

$$\bar{\phi}(\rho) = f(\rho - R) , \tag{3.15}$$

where R was a number inversely proportional to ϵ, and f was a function independent of ϵ, such that

$$f(\pm\infty) = \phi_{\pm}. \tag{3.16}$$

Thus, in this limit, the bounce looks like a large four-dimensional spherical bubble of true vacuum separated by a thin wall from a sea of false vacuum.

In this situation, because both $U''(\phi_+)$ and $U''(\phi_-)$ are positive, the only possible eigenfunctions of negative eigenvalue are those that are bound to the bubble wall. For such eigenfunctions (if they exist), the centrifugal potential in Eq. (3.14) can reasonably be approximated by a constant, and

$$\lambda_{nj} = \lambda_n + \frac{8j(j+1)+3}{4R^2}, \tag{3.17}$$

where λ_n is a number independent of j. Furthermore, as R goes to infinity, λ_n goes to the nth eigenvalue of the one-dimensional Schrödinger operator,

$$-\frac{d^2}{dx^2} + U''(f(x)). \tag{3.18}$$

We already know that, for $j=\frac{1}{2}$, the lowest eigenvalue is zero. Equation (3.17) tells us that, corresponding to this, there is a negative eigenvalue, minus $3/2R^2$, for $j=0$. However, every other eigenvalue for $j=\frac{1}{2}$ must have a positive limit as R goes to infinity. Thus, for large R, none of these can correspond to negative eigenvalues for $j=0$. This completes the argument.

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[1]S. Coleman, Phys. Rev. D 15, 2929 (1977); 16, 1248(E) (1977). This paper contains references to earlier work on the subject.

[2]An earlier and unpublished version of this work obtained the same results by direct integration of the multidimensional Schrödinger equation, using the WKB methods of T. Banks, C. Bender, and T. T. Wu, [Phys. Rev. D 8, 3346 (1973); 8 3366 (1973)]. The much simpler functional method described in the text of this paper is essentially a straightforward field-theoretic adaptation of the method introduced by J. S. Langer [Ann. Phys. (N.Y.) 41, 108 (1967)] in his analysis of the droplet model of statistical mechanics. One of us (S.C.) thanks E. Brezin and A. Linde for calling this work to his attention, and is especially grateful to M. Peskin for a patient explanation of the factor of $\frac{1}{2}$ that appears in Eq. (2.22).

[3]See, for example, R. Feynman and A. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw-Hill, New York, 1965).

[4]We depart from the notation of Ref. 1 here. Throughout this paper we work exclusively in Euclidean space; therefore we do not bother to use special symbols to distinguish Euclidean quantities from their Minkowskian counterparts.

[5]This is, of course, nothing but the standard prescription for treating collective coordinates in soliton problems. See J. L. Gervais and B. Sakita, Phys. Rev. D 11, 2943 (1975).

[6]See R. Jackiw, Phys. Rev. D 9, 1686 (1974). We (unconventionally) denote the effective action by a lowercase γ to avoid confusion with the decay probability, Γ.