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Interaction of isovector scalar mesons with simple sources*

M. Bolsterli
T Division, Los Alamos Scientific Laboratory, Los Alamos, New Mexico 87545
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A variational procedure using coherent states is shown to be superior to previous methods for treating the interaction of charged or isovector mesons with a static source. The technique is then applied to interaction with a Schrödinger field.

I. INTRODUCTION

Since perturbation theory is irrelevant to strong-coupling field theories and relativistic field theories have divergences that prevent consistent treatment by other than perturbation methods, theoretical consideration of strong coupling has largely been in terms of noncovariant models. For neutral Bose fields, the case of interaction with a static source has a very simple solution[1]; interaction with a recoiling source is not simple and has been widely treated.[2-4] When the Bose field is charged, even the static source is complicated[5-7]; nonstatic models have hardly been considered.

In Ref. 5, the charged field interaction with a static source was treated by a canonical transformation method that seems not to be useful for nonstatic sources. Reference 7 obtained the same results as Ref. 5 by a self-consistent field method and treated isovector, as well as charged, Bose fields. The connection between the self-consistent field method and the more powerful method of coherent states has been elaborated in Ref. 3; in particular, it was shown that coherent-state methods are useful with nonstatic sources.

In this paper, coherent states are used as the basic tool for treating both static and nonstatic sources interacting with an isovector scalar field. Section II describes a natural set of coherent states for variational calculations in the case of a static source. The trial states have the advantage of being eigenstates of the isospin operators T^2 and T_3; in the self-consistent field[7] treatment the trial states were not isospin eigenstates. In Sec. II it is shown that for weak coupling the coherent state gives an energy that is three times as good as the self-consistent field energy; for strong coupling both methods give the same energy.

Besides the natural coherent state of Sec. II, a special coherent state can be constructed to be an eigenstate of all the appropriate annihilation operators. Section III shows that the same energies are obtained with this state in the strong- and weak-coupling limits.

The same coherent-state technique can then be used for nonstatic sources by applying the general method described in Ref. 3; details are given in Sec. IV. The final section contains a few remarks.

II. STATIC SOURCE

The Hamiltonian is, as in Ref. 7,

$$H=\sum_{i=1}^{3}\left(\int \omega(k)a_i^\dagger(\vec{k})a_i(\vec{k})d\vec{k} - \tau_i\int [W^*(k)a_i(\vec{k})+a_i^\dagger(\vec{k})W(k)]d\vec{k}\right)$$
$$=\sum_{\alpha=-1}^{1}\left(\int \omega(k)a_\alpha^\dagger(\vec{k})a_\alpha(\vec{k})d\vec{k} - \tau_\alpha\int [W^*(k)a_\alpha(\vec{k})+a_{-\alpha}^\dagger(\vec{k})W(k)]d\vec{k}\right), \quad (2.1)$$

with

$$\tau_\pm=(\tau_1\pm i\tau_2)/\sqrt{2}\,,\quad \tau_0=\tau_3\,,$$
$$a_\pm=(a_1\pm ia_2)/\sqrt{2}\,,\quad a_0=a_3\,,$$
$$(a^\dagger)_\alpha=a_{-\alpha}^\dagger\,,$$
$$\tau\cdot a=\sum_{i=1}^{3}\tau_i a_i=\sum_{\alpha=-1}^{1}\tau_\alpha a_\alpha\,, \quad (2.2)$$
$$\tau\cdot a^\dagger=\sum_{\alpha=-1}^{1}\tau_\alpha a_{-\alpha}^\dagger\,,$$
$$\omega(k)=(k^2+m^2)^{1/2}\,.$$

The function $W(k)$ is the form factor of the static source; if W is spherically symmetric, as here, only s-wave mesons interact with the source.

From the point of view of coherent states, it is natural to use the trial vector $|\frac{1}{2},m;b\rangle_e$,

$$|\tfrac{1}{2},m;b\rangle_e\equiv\exp[\tau\cdot\int b(k)a^\dagger(\vec{k})d\vec{k}]\,|\tfrac{1}{2},m\rangle\,, \quad (2.3)$$

where $|\frac{1}{2},m\rangle$ is the bare source state, an eigenvector of T^2 and T_3. Since $\tau\cdot a^\dagger$ is invariant under isorotations, it commutes with T and T_3; the state $|\frac{1}{2},m;b\rangle$ has $T=\frac{1}{2}$, $T_3=m$. The function $b(k)$ is the variational "wave function of the isovector field"; it is chosen so as to minimize the energy functional

$F\{b\}$

$$F\{b\} = \frac{{}_e\langle \frac{1}{2}, m; b | H | \frac{1}{2}, m; b \rangle_e}{{}_e\langle \frac{1}{2}, m; b | \frac{1}{2}, m; b \rangle_e} . \tag{2.4}$$

The algebra is given in the Appendix. The result is

$$\begin{aligned} &{}_e\langle \tfrac{1}{2}, m; b | \tfrac{1}{2}, m; b \rangle_e = d_e(B) , \\ &{}_e\langle \tfrac{1}{2}, m; b | H | \tfrac{1}{2}, m; b \rangle_e = -a\{b\} d_e'(B) , \\ &B\{b\} = \int |b(k)|^2 d\vec{k} , \\ &d_e(B) = (1+B)e^B + \sinh B , \\ &a\{b\} = \int [W^*(k)b(k) + b^*(k)W(k) - \omega(k) |b(k)|^2] d\vec{k} , \end{aligned} \tag{2.5}$$

so that

$$\begin{aligned} &F\{b\} = -a\{b\} g_e(B) , \\ &g_e(B) = d_e'(B)/d_e(B) . \end{aligned} \tag{2.6}$$

[Note that the neutral scalar case can be written in the same form with $d_e(B) = e^B$ and $g_e(B) = 1$.] Now the function b that minimizes $F\{b\}$ is easily seen to be

$$\begin{aligned} &b(k) = \frac{W(k)}{\omega(k) + aj(B)} , \\ &j(B) = -g_e'(B)/g_e(B). \end{aligned} \tag{2.7}$$

So that the equations to be solved are mostly simply expressed in terms of

$$K = a\{b\} j(B) \tag{2.8}$$

and are

$$\begin{aligned} &B = \int \frac{|W(k)|^2 d\vec{k}}{[\omega(k) + K]^2} , \\ &K = j(B) \int |W(k)|^2 \frac{\omega(k) + 2K}{[\omega(k) + K]^2} d\vec{k} \end{aligned} \tag{2.9}$$

Once B and K are determined by solving Eqs. (2.9), then the energy in this approximation is given by

$$E = -K g_e(B)/j(B). \tag{2.10}$$

First consider weak coupling. For W small enough, both K and B are small; $j(B)$ can be replaced by $j(0) = 2$, $g_e(B)$ by $g_e(0) = 3$, and

$$E_{\text{WC}} = -3 \int \frac{|W(k)|^2}{\omega(k)} d\vec{k}. \tag{2.11}$$

This is just the result of second-order perturbation theory. It is three times the result of Ref. 7 for weak coupling. (It can be verified that for charged mesons the coherent-state result is twice the result of Refs. 5 and 7.)

For strong coupling, B becomes large and $j(B)$ can be replaced by its asymptotic value B^{-2}, while $g(B)$ is approximately 1. Since K is proportional to B^{-2}, it is small and

$$\begin{aligned} &K_{\text{SC}} = \frac{1}{B_{\text{SC}}^2} \int \frac{|W(k)|^2}{\omega(k)} d\vec{k}, \\ &E_{\text{SC}} = -\int \frac{|W(k)|^2}{\omega(k)} d\vec{k}. \end{aligned} \tag{2.12}$$

This is the same as the result of Ref. 7 for strong enough coupling, and is also equal to the result for an isoscalar Bose field.

III. SPECIAL COHERENT STATE

The coherent state of the previous section has the property that the meson hole state

$$a(\vec{q}) | \tfrac{1}{2}, m; b \rangle_e \tag{3.1}$$

has components orthogonal to the coherent state. This is unlike the static isoscalar scalar case, where the meson hole state is a multiple of the coherent state. Meson hole states are a nuisance in higher approximations; it is useful to define a special coherent state that has no meson hole states. Despite the fact that the operators $\tau \cdot a(\vec{q})$ for various $\vec{q}$ do not commute, it is possible to find a simultaneous eigenstate of the $\tau \cdot a(\vec{q})$ operators. Let the power series $d(x)$ be defined by

$$\begin{aligned} d(x) &= \sum_{n=0}^{\infty} \frac{n+1+\delta_{n,\text{even}}}{(n+2)!} x^n \\ &= \sum_{n=0}^{\infty} \frac{x^n}{\prod_{j=1}^{n} K_j} \\ &= \frac{e^x}{x} - \frac{\sinh x}{x^2} , \end{aligned} \tag{3.2}$$

where the numbers K_n are defined in the Appendix. Then the state

$$| \tfrac{1}{2}, m; b \rangle \equiv d\left(\tau \cdot \int b(k) a^\dagger(\vec{k}) d\vec{k} \right) | \tfrac{1}{2}, m \rangle \tag{3.3}$$

is easily seen to be an eigenstate of $\tau \cdot a(q)$.

$$\tau \cdot a(\vec{q}) | \tfrac{1}{2}, m; b \rangle = b(q) | \tfrac{1}{2}, m; b \rangle. \tag{3.4}$$

Then, algebra as before gives

$$\begin{aligned} &\langle \tfrac{1}{2}, m; b | \tfrac{1}{2}, m; b \rangle = d(B), \\ &\langle \tfrac{1}{2}, m; b | H | \tfrac{1}{2}, m; b \rangle \\ &\qquad = \int \omega(k) |b(k)|^2 d\vec{k} d'(B) \\ &\qquad\quad - \int [W^*(k) b(k) + b^*(k) W(k)] d\vec{k} d(B), \\ &B = \int |b(k)|^2 d\vec{k}. \end{aligned} \tag{3.5}$$

Minimization with respect to $b(k)$ gives

$$b(k)=\frac{W(k)}{g(B)\omega(k)+g'(B)K},$$
$$K=\int\frac{\omega(k)|W(k)|^2}{[g(B)\omega(k)+g'(B)K]^2},\qquad(3.6)$$
$$B=\int\frac{|W(k)|^2}{[g(B)\omega(k)+g'(B)K]^2},$$
$$g(B)=d'(B)/d(B).$$

Now the weak-coupling case occurs for $B\to 0$, $g(B)\to\frac{1}{3}$, and gives

$$E_{\mathrm{WC}}=-3\int\frac{|W(k)|^2}{\omega(\vec{k})}d\vec{k}\ .\qquad(3.7)$$

In strong coupling, $g=1$ and

$$E_{\mathrm{WC}}=-\int\frac{|W(k)|^2}{\omega(\vec{k})}d\vec{k}.\qquad(3.8)$$

Both results are as in the previous section, but Eq. (3.4) is very useful for going to higher order approximations.

IV. NONSTATIC SOURCE

If $\tilde{\psi}(p)$ is (the Fourier transform of) a Fermi field with isospin $\frac{1}{2}$, the general nonstatic Hamiltonian is

$$H=\sum_{i=1}^{3}\left\{\int\omega(k)a_i^\dagger(\vec{k})a_i(\vec{k})d\vec{k}-\int[W^*(\vec{p},\vec{q})a_i(\vec{k})+W(\vec{q},\vec{p})a_i^\dagger(-\vec{k})]\times\tilde{\psi}^\dagger(\vec{p})\tau_i\tilde{\psi}(\vec{q})\delta(\vec{p}-\vec{q}-\vec{k})d\vec{p}\,d\vec{q}\,d\vec{k}\right\}+\int\tilde{\psi}^\dagger(\vec{p})\epsilon(\vec{p})\tilde{\psi}(\vec{p})d\vec{p}.\qquad(4.1)$$

The low momentum-transfer approximation to a local relativistic theory[8] gives

$$W_{\mathrm{LR}}(\vec{p},\vec{q})=\frac{gM}{[16\pi^3\omega(\vec{p}-\vec{q})]^{1/2}\epsilon((\vec{p}+\vec{q})/2)},\qquad(4.2)$$

and the corresponding Hamiltonian for the isoscalar case is discussed in Ref. 8.

The localized special coherent state appropriate to the Hamiltonian of Eq. (4.1) is

$$|\vec{x};b,f\rangle=U_{\vec{x}}^\dagger\{b\}\int e^{-i\vec{p}\cdot\vec{x}}\tilde{\psi}^\dagger(\vec{p})\tilde{f}(\vec{p})d\vec{p}\,|\Omega\rangle,\qquad(4.3)$$
$$U_{\vec{x}}^\dagger\{b\}=d\left(\tau\cdot\int b(k)a^\dagger(\vec{k})e^{-i\vec{k}\cdot\vec{x}}d\vec{k}\right);$$

Ref. 3 gives a general discussion of the plane-wave phase factors in Eq. (4.3). Now the algebra of the Appendix again applies. Let

$$B(\vec{x})=\int|b(k)|^2e^{i\vec{k}\cdot\vec{x}}d\vec{k}.\qquad(4.4)$$

Then

$$\langle\vec{y};b,f|\vec{x};b,f\rangle=D(\vec{y}-\vec{x})=D_F(\vec{y}-\vec{x})D_B(\vec{y}-\vec{x}),$$
$$D_B(\vec{x})=d(B(\vec{x})),\qquad(4.5)$$
$$D_F(\vec{x})=\int e^{i\vec{p}\cdot\vec{x}}|\tilde{f}(\vec{p})|^2d\vec{p},$$

with $d(B)$ given by Eq. (2.5). Similarly,

$$\langle\vec{y};b,f|H|\vec{x};b,f\rangle=A(\vec{y}-\vec{x}),$$
$$A(\vec{x})=d(B(\vec{x}))\left\{t(x)-\int[b^*(k)\tilde{\rho}(\vec{k},\vec{x})+\tilde{\rho}^*(\vec{k},-\vec{x})b(x)]d\vec{k}\right\}+d'(B(\vec{x}))a_1(\vec{x}),\qquad(4.6)$$
$$t(\vec{x})=\int\epsilon(\vec{p})\tilde{f}^\dagger(\vec{p})\tilde{f}(\vec{p})e^{i\vec{p}\cdot\vec{x}}d\vec{p},$$
$$a_1(\vec{x})=\int\omega(k)|b(k)|^2e^{i\vec{k}\cdot\vec{x}}d\vec{k},$$
$$\tilde{\rho}(\vec{k},\vec{x})=\int\delta(\vec{p}-\vec{q}+\vec{k})\tilde{f}^\dagger(\vec{p})\tilde{f}(\vec{q})e^{i\vec{q}\cdot\vec{x}}W(\vec{q},\vec{p})d\vec{p}\,d\vec{q}.$$

[Note that in the case $d(B)=d'(B)=e^B$, the functions $A(x)$ and $D(x)$ go over into the corresponding functions for the isoscalar case.[8]]

As in Refs. 3 and 8, the weak-coupling approximation is obtained by taking $\tilde{f}(\vec{p})$ to be constant in the translated-localized-state (TLS) energy functional,

$$F_{\mathrm{TLS}}=\frac{\int A(\vec{x})d\vec{x}}{\int D(\vec{x})d\vec{x}};\qquad(4.7)$$

the result is the same as second-order perturbation theory, namely,

$$E_{\mathrm{WC}}=-3\int\frac{|W(0,\vec{k})|^2}{\omega(\vec{k})+\epsilon(-\vec{k})}d\vec{k}.\qquad(4.8)$$

For strong coupling, the localized-state functional F_{LS},

$$F_{\mathrm{LS}}=A(0)/D(0),\qquad(4.9)$$

can be used:

$$F_{\mathrm{LS}}=g(B)a_1(0)+t(0)-\int[\tilde{\rho}^*(\vec{k},0)b(k)+b^*(k)\tilde{\rho}(\vec{k},0)]d\vec{k}.$$

Thus, the procedure of the preceding section gives for strong coupling

$$F_{\mathrm{LS}}\{f\}=\int\epsilon(\vec{p})\tilde{f}^\dagger(\vec{p})\tilde{f}(\vec{p})d\vec{p}-\int\frac{|\tilde{\rho}(k,0)|^2}{\omega(\vec{k})}d\vec{k}.\qquad(4.10)$$

This is the same functional of f that was obtained for the isoscalar field; some properties are considered in Ref. 8.

V. REMARKS

(*a*) *Isoscalar pseudoscalar case*. Here the Hamiltonian is written

$$H=\int \omega(k)a^\dagger(\vec{k})a(\vec{k})d\vec{k}$$
$$-\sqrt{3}\int [W^*(k)a(\vec{k})+a^\dagger(\vec{k})W(k)]\vec{\sigma}\cdot\hat{k}d\vec{k};$$

the coherent state has the Bose part

$$U_e^\dagger\{b\}=\exp\left[\sqrt{3}\vec{\sigma}\cdot\int \hat{k}b(k)a^\dagger(\vec{k})d\vec{k}\right].$$

It is left to the reader as a not completely trivial exercise to show that Eqs. (2.4)–(2.13) hold in this case, and also the corresponding results for the special coherent state and for the nonstatic source. The isovector pseudoscalar case is more complicated.

(*b*) *Many-particle sources*. In contrast to the isoscalar scalar case, where the source is merely a density, the isovector field depends on the isospin coupling of the sources. A source with zero isospin density has no interaction; others are less tractable.

(*c*) *Meson scattering and isobars*. In contrast to the mean field treatment of Ref. 7, where approximate state vectors and energies for the lowest state of every isospin were computed, the present treatment has only produced an approximation for the lowest $T=\frac{1}{2}$ state. In the coherent-state treatment, higher-isospin states and meson scattering are closely linked. Their treatment will be considered in a subsequent paper.

APPENDIX

Let

$$A_i^\dagger=\int b(k)a_i^\dagger(\vec{k})d\vec{k}\ ;$$

so that

$$|\tfrac{1}{2},m;b\rangle_e=e^{\tau\cdot A^\dagger}|\tfrac{1}{2},m\rangle\ ,$$
$$[A_i,A_j^\dagger]=B\delta_{ij}\ ,$$
$$B=\int |b(k)|^2d\vec{k}\ .$$

Let

$$X_i=\int x^*(k)a_i(\vec{k})d\vec{k}\ ,$$
$$Y=\int x^*(k)b(k)d\vec{k}\ ;$$

then

$$\tau\cdot X\tau\cdot A^\dagger=-\tau\cdot A^\dagger\tau\cdot X+2A^\dagger\cdot X+3Y\ ,$$
$$A^\dagger\cdot X\tau\cdot A^\dagger=\tau\cdot A^\dagger(A^\dagger\cdot X+Y)\ ,$$

$$\tau\cdot X(\tau\cdot A^\dagger)^n|\tfrac{1}{2},m\rangle$$
$$=\sum_{j=0}^{n-1}(-)^j(\tau\cdot A^\dagger)^j(2A^\dagger\cdot X+3Y)(\tau\cdot A^\dagger)^{n-j-1}|\tfrac{1}{2},m\rangle$$
$$=K_nY(\tau\cdot A^\dagger)^{n-1}|\tfrac{1}{2},m\rangle\ ,$$

$$K_n=\sum_{j=0}^{n-1}(-)^j(2n-2j+1)=\begin{cases}n, & n \text{ even}\\ n+2, & n \text{ odd}\end{cases}$$

$$\langle\tfrac{1}{2},m|e^{\tau\cdot A}e^{\tau\cdot A^\dagger}|\tfrac{1}{2},m\rangle$$
$$=\sum_{n=0}^\infty\langle\tfrac{1}{2},m|(\tau\cdot A)^n(\tau\cdot A^\dagger)^n|\tfrac{1}{2},m\rangle/(n!)^2$$
$$=\sum_0^\infty\frac{B^n}{n!}\frac{\prod_{j=1}^n K_j}{n!}$$
$$=\sum_0^\infty\frac{B^n}{n!}(n+1+\delta_{n,\text{odd}})$$
$$=(1+B)e^B+\sinh B=d_e(B)\ ,$$

$$\langle\tfrac{1}{2},m|e^{\tau\cdot A}\tau\cdot Xe^{\tau\cdot A^\dagger}|\tfrac{1}{2},m\rangle$$
$$=\sum_{n=0}^\infty\frac{\langle\tfrac{1}{2},m|(\tau\cdot A)^n\tau\cdot X(\tau\cdot A)^{n+1}|\tfrac{1}{2},m\rangle}{n!(n+1)!}$$
$$=\sum_0^\infty\frac{YB^n}{n!}\frac{\prod_{j=1}^{n+1}K_j}{(n+1)!}$$
$$=\sum_0^\infty\frac{YB^n}{n!}(n+2+\delta_{n,\text{even}})$$
$$=Y[(2+B)e^B+\cosh B]=Yd_e'(B)\ ,$$

$$\left[\tau\cdot A,\int\omega(k)a^\dagger(\vec{k})\cdot a(\vec{k})d\vec{k}\right]=\tau\cdot U\ ,$$

$$U_i=\int\omega(k)b^*(k)a_i(\vec{k})d\vec{k}\ ,$$

$$\langle\tfrac{1}{2},m|e^{\tau\cdot A}\int\omega(k)a^\dagger(\vec{k})\cdot a(\vec{k})d\vec{k}e^{\tau\cdot A^\dagger}|\tfrac{1}{2},m\rangle$$
$$=\sum_{n=1}^\infty\sum_{j=0}^{n-1}\frac{\langle\tfrac{1}{2},m|(\tau\cdot A)^{n-j-1}\tau\cdot U(\tau\cdot A)^j(\tau\cdot A^\dagger)^n|\tfrac{1}{2},m\rangle}{(n!)^2}$$
$$=\sum_{n=1}^\infty\frac{\Gamma B^{n-1}n\prod_1^n K_j}{n!n!}=\Gamma\sum_{n=0}^\infty\frac{B^n}{n!}(n+2+\delta_{n,\text{even}})$$
$$=\Gamma d_e'(B)\ ,$$

$$\Gamma=\int\omega(k)|b(k)|^2d\vec{k}.$$

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