Spinor Yang-Mills superfields*

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Following the example provided by the ordinary derivative and vector Yang-Mills fields, we give an argument which demonstrates that spinor superfields may be combined with the fermionic components of the invariant superspace gradient in order to define a fermionic, supersymmetric Yang-Mills covariant derivative. Additionally, it is shown that this fermionic, covariant gradient is the "square root" of a bosonic covariant gradient. Thus, it is possible to define a "supercovariant derivative" in superspace. We also discuss some general aspects of supersymmetric, Yang-Mills theories.

I. INTRODUCTION AND SUMMARY

The spinor superfield has not been studied very extensively thus far. In the work of Adjei and Akyeampong,[1] a Lagrangian for the interaction of a chiral, spinor superfield with a chiral, scalar superfield was examined. It was found that that Lagrangian is nonrenormalizable. As far as we are aware, this is the only work that has been done in this direction.

In supersymmetric theories, there exists a differential operator known as the covariant derivative. In order to avoid confusion, we will henceforth refer to this operator as the fermionic components of the invariant supergradient or, more succinctly, the fermionic gradient. This derivative transforms as a relativistic spinor under the Lorentz group. This suggests that perhaps the spinor superfield may be able to play a role that is analogous to that played by gauge vector fields in ordinary theories. We shall see that in exact analogy with the covariant derivative of usual Yang-Mills theories one may define a "supercovariant derivative" in the fermionic sector of superspace. More remarkably, the existence of this fermionic Yang-Mills covariantized derivative implies the existence of a bosonic Yang-Mills covariantized derivative. The truly remarkable feature about this relation is that it does not require the introduction of independent vector superfields for the bosonic components of the supercovariant derivative.

The organization of this paper is as follows. In Sec. II, we give a brief discussion of supersymmetry and introduce several important operators. In Sec. III, we recall some very familiar results from ordinary gauge theories. In Sec. IV, we present our argument, which is guided by the experience gained from the usual Yang-Mills–type theory. Here we introduce the "supercovariant derivative," which will allow for local invariances in a manifestly supersymmetric manner. We also use this operator to deduce the form of the pure gauge Lagrangian for the spinor superfields. We present an example for the interaction of the gauge superfields with matter superfields. The final section is a discussion of Yang-Mills invariance in both ordinary space and superspace. We show here a possible connection between this work and earlier works on Yang-Mills invariance and supersymmetry.

An important question we shall not address here is whether the example given is renormalizable. This we shall do in a future paper. There is a hint that our example may be renormalizable. It has been observed[1,2] that in supersymmetric theories, as in ordinary theories, the presence of coupling constants which possess dimensions which are positive powers of inverse mass usually indicates nonrenormalizability. Our example contains no such coupling constants.

II. SUPERSYMMETRIC GEOMETRY

By starting from a pseudogeometric viewpoint, we may think of the supersymmetric group of Wess and Zumino[3] arising from an extended spacetime $\{X\}$ which has elements which are of the form

$$X^M = (x^\mu, \theta^m)\,, \tag{1}$$

where x^μ are the usual coordinates of spacetime and θ^m are, in the simplest instance, the coordinates in a four-dimensional fermionic space. The coordinates θ^m anticommute among themselves, commute with the bosonic coordinates, and are further restricted to transform as a Majorana, Dirac spinor under the Lorentz group. If we choose a representation where charge conjugation is equivalent to the complex conjugation of a Dirac spinor, then the fermionic coordinates are real. As a result of these requirements, the generators of Lorentz transformations on the extended spacetime are given by

$$M^{\alpha\beta}=(L^{\alpha\beta})^{\mu}{}_{\nu}x^{\nu}\partial_{\mu}-\tfrac{1}{2}(\sigma^{\alpha\beta})^{m}{}_{n}\theta^{n}\Delta_{m}\,, \tag{2}$$

where

$$(\sigma^{\alpha\beta})^{m}{}_{n}\equiv(i\tfrac{1}{2}[\gamma^{\alpha},\gamma^{\beta}])^{m}{}_{n}\,,$$
$$(L^{\alpha\beta})^{\mu}{}_{\nu}\equiv i(\eta^{\alpha\mu}\delta^{\beta}_{\nu}-\eta^{\beta\mu}\delta^{\alpha}_{\nu})\,,$$
$$\Delta_{m}\equiv\frac{\partial}{\partial\theta^{m}}\,.$$

The supersymmetric generators of supertranslations are given by

$$S^{l}=-i[(\gamma^{0})^{lm}\Delta_{m}+i\tfrac{1}{2}(\not\partial\theta)^{l}]\,. \tag{3}$$

In writing this expression, we have asserted that $(\gamma^{0})^{lm}$ acts as a metric for the fermionic coordinates in much the same way as $\eta_{\mu\nu}$ does for the bosonic coordinates. We have also redefined complex conjugation so that in addition to its usual action ($i\to -i$), it also reverses the order of all products of fermionic factors. This last statement is important since it ensures that the real extended spacetime will be mapped into a real extended spacetime under the action of $\exp(i\bar{\epsilon}S)$, where ϵ, like θ, is a real, anticommuting spinor.

Following Salam and Strathdee,[4] we may also introduce the supersymmetric covariant derivative via the definition

$$D^{l}=(\gamma^{0})^{lm}\Delta_{m}-i\tfrac{1}{2}(\not\partial\theta)^{l}\,, \tag{4}$$

and verify that it satisfies the relations

$$\begin{aligned}&\{S^{l},\ D^{m}\}=0\,,\\&D^{l}D^{m}=i\tfrac{1}{2}(\not\partial\gamma^{0})^{lm}+\tfrac{1}{4}(\gamma^{0})^{lm}(\bar{D}D)\\&\qquad+\tfrac{1}{4}(\gamma^{5}\gamma^{\mu}\gamma^{0})^{lm}(\bar{D}\gamma^{5}\gamma_{\mu}D)+\tfrac{1}{4}(\gamma^{5}\gamma^{0})^{lm}(\bar{D}\gamma^{5}D)\,,\end{aligned} \tag{5}$$

where

$$\bar{D}_{l}=(\gamma^{0})_{lm}D^{m}\,.$$

If we restrict ourselves to the Poincaré group and the supertranslations,[5] then we find a Lie algebra which closes. In addition to the usual commutators of the generators of the Poincaré group, we need the following relations:

$$\begin{aligned}&[S^{l},P_{\mu}]=0\,,\\&[M^{\alpha\beta},S^{l}]=-\tfrac{1}{2}(\sigma^{\alpha\beta}S)^{l}\,,\\&\{S^{l},S^{m}\}=-(\gamma^{\mu}\gamma^{0})^{lm}P_{\mu}\,,\end{aligned} \tag{6}$$

where

$$P_{\mu}\equiv-i\partial_{\mu}\,.$$

III. RESULTS FROM ORDINARY GAUGE THEORIES

We would like to recall some results from other theories which possess gauge invariances. We begin from the simplest of gauge theories, quantum electrodynamics. In this theory, by the minimal-coupling prescription, we introduce a covariant derivative via the definition

$$D_{\mu}\equiv\partial_{\mu}+ieA_{\mu}\,. \tag{7}$$

It is natural to introduce a vector field since the operator ∂_{μ} transforms as a vector under the Lorentz group. With this definition of the covariant derivative, we can ensure the existence of a local invariance under redefinition of the phase of the electron field. The well-known transformation is given by

$$\begin{aligned}&\psi_{e}'=\exp[ie\Lambda(x)]\psi_{e}\,,\\&A_{\mu}'=A_{\mu}-\partial_{\mu}\Lambda(x)\,,\end{aligned} \tag{8}$$

where $\Lambda(x)$ is an arbitrary local function. Next, we need an expression for the kinetic energy of the gauge field A_{μ}, which is invariant under the gauge transformation. This is done by defining the field strength tensor

$$F_{\mu\nu}=\partial_{\mu}A_{\nu}-\partial_{\nu}A_{\mu} \tag{9}$$

and contracting it with itself. With these definitions, we observe that the identities

$$[D_{\mu},D_{\nu}]=ieF_{\mu\nu}\,, \tag{10a}$$
$$F^{\mu\nu}=-i(L^{\mu\nu})^{\alpha}{}_{\beta}\partial_{\alpha}A^{\beta} \tag{10b}$$

are valid.

If we now consider some non-Abelian group,[6] we introduce a multiplet of vector gauge fields which transform as the adjoint representation of the group. The covariant derivative in Eq. (7) is redefined so that

$$D_{\mu}\equiv\partial_{\mu}+igT_{a}A^{a}{}_{\mu}\,, \tag{11}$$

where T_{a} is some representation of the group. In analogy with Eq. (10), we find

$$[D_{\mu},D_{\nu}]=igT_{a}F^{a}{}_{\mu\nu}\,, \tag{12a}$$
$$\begin{aligned}F^{a}{}_{\mu\nu}&=\partial_{\mu}A^{a}{}_{\nu}-\partial_{\nu}A^{a}{}_{\mu}-gf^{a}{}_{bc}A^{b}{}_{\mu}A^{c}{}_{\nu}\\&=-i(L_{\mu\nu})^{\alpha\beta}(\partial_{\alpha}A^{a}{}_{\beta}-\tfrac{1}{2}gf^{a}{}_{bc}A^{b}{}_{\alpha}A^{c}{}_{\beta})\,,\end{aligned} \tag{12b}$$

where $f^{a}{}_{bc}$ are the totally antisymmetric structure constants of some Lie algebra.

IV. GAUGE SPINOR SUPERFIELDS

We can easily see that the covariant derivative of supersymmetry transforms like a Dirac spinor under the Lorentz group, for we find the relation

$$[M^{\alpha\beta},D^{l}]=-\tfrac{1}{2}(\sigma^{\alpha\beta}D)^{l} \tag{13}$$

is satisfied. Thus, if we think about the covariant derivative, D, as being a projection of a superspace gradient, $\partial/\partial X^{M}$, onto the fermion sector of superspace and the ordinary derivative, ∂_μ, the projection onto the boson sector, then, in the case where superfields[7] possess some internal symmetry, it does not seem unreasonable to add to the supergradient the quantity

$$V_M = [G^a{}_\mu, \tfrac{1}{2}(\gamma^0\Lambda^a)_m]\, T_a \tag{14}$$

to form a supercovariant gradient

$$\mathfrak{D}_M \equiv \frac{\partial}{\partial X^M} + ig V_M . \tag{15}$$

In Eq. (14) the multiplet of spinor superfields, $\Lambda^a(X)$, must transform as the adjoint representation of the internal-symmetry group. The vector superfields $G^a{}_\mu$ are defined by the equation

$$G^a{}_\mu \equiv -i\,\tfrac{1}{4}(\bar{D}\gamma_\mu\Lambda^a + \tfrac{1}{4}gf^a{}_{bc}\bar{\Lambda}^b\gamma_\mu\Lambda^c) . \tag{16}$$

The quantities T_a have their usual meanings. We may require that the spinor superfields are constrained to be real. With our conventions, this implies that the vector superfields are also real. *A priori*, in Eq. (14) we could assume that the vector superfields are independent of the spinor superfield. We will have to justify Eq. (16) below. The analogy between Eqs. (11) and (14) is more striking if we recall that an arbitrary spinor superfield contains a spinor superfield which is the covariant derivative of a scalar superfield. By thinking of this as the analog of the transformation of the photon field in Eq. (8), we are led to require that the Lagrangian for the gauge, spinor superfields be invariant under the transformation

$$\Lambda^a \to \Lambda^a - 2D\delta\Phi^a - gf^a{}_{bc}\delta\Phi^b\Lambda^c , \tag{17}$$

where $\delta\Phi^a$ is an infinitesimal multiplet of scalar superfields. Under this transformation, the vector superfields $G^a{}_\mu$ change as

$$G^a{}_\mu \to G^a{}_\mu - \partial_\mu\delta\Phi^a - gf^a{}_{bc}\delta\Phi^b\, G^c{}_\mu , \tag{18}$$

which justifies the identification made in Eq. (16). Next, we need to construct the Lagrangian for the gauge-spinor superfield. To this end we need to employ the generalized Lie bracket. This Lie bracket is defined by the relation

$$[A, B\} \equiv AB - (-1)^{\alpha_A\alpha_B} BA , \tag{19}$$

where $\alpha_A = (0, 1)$ depending on whether A is a Bose or Fermi operator. Using this operator on the supercovariant gradient, we arrive at

$$([\mathfrak{D}_M, \mathfrak{D}_N\})^i{}_j = ig(R_{MN})^i{}_j - (\mathfrak{M}^L{}_{MN})(\mathfrak{D}_L)^i{}_j , \tag{20}$$

where we have used the following definitions:

$$(R_{MN})^i{}_j \equiv \begin{bmatrix} -\frac{i}{2}(\gamma^0\sigma^{\mu\nu})_{mn}\, E^a{}_{\mu\nu} & F^a{}_{m\nu} \\ F^a{}_{\mu n} & G^a{}_{\mu\nu} \end{bmatrix} (T_a)^i{}_j ,$$

$$E^a{}_{\mu\nu} \equiv -\frac{i}{4}(\bar{D}\sigma_{\mu\nu}\Lambda^a + \tfrac{1}{4}gf^a{}_{bc}\bar{\Lambda}^b\sigma_{\mu\nu}\Lambda^c) ,$$

$$F^{am\,\nu} \equiv D^m G^{a\nu} - \tfrac{1}{2}\partial^\nu\Lambda^{am} - \tfrac{1}{2}gf^a{}_{bc}\Lambda^{bm}G^{c\nu} , \tag{21}$$

$$F^{am\,\nu} = -F^{a\nu m} ,$$

$$G^a{}_{\mu\nu} \equiv \partial_\mu G^a{}_\nu - \partial_\nu G^a{}_\mu - gf^a{}_{bc}\, G^b{}_\mu G^c{}_\nu ,$$

$$(\mathfrak{M}^L{}_{MN}) \equiv \begin{cases} i(\gamma^0\gamma^\lambda)_{mn} & \text{if } L=\lambda,\ M=m,\ N=n \\ 0 & \text{otherwise.} \end{cases}$$

The term proportional to $\mathfrak{D}_L$ on the right-hand side of Eq. (20) might be called the "anomalous term." It is anomalous in the sense that it does not have an analog in Eqs. (10) and (12). But the presence of such a term has an interesting interpretation within the context of differential geometry. Such a term can arise from the fact that we are describing superspace in terms of a noncommuting coordinate basis and therefore the components of the invariant superspace gradient are the directional derivatives of such a basis.

There are no nonzero scalars which may be formed from $(R_{MN})^i{}_j$. Therefore we may form a quadratic and take the trace over the internal elements to obtain

$$R^a{}_{KL} R^b{}_{MN}\delta_{ab} . \tag{22}$$

Therefore we may take as the gauge Lagrangian the expression

$$\mathcal{L}'_{\text{gauge}} = \tfrac{1}{4}(\bar{D}D)^2\{R^a{}_{KL} R^b{}_{MN}\}\delta_{ab}A^{KLMN} , \tag{23}$$

where A^{KLMN} is the most general constant tensor such that $\mathcal{L}'_{\text{gauge}}$ is invariant. Thus, we have constructed a manifestly supersymmetric Lagrangian for the gauge-spinor superfields. As can be seen, there remains quite a bit of ambiguity in this equation. We expect, however, that the requirement of renormalizability will place further restrictions on the arbitrary supertensor. We may note that the various sectors of the superfield strength tensor $E^a{}_{\mu\nu}$, $E^a{}_{m\nu}$, and $G^a{}_{\mu\nu}$ have dimensionalities of $d+\frac{1}{2}$, $d+1$, and $d+\frac{3}{2}$, respectively, where $d=\frac{1}{2}$ is the dimensionality of the spinor superfield in units of mass. Therefore, various sectors of A must differ by powers of inverse mass. Thus, we may argue that A must be chosen so that $\mathcal{L}'_{\text{gauge}}$ is proportional only to the square of the fermion-fermion sector of the superfield strength tensor. So we may assume that

$$\mathcal{L}'_{\text{gauge}} = \tfrac{1}{4}(\bar{D}D)^2\{E^a{}_{\mu\nu}E_a{}^{\mu\nu}\}c_0 \tag{24}$$

is the form of the gauge Lagrangian.

However, when this expression is expanded in terms of component fields, it is found not to contain a term which may be interpreted as the kinetic energy of a vector gauge field. Thus, by following the procedure which leads to a gauge theory in ordinary Minkowski space, we have not, as yet, a complete gauge Lagrangian. On the other hand, the expansion of the quantity

$$\mathcal{L}''_{\text{gauge}} = \tfrac{1}{4}(\bar{D}D)^2\{G^a{}_\mu G_a{}^\mu\} \tag{25}$$

is found to contain the kinetic-energy term of a gauge vector field, but it is not invariant under a gauge transformation. Under an infinitesimal gauge transformation this quantity is changed by an amount

$$-\tfrac{1}{2}(\bar{D}D)^2\{G^a{}_\mu \partial^\mu(\delta\Phi_a)\}. \tag{26}$$

Therefore in order to have a Lagrangian which is gauge invariant, we must add an additional term to Eq. (25). This additional term should have the same dimensionality as Eq. (25). We note that in Eqs. (24) and (25) two powers of the fermionic gradient act on the gauge superfield. We also know from Eq. (5) that two powers of the fermionic derivative may be combined to yield the bosonic derivative. This suggests that we may try to add Eq. (25) a term which is linear in ∂_μ. The simplest such term is of the form

$$\tfrac{1}{4}(\bar{D}D)^2\{\bar{\Lambda}^a \not\partial \Lambda^b\}\delta_{ab}. \tag{27}$$

We may subject this to the gauge transformation, and we find it is changed by the amount below plus two pure divergence terms:

$$i4(\bar{D}D)^2\{G^a{}_\mu \partial^\mu(\delta\Phi_a)\}. \tag{28}$$

Thus, it is clear that the expression

$$\mathcal{L}''_{\text{gauge}} = \tfrac{1}{4}(\bar{D}D)^2\{G^a{}_\mu G_a{}^\mu - i\tfrac{1}{8}\bar{\Lambda}^a \not\partial \Lambda_a\} \tag{29}$$

will change by pure divergences under a gauge transformation. But this is exactly the manner in which a supersymmetric Lagrangian transforms under a fermionic translation. Therefore, the gauge Lagrangian for the spinor superfield is

$$\mathcal{L}_{\text{gauge}} = \tfrac{1}{4}(\bar{D}D)^2\{c_1[G^a{}_\mu G_a{}^\mu - i\tfrac{1}{8}\bar{\Lambda}^a \not\partial \Lambda_a] + c_0 E^a{}_{\mu\nu} E_a{}^{\mu\nu}\}. \tag{30}$$

Using either the bosonic or fermionic sectors of the "supercovariant derivative" we may couple the gauge fields to matter superfields, provided that the pure kinetic terms for the matter superfields are only expressable with the use of fermionic and/or bosonic components of the invariant supergradient. An example of an interacting model is provided by

$$\mathcal{L}_{\text{gauge}} + \tfrac{1}{4}(\bar{D}D)^2\{\Phi^\dagger[(\mathfrak{D}^a(\gamma^0)_{ab}\mathfrak{D}^b) + M_0]\Phi\}, \tag{31}$$

where Φ is a complex scalar superfield belonging to some representation of the group.

Thus, formally at least, it appears that we have a solution to the problem of implementing Yang-Mills invariance for nonchiral superfields. The gauge-spinor superfields allow the Yang-Mills transformation of the gauge superfields to be realized linearly in a manner that is consistent with global supersymmetry. In previous works done on supersymmetry and Yang-Mills invariance by Salam and Strathdee[4] and Ferrara and Zumino,[8] the Yang-Mills transformation of the gauge superfield is implemented nonlinearly with respect to the supermultiplet, by introducing the gauge fields as components of a multiplet of real pseudoscalar superfields, $V^a(X)$. This allows the definition of two "phase factors" via the equations

$$\exp[\pm g V(X)], \tag{32}$$

where $V(X) \equiv V^a(X)T_a$. Using chiral matter superfields permits the gauge and matter superfields to be coupled:

$$\tfrac{1}{4}(\bar{D}D)^2\{\Phi_+^\dagger \exp[gV]\Phi_+ + \Phi_-^\dagger \exp[-gV]\Phi_-\}. \tag{33}$$

We will return to this point at the end of the next section. It remains to be seen whether our linear approach will prove as useful in model building as the nonlinear one. We are presently studying this question.

V. YANG-MILLS BASIS VECTORS AND SUPERSYMMETRY

In ordinary Yang-Mills theories, we have a set of gauge fields, $A^a_\mu(x)$, and set of generators, T_a, which belong to some representation of a compact, semisimple Lie algebra. At each point in space-time, we associate a set of internal basis vectors, $e^i_I(x)$, which are given by the expression

$$e^i{}_I(x) \equiv \left\{\exp\left[-ig\int_0^x dy^\mu A^a{}_\mu(y)T_a\right]\right\}^i{}_I, \tag{34}$$

where we have explicitly exhibited the matrix indices i and I. That we should recognize this as the set of basis vectors for the internal space is made plausible by observing that if we define a connection $\Gamma^b_{a\mu}$ in the usual manner,

$$de^i{}_I = dx^\mu \Gamma_\mu{}^J{}_I e^i{}_J, \tag{35}$$

the equation below then follows:

$$\begin{aligned}0 &= dx^\mu D_\mu e^i{}_J \\ &= dx^\mu(\partial_\mu \delta^J_I - \Gamma_\mu{}^J{}_I)e^i{}_J.\end{aligned} \tag{36}$$

Now we may perform the differentiation indicated in Eq. (29) and substitute the result into Eq. (30) to find

$$D_\mu = \partial_\mu + igA^a{}_\mu T_a, \tag{37}$$

which is the usual expression for the covariant derivative in a Yang-Mills theory. Now we make the observation that the internal-basis-vector concept easily generalizes in a flat, Fermi-Bose superspace. Indeed, we may replace Eq. (34) by the expression

$$e^i{}_I(X) \equiv \left\{\exp\left[-ig\int_0^X(\tfrac{1}{2}d\bar{\theta}\,\Lambda^a + dx^\mu G^a{}_\mu)T_a\right]\right\}^i{}_I, \tag{38}$$

where X is some point in the superspace. Here we can see that it is crucial that both spinor and vector superfields are present in order to define the supersymmetric generalization of the line integral.

Equation (38) is reminiscent of the phase factor in Eq. (33). It appears that we may make some identification between V^a and the supersymmetric line integral. The fact that chiral superfields couple to the basis vectors is analogous to the coupling of ordinary spinors to the *vierbein* fields of a gravitational theory. Thus, chiral scalar superfields may be viewed as Yang-Mills spinors.

It is obvious that the internal basis vectors are nothing but Yang's gauge phase factors.[9] This in turn implies that the usual supersymmetric "phase factors" discussed in the previous section may also be identified as a supersymmetric version of the Yang gauge phase factor for chiral theories.

This viewpoint suggests a whole class of supersymmetric, chiral gauge models which have not, as yet, been explored. One could consider a chiral model where the matter superfields are chiral spinor superfields. The gauge superfields may couple to these matter superfields through the chiral, gauge vector superfields. An example of such a model is given by

$$\mathcal{L} = \tfrac{1}{2}\mathrm{Tr}\{(\bar{D}D)^2(V^\mu V_\mu + V^{\dagger\mu}V^\dagger{}_\mu)\} + \tfrac{1}{4}(\bar{D}D)\{\bar{\Psi}_-[i\gamma^\mu(\partial_\mu + V_\mu) - M_0]\Psi\} + \mathrm{H.c.}, \tag{39}$$

where for simplicity we have used the notation of Salam and Strathdee.[4] In this expression, Ψ_+ and Ψ_- are independent chiral spinor superfields which belong to a representation of the group. An interesting point about such a model is that it easily admits the existence of a conserved fermion number. It would also be of some interest to see if this model is renormalizable in view of the model of Adjei and Akyeampong.[1] It is clear that the free propagator for the chiral spinor superfield here is just the Dirac propagator. This is to be compared with the propagator for the aforementioned model. Therefore, naively, we might suspect that the model of Eq. (39) may be renormalizable.

Note added in proof. At the completion of this work, we were informed by Dr. V. I. Ogievetskii and Dr. E. Sokachev that similar conclusions were reached about the spinor superfield by them (Zh. Eksp. Teor. Fiz. Pis'ma Red. 23, 66 (1976) [JETP Lett. 23, 58 (1976)]). We thank these authors for bringing this work to our attention.

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APPENDIX A: REPRESENTATION AND CONVENTIONS

We use a representation of the Dirac matrices where

$$\gamma^\mu \equiv (\sigma^3\otimes\sigma^2, iI\otimes\sigma^1, i\sigma^2\otimes\sigma^2, iI\otimes\sigma^3),$$

$$\gamma^5 \equiv i\,\frac{1}{4!}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma = i\gamma^0\gamma^1\gamma^2\gamma^3 = \sigma^1\otimes\sigma^2,$$

$$\sigma^{\mu\nu} \equiv i\tfrac{1}{2}[\gamma^\mu,\gamma^\nu].$$

Then the full set of Dirac matrices is given by $\underline{1}$, γ^μ, $\sigma^{\mu\nu}$, $\gamma^5\gamma^\mu$, and γ^5. In this representation we find

$$\{\gamma^\mu,\gamma^\nu\} = -2\eta^{\mu\nu},$$

$$\mathrm{diag}(\eta^{\mu\nu}) = (-1,1,1,1).$$

Under Hermitian conjugation, we find

$$\underline{1}^\dagger = \underline{1}, \quad \gamma^{5\dagger} = \gamma^5,$$

$$\gamma^{\mu\dagger} = -\sum_{\mu'}\eta^{\mu\mu'}\gamma^{\mu'},$$

$$(\gamma^5\gamma^\mu)^\dagger = \sum_{\mu'}\eta^{\mu\mu'}(\gamma^5\gamma^{\mu'}),$$

$$(\sigma^{\mu\nu})^\dagger = \sum_{\mu',\nu'}\eta^{\mu\mu'}\eta^{\nu\nu'}(\sigma^{\mu'\nu'}).$$

For this representation, we have an orthogonality relation,

$$\tfrac{1}{4}\mathrm{Tr}\{\Gamma_A\Gamma_B^\dagger\} = \delta_{AB},$$

if we restrict $\sigma^{\mu\nu}$ so that $\mu \le \nu$. This in turn implies a completeness relation given by

$$(\underline{1})^a{}_j(\underline{1})^i{}_b = \tfrac{1}{4}\sum_A(\Gamma_A)^a{}_b(\Gamma_A^\dagger)^i{}_j.$$

We may use this to derive the following Fierz rearrangement matrix:

$$-\tfrac{1}{4}\begin{array}{c|ccccc|c} & S & V & T & A & P & \\ \hline & 1 & -1 & 1 & 1 & 1 & S\\ & -4 & -2 & 0 & -2 & 4 & V\\ & 6 & 0 & -2 & 0 & 6 & T\\ & 4 & -2 & 0 & -2 & -4 & A\\ & 1 & 1 & 1 & -1 & 1 & P\end{array},$$

where

$$S(1234)\equiv(\bar{\psi}_1\psi_2)(\bar{\psi}_3\psi_4)\,,$$
$$V(1234)\equiv(\bar{\psi}_1\gamma^{\mu}\psi_2)(\bar{\psi}_3\gamma_{\mu}\psi_4)\,,$$
$$T(1234)\equiv\tfrac{1}{2}(\bar{\psi}_1\sigma^{\mu\nu}\psi_2)(\bar{\psi}_3\sigma_{\mu\nu}\psi_4)\,,$$
$$A(1234)\equiv(\bar{\psi}_1\gamma^5\gamma^{\mu}\psi_2)(\bar{\psi}_3\gamma^5\gamma_{\mu}\psi_4)\,,$$
$$P(1234)\equiv(\bar{\psi}_1\gamma^5\psi_2)(\bar{\psi}_3\gamma^5\psi_4)\,.$$

Note that the minus sign preceding the matrix is the consequence of the anticommutivity of ψ_1, ψ_2, ψ_3, and ψ_4. Furthermore, the sum on μ and ν in the tensor, T, is unrestricted.

Under charge-conjugation, time-reversal, and parity transformations, we define the following transformations for a Dirac field $\psi(t,\vec{x})$:

$$\psi^C(t,\vec{x})=(-\gamma^0)[\bar{\psi}(t,\vec{x})]^t=\psi^*(t,\vec{x})\,,$$
$$\psi^T(t,\vec{x})=\gamma^5\gamma^0\psi^*(-t,\vec{x})\,,$$
$$\psi^P(t,\vec{x})=i\gamma^0\psi(t,-\vec{x})\,.$$

With these conventions, we find that

$$\psi^{CTP}(t,\vec{x})=i\gamma^5\psi(-t,-\vec{x}).$$

Additionally, the matrices γ^0, $\gamma^0\gamma^5\gamma^{\mu}$, and $\gamma^0\gamma^5$ are antisymmetric, while $\gamma^0\gamma^{\mu}$ and $\gamma^0\sigma^{\mu\nu}$ are symmetric.

APPENDIX B: COMPONENTS OF THE SUPERFIELDS

The spinor superfield multiplet $\Lambda^a(\theta,x)$ may be written in the form

$$\Lambda^a(\theta,x)=2\phi^a(x)+\alpha^a(x)\theta+\tfrac{1}{4}\bar{\theta}\theta\zeta'^a(x)$$
$$-\tfrac{1}{4}\bar{\theta}\gamma^5\theta(\gamma^5\eta'^a(x))+\tfrac{1}{4}\bar{\theta}\gamma^5\gamma^{\lambda}\theta(-i\gamma^5\psi'^a_{\lambda}(x))$$
$$+\tfrac{1}{4}\bar{\theta}\theta\beta^a(x)\theta+\tfrac{1}{32}(\bar{\theta}\theta)^2\pi'^a(x)\,,$$

where a is an internal index. If we ignore this momentarily, then we see that this superfield contains a Rarita-Schwinger field Ψ'_{λ}, four Dirac fields ϕ, ζ', η', and π', and two sixteen-component matrix fields α and β. These matrix fields may also be expanded:

$$\alpha(x)=s(x)-i\gamma^{\mu}v_{\mu}(x)+i\tfrac{1}{2}\sigma^{\mu\nu}t_{\mu\nu}(x)+\gamma^5\gamma^{\mu}a_{\mu}(x)+i\gamma^5p(x)\,,$$
$$\beta(x)=\mathcal{S}(x)-i\gamma^{\mu}[\mathcal{V}_{\mu}(x)+\partial_{\mu}s(x)+\partial^{\kappa}t_{\kappa\mu}(x)]+i\tfrac{1}{2}\sigma^{\mu\nu}[\mathcal{T}_{\mu\nu}(x)-\partial_{\mu}v_{\nu}(x)+\partial_{\nu}v_{\mu}(x)+\epsilon_{\mu\nu\kappa\lambda}\partial^{\kappa}a^{\lambda}(x)]$$
$$+\gamma^5\gamma^{\mu}[\mathcal{A}_{\mu}(x)+\partial_{\mu}p(x)+\tfrac{1}{2}\epsilon_{\mu\alpha\beta\gamma}\partial^{\alpha}t^{\beta\gamma}(x)]+i\gamma^5[\mathcal{P}(x)+\partial^{\mu}a_{\mu}(x)]\,.$$

So, additionally the spinor superfields contain two scalars, two vectors, two antisymmetric tensors, two axial vectors, and two pseudoscalars. With our conventions, the reality of the spinor superfield implies that all boson components are real and that all fermion components are Majorana fields.

Furthermore, it may be convenient to express some of the fermion components as linear combinations of other fermion fields:

$$\zeta'^a=\zeta^a-i\gamma^{\lambda}\psi^a{}_{\lambda}+\eta^a+i\partial\!\!\!/\phi^a\,,$$
$$\eta'^a=\eta^a+i\partial\!\!\!/\phi^a\,,$$
$$\psi'^a{}_{\lambda}=\psi^a{}_{\lambda}-i\tfrac{1}{2}\gamma_{\lambda}\zeta^a-i\gamma_{\lambda}\eta^a+\partial\!\!\!/\gamma_{\lambda}\phi^a\,,$$
$$\pi^a=\pi^a+i\partial\!\!\!/\zeta^a+2\partial^{\lambda}\psi^a{}_{\lambda}+i2\partial\!\!\!/\eta^a\,.$$

The scalar superfield may be expanded in component form as

$$\Phi^a(\theta,x)=A^a(x)+\bar{\theta}\psi^a(x)+\tfrac{1}{4}\bar{\theta}\theta F^a(x)+i\tfrac{1}{4}\bar{\theta}\gamma^5\theta G^a(x)$$
$$+\tfrac{1}{4}\bar{\theta}\gamma^5\gamma^{\mu}\theta A^a_{\mu}(x)+\tfrac{1}{4}\bar{\theta}\theta\bar{\theta}\chi^a(x)+\tfrac{1}{32}(\bar{\theta}\theta)^2D^a(x)\,,$$

where A^a_{μ} is an axial-vector multiplet, A^a, F^a, and D^a are multiplets of scalar fields, G^a is a multiplet of a pseudoscalar field, and Ψ^a and X^a are multiplets of Dirac fields. Once again, the reality of the boson components and the Majorana properties of the fermion components are fixed by the requirement that the scalar superfield be real.

With these expansions, we may express the transformation law of Eq. (17) in terms of the component fields. In particular, for the fermion fields ϕ^a and the vector fields $v^a{}_{\mu}$ we find

$$\phi^a\to\phi^a-\delta\psi^a\,,$$

$$v^a{}_{\mu}\to v^a{}_{\mu}-\partial_{\mu}A^a-gf^a{}_{bc}A^b\,v^c{}_{\mu}-i2f^a{}_{bc}\delta\bar{\psi}^b\gamma_{\mu}\phi^c\,.$$

The first of these implies that there are gauges where ϕ^a vanishes, and in such a gauge the second of these transformation laws reduces to the usual transformation law of a Yang-Mills vector field.

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