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Classical null strings*

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Free Nambu strings in space-time are the two-dimensional sheet analog of one-dimensional curves which are timelike or spacelike geodesics. It is pointed out that there is also a sheet analog of one-dimensional curves which are null geodesics.

In the flat Minkowski space-time of special relativity theory, with metric $g_{\mu\nu} = \mathrm{diag}(-1, -1, -1, 1)$, consider the two action or variational principles

$$\delta\left(m\int_{u^*}^{u^{**}}(\dot{x}^2)^{1/2}du\right) = 0\,, \tag{1a}$$

$$\delta\left(\int_{u^*}^{u^{**}}\tfrac{1}{2}\dot{x}^2du\right) = 0\,, \tag{1b}$$

for a smooth curve $x(u)$, where m is the mass. The two end points are to be held fixed, i.e., $\delta x(u^*) = \delta x(u^{**}) = 0$. We have suppressed space-time indices, so that x, $\dot{x} = dx/du$, $\partial/\partial\dot{x}$ respectively stand for x^μ, $\dot{x}^\mu = dx^\mu/du$, $\partial/\partial\dot{x}^\mu$, and $\dot{x}^2 = \dot{x}^\mu\dot{x}_\mu = g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$; more generally we shall use scalar-product notation, $A\cdot B = A^\mu B_\mu = g_{\mu\nu}A^\mu B^\nu$ for two vectors A and B. It is well known that both variational principles (1) give geodesics, but there is an important difference:

The Euler-Lagrange equations for (1a) are[1]

$$m\frac{d}{du}\frac{\partial(\dot{x}^2)^{1/2}}{\partial\dot{x}} \equiv m\frac{d}{du}\left[(\dot{x}^2)^{-1/2}\dot{x}\right] = 0 \Leftrightarrow \ddot{x} = \lambda\dot{x}\,, \quad \text{for } \dot{x}^2 \neq 0\,, \tag{2a}$$

where λ is a function of u which depends on the parametrization. The left side of (2a) is meaningful only for $\dot{x}^2 \neq 0$. Thus the variational principle (1a) gives only timelike and spacelike geodesics. It does not determine the parameter u, as is also obvious from the invariance of $(\dot{x}^2)^{1/2}du$ under changes of the parameter $u' = u'(u)$.

The Euler-Lagrange equations for (1b) are

$$\frac{d}{du}\frac{\partial(\frac{1}{2}\dot{x}^2)}{\partial\dot{x}} \equiv \ddot{x} = 0\,. \tag{2b}$$

They have the first integral $\dot{x}^2 = \text{constant}$. Thus the variational principle (1b) gives all three classes of geodesics, timelike, spacelike, and null (for $\dot{x}^2$ positive, negative, or zero). It restricts u to be an affine parameter[2] which is unique to within linear transformations, $du'/du = \text{constant}$. We can always choose the affine parameter so that $\dot{x}$ is the 4-momentum of a free massive particle, a free tachyon, or a free particle of zero rest mass, which has the geodesic as its world line[3]; u is then unique to within an additive constant, $du'/du = 1$. In the non-null cases, this is done by writing the first integral in the form $\dot{x}^2 = \pm m^2$, where m is the mass of the particle; then $\dot{x} = m\,dx/ds$ for the timelike case, where s is the Minkowski arc length or proper time along the geodesic, and $\dot{x} = m\,dx/d\tau$ for the spacelike (or tachyon) case with $d\tau^2 = -ds^2 > 0$. It would thus appear that the variational principle (1b) is superior to (1a), since it is both simpler and more general.

Nambu and others[4] have proposed massless strings with tension as classical models, whose quantized states are identified with mesonic resonances which mediate the hadronic interactions. Mandelstam[5] has shown that the Veneziano amplitudes[6] can be obtained by breaking and joining open strings.

The history of a classical string is a sheet[7] in space-time, which can be described by $x = x(u^a) = x(u^1, u^2)$, where u^1 and u^2 are two parameters. Latin indices will be used throughout to label the two parameters; they range and sum over 1 and 2. We shall use the notation $x_a = \partial x/\partial u^a$ and $\partial/\partial u^a$ will denote the partial-derivative operator along a parameter curve on the sheet. An element of the sheet is characterized by the bivector (skew-symmetric tensor), σdu^1du^2, where $\sigma = x_1\wedge x_2 = x_1x_2 - x_2x_1$ and stands for

$$\sigma^{\mu\nu} = (\partial x^\mu/\partial u^1)(\partial x^\nu/\partial u^2) - (\partial x^\mu/\partial u^2)(\partial x^\nu/\partial u^1)\,.$$

Consider the two action or variational principles

$$\delta\left(\mu\int_{C^*}^{C^{**}}\int_{C^-}^{C^+}(\sigma^2)^{1/2}du^1du^2\right) = 0\,, \tag{3a}$$

$$\delta\left(\int_{C^*}^{C^{**}}\int_{C^-}^{C^+}\tfrac{1}{2}\sigma^2du^1du^2\right) = 0\,, \tag{3b}$$

for a smooth string, where $\sigma^2 = -\frac{1}{2}\sigma^{\mu\nu}\sigma_{\mu\nu}$ and μ is the tension. The initial and final curves C^* and C^{**} (Fig. 1) are to be held fixed, i.e., $\delta x = 0$ for x on C^* or C^{**}. Equations (3a) and (3b) are the two-dimensional analogs of (1a) and (1b), respectively.

We first consider the simpler case of a closed

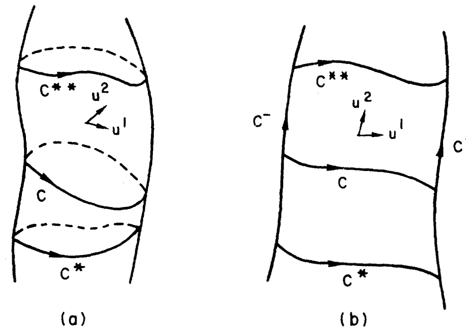

FIG. 1. Strings: (a) closed string; (b) open string.

string [Fig. 1(a)]. Its sheet has the cylindrical topology $S^1 \times R$, and $\int_{C^-}^{C^+}$ in Eqs. (3) must be interpreted as an integration $\oint$ around the cylinder.

The Euler-Lagrange equations for (3a) are the well-known string equations

$$\mu\frac{\partial}{\partial u^a}\frac{\partial(\sigma^2)^{1/2}}{\partial x_a} \equiv \mu\frac{\partial}{\partial u^a}[-(\sigma^2)^{-1/2}\epsilon^{ab}\sigma\cdot x_b]=0\,, \quad (4a)$$

where $\epsilon^{12}=-\epsilon^{21}=1$, $\epsilon^{11}=\epsilon^{22}=0$, and $\sigma\cdot x_b$ denotes the space-time vector $\sigma_{\mu\nu}x_b^\nu=\sigma_{\mu\nu}\partial x^\nu/\partial u^b$. The left-hand side of (4a) is meaningful only for $\sigma^2\neq 0$. Thus the variational principle (3a) gives only timelike ($\sigma^2>0$) and spacelike or tachyon ($\sigma^2<0$) (Ref. 8) strings. It does not determine the parameters u^a, as is also obvious from the invariance of $(\sigma^2)^{1/2}du^1du^2$ under changes of the parameters $u'^a=u'^a(u^1,u^2)$.

The Euler-Lagrange equations for (3b) are

$$\frac{\partial}{\partial u^a}\frac{\partial(\frac{1}{2}\sigma^2)}{\partial x_a} \equiv \frac{\partial}{\partial u^a}(-\epsilon^{ab}\sigma\cdot x_b)=0\,. \quad (4b)$$

Using the identity $x_b\cdot[\partial(\frac{1}{2}\sigma^2)/\partial x_a]\equiv\delta_b^a\sigma^2$, which is easily established, Eq. (4b) implies

$$\begin{aligned}0&=x_b\cdot(\partial/\partial u^a)[\partial(\tfrac{1}{2}\sigma^2)/\partial x_a]\\&=(\partial/\partial u^a)\{x_b\cdot[\partial(\tfrac{1}{2}\sigma^2)/\partial x_a]\}-[\partial(\tfrac{1}{2}\sigma^2)/\partial x_a]\cdot(\partial x_a/\partial u^b)\\&=\partial(\sigma^2)/\partial u^b-\partial(\tfrac{1}{2}\sigma^2)/\partial u^b\\&=\partial(\tfrac{1}{2}\sigma^2)/\partial u^b\,.\end{aligned}$$

Thus (4b) implies $\sigma^2=$ constant. When this constant is nonzero, we see immediately that (4b) implies (4a), so that the variational principle (3b) gives the timelike and spacelike strings (for σ^2 positive or negative).

However, (4b) also gives a *new third class* of strings when

$$\sigma^2=0\,. \quad (5)$$

We shall call them *null strings*, in analogy to null geodesics.

It is also easy to see that the variational principle (3b) restricts the parameters u^a to a preferred class, which we shall call the class of *affine parameters*. In the non-null cases, the affine parameters are unique to within parameter transformations with a constant Jacobian, $\partial(u'^1,u'^2)/\partial(u^1,u^2)=$ constant.

Let us now consider the case of open strings [Fig. 1(b)].

In addition to the equations of motion (4), which hold in the interior of the sheet, the variational principles (3) also give boundary conditions which hold at the free ends of the open string:

$$\mu(\sigma^2)^{-1/2}\sigma\cdot\mathring{x}=0\,,\quad \text{for } x \text{ on } C^- \text{ or } C^+\,, \quad (6a)$$

$$\sigma\cdot\mathring{x}=0\,,\quad \text{for } x \text{ on } C^- \text{ or } C^+\,, \quad (6b)$$

where the parametric equations of C^- and C^+ are respectively $x=x^-(u^-)$ and $x=x^+(u^+)$, and where $\mathring{x}=dx^-/du^-$ for x on C^- and $\mathring{x}=dx^+/du^+$ for x on C^+. This is easily seen by performing the first variations in Eqs. (3) and applying Stokes's theorem. If the parametrization (u^1,u^2) is smooth, i.e., if x_1, x_2 are independent, finite, and continuous on the whole sheet, including the boundaries $C^\pm$, then Eqs. (6) imply that $\mathring{x}$ and σ are both null on C^- and C^+:

$$\mathring{x}^2=0\,,\quad \text{for } x \text{ on } C^- \text{ or } C^+\,, \quad (7)$$

$$\sigma^2=0\,,\quad \text{for } x \text{ on } C^- \text{ or } C^+\,. \quad (8)$$

The case of an *open null string* is completely straightforward. The action principle is given by Eq. (3b); the equations of motion are (4b), with subsidiary conditions $\sigma^2=0$ which hold both in the interior of the sheet [Eq. (5)] and at its free edges C^- and C^+ [Eq. (8)]; finally, C^- and C^+ are null lines [Eq. (7)] and, as we shall show shortly, are in fact null geodesics.

The case of an open string which is not null is a bit more complicated. If we use the variational principle (3a) and general smooth parameters u^a, then at first sight Eq. (6a) seems to be self-contradictory, because $\sigma\cdot\mathring{x}=0$ implies $\sigma^2=0$ and thus $(\sigma^2)^{-1/2}=\infty$. This difficulty is avoided by interpreting the left-hand side of (6a) as a limit as x approaches the boundary $C^\pm$ from the interior of the sheet; then (6a) implies (6b), (7), and (8). If we use the variational principle (3b) for affine parameters u^a, then $\sigma^2=\text{const}\neq 0$ in the interior contradicts $\sigma^2=0$ on $C^\pm$ [Eq. (8)]. This is again avoided by choosing judicious affine parameters and interpreting the left-hand side of (6b) as a limit; then (6b) can be satisfied, $\sigma^2=\text{const}\neq 0$ in the interior of the sheet and on $C^\pm$, and $\mathring{x}^2=0$ on $C^\pm$. Thus Eqs. (6b) and (7) are valid. However, the affine parameters are no longer smooth on $C^\pm$, so that Eq. (8) does not apply. Let us illustrate this behavior by a well-known example:

A string of length 2 rotates rigidly and uniformly about its midpoint in the x^1x^2 plane. The corresponding sheet is given by $x=(r\cos t, r\sin t, 0, t)$, with $-1\leq r\leq 1$ and $-\infty<t<\infty$. Choose $u^2=t$, and $u=u^2=t$ on the boundary $C^{\pm}$, given by $r=\pm 1$. First choose $u^1=r$; the parameters u^a are then nonaffine. In the interior of the sheet $(-1<r<1)$ the equations of motion (4a) are satisfied, and $\sigma^2={x_2}^2=(1-r^2)$, $(\sigma^2)^{-1/2}\sigma\cdot x_2=(1-r^2)^{1/2}(\cos t, \sin t, 0, 0)$; thus in the limit as $r\to\pm 1$ and $x_2\to\dot{x}$, $\sigma^2=\dot{x}^2=0$ and $(\sigma^2)^{-1/2}\sigma\cdot\dot{x}=0$ on $C^{\pm}$. Next choose $u^1=(2\mu)^{-1}[r(1-r^2)^{1/2}+\sin^{-1}r]$, so that $dr/du^1=\mu(1-r^2)^{-1/2}$; the parameters u^a are then affine. In the interior of the sheet $(-1<r<1)$ the equations of motion (4b) are satisfied, and $\sigma^2=\mu^2$, ${x_2}^2=(1-r^2)$, $\sigma\cdot x_2=\mu(1-r^2)^{1/2}(\cos t, \sin t, 0, 0)$ and $x_1=\mu(1-r^2)^{-1/2}(\cos t, \sin t, 0, 0)$; thus in the limit as $r\to\pm 1$ and $x_2\to\dot{x}$, $\sigma^2=\mu^2$, $\dot{x}^2=0$, $\sigma\cdot\dot{x}=0$ on $C^{\pm}$; but $x_1\to\infty$, so that the affine parameters u^1, u^2 are not smooth on $C^{\pm}$.

The equations of motion (4) and, for open strings, the boundary conditions (6) imply that the vector

$$P=\mu\int_C |\sigma^2|^{-1/2}\sigma\cdot dx\,, \tag{9a}$$

$$P=\int_C \sigma\cdot dx \tag{9b}$$

is conserved, in the sense that the integral is independent of the choice of the curve C which circles the closed sheet [Fig. 1(a)] or runs from C^- to C^+ for open sheets [Fig. 1(b)].

For the action principle (3a), the vector P, given by Eq. (9a), is the 4-momentum of the string. For the action principle (3b), we can always choose the affine parameters u^a so that P, given by Eq. (9b), is the 4-momentum of the string. In the non-null cases, comparison of Eqs. (9a) and (9b) shows that this is done by writing the first integral of the equations of motion (4b) in the form

$$\sigma^2=\pm\mu^2\,, \tag{10}$$

where μ is the tension, and the plus sign applies to timelike sheets, while the minus sign applies to spacelike sheets. It is clear that, in the non-null cases, the affine parameters are unique to within parameter transformations with a unit Jacobian, $\partial(u'^1, u'^2)/\partial(u^1,u^2)=1$.

We shall now study the geometrical characterization of null strings, which satisfy Eqs. (4b) and (6b) and have the first integral $\sigma^2=0$.

A submanifold of space-time is called a *null submanifold* if, at every one of its points, it is tangent to the null cone with vertex at that point. Equivalently, a null submanifold is characterized by the property that, at every one of its points, the tangent space contains exactly one null direction N $(N^2=0)$ and independent spacelike directions S $(S^2<0)$ orthogonal to N $(S\cdot N)=0$); it follows that every tangential direction other than N is also spacelike and orthogonal to N, since it must be of the form $\sum\alpha S+\beta N$, with the α's not all zero. Thus a null submanifold determines a unique congruence of null curves in it, the curves which have the null directions N as tangents. If these null curves are geodesics, then the submanifold is called a *geodesic null submanifold.*

It is well known that every null hypersurface is geodesic.[9] This is not true for null sheets, as the following example shows: Choose a curve γ in the x^1x^2 plane which is not a straight line; let a particle move along γ with the speed of light; its world line Γ is then a nongeodesic null curve in the $x^1x^2x^4$ space; displace Γ along the x^3 direction; then Γ sweeps out a null sheet which is not geodesic.

We shall now show that the *history of a null string is a geodesic null sheet* and that, conversely, every geodesic null sheet satisfies the string equations (4b). It follows immediately that the free ends of an open null string move along null geodesics.

Consider a null sheet, the history of a null string, with the integral of the motion $\sigma^2=0$. This implies that σ may be written in the form $\sigma=S\wedge N=SN-NS$, where $S^2<0$, $N^2=0$, and $S\cdot N=0$. Any tangent vector S' to the null sheet, that is not in the direction of N, is of the form $S'=\alpha S+\beta N$, with $\alpha\neq 0$. Thus S' is spacelike since $S'^2=\alpha^2S^2<0$, and orthogonal to N since $S'\cdot N=0$. Also, $\sigma\cdot S'=-\alpha S^2 N$ is a nonzero null vector and $\sigma\cdot N=0$.

Let the null sheet be given by $x=x(u^1,u^2)$, where u^a are affine parameters. Since x_1 and x_2 are independent, at least one of them is spacelike, say x_1 so that ${x_1}^2<0$. Let the one-parameter family of null curves on the null sheet be given by $\phi(u^1,u^2)=\text{constant}$, with $\phi_1\neq 0$. Define the tangential null vector field N by $\sigma\cdot x_1=\phi_1 N$; along each of the null curves $x(u)$ on the null sheet, define the parameter u by $\dot{x}=N$, where the overdot denotes d/du. Since $\dot{x}=x_1\dot{u}^1+x_2\dot{u}^2$ and $\dot{x}=(\phi_1)^{-1}\sigma\cdot x_1=(\phi_1)^{-1}x_1\cdot x_2x_1-(\phi_1)^{-1}{x_1}^2x_2$, comparison of the coefficients of x_2 shows that $\dot{u}^2=-(\phi_1)^{-1}{x_1}^2\neq 0$. We have $0=\sigma\cdot\dot{x}=\sigma\cdot x_1\dot{u}^1+\sigma\cdot x_2\dot{u}^2$ and $0=\dot{\phi}=\phi_1\dot{u}^1+\phi_2\dot{u}^2$; it follows that $\sigma\cdot x_2=\phi_2 N$ and that $\sigma\cdot x_a=\phi_a N$.

We can now compute $-(\partial/\partial u^a)(\epsilon^{ab}\sigma\cdot x_b)=-(\partial/\partial u^a)(\epsilon^{ab}\phi_b N)=-\epsilon^{ab}\phi_b\partial N/\partial u^a=N_2\phi_1-N_1\phi_2=(\phi_1/\dot{u}^2)(N_2\dot{u}^2+N_1\dot{u}^1)=(\phi_1/\dot{u}^2)\dot{N}$. Thus the equations of motion (4b) state precisely that

$$\dot{N}=0\,. \tag{11}$$

This proves that the null curves are geodesic, that u is an affine parameter along each null geodesic, and that the history of a null string is a geodesic null sheet.

For null strings we have a stronger conservation law than Eq. (9b). Not only is the total 4-momentum P conserved, but the element of 4-momentum between two neighboring null geodesics,

$$dP = \sigma \cdot dx = \sigma \cdot x_a du^a = N\phi_a du^a = N d\phi\,, \tag{12}$$

is also conserved. This can also be seen from the fact that an infinitesimal strip of a geodesic null sheet bounded by two neighboring null geodesics can be regarded as the history of an infinitesimal open null string with the boundary conditions (6b) satisfied at its ends.

We shall therefore regard a null string as being characterized both by the geodesic null sheet which gives its history, and by the distribution of 4-momentum dP along its length. This is illustrated in Fig. 2(a).

The general null string can be obtained by the construction illustrated in Fig. 2(b). Choose a spacelike curve C with parameter ϕ, $x = x(\phi)$, and along C choose a field of null vectors $N(\phi)$ orthogonal to C, so that $N^2 = 0$ and $N \cdot (dx/d\phi) = 0$. Construct the null geodesics with N as tangents, and along each null geodesic choose an affine parameter u which is zero on C and such that $N = \partial x/\partial u$ on C. The resulting geodesic null sheet has points $x = x(\phi, u)$. Using the notation that a prime denotes $\partial/\partial\phi$, and an overdot denotes $\partial/\partial u$, we have

$$\dot{x}^2 = 0, \quad \dot{x} \cdot x' = 0, \quad \ddot{x} = 0\,, \tag{13}$$

$$\dot{x} = N, \quad dP = N d\phi\,. \tag{14}$$

It remains to be shown that affine parameters u^a can be found so that the equations of motion (4b) are satisfied and $dP = \sigma \cdot dx$, with $\sigma = x_1 \wedge x_2$. This is done by choosing the special affine coordinates $u^1 = \phi$, $u^2 = \int_0^u (-x'^2)du$, where the integration is

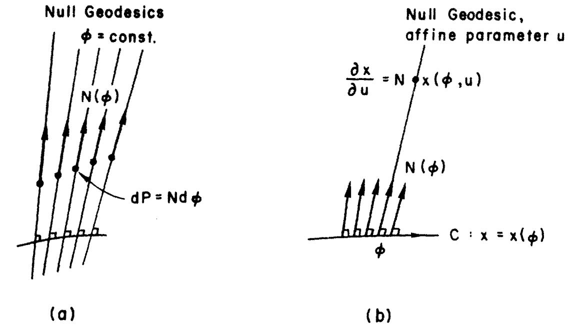

FIG. 2. The null string.

along a null geodesic $u^1 = \phi = \text{const}$. For a null string with a given distribution of 4-momentum, the affine parameters are unique to within parameter transformations with a unit Jacobian, $\partial(u'^1, u'^2)/\partial(u^1, u^2) = 1$.

This concludes our analysis of classical null strings. It is hoped that their quantization will contribute to our understanding of elementary particles.

The following generalizations are possible:

All our results are valid for an n-dimensional flat space-time with normal-hyperbolic signature, i.e., with metric $g_{\mu\nu} = \text{diag}(-1, -1, \ldots, -1, +1)$, if the term "sheet" denotes a two-dimentional submanifold and the term "hypersurface" an $(n-1)$-dimensional submanifold.

Our results are easily extended to systems of strings with interactions. An example of this are Nambu-Kalb-Ramond strings with interactions given by a Fokker-type action principle.[10] The action principles (3) are replaced by

$$\text{Lim}\,\delta\Big(\sum \mu \int_{C^*}^{C^{**}} \int_{C^-}^{C^+} (\sigma^2)^{1/2} du^1 du^2 + \sum\sum \tfrac{1}{2} e\bar{e} \int_{C^*}^{C^{**}} \int_{C^-}^{C^+} du^1 du^2 \int_{\bar{C}^*}^{\bar{C}^{**}} \int_{\bar{C}^-}^{\bar{C}^+} d\bar{u}^1\, d\bar{u}^2 \Lambda\Big) = 0\,, \tag{15a}$$

$$\text{Lim}\,\delta\Big(\sum \int_{C^*}^{C^{**}} \int_{C^-}^{C^+} \tfrac{1}{2}\sigma^2 du^1 du^2 + \sum\sum \tfrac{1}{2} e\bar{e} \int_{C^*}^{C^{**}} \int_{C^-}^{C^+} du^1 du^2 \int_{\bar{C}^*}^{\bar{C}^{**}} \int_{\bar{C}^-}^{\bar{C}^+} d\bar{u}^1 d\bar{u}^2 \Lambda\Big) = 0\,. \tag{15b}$$

Here the strings of the system are described by $x = x(u^1, u^2)$, $\bar{x} = \bar{x}(\bar{u}^1, \bar{u}^2), \ldots$, tensions $\mu, \bar{\mu}, \ldots$, and coupling constants $e, \bar{e}, \ldots$, and $\sum$ denotes a summation over the strings of the system. The interaction function Λ is a Lorentz-invariant function of the form $\Lambda = \Lambda(x - \bar{x}; x_1, x_2; \bar{x}_1, \bar{x}_2)$ which is assumed to be homogeneous of degree one in each of the variables x_1, x_2, $\bar{x}_1$ and $\bar{x}_2$; it may also be assumed, without loss of generality, to be symmetric in the two strings, i.e., $\Lambda(x - \bar{x}; x_1, x_2; \bar{x}_1, \bar{x}_2) = \Lambda(\bar{x} - x; \bar{x}_1, \bar{x}_2; x_1, x_2)$. The variations δx, $\delta \bar{x}, \ldots$, are to have *finite support*, but are otherwise arbitrary; after such a variation is performed, "Lim" in Eqs. (15) denotes that the limit is to be taken as all initial curves $C^*, \bar{C}^*, \ldots$ recede to the infinite past, and all final curves $C^{**}, \bar{C}^{**}, \ldots$ recede to the infinite future. It can be shown that the equations of motion obtained from the action principle (15b) have first integrals $\sigma^2 = \text{constant}$ for each string, and that for non-null strings ($\sigma^2 \neq 0$) the identification $\sigma^2 = \pm\mu^2$ ensures the equivalence of (15a) and (15b). However, the action principle (15b) also admits null strings for which $\sigma^2 = 0$. Expressions can be obtained[11] for the total 4-momentum P of the system, which is a generalization of Eqs. (9), and for the total angular momentum of the system about an event O, which is described by a bivector $L_{(0)}$; P and $L_{(0)}$ are conserved as a consequence of the equations of motion.

Except for Eqs. (9) and (15), all our results can be generalized immediately to the curved space-times of general relativity theory and to general coordinates x^μ, by using a metric tensor field $g_{\mu\nu}(x)$ of normal-hyperbolic signature and replacing partial derivatives $\partial/\partial u^a$ by absolute (or covariant) derivatives $D/\partial u^a$ along the parameter curves. Thus, for example, σ^2 is defined by $\sigma^2 = -\frac{1}{2}g_{\mu\alpha}g_{\nu\beta}\sigma^{\mu\nu}\sigma^{\alpha\beta}$, and the Euler-Lagrange equations (4b) are replaced by

$$\frac{\delta(\frac{1}{2}\sigma^2)}{\delta x^\mu} \equiv \frac{\partial}{\partial \mu^\mu}\frac{\partial(\frac{1}{2}\sigma^2)}{\partial x^\mu_a} - \frac{\partial(\frac{1}{2}\sigma^2)}{\partial x^\mu} \tag{16}$$

$$\equiv \frac{D}{\partial u^a}\frac{\partial(\frac{1}{2}\sigma^2)}{\partial x^\mu_a} \equiv \frac{D}{\partial u^a}(-\epsilon^{ab}\sigma_{\mu\nu}x^\nu_b) = 0\,. \tag{17}$$

The expressions (16) and the first expression (17) are vectors, which agree at the origin of geodesic (or Riemannian) coordinates,[12] and hence agree in all coordinate systems; this proves that Eqs. (17) are the equations of motion for the action principle (3b).

Note Added. Lagrange principles, such as (1b) and (3b), which are not invariant under arbitrary reparametrization, may lead to difficulties if one attempts a Hamiltonian formulation which is a prerequisite for quantization. An alternative approach for the quantization of a massless system is to start with a massive system and then let the masses tend to zero. This has been studied by Bardeen, Bars, Hanson, and Peccei,[14] and is also discussed in the Tbilisi review article by Nielsen.[15]

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[1] $A \Leftrightarrow B$ is understood to mean that A implies B, that B implies A, and that both A and B are true.

[2] The term "affine parameter u along a geodesic" is defined precisely by the fact that the equation of the geodesic has its simplest form $d^2x/du^2 = 0$. In the non-null cases, it is essentially the arc length to within a constant factor.

[3] A. Schild, Ann. Phys. (N.Y.) 93, 88 (1975); 99, 434 (1976).

[4] Y. Nambu, in *Symmetries and Quark Models*, edited by R. Chand (Gordon and Breach, New York, 1970); for other references see P. Ramond, review article in *Particles and Fields*—1974, proceedings of the 1974 Williamsburg meeting of the Division of Particles and Fields of the American Physical Society, edited by Carl E. Carlson (AIP, New York, 1975).

[5] S. Mandelstam, Nucl. Phys. B64, 205 (1973); Phys. Lett. 46B, 447 (1973); Nucl. Phys. B69, 77 (1974).

[6] G. Veneziano, Nuovo Cimento 57A, 190 (1968).

[7] Throughout, we shall use the term "sheet" to denote a two-dimensional submanifold of four-dimensional space-time, and the term "hypersurface" to denote a three-dimensional submanifold.

[8] Tachyon strings with spacelike sheets may be physically uninteresting. They are included here for the sake of completeness.

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