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A "superposition" of static, cylindrically symmetric solutions of the Einstein-Maxwell field

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The already unified field theory of Rainich, Misner, and Wheeler is used to examine the solution of a geometry which is locally a static cylindrically symmetric geometry but whose global properties preclude this interpretation. It is shown that this geometry can be interpreted as a "superposition" of two static cylindrically symmetric universes. In particular, the fields found are those exterior to a line of current with mass and interior to a coaxial solenoidal current. Physically reasonable sources are shown to exist.

I. INTRODUCTION

The solution of the Einstein-Maxwell problem with a static cylindrically symmetric line element has previously been investigated using the already unified field theory of Rainich, Misner, and Wheeler (RMW).[1-4] Discarding solutions which require magnetic monopoles or magnetic currents, there are only three possible static cylindrical solutions:

I. Case I consists of a magnetic field about the axis of symmetry. Thus it corresponds to the field exterior to a line current. If the magnetic field is nonzero there must be mass along the axis.

II. Case II consists of a magnetic or electric (or combination) field parallel to the symmetry axis. It can be subdivided into either an electric or magnetic field but any superposition is also included as a solution. There may or may not be mass along the axis.

A. Case II A is a magnetic field parallel to the symmetry axis. Physically it could correspond to the field interior to a solenoidal current whose axis is the symmetry axis. However, there may be no apparent source of the magnetic field at finite distances. If so and if there is no mass along the symmetry axis we have what is termed a magnetic geon.

B. Case II B is an electric field parallel to the symmetry axis. Physically it could correspond to the field interior to a parallel-plate capacitor. Again an electric geon exists.

III. Case III consists of an electric field perpendicular to the symmetry axis. Thus it corresponds to the field exterior to a line of charge.

It was surprising to me that fields produced by superpositions of the sources for cases I, II, or III were not static cylindrically symmetric. I have therefore set out on a search for the proper metrics to use to find the "superposition" of the fields of static cylindrical symmetry. I will use the RMW theory as a tool to be certain that all possible Einstein-Maxwell source-free solutions are found.

In this paper I will investigate a line element which, as it turns out, produces an electromagnetic field which can be interpreted as a superposition of the sources of cases I and II A. Although arguments are presented to suggest that this is the proper interpretation of the line element, I use the RMW theory to deduce the physical interpretation of the fields and sources. It is believed that such an approach contains only a "symmetry" assumption and hence the solutions found are the most general for that symmetry.

However, when using an explicit coordinate system we must be careful to distinguish local symmetry from global symmetry. In particular I will use a coordinate system where the three spacelike coordinates can be interpreted as a space distance (ρ) perpendicular to the symmetry axis, an angle (ϕ) about the symmetry axis, and a distance (z) along the symmetry axis. The solution I find corresponds to a ϕ-directed magnetic field and a z-directed magnetic field. It is always possible to locally define $\phi' = \phi'(\phi, z)$ and $z' = z'(\phi, z)$ such that the ϕ' or z' axis lies along the direction of the total magnetic field. However, it is no longer possible to interpret ϕ' as an angle so the overall global symmetry becomes hidden. This was also indicated in cases I and II where the metric of either can be obtained from the other by interchanging the metric component associated with the angular direction with that associated with the axial direction.

In the next section I summarize the RMW formalism and apply it to the metric form to be studied. Without actually solving any of the field equations I am able to determine what electromagnetic fields and what sources are present. This shows the advantage of the RMW method as a calculational tool over the normal Einstein-Maxwell approach. I find that the physical field present is a pure magnetic field lying in a surface defined by being at a constant distance from the symmetry axis, i.e., the ϕ, z surface of the preceding paragraph.

Section III relates the solution of the problem to the known solutions of cases I and II. I find that since the problem is locally the same as that of cases I and II, I am able to express the solutions in terms of cases I and II and a single parameter. This parameter, which can be interpreted as a constant angle, gives us a useful measure of the relative strengths of the magnetic field in the z and ϕ directions. As this angle is adjusted from 0 to $\pi/2$ the nature of the universe changes from pure case I to pure case I to pure case II A.

Section IV discusses the motion of test particles. I show that this motion can be analyzed in terms of the already known motions of cases I and IIA. The final section discusses a source model which could give rise to the fields considered.

II. RAINICH, MISNER, AND WHEELER SOLUTIONS FOR A LOCALLY STATIC CYLINDRICALLY SYMMETRIC METRIC

The Einstein-Maxwell equations for a combined gravitational and electromagnetic field in the absence of electromagnetic sources are[5]

$$R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R=-\tfrac{1}{2}T_{\mu\nu}\equiv-\tfrac{1}{4}\omega_{\mu\alpha}\omega_{\nu}{}^{\alpha}, \tag{2.1}$$

$$\omega_{\alpha\beta,\gamma}+\omega_{\beta\gamma,\alpha}+\omega_{\gamma\alpha,\beta}=0, \tag{2.2}$$

$$\omega_{\mu\nu}\equiv f_{\mu\nu}+\tfrac{1}{2}i(-g)^{1/2}\epsilon_{\mu\nu\alpha\beta}f^{\alpha\beta}. \tag{2.3}$$

$f_{\mu\nu}$ is the antisymmetric electromagnetic field which in a local Minkowski frame is determined by $f_{01}=E_x$, $f_{12}=H_z$, etc.; $\epsilon_{\mu\nu\sigma\tau}$ is the completely antisymmetric Levi-Civita tensor density and g is $\det g_{\mu\nu}$. In the presence of an electromagnetic-charge-current vector j^μ, Eq. (2.2) can be written in the integral form

$$\int\!\!\int_{\mu<\nu}\omega_{\mu\nu}d(x^{(\mu)},x^{(\nu)})$$

$$=i\int\!\!\int\!\!\int_{\mu<\sigma<\tau}(-g)^{1/2}\epsilon_{\mu\nu\sigma\tau}j^\mu d(x^{(\nu)},x^{(\sigma)},x^{(\tau)}). \tag{2.4}$$

The integral on the left-hand side is a surface integral taken over a closed 2-dimensional surface. If the surface is described by the parameters μ_1 and μ_2, we have

$$d(x^{(\mu)},x^{(\nu)})=\frac{\partial(x^{(\mu)},x^{(\mu)})}{\partial(\mu_1,\mu_2)}d\mu_1 d\mu_2. \tag{2.5}$$

The integral on the right-hand side of (2.4) is the volume integral over the 3-dimensional volume enclosed by the surface over which the left-hand side was integrated.

The theory described by (2.1), (2.2), and (2.3) is equivalent to a purely geometrical theory for which the following equations are valid[2]:

$$R\equiv R_\mu{}^\mu=0, \tag{2.6}$$

$$R_\mu{}^\alpha R_\alpha{}^\nu=\tfrac{1}{4}R_{\alpha\beta}R^{\alpha\beta}\delta_\mu{}^\nu, \tag{2.7}$$

$$\alpha_{\mu,\nu}-\alpha_{\nu,\mu}=0, \tag{2.8}$$

$$\alpha_\sigma\equiv(-g)^{1/2}\epsilon_{\sigma\nu\lambda\mu}R^{\lambda\gamma;\mu}R_\gamma{}^\nu/R_{\alpha\beta}R^{\alpha\beta}, \tag{2.9}$$

$$R_{\mu\nu}\nu^\mu\nu^\nu\geq 0. \tag{2.10}$$

Equation (2.10) must be true for any timelike vector ν^μ. Actually, the geometrical theory (2.6)–(2.10) is equivalent to the usual theory (2.1)–(2.3) only for non-null fields, i.e., fields for which $R_{\alpha\beta}R^{\alpha\beta}$ do not vanish. I will only deal with non-null fields in this paper.

If Eqs. (2.6) and (2.7) are valid, it can be shown that there are two null eigenvectors, k_μ and l_μ of $R_{\mu\nu}$ normalized such that $k_\mu l^\mu=\rho$, where $\rho^2=\tfrac{1}{4}R_{\alpha\beta}R^{\alpha\beta}$. $\omega_{\mu\nu}$ can be determined from k_μ and l_μ as follows:

$$R_\mu{}^\nu k_\nu=\rho k_\mu,\quad R_\mu{}^\nu l_\nu=\rho l_\mu, \tag{2.11}$$

$$\omega_{\mu\nu}=\rho^{-1/2}[l_\mu k_\nu-k_\mu l_\nu+i(-g)^{1/2}\epsilon_{\mu\lambda\nu\sigma}k^\lambda k^\sigma]e^{i\theta}. \tag{2.12}$$

With ω chosen in this way and θ an arbitrary function of the coordinates, (2.1) is satisfied. If and only if (2.8) is satisfied, θ can be chosen up to a constant by a line integral[2] of α_μ so that the resultant ω satisfies (2.2). If there are null fields present the complexion can still be satisfactorily handled as has been shown by Geroch.[6]

The sources given by Eq. (2.4) are expressed in a more obvious form by using what is termed the physical components of the field. Let $\lambda^\mu_{(\alpha)}$ be an orthonormal tetrad with $\lambda^\mu_{(0)}$ timelike and future pointing. From $f_{\alpha\beta}$ the invariants

$$f_{(\alpha\beta)}=f_{\mu\nu}\lambda^\mu_{(\alpha)}\lambda^\nu_{(\beta)} \tag{2.13}$$

are formed. These invariants are referred to as physical components.

Then the source of the fields is given by

$$\text{source}=-\int_{v_3}E(L)L_\mu J^\mu d_3V$$

$$=\oint E(N)E(M)f_{\mu\nu}M^\mu N^\nu d_2V. \tag{2.14}$$

d_2V is an invariant element of area, d_3V is an invariant element of volume, V_2 is the two-surface bounding V_3, V_2, M, and N in order form a right-handed orthonormal tetrad, L is a unit vector orthogonal to V_3, and $E(N)$ is +1 (−1) if N is spacelike (timelike). J^μ is the current 4-vector and $f_{\mu\nu}N^\mu N^\nu$ is the physical field (2.13).

I take as the line element of distance

$$ds^2 = e^{2\gamma-2\psi}(-dt^2 + d\rho^2) + \rho^2 e^{-2\psi} d\phi^2 + e^{2\psi+2\mu} dz^2 + 2\beta\rho e^{\mu} d\phi dz$$
$$= g_{11}(-dt^2 + d\rho^2) + g_{22} d\phi^2 + g_{33} dz^2 + 2g_{23} d\phi dz , \tag{2.15}$$
$$-\infty < t = x^0 < \infty , \quad 0 \le \phi = x^2 < 2\pi ,$$
$$-\infty < z = x^3 < \infty , \quad 0 \le \rho = x^1 < \infty ,$$

where γ, ψ, μ, and β are functions of ρ alone. Although a local rotation in the ϕ-z plane would diagonalize this metric, I will not do such a rotation since we could no longer interpret ϕ as an angle. We know the solutions of this problem when $\beta \equiv 0$ (static cylindrically symmetric) when ϕ can be interpreted as an angle. What I want then, is to keep our coordinate system fixed and see what effect the function β has. This is what I meant earlier by the distinction between local "symmetry" and global "symmetry." I will show in the next section that the local symmetry enables me to present an exact solution to this metric.

A calculation of $R^\mu{}_\nu$ for the line element of the form (2.15) shows that all $R^\mu{}_\nu = 0$ for $\mu \ne \nu$ except $R^2{}_3$ and $R^3{}_2$. Thus Eqs. (2.6)–(2.7) give

$$R^0{}_0 + R^1{}_1 = 0 = R^2{}_2 + R^3{}_3 ,$$
$$(R^0{}_0)^2 = (R^2{}_2)^2(1+\Delta), \quad \Delta = R^2{}_3 R^3{}_2/(R^2{}_2)^2 , \tag{2.16}$$

and a complexion

$$\alpha_0 = \text{constant}$$
$$= [(-g)^{1/2}/(R_{\alpha\beta}R^{\alpha\beta})]$$
$$\times (2R_{23;1}R^2{}_2 + R_{33;1}R^3{}_2 - R_{22;1}R^2{}_3) ,$$

where $R_{23;1}$ is the covariant derivative of R_{23} with respect to ρ.

I choose an orthonormal tetrad as

$$\lambda^\mu_{(0)} = ((g_{11})^{-1/2}, 0, 0, 0) ,$$
$$\lambda^\mu_{(1)} = (0, (g_{11})^{-1/2}, 0, 0) ,$$
$$\lambda^\mu_{(2)} = (0, 0, (g_{22})^{-1/2}, 0) , \tag{2.17}$$
$$\lambda^\mu_{(3)} = (0, 0, -\beta(1-\beta^2)^{-1/2}(g_{22})^{-1/2} , (1-\beta^2)^{-1/2}(g_{33})^{-1/2}) ,$$

so even for this tetrad coordinate (2) can be interpreted as an angle. There are two cases possible:

(a) $R^0{}_0 = +R^2{}_2(1+\Delta)^{1/2}$. Then the solutions of Eq. (2.11) are

$$l^\mu = (l^0, 0, Al^0, BAl^0) ,$$
$$k^\mu = (k^0, 0, -Ak^0, -BAk^0) ,$$
$$B = R^3{}_2/(R^0{}_0 + R^2{}_2), \tag{2.18}$$
$$A^2 = (g_{11}/g_{22})\lambda^2,$$
$$1/\lambda^2 = 1 + 2Bg_{23}/g_{22} + B^2 g_{33}/g_{22} ,$$
$$k^0 l^0 = -|R^0{}_0|/(2g_{11}) .$$

The fields, $f_{\mu\nu}$, and the sources are calculated from (2.11) and (2.14). To avoid magnetic monopoles I choose $\theta = 0$ and hence

$$f_{12} = -Bf_{13} = (-B)(g_{11}/g_{22})^{1/2}(g_{22}g_{33} - g_{23}{}^2)^{1/2} \times |R^0{}_0|^{1/2}\lambda .$$

So

$$f_{(12)} = -B(g_{33}/g_{22})^{1/2}(1-\beta^2)^{1/2}|R^0{}_0|^{1/2}\lambda = B_z , \tag{2.19}$$

$$f_{(13)} = (1 + Bg_{23}/g_{22})|R^0{}_0|^{1/2}\lambda$$
$$= -B_\phi$$
$$= -[I/(2\pi\rho)]e^{2\psi-\gamma} , \tag{2.20}$$

where

$$I/2\pi = -(g_{11}g_{22})^{1/2}(1 + Bg_{23}/g_{22})|R^0{}_0|^{1/2}\lambda \tag{2.21}$$

is found from Eq. (2.14).

Since $\theta = 0$ we must have $\alpha_0 = 0$ as the only physically acceptable solution. Thus the physical fields and sources are a "sum" of case I and case IIA. Obviously, as $\beta(\rho) \to 0, B \to 0$ and the solution reduces to case *I*.

(b) $R^0{}_0 = -R^2{}_2(1+\Delta)^{1/2}$. The analogs of Eq. (2.15) are

$$l^\mu = (l^0, 0, BAl^0, Al^0) ,$$
$$k^\mu = (k^0, 0, -BAk^0, -Ak^0) ,$$
$$B = R^2{}_3/(R^0{}_0 - R^2{}_2) , \tag{2.22}$$
$$A^2 = (g_{11}/g_{33})\lambda^2 ,$$
$$1/\lambda^2 = 1 + 2Bg_{23}/g_{33} + B^2 g_{22}/g_{33} ,$$
$$k^0 l^0 = -|R^0{}_0|/(2g_{11}) .$$

Again $\theta = 0$ and

$$f_{(12)} = (1-\beta^2)^{1/2}|R^0{}_0|^{1/2}\lambda = B_z , \tag{2.23}$$
$$f_{(13)} = -(g_{22}/g_{33})^{1/2}(B + g_{23}/g_{22})|R^0{}_0|^{1/2}\lambda$$
$$= -B_\phi$$
$$= -(I/2\pi\rho)e^{2\psi-\gamma} \tag{2.24}$$

with

$$I/2\pi = (g_{11}/g_{33})^{1/2}(g_{23} + Bg_{22})|R^0{}_0|^{1/2}\lambda . \tag{2.25}$$

Again $\alpha_0 = 0$ is the only physically acceptable solution. Hence, for (b) the physical fields and sources are a "sum" of cases IIA and I. Once again the limit $\beta \to 0$ is well defined. Now $\beta \to 0$ yields case IIA as the limit.

It would appear that to proceed further I must solve Eqs. (2.16) for the functions, γ, ψ, μ, and β. At this point we see the disadvantage of the RMW approach. Some of these equations are, in general, quadratic in the second derivatives of $g_{\mu\nu}$. However, for this problem the solutions become trivial if I make use of the local symmetry and if I can show that the rotation angle in the ϕ-z plane is independent of the radial coordinate ρ.

III. SOLUTIONS USING LOCAL SYMMETRY

I now solve Eqs. (2.16) using the local symmetry. The metric (2.15) is diagonalized by the transformation

$$\rho\, d\bar{\phi} = \cos\epsilon\, \rho\, d\phi - \sin\epsilon\, dz\,, \qquad d\bar{z} = \sin\epsilon\, \rho\, d\phi + \cos\epsilon\, dz\,. \tag{3.1}$$

I later show the consistency of ϵ = constant. The new line element is written in the form

$$ds^2 = -e^{2\bar{\gamma}-2\bar{\psi}}(dt - d\rho^2) + \rho^2 e^{2\bar{\psi}} d\bar{\psi}^2 + e^{2\bar{\mu}+2\bar{\psi}} d\bar{z}^2\,. \tag{3.2}$$

Consider now case (a) of the preceding section; the solution to (3.2) is just of the form of case I of the previous work by Witten and by Safko and Witten.[1,3] Hence,

$$\begin{aligned} ds^2 &= -\rho^{2c^2+2c}(1+k\rho^{-2c})^2 e^{2a}(dt^2 - d\rho^2) \\ &\quad + \rho^{2c+2}(1+k\rho^{-2c})^2 d\bar{\phi}^2 + \rho^{-2c}(1+k\rho^{-2c})^{-2} d\bar{z}^2 \\ &= -\bar{g}_{11}(dt^2 - d\rho^2) + \bar{g}_{22} d\bar{\phi}^2 + \bar{g}_{33} d\bar{z}^2\,. \end{aligned} \tag{3.3}$$

However, at this point I cannot take the previous interpretations[3,4] of c, k, and a since $\bar{\phi}$ is not an angle. Since the relationships among the quantities in (2.16), the Ricci tensors, etc., calculated from (2.16) and the equivalent quantities in the frame of (3.3) (such quantities denoted by an overbar) are related by the angle ϵ, we have

$$\begin{aligned} & g_{11} = \bar{g}_{11}, \quad g_{22} = \bar{g}_{22}\cos^2\epsilon + \bar{g}_{33}\rho^2 \sin^2\epsilon\,, \\ & g_{33} = \bar{g}_{22}\frac{\sin^2\epsilon}{\rho^2} + \bar{g}_{33}\cos^2\epsilon\,, \\ & g_{23} = (-\bar{g}_{22}/\rho + \rho\bar{g}_{33})\cos\epsilon \sin\epsilon\,, \\ & R^2{}_2 = \bar{R}^2{}_2 \cos(2\epsilon) = -R^3{}_3\,, \\ & R^2{}_3 = R^3{}_2 = -\sin(2\epsilon)\bar{R}^2{}_2\,, \\ & B = -\tan\epsilon\,. \end{aligned} \tag{3.4}$$

Equation (3.4) satisfies Eq. (2.16) and a direct calculation shows $\alpha_0 \equiv 0$. So the complexion Eqs. (2.9) provide no restriction on the solution.

In terms of ϵ, and the $\bar{g}_{\mu\nu}$,

$$\begin{aligned} B_\phi &= -2c(k)^{1/2}(\bar{g}_{11}\bar{g}_{22})^{-1/2} \\ &\quad \times [\cos^2\epsilon + (\bar{g}_{33}/\bar{g}_{22})\sin^2\epsilon]^{-1/2}\cos\epsilon \\ &= (B_\phi)_{\rm I}[\cos^2\epsilon + (\bar{g}_{33}/\bar{g}_{22})\sin^2\epsilon]^{-1/2}\cos\epsilon\,, \end{aligned} \tag{3.5}$$

$$\begin{aligned} B_z &= +2c(k)^{1/2}(g_{11}g_{22})^{-1}(\bar{g}_{33})^{1/2} \\ &\quad \times [\cos^2\epsilon + (\bar{g}_{22}/\bar{g}_{33})\sin^2\epsilon]^{-1/2}\sin\epsilon \\ &= (B_z)_{\rm II}[\sin^2\epsilon + (g_{33}/g_{22})\cos^2\epsilon]^{-1/2}\sin\epsilon\,, \end{aligned} \tag{3.6}$$

where

$$I/2\pi = -2c(k)^{1-2}\cos\epsilon \tag{3.7}$$

is a current along the z axis.

Since the current along the axis must be independent of the size of the pillbox used in Eq. (2.14), I must be independent of ρ so (3.7) forces ϵ = constant. So four constants describe the solution: c, a, $k>0$, and ϵ. It is possible that $c = c(\epsilon)$. When $\epsilon = 0$, and weak fields $\frac{1}{2}c$ can be identified as the mass per unit length of the Newtonian theory, k as a measure of the magnetic field in the ϕ direction, and a as a matching parameter for a real physical mass distribution extending a distance ρ_0 from the axis. Also, a is needed as a measure of some parameter of a physical system which is initially and finally stationary but undergoes radial motions in between.

As $\epsilon \to \pi/2$, $A \to 0$ but $BA \to (\bar{g}_{11}/\bar{g}_{22})^{1/2}$, hence the k^μ and l^μ of Eq. (2.15) become the appropriate ones for case IIA. Also as $\epsilon \to \pi/2$, $g_{22} \to \bar{g}_{33}$ and $g_{33} \to \bar{g}_{22}$, so the limit $\epsilon \to \pi/2$ is well defined (case IIA) and we are able to describe any superposition of B_ϕ and B_z with form a. The only ϵ that might give trouble, $\epsilon = \pi/4$, can be handled by taking $A^2 = g_{11}/2(g_{22}+g_{33})$.

We might expect that the same technique applied to (b) would produce an ϵ = constant also. I find

$$\begin{aligned} B_z &= (B_z)_{\rm II}[\cos^2\epsilon + (\bar{g}_{33}/g_{22})\sin^2\epsilon)^{-1/2}\cos\epsilon\,, \\ B_\phi &= (B_\phi)_{\rm I}(\bar{g}_{33}/\bar{g}_{22})^{1/2}[\cos^2\epsilon + (\bar{g}_{33}/\bar{g}_{22})\sin^2\epsilon]^{-1/2} \\ &\quad \times \sin\epsilon\cos(2\epsilon)\,, \end{aligned} \tag{3.8}$$

and

$$I/2\pi = 2c\sqrt{k}\cos(2\epsilon)\sin(\epsilon) \tag{3.9}$$

is a current along the z axis. Let $(B_\phi)_{\rm I}$ and $(B_z)_{\rm II}$ indicate the fields for pure case I and pure case IIA:

$$\begin{aligned} (B_\phi)_{\rm I} &= 2c(k)^{1/2}e^a\rho^{-(c+1)^2}(1+k\rho^{-2c})^{-2}\,, \\ (B_z)_{\rm II} &= -2c(k)^{1/2}e^{-a}\rho^{(c+1)^2}(1+k\rho^{-2c})^2 \end{aligned} \tag{3.10}$$

Thus I can either expand in terms of (a) or in terms of (b). As expected for (a) if $\epsilon \sim 0$, $c \sim 0$ for weak fields and for (b) for $\epsilon \sim 0$, $c \sim -1$ as was the case for pure cases I or IIA.

It is reasonable to use (a) with $c>0$ when $\epsilon < \pi/4$ and (b) with $c<-1$ when $\epsilon > \pi/4$ since we will then be using the more natural forms with the c appropriate for the weak-field approximation. However, each representation (a) and (b) contains the other; so we may concentrate our attention on one, say (a). If (a) behaves properly we may justifiably expect (b) to behave properly also.

This also suggests the use of some constants other than k and ϵ. I do this by defining

$$\sin\epsilon = k_2/k\,, \qquad \cos\epsilon = k_1/k \tag{3.11}$$

with

$$k^2 = k_1{}^2 + k_2{}^2 . \qquad (3.12)$$

Equations (3.11) allow us to rewrite Eqs. (3.4)–(3.7). The relevant equations are

$$g_{11} = g_{11}, \quad g_{22} = \bar{g}_{22}(k_1/k)^2 + \bar{g}_{33}\rho^2(k_2/k)^2 ,$$

$$g_{33} = \bar{g}_{22}(1/\rho^2)(k_2/k)^2 + \bar{g}_{33}(k_1/k)^2 , \qquad (3.13)$$

$$g_{23} = (-\bar{g}_{22}/\rho + \rho\bar{g}_{33}) ,$$

$$B_\phi = (B_\phi)_{\rm I}[k_1{}^2 + (\bar{g}_{33}/\bar{g}_{22})k_2{}^2]^{-1/2}k_1 ,$$

$$B_z = (B_z)_{\rm II}[k_2{}^2 + (\bar{g}_{33}/\bar{g}_{22})k_1{}^2]^{-1/2}k_2 , \qquad (3.14)$$

$$I/2\pi = -2c(k)^{-1/2}k_1 , \qquad (3.15)$$

where $\bar{g}_{\mu\nu}$ is given by Eq. (3.3) with k defined by (3.11). An analogous behavior follows for (3.8) to (3.10).

These equations [(3.13)–(3.15)] are preferred to (3.5)–(3.7) since they show the independent behavior of B_ϕ and B_z explicitly. Equations (3.14) and (3.15) strongly suggest that k_1 is related to the strength of the magnetic field along the ϕ direction while k_2 is related to the strength along the z axis. I can best investigate the meaning of the parameter by considering the motions of test particles.

IV. MOTION OF TEST PARTICLES

In this section the first integrals of the equation of motion of test particles (both charged and neutral particles) are derived for case (a) of the previous section. By test particles I mean that we ignore radiation and other electromagnetic and gravitational effects produced by the charge and mass of the test particles. For neutral test particles the motion is along geodesics while for charged test particles we assume a Lorentz-force law. Thus the equations of motion for a test particle of charge e and mass m are

$$m\, D^2x^\mu/D^2s = ie(dx^\nu/ds)f_\nu{}^\mu . \qquad (4.1)$$

The right-hand side is pure imaginary since $(ds)^2 = -(d\tau)^2$, where τ is the proper time of the particle. The analysis of (4.1) can be carried out in a fashion very similar to that done for the static cylindrically symmetric problem by Safko and Witten.[3]

There were, for case I (our frame denoted by the overbar, $\bar{g}_{\mu\nu}$), three constants of integration E, J, and L. E was identified as an energy, J an angular momentum, and L a linear momentum in the axial direction. Now, of course, we cannot give those interpretations for J and L since $\bar{\phi}$ is no longer an angular coordinate. Three of the solutions of (4.1) are

$$g_{00}(dt/ds) \equiv iE , \qquad (4.2)$$

$$g_{22}(d\phi/ds) \equiv i\{(k_1/k)^2[(k_1/k)J + (\bar{g}_{22}/\bar{g}_{33})(k_2/k)\bar{\mathfrak{L}}] + (k_2/k)^2[(k_2/k)\bar{\mathfrak{L}} + (\bar{g}_{33}/\bar{g}_{22})(k_1/k)J]\} \equiv i\mathfrak{J}(\rho) , \qquad (4.3)$$

$$g_{33}(dz/ds) \equiv i\{(k_1/k)^2[(k_1/k)\bar{\mathfrak{L}} - (\bar{g}_{33}/\bar{g}_{22})(k_2/k)J] - (k_2/k)^2[(k_2/k)J - (\bar{g}_{22}/\bar{g}_{33})(k_1/k)\bar{\mathfrak{L}}]\} \equiv i\mathfrak{L}(\rho) , \qquad (4.4)$$

where

$$\bar{\mathfrak{L}} = L + 2(e/m)e^a(k)^{-1/2}(1 + k\rho^{-2c})^{-1}\psi(k, c) .$$

ψ is a function which is equal to zero when either k or c equals zero and is equal to one otherwise. The fourth integral of Eqs. (4.1) is just given by the line element. For timelike motion E, J, or L are all real, for spacelike motion they are all purely imaginary.

The physical significance of E, J, L are again deduced in the weak-field, low-velocity limit for a neutral particle. Define the velocity v by

$$ds^2 = g_{00}dt^2(1 - v^2) . \qquad (4.5)$$

Then from (3.2)

$$E = \rho^{c(c+1)}(1 + k\rho^{-2c})e^a(1 - v^2)^{-1/2} . \qquad (4.6)$$

Now the total energy U of a particle of mass M and velocity v moving in the gravitational field of an axial distribution of mass m per unit length is, in the appropriate units,

$$U/M = 1 + 2m \ln\rho + \tfrac{1}{2}v^2 ,$$

while (3.6) becomes for small $c(c+1)$, k_1, k_2, a, and v^2

$$E = 1 + a + v^2/2 + c(c+1)\ln\rho + k\rho^{-2c} .$$

This suggests the definition

$$c = \delta\theta(1 - b) - (1 + \delta)\theta(b - 1) ,$$

$$\theta = \text{step function} , \qquad (4.7)$$

$$b = k_2/k_1 ,$$

and for sufficiently small k_1, k_2, δ, and ν the identifications

$$U/M = E - a - k\theta(1 - |k_2/k_1|) \qquad (4.8)$$

and

$$m = \delta . \qquad (4.9)$$

This interpretation can only be made for ρ sufficiently small that second-order terms are ignorable compared to first-order terms. In ordinary units (3.9) gives us $m = 2.2 \times 10^{18}(\delta)$ kg/m. The substitution (4.7) will be made even when $c(c+1)$ is not small, but the identification (4.9) is only good for δ small and in the limit $k_1, k_2 \to 0$. The term k_2/k in (3.7) is needed so that $g_{22} \to \rho^2$ and $g_{33} \to 1$ in the limit $\delta, k_1, k_2, a \to 0$. This, of course, holds if either k_1 or k_2 goes to zero faster than the other. If they go to zero together we must redefine ρ, in order to make the identification (3.9) for small δ. The quantities $\mathcal{J}$ and $\mathcal{L}$ will again reduce to angular and linear momentum in the zero-order limit.

There is no need to carry out the analysis of many of the types of charged and uncharged motions possible since this analysis is completely local. Thus the results of Safko and Witten[3] can be taken even for $\bar{\phi}$ and $\bar{z}$. Orbits which are constant $\bar{z}$ now become spirals about the axis in ϕ, z as do orbits which are constant in $\bar{\phi}$. This is entirely as expected; there is a smooth transition from case I to case IIA as k_1 goes from k to 0 for fixed k. We could proceed to analyze other particle motions, for example motions with constant ρ, ϕ; ρ, z; and ϕ, z in analogy to what was done for case I and IIA. Alternatively, we can use Eq. (3.1) to allow us to relate

(a) ρ, ϕ constant $\to \rho$ constant, $d\bar{\phi}/d\bar{z} = -1$;
(b) ρ, z constant $\to \rho$ constant, $d\bar{\phi}/d\bar{z} = 1$;
(c) ϕ, z constant $\to \bar{\phi}, \bar{z}$ constant.

It is clear that (c) is unchanged from the case I and case IIA analysis while (a) and (b) do not permit charged-particle motion. For neutral test particles (a) and (b) both give

$$E^2/J^2 = [(g_{11})^2/g_{11,\rho}][\bar{g}_{22,\rho}/(\bar{g}_{22})^2 + \bar{g}_{33,\rho}/(\bar{g}_{22})^2]$$
$$= e^{2a}\rho^{2c^2-2}(1/c)\left[\frac{(c+1)+(1-c)k\rho^{-2c}}{(c+1)+(c-1)k\rho^{-2c}} + c\rho^{-4c-2}(1+k\rho^{-2c})^{-4}\frac{2k-\rho^{2c}(1+k\rho^{-2c}}{2k-(c+1)\rho^{2c}(1+k\rho^{-2c})}\right] \tag{4.10}$$

and

$$\bar{\mathcal{L}} = \pm(\bar{g}_{33}/\bar{g}_{22})J. \tag{4.11}$$

I have assumed that both k_1 and k_2 are nonzero so I do not consider (4.10) in the limit $k \to 0$. For that limit the results of cases I and IIA apply directly as discussed previously.[3] The nature of the motion is determined by the line element (2.15), which can now be written as

$$-(d\tau/ds)^2 = \frac{J^2}{\bar{g}_{22}}\left[\frac{(1-c^2)(1+k\rho^{-2c}) + c^2\rho^{-6c-2}(1+k\rho^{-2c})^{-4}}{(c+1)+(c-1)k\rho^{-2c}}\right].$$

The sign of the term inside the large square brackets determines whether the motion is timelike, spacelike, or null as

$$[\cdots]\begin{cases} >0 \text{ timelike} \\ =0 \text{ null} \\ <0 \text{ spacelike.}\end{cases}$$

From this I conclude the following:

$0 < c < 1$. All motions are timelike with $\rho_1 > \rho > \rho_0$, where

$$\rho_0^{2c} = [(1-c)/(1+c)]k$$

and

$$\rho_1^{2c} = k + (1/c)\rho_1^{2-4c}(k+\rho_1^{2c})^4[(c+1)\rho_1^{2c} + (1-c)k].$$

$1 < c$. If $\rho_2 < \rho_3$ motions are timelike for $\rho_2 < \rho < \rho_3$, null for $\rho = \rho_3$, spacelike if $\rho > \rho_3$ and if $\rho_3 < \rho_2$, then all motions are spacelike with $\rho_2 < \rho$, where

$$(c-1)k + c\rho_2^{6c-2}(k+\rho_2^{2c})^{-4} = (c+1)\rho_2^{2c} + kc\rho_2^{4c-2}(k+\rho_2^{2c})^{-4}$$

and

$$(c^2-1)(k+\rho_3^{2c})^5 = c^2\rho_3^{4c-2}.$$

$-1 < c < 0$ (*negative mass*). All motions are spacelike and ρ must be larger than the greater of ρ_4 or ρ_5, where

$$k\rho_4^{-2c} = (1+c)/(1-c),$$
$$(1+c) + (1-c)k\rho_5^{-2c} = (-c)\rho_5^{-2-4c}(1+k\rho_5^{-2c})^{-4} \times (1 \mp k\rho_5^{-2c}).$$

ρ_5 is greater than ρ_4 provided $k^{2+1/c} > (-c)(1-c^2)(1-c)^{1-1/c}(1+c)^{1/c}$.

$c < -1$. Define ρ_6, ρ_7, ρ_8, and ρ_9 as

$$k\rho_6^{-2c} = 1,$$
$$-(c+1) + (-c)\rho_7^{-4c-2}(1+k\rho_7^{-2c})^{-4}(1-k\rho_7^{-2c}) = (1-c)k\rho_7^{-2c},$$
$$-(c+1) = (1-c)k\rho_8^{-2c} + (-c)\rho_8^{-4c-2}(1+k\rho_8^{-2c})^{-4}(k\rho_8^{-2c}-1),$$
$$(c^2-1)(1+k\rho_9^{-2c}) = c^2\rho_9^{-6c-2}(1+k\rho_9^{-2c})^{-4}.$$

If $0 < \rho_7 < \rho_6$ motion is allowed for $\rho > \rho_7$. If $\rho_6 < \rho_8$ then motion is allowed for $\rho > \rho_8$. In any event the motion is null if $\rho = \rho_9$, timelike for $\rho < \rho_9$ and spacelike for $\rho > \rho_9$.

In general it is apparent for a positive mass density along the axis, $c > 0$ or $c < -1$, there is a certain amount of energy E which can be divided between angular momentum $\mathcal{J}$ and linear $\mathcal{L}$. If too much is given to one there is not enough left to provide for the other. Thus it is not possible to have timelike orbits of the type considered, i.e., too close or too far from the axis. In summary the overall behavior of these motions is entirely as expected after consideration of the motions allowed by cases I and IIA.

The results of analyzing case (b) of Sec. III give comparable behavior.

V. MODEL OF A SOURCE

I now show that there are physically reasonable sources which could produce the field I have discussed. The situation I consider is that of a perfectly conducting wire coaxial with a solenoidal current as shown in Fig. 1. Space is divided into four regions denoted by A, B, C, and D. I will examine case (a) (I and IIA) in detail. The analysis for case (b) (IIA + I) follows in an identical fashion.

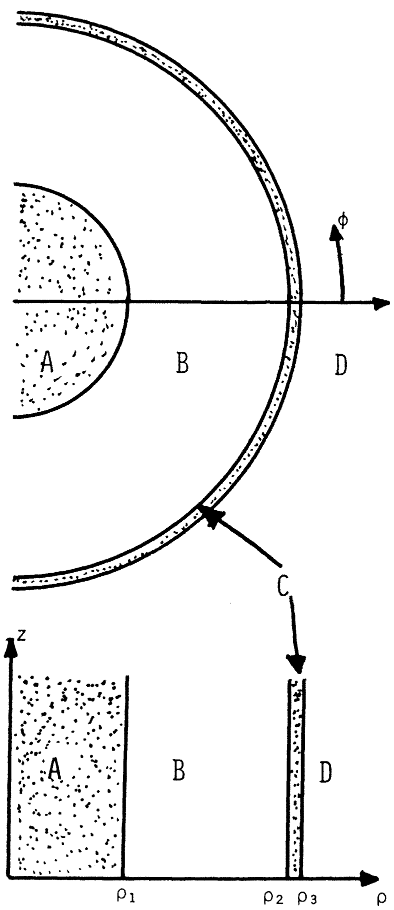

FIG. 1. The model configuration showing side and top views. The shading indicates the distribution of matter. ρ is the distance from the axis of symmetry, z is the symmetry axis, and ϕ an angular measure about the axis.

Region A is a wire of infinite conductivity. The stress-energy momentum tensor is taken to be

$$T_{\mu\nu}=\begin{bmatrix} D & 0 & 0 & 0 \\ 0 & P_1 & 0 & 0 \\ 0 & 0 & P_2 & S \\ 0 & 0 & S & P_3 \end{bmatrix} \qquad (5.1)$$

with an internal line element given by

$$ds_A{}^2=-e^{2\gamma-2\psi}(dt^2-d\rho^2)+\rho^2 e^{-2\psi}d\phi^2+e^{2\psi+2\mu}dz^2 + 2g_{23}d\phi dz \qquad (5.2)$$

with γ, ψ, μ, and g_{23} functions of ρ only.

In region B there is the mixture of B_z and B_ϕ I have been considering. That is, if I rotate my coordinates in ρ-z by ϵ given by (3.1), I obtain

$$d\bar{s}_B{}^2=-(\rho+\rho_0)^{2c^2-2c}[k+(\rho+\rho_0)^{2c}]^2 e^{2\bar{\gamma}_0-2\bar{\psi}_0}(dt^2-d\rho^2) + (\rho+\rho_0)^{2-2c}[k+(\rho+\rho_0)^{2c}]^2 e^{-2\bar{\psi}_0}d\bar{\phi}^2 + (\rho+\rho_0)^{2c}[k+(\rho+\rho_0)^{2c}]^{-2}e^{2\bar{\psi}_0+2\bar{\mu}_0}d\bar{z}^2, \qquad (5.3)$$

where $\bar{\rho}_0$, $\bar{\gamma}_0$, $\bar{\psi}_0$, and $\bar{\mu}_0$ are constants. I will choose a scaling $\bar{\psi}_0=\bar{\mu}_0=0$, $\bar{\gamma}_0=a$. Since region A is of perfect conductivity, we expect that there will exist a surface current of strength

$$I_{\bar{z}}=\pm 2\pi c k^{1/2} \qquad (5.4)$$

in the $\bar{z}$ direction, i.e., at an angle ϵ to z on the surface at ρ_1.

At ρ_2 there is the region C which has infinite conductivity in the ϕ direction and zero conductivity in the z direction. We can imagine a solenoid of wire with the wire diameter very small compared to the accuracy of measurement. At this surface there is a discontinuity in B_z while B_ϕ is continuous. I expect a current of

$$I_\phi = 2ck^{1/2}(\bar{g}_{33}/\bar{g}_{22})_{|\rho_2}[\cos^2\epsilon+(\bar{g}_{22}/\bar{g}_{33})\sin^2\epsilon]^{1/2} \times[\cos^2\epsilon+(\bar{g}_{33}/\bar{g}_{22})\sin^2\epsilon]^{-1}\sin\epsilon \qquad (5.5)$$

in the ϕ direction at ρ_2. In region C there will be only f_{13}. Thus in this region

$$T_\mu{}^\nu{}_{\text{tot}}=T_\mu{}^\nu+T_\mu{}^\nu{}_{\text{em}}, \qquad (5.6)$$

where $T_\mu{}^\nu$ will be of the form (5.1) with a line element (5.2), while

$$T_\mu{}^\nu{}_{\text{em}}=f_{\mu\alpha}f^{\alpha\nu}-\tfrac{1}{4}\delta_\mu{}^\nu f_{\alpha\beta}f^{\alpha\beta}. \qquad (5.7)$$

At the surface ρ_3 the components of f_{13} will be con continuous. The external line element can be taken similar to (5.3) but I cannot take $\mu_0=\psi_0=0$. Thus

$$ds_D{}^2=-(\rho+\rho_e)^{2C^2-2C}[K+(\rho+\rho_e)^{2C}]^2 e^{2\gamma_0-2\psi_0}(dt^2-d\rho^2) + (\rho+\rho_e)^{2-2C}[K+(\rho+\rho_e)^{2C}]^2 e^{-2\psi_0}d\phi^2 + (\rho+\rho_e)^{2C}[K+(\rho+\rho_e)^{2C}]^{-2}e^{2\psi_0+2\mu_0}dz^2 \qquad (5.8)$$

with a physical component of the magnetic field given by

$$B_\phi = \pm \frac{2Ck^{1/2}e^{\gamma_0-\mu_0}}{(\rho+\rho_e)^{(C+1)^2}[k+(\rho+\rho_e)^{2C}]^2}. \qquad (5.9)$$

I now argue that the components of $T_{\mu\nu}$ in regions A and C can be chosen to be physically reasonable. There are three local conditions[4]:

1. The components D, P_1, P_2, P_3, and S of the stress-energy-momentum tensor are finite at all points.
2. The energy density $D \geq 0$ within the matter.
3. For weak fields (small c and $k \ll c$), D is of the order of the mass density and P_1, P_2, P_3, and S are of the order of the square of the mass density sity:

$$D = O(c), \quad P_i = O(c^2), \quad S = O(c^2)$$
$$\text{for } c \ll 1, \quad k = O(c^2).$$

In addition for region A there is a further condition:

4. We have the condition of elementary flatness. If R is the ratio of the circumference of a circle (coordinate radius ρ) to its radius, $R \to 2\pi$ as $\rho \to 0$, for all circles centered on the axis and perpendicular to it.

All solutions are chosen so that the first and second fundamental forms are continuous across the boundaries.

It has been shown by Safko and Witten[4] that there exist sources satisfying conditions 1–4 for either pure case-I or pure case-IIA external fields. For example with an external case-I field and an internal metric of the form of (5.2) with $g_{23} \equiv 0$, they found a $T_{\mu\nu}$ with $S \equiv 0$ which satisfied conditions 1–4 if they took

$$\mu = \alpha_0 - \frac{\alpha}{n+1}\left(\frac{\rho}{\rho_1}\right)^{n+1}, \qquad (5.10a)$$

$$\psi = -\mu + \beta_0 - \frac{\beta}{m+1}\left(\frac{\rho}{\rho_1}\right)^{m+1} = -\mu + \nu, \qquad (5.10b)$$

$$\gamma = -\mu + \epsilon_0 + \frac{\epsilon}{q+1}\left(\frac{\rho}{\rho_1}\right)^{q+1} = -\mu + \eta, \qquad (5.10c)$$

with

$$2\beta(n+1) = \alpha(n+2), \quad m = n, \quad \alpha_0 = \epsilon_0. \qquad (5.10d)$$

For small c and $k = O(c^2)$

$$\rho_0 = 2\frac{n+1}{n+2}c\rho_1. \qquad (5.10e)$$

The quantities α_0, α, β_0, and ϵ are determined by the matching conditions across $\rho = \rho_1$. Elementary flatness determines $\alpha_0 = \epsilon_0$ and the relation between α and β is fixed by the requirement $P_3 = O(c^2)$. I now show that the existence of this solution enables me to construct the solution I desire.

Take g_{23} in region A to be of the form

$$g_{23} = (1 - e^{2\lambda})^{1/2}\rho e^{\mu} \qquad (5.11)$$

as given in the Appendix and further assume in addition to (5.10) that

$$\lambda(\rho) = \frac{\sigma}{p+2}\left(\frac{\rho}{\rho_1}\right)^{p+2}, \quad p \geq 0 \qquad (5.12)$$

and use an overbar to denote the values when $\epsilon = 0$ (i.e., the solution of Safko and Witten[4]), then it follows that

$$\alpha = \bar{\alpha}\{1 + \bar{R}x^2[\bar{R}x^2 - (1 - x^4)]^{-1}/(n+1)\}, \qquad (5.13a)$$

$$\beta = \bar{\beta}[1 + Rx^2 n/(n+2)](1 + Rx^2)^{-1}, \qquad (5.13b)$$

$$\epsilon = \bar{\epsilon} + \bar{\alpha}Rx^2(1 + Rx^2)^{-1}/(n+1), \qquad (5.13c)$$

$$\sigma = \bar{g}_{33}(R - 1)e^{\alpha/(n+1) - \alpha_0}x/(n+1), \qquad (5.13d)$$

where

$$x = \sin\epsilon/\cos\epsilon, \qquad (5.14a)$$

$$R = \bar{g}_{22}/(\rho_1^2 \bar{g}_{33}), \qquad (5.14b)$$

$$\bar{R} = R^{-1} - R. \qquad (5.14c)$$

For c small, and $k \ll c$,

$$R \approx 1 + 4c[2\ln\rho_1 + (n+1)/(n+2)]$$
$$= 1 + 4c\Delta. \qquad (5.15)$$

Inserting (5.15) into (5.13) we obtain for ϵ small (an unessential simplification)

$$\alpha \approx \bar{\alpha}(1 - 8cx^2\Delta), \qquad (5.13a')$$

$$\beta \approx \bar{\beta}[1 + x^2 n/(n+2)], \qquad (5.13b')$$

$$\epsilon \approx \bar{\epsilon} + \bar{\alpha}x^2/(n+1), \qquad (5.13c')$$

$$\sigma = 4c\Delta\bar{g}_{33}c^{\alpha/(n+1) - \bar{\alpha}_0}x/(n+1). \qquad (5.13d')$$

I can make σ as small as desired by choosing n sufficiently large. Substituting these into the expressions for the energy density, D, and pressures, P_i and S, given in the Appendix, $D = O(c)$, $P_i = O(c^2)$, and $S = O(c^2)$ as required for physical acceptability. Thus the matter in region A can be chosen to be physically reasonable.

At the inner surface of region C, I require that B_ϕ be continuous so at ρ_2 within region C the nonvanishing component of $f_{\mu\nu}$ is

$$f_{13}(\rho_2) = 2ck^{1/2}\bar{g}_{33}^{1/2}[\cos^2\epsilon + (\bar{g}_{22}/\bar{g}_{22})\sin^2\epsilon]^{-1}$$
$$\times \cos\epsilon,$$

where $\bar{g}_{22}$ and $\bar{g}_{33}$ are calculated in region B at $\rho = \rho_2$. Since the metric in region C is nondiagonal there may be several components of $f_{\mu\nu}$ and $f^{\mu\nu}$. I again choose a power-law expansion for μ, ψ, γ, and λ in which the forms are the same as (5.10) and (5.11), except ρ/ρ_1 is replaced by $(\rho_3 - \rho)/(\rho_3 - \rho_2)$.

Neglecting terms of order k, and hence $T_{\mu\nu\,\mathrm{em}}$, I can show once again physical acceptability.

In the exact solution it is not possible to neglect the terms of order k. In this case we express the components of $f_{\mu\nu}$ at $\rho=\rho_2$ in terms of a power law similar to (5.10) again in terms of $(\rho_3-\rho)/(\rho_3-\rho_2)$. All the $f_{\mu\nu}$ are chosen to vanish at $\rho=\rho_3$ except f_{13}. The four Maxwell equations that are not trivially satisfied provide the additional constraints to lead to an exact solution in terms of the boundary conditions at $\rho=\rho_2$ and $\rho=\rho_3$. Solutions are only possible if the parameter C of Eq. (5.8) is greater than the c of Eqs. (5.3) (neglecting k again).

If we choose to expand in terms of solution II instead of I in region B, the entire argument carries through with the appropriate changes. So it is possible to construct physically reasonable sources and solutions for a wire containing a current coaxial with a cylinder containing a solenoidal current.

APPENDIX

I take a nondiagonal metric of the form

$$ds^2 = e^{2\gamma-2\psi}(d\rho^2 - dt^2) + \rho^2 e^{-2} d\phi^2 + e^{2\psi+2\mu} dz^2 + 2\rho e^{\mu} \left| e^{2\lambda} - 1 \right|^{1/2},$$

where γ, ψ, μ, and λ are functions of ρ alone. The form of g_{23} is chosen to simplify later calculations. With this metric form the Ricci tensor has the following nonvanishing components:

$$R_{00} = +\mu'\gamma' - \mu'\psi' + \gamma'' - \psi'' - \psi'/\rho + \gamma'/\rho + \lambda'(\gamma' - \psi'),$$

$$R_{11} = -\mu'' - \tfrac{1}{2}(e^{-2\lambda}+1)\mu'^2 - (2e^{-2\lambda}+1)\mu'\psi' + \mu'\gamma' - \gamma'' - 2e^{-2\lambda}\psi'^2 + \psi'' + (2e^{-2\lambda}-1)\psi'/\rho + \gamma'/\rho$$
$$+ (1-e^{-2\lambda})[1/(2\rho^2) - \mu'/\rho] - \lambda'(\psi' + \gamma' + \mu' + 1/\rho) - \lambda'' - (e^{2\lambda}-2)(e^{2\lambda}-1)^{-1}\lambda'^2,$$

$$e^{2\gamma}R_{22}/\rho^2 = (2e^{-2\lambda}-1)\mu'\psi' + \psi'' - e^{-2\lambda}\mu'/\rho + (3-2e^{-2\lambda})\psi'/\rho$$
$$-(1-e^{-2\lambda})(\mu'^2/2 + 2\psi'^2 - 1/\rho^2) + \lambda'(\mu' + \psi') - \tfrac{1}{2}(1-e^{-2\lambda})^{-1}\lambda'^2,$$

$$2R_{23}e^{2\gamma-2\psi-\mu} = (e^{2\lambda}-1)^{1/2}[4e^{-2\lambda}\mu'\psi' + 4e^{-2\lambda}\psi'^2 - 4e^{-2\lambda}\psi'/\rho + e^{-2\lambda}/\rho^2 - 2(e^{-2\lambda}+1)\mu'/\rho - \mu'' - (1-e^{-2\lambda})\mu'^2$$
$$-(2e^{2\lambda}-1)(e^{2\lambda}-1)\lambda'\mu' - (e^{2\lambda}-2)(1-e^{-2\lambda})^{-1}\lambda'^2 - (1-e^{-2\lambda})^{-1}\lambda''],$$

$$R_{33}e^{2\gamma-4\psi-2\mu} = -\mu'' - \tfrac{1}{2}(3-e^{-2\lambda})\mu'^2 - (3-2e^{-2\lambda})\mu'\psi' - \psi'' - e^{-2\lambda}\mu'/\rho + (1-2e^{-2\lambda})\psi'/\rho$$
$$+(1-e^{-2\lambda})[1/(2\rho^2) - 2\psi'^2] + \lambda'(\psi' - 1/\rho) - \tfrac{1}{2}(1-e^{-2\lambda})^{-1}\lambda'^2.$$

If these values for $R_{\mu\nu}$ are substituted into Einstein's equations with $T_{\mu\nu}$ of the form (5.1) and with the definitions of ν and η as given in Eqs. (5.10) I obtain

$$P_1 - D = \mu'' + \mu'^2 + 2\mu'/\rho + \lambda'(4\mu' - 2\eta' + 2/\rho) + \lambda'' + \tfrac{1}{2}(1-e^{-2\lambda})^{-1}\lambda'^2,$$

$$P_1 = \eta'\mu' + \eta'/\rho - \nu'^2 + (e^{-2\lambda}-1)[-\mu'^2/4 - \nu'/\rho + \mu'\nu' + \mu'/(2\rho) - \nu'^2 + 1/(4\rho)]$$
$$+\lambda'[5\mu'/2 - \nu' - \eta' + 1/(2\rho)] - \tfrac{1}{4}(e^{2\lambda}-4)(e^{2\lambda}-1)^{-1}\lambda'^2,$$

$$P_2 e^{2\gamma}/\rho^2 = \eta'' + \nu'^2 + (e^{-2\lambda}-1)[3\mu'^2/4 - 3\mu'\nu' + 3\nu'^2 + 3\mu'/(2\rho) - 3\nu'/\rho - 5/(4\rho^2)]$$
$$+\lambda'[3\mu'/4 - \nu' + 3/(2\rho)] + \lambda'' + \tfrac{1}{4}(e^{2\lambda}-4)(e^{2\lambda}-1)^{-1}\lambda'^2,$$

$$P_3 e^{2\eta-4\nu} = \eta'' - 2\nu'' + \nu'^2 - 2\mu'\nu' - 2\nu'/\rho + (\mu'' + \mu'^2 + 2\mu'/\rho$$
$$+(e^{-2\lambda}-1)[-3\nu'^2 - 3\mu'\nu' + 3\mu'^2/4 + 3\mu'/(2\rho) - 3\nu'/\rho - 1/(4\rho^2)]$$
$$+\lambda'[-5\mu'/2 + \nu' + 1/(2\rho)] + \lambda'' + \tfrac{3}{4}(e^{2\lambda}-2)(e^{2\lambda}-1)^{-1}\lambda'^2,$$

$$2Se^{2\gamma-2\psi-\mu} = (e^{2\lambda}-1)^{1/2}\{\mu'' + (e^{2\lambda}-1)^{-1}[(e^{2\lambda}-2)\lambda'' + (\tfrac{5}{2}e^{2\lambda}+1)\lambda'^2 + (e^{2\lambda}+1)\lambda'\mu' + (e^{2\lambda}-2)\lambda'/\rho]$$
$$+(5e^{-2\lambda}-3)\mu'^2/2 + 6e^{-2\lambda}\psi'(\mu' + \psi' - 1/\rho) + 2\gamma'' - 2\psi'' + (1-3e^{-2\lambda})\mu'/\rho + (3e^{-2\lambda}-1)/(2\rho^2)\}.$$

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[1] *Gravitation: An Introduction to Current Research*, edited by L. Witten (Wiley, New York, 1962), Chap. 9.

[2] J. L. Safko, Ann. Phys. (N.Y.) 58, 352 (1970).

[3] J. L. Safko and L. Witten, J. Math. Phys. 12, 257 (1971).

[4] J. L. Safko and L. Witten, Phys. Rev. D 5, 293 (1972).

[5] We use the sign conventions of C. Misner, K. Thorne, and A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).

[6] R. P. Geroch, Ann. Phys. (N.Y.) 36, 147 (1966).