Magnetic support against gravitational collapse: A static axisymmetric interior solution of the Einstein-Maxwell equations

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It is shown that gravitationally collapsed bounded systems which are too massive to be supported by their pressure may be held in equilibrium by self-induced magnetic stresses. A three-parameter family of solutions of the static axisymmetric interior Einstein-Maxwell equations is derived which describes matter distributions that are solely balanced by magnetic forces. The mass and the magnetic moment associated with these systems are then determined directly from the exterior field equations and the appropriate fluxes at the boundary of each system's interior. Some physical implications of the derived solutions—none of which contain a singularity or an event horizon—have also been discussed.

I. INTRODUCTION

The gravitational collapse of a star whose mass exceeds a certain limit ($\sim 3M_{\odot}$) cannot, according to the general theory of relativity, be halted by the degeneracy pressure of neutron matter. Since the discovery of this fact by Oppenheimer and his collaborators[1,2] many attempts have been made to find an alternative to the notion of continued collapse both by postulating the existence of other forces[3] and by invoking other versions of the theory of gravitation.[4] Recently, observations of binary x-ray sources have lent additional significance to this inquiry by providing some evidence for the presence in our galaxy of collapsed stellar objects whose masses are in excess of the theoretical upper limit of the mass of a neutron star.[5] It is therefore of current interest[6] to ascertain whether such objects exemplify the state reached by continued contraction and should accordingly be identified as black holes, or whether in fact they represent a state of equilibrium in the evolution of certain stars which are supported against collapse by a known force.

The purpose of the present paper is to demonstrate the possibility of balancing a gravitationally bound system by means of self-induced magnetic stresses. Since the systems of interest are collapsed objects which are too massive to be supported by pressure forces, the corresponding mathematical task of the paper is therefore to seek merely those solutions of the static axisymmetric Einstein-Maxwell field equations in which the gravitational attraction is balanced solely by the magnetic force. Viewed from the standpoint of the Newtonian theory, it is at first by no means clear whether the topologies of the magnetic and the gravitational forces can within a bounded system be sufficiently concordant for solutions of this kind to exist. The present analysis, however, shows that not only is there a solution to the differential equations demanding the local balance of gravitational and magnetic forces, but that this solution—without being subjected to any conditions other than the finiteness of matter density at the origin—of itself turns out to describe a distribution of matter which is also globally balanced, i.e., is spatially bounded.

In the course of the mathematical formulation of this problem a number of simplifications occur which enable us to reduce the set of interior Einstein-Maxwell equations to a single linear equation for one of the components of the metric tensor. Weyl's metric in its vacuum form happens to be applicable to the present case, and one of the conservation laws—which in the Newtonian limit requires the magnetic lines of force to lie on the equipotential surfaces of the gravitational field—can for the cases of nonvanishing matter density be explicitly integrated. In matter-free space where this integration cannot be performed, the corresponding field equations no longer lend themselves to similar simplifications. However, despite the absence of an exterior solution, the mass and the magnetic moment of each equilibrium configuration can in the present instance be determined directly from the exterior field equations and the appropriate fluxes at the boundary of each system's interior. This is made possible by the fact that each member of the three-parameter family of interior solutions derived here is bounded by a surface at which both the matter and the electric current densities vanish. To give an example of the masses thus obtained, for a central density of 10^{12} g cm^{-3}, a central magnetic field of 10^{15} G, and a wide range of values of the third parameter characterizing these interior solutions, the mass is of the order of 10 solar masses.

The interior solutions derived in this paper contain neither a singularity nor an event horizon. That in principle there is also no difficulty in supplementing these solutions with exterior metrics has not been proved here—though it is affirmed both by the asymptotic expansion of the field equations at large distances and by the global topology of the equipotentials of the gravitational field in the post-Newtonian limit. However, should an exterior solution exist, as we have assumed in this paper, then it too can be shown to be free of any singularity or event horizon. In fact, none of the essential features of the systems considered here (except possibly for the cutoff in matter density) is purely relativistic; magnetically-supported self-gravitating objects are possible also within the framework of the Newtonian theory.

The plan of the paper is as follows: Secs. II and III are concerned with solving the interior Einstein-Maxwell equations for the metric and the electromagnetic field tensors and their invariants. Section IV deals with the derivations of mass and magnetic moment, and Sec. V is devoted to a discussion of the physical content of the obtained results. In addition, Appendixes A, B, and C present, respectively, the Newtonian version of the analysis of Sec. II, the derivation of the continuity of the electromagnetic field and the energy-momentum tensors across the boundary of the system, and the asymptotic expansion of the exterior solution.

II. THE INTERIOR FIELD EQUATIONS

In this section we shall consider the Einstein-Maxwell equations for a distribution of matter in an advanced stage of gravitational collapse which, too dense to be supported by its pressure, is held in equilibrium by means of self-induced magnetic stresses. The pressure in such a system can be regarded as negligible relative to the magnetic energy density, and accordingly the field equations may be written

$$R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R=-\frac{8\pi G}{c^2}\left[\frac{1}{4\pi c^2}(F_{\mu\lambda}F^{\lambda}{}_{\nu}+\tfrac{1}{4}g_{\mu\nu}F_{\lambda\sigma}F^{\lambda\sigma})+\rho u_\mu u_\nu\right], \tag{1a}$$

where

$$F_{\mu\nu}=A_{\nu,\mu}-A_{\mu,\nu}, \tag{1b}$$

and where the notation adopted is that of Ref. 7. Furthermore, neglecting rotation, we shall employ a comoving coordinate system whose metric is static and axisymmetric and in which the matter is electrically neutral. For such a system, the general form of the metric in cylindrical coordinates is well known[8]:

$$ds^2=\gamma^2dt^2-\alpha^2(dr^2+dz^2)-\beta^2d\varphi^2, \tag{2}$$

where α, β, and γ are independent of the coordinate φ. Likewise, the electromagnetic potential of the system in a suitable gauge is

$$A_\mu=(0,0,A_2,A_3).$$

However, we shall here further restrict ourselves to the case of $A_2=0$ corresponding (in the Newtonian limit) to a purely poloidal magnetic field, in order to realize the condition

$$R_0{}^0+R_3{}^3=0. \tag{3}$$

Just as in the vacuum case,[9,10] this condition would then guarantee that the metric can be reduced to a form involving only two unknown functions:

$$ds^2=c^2e^{2\lambda}dt^2-e^{2(\nu-\lambda)}(dr^2+dz^2)-r^2e^{-2\lambda}d\varphi^2. \tag{4}$$

With the above expression for the metric, and with the following particular forms of the electromagnetic potential and the four-velocity,

$$A_\mu=(0,0,0,A),\quad u_\mu=(g_{00}{}^{1/2},0,0,0), \tag{5}$$

the field equations (1) can now be written out:

$$\nabla^2\lambda=\kappa\left[\frac{e^{2\lambda}}{r^2}|\vec{\nabla}A|^2+4\pi\rho c^2e^{2(\nu-\lambda)}\right], \tag{6a}$$

$$\nabla^2(\lambda-\nu)+\frac{2}{r}\nu_r-2\lambda_r{}^2=\kappa\left[\frac{e^{2\lambda}}{r^2}(A_r{}^2-A_z{}^2)+4\pi c^2\rho e^{2(\nu-\lambda)}\right], \tag{6b}$$

$$\nabla^2(\lambda-\nu)-2\lambda_z{}^2=-\kappa\left[\frac{e^{2\lambda}}{r^2}(A_r{}^2-A_z{}^2)-4\pi\rho c^2e^{2(\nu-\lambda)}\right], \tag{6c}$$

$$\frac{1}{r}\nu_z-2\lambda_r\lambda_z=2\kappa\frac{e^{2\lambda}}{r^2}A_rA_z, \tag{6d}$$

where $\kappa=G/c^4$, $\vec{\nabla}$ and ∇^2 are the usual gradient and Laplacian operators in the cylindrical coordinates (r,z,φ), and $\lambda_r=\partial\lambda/\partial r$, $\lambda_z=\partial\lambda/\partial z$, etc. The conservation laws, which are of course directly derivable from the above field equations, read

$$\frac{\partial}{\partial r}\left[\frac{e^{2\lambda}}{r}(A_r{}^2-A_z{}^2)\right]+\frac{\partial}{\partial z}\left(\frac{2e^{2\lambda}}{r}A_rA_z\right)-\frac{e^{2\lambda}}{r^2}|\vec{\nabla}A|^2+2r\lambda_r\chi=0, \tag{7a}$$

$$\frac{\partial}{\partial z}\left[\frac{e^{2\lambda}}{r}(A_r{}^2-A_z{}^2)\right]-\frac{\partial}{\partial r}\left(\frac{2e^{2\lambda}}{r}A_rA_z\right)-2r\lambda_z\chi=0, \tag{7b}$$

where

$$\chi=\frac{e^{2\lambda}}{r^2}|\vec{\nabla}A|^2+4\pi\rho c^2e^{2(\nu-\lambda)}. \tag{7c}$$

Curiously, the system of interior equations (6) happens to be much more amenable to analysis than the corresponding set of matter-free equations that have already been studied in the literature.[11,12] A simplifying feature of the present set of equations is that it requires A and λ to be functionally dependent. This result is readily obtained by rewriting the conservation laws (7) with the aid of Eq. (6a),

$$A_z\left(\nabla^2 A-\frac{2}{r}A_r+2\vec{\nabla}\lambda\cdot\vec{\nabla}A\right)+\lambda_z(4\pi r^2\rho c^2 e^{2(\nu-2\lambda)})=0\,, \tag{8a}$$

$$A_r\left(\nabla^2 A-\frac{2}{r}A_r+2\vec{\nabla}\lambda\cdot\vec{\nabla}A\right)+\lambda_r(4\pi r^2\rho c^2 e^{2(\nu-2\lambda)})=0\,, \tag{8b}$$

and observing that they do not possess any nontrivial (i.e., $\rho\neq 0$) solutions unless the following constraint is satisfied:

$$\lambda_r A_z - A_r\lambda_z=0\,, \tag{9}$$

i.e., unless A is a function of λ. Therefore, upon setting

$$A'=e^{-\lambda}f(\lambda) \tag{10}$$

and eliminating ρ by means of Eq. (6a), we are led from the conservation law (8) to the single equation

$$\left(1+\frac{r^2}{\kappa f^2}\right)\nabla^2\lambda-\frac{2}{r}\lambda_r+\frac{f'}{f}|\vec{\nabla}\lambda|^2=0\,, \tag{11}$$

where the primes designate differentiation with respect to λ. When interpreted in the Newtonian limit, Eq. (9) states that the magnetic field lines lie in the equipotential surfaces of the gravitational field (see Appendix A). Inasmuch as the gravitational force is here solely balanced by the magnetic force, this is clearly an expected result.

Because it contains the arbitrary function f, Eq. (11) should in principle admit of a wide class of solutions. However, since our aim here is not to pursue the general case but rather to find specific solutions of physical interest, we shall here set $f=\alpha=\text{const}$, i.e., let

$$A'=\alpha e^{-\lambda}\,, \tag{12}$$

so that Eq. (11) assumes the linear form

$$\hat{\vec{\nabla}}\cdot\left[\left(1+\frac{1}{\hat{r}^2}\right)\hat{\vec{\nabla}}\lambda\right]=0\,, \tag{13}$$

and the magnitude of electrical current density becomes proportional to ρ (see Sec. III C). Equation (13) has here been expressed in terms of the dimensionless quantities $\hat{r}=r/r_0$, $\hat{z}=z/r_0$, and $\hat{\vec{\nabla}}\lambda=r_0\vec{\nabla}\lambda$, which are based on the length scale

$$r_0=\alpha G^{1/2}/c^2\,. \tag{14}$$

To simplify the notation, however, *henceforth in the paper r_0 will be adopted as the unit of length and the quantities $\hat{r}$, $\hat{z}$, and $\hat{\vec{\nabla}}$ will consistently be written r, z, and $\vec{\nabla}$.*

Clearly, crucial to the derivation of a single linear equation for λ has been the constraint expressed in Eq. (9). In matter-free space where this constraint is removed, the field equations no longer lend themselves to such a reduction. Therefore, to elucidate the physical content of Eq. (9) and to emphasize its interpretation as a Bernoulli-type conservation law, we have presented the Newtonian version of the analysis of this section in Appendix A.

III. SOLUTION OF THE INTERIOR FIELD EQUATIONS

A. Components of the metric tensor

Equation (13) can be readily solved by separation of variables. Setting $\lambda=u(r)v(z)$ and adopting the separation constant k^2 we obtain

$$v''+k^2v=0\,, \tag{15}$$

$$u''-\frac{1}{r}\frac{1-r^2}{1+r^2}u'-k^2u=0\,, \tag{16}$$

and hence,

$$\lambda=\sum_k a_k e^{ikz}u_k(r)\,, \tag{17}$$

in which the primes designate differentiation, a_k are arbitrary constants, and u_k is the solution of Eq. (16) corresponding to a specific value of k. The boundary conditions for u_k follow from the requirement that the invariant quantity ρ should be finite at $r=0$. Formulated in terms of the behavior of u_k in the neighborhood of $r=0$, this requirement is

$$u_k\to ar^2\quad(r\to 0)\,, \tag{18}$$

where a is an arbitrary constant whose values, as we shall see later, should be in the interval between zero and unity.

Before discussing the results of a numerical integration of Eq. (16), it is instructive to examine the following alternative form of this equation:

$$\frac{d^2u}{d\xi^2}=\frac{k^2}{4}\frac{e^{2\xi}}{e^{\xi}-1}u\,, \tag{19a}$$

where

$$\xi=\ln(1+r^2)\,. \tag{19b}$$

Near the point $r=0$, solution (18) starts with both $u_k\geq 0$ and $du_k/dr\geq 0$. Therefore, not only is the right-hand side of Eq. (19a) initially positive, but also it persists in remaining positive for all r. Stated differently, solution (18) corresponds to a function $u_k(r)$ which together with its derivative $du_k/d\xi=\frac{1}{2}(r+1/r)du_k/dr$ is monotonically increasing everywhere.

Clearly, solution (17) by virtue of containing an infinite number of arbitrary constants offers the possibility of satisfying a prescribed set of boundary conditions on λ (and/or A). However, we are not in this paper concerned with satisfying any specific conditions other than those which are required for demonstrating the existence of a well-behaved interior solution. So as to simplify the ensuing calculations, therefore, we shall henceforth base our analysis on the following particular solution of Eq. (13):

$$\lambda = u_k(r)\cos(kz)\,, \tag{20}$$

where u_k is that solution of Eq. (16) which satisfies condition (18). It should be noted that with the choice $\lambda(r=0)=0$ made here, the function λ will in general assume a nonzero value at infinity.

To determine ν for the above choice of λ, and thus to complete the solution of the field equations for the metric tensor, we can solve the set of Eqs. (6) for ν_r and ν_z:

$$\nu_r = r(\lambda_r^{\,2} - \lambda_z^{\,2}) + \kappa\frac{e^{2\lambda}}{r}(A_r^{\,2} - A_z^{\,2})\,, \tag{21a}$$

$$\nu_z = 2r\lambda_r\lambda_z + 2\kappa\frac{e^{2\lambda}}{r}A_r A_z\,, \tag{21b}$$

and employ Eqs. (12) and (20) both to check their consistency, and to evaluate their integral:

$$\nu = \left(r + \frac{1}{r}\right)u_k u_k' \cos^2 kz - k^2\int_0^r \left(r' + \frac{1}{r'}\right)u_k^{\,2}(r')dr'\,. \tag{22}$$

Here, as in the case of λ, we have chosen $\nu(r=0)=0$. The reference frame thus obtained is therefore locally inertial for all interior points along the z axis.

A straightforward though lengthy calculation now shows that none of the invariant quantities $R_\mu{}^\mu$, $R_{\mu\nu}R^{\mu\nu}$, and $R_{\mu\nu\sigma\lambda}R^{\mu\nu\sigma\lambda}$ is singular for solution (20). Furthermore, since in the present case the equation describing an axially symmetric time-independent null hypersurface reduces to $g^{11}=0$, we can see that there is no event horizon in the domain of validity of solution (20).

B. Rest-mass density

With the functional relation (12) between A and λ, Eq. (6a) yields the following expression for the rest-mass density:

$$\frac{\rho}{\rho_0} = e^{2(\lambda-\nu)}\left(\nabla^2\lambda - \frac{1}{r^2}|\vec{\nabla}\lambda|^2\right), \tag{23a}$$

in which

$$\rho_0 = \frac{c^2}{4\pi G r_0^{\,2}}\,, \tag{23b}$$

and λ and ν are given by Eqs. (20) and (22), respectively. Along the z axis, this expression reduces to

$$\left.\frac{\rho}{\rho_0}\right|_{r=0} = 4a\cos kz(1 - a\cos kz)\,, \tag{24}$$

which implies that the central density is

$$\rho_c = 4a(1-a)\rho_0 \tag{25}$$

and indicates that ρ drops to zero at $z=\pi/2k$. The fact that the value of a is limited to the interval between zero and unity (mentioned in the preceding subsection), as well as the fact that the value of ρ_c can never exceed that of ρ_0 is now evident from Eq. (25).

As shown by Eq. (24), the relative contributions of magnetic energy density and rest-mass density as sources of the gravitational field are such that beyond a certain value of z the source of gravity is solely magnetic. This property of solution (20) is in fact not limited to the axis of symmetry; from a numerical computation based on Eq. (23a), we find that there is a cutoff in ρ for all directions away from the center. Solution (20) which was derived from a set of differential equations demanding only the local balance of gravitational and magnetic forces—without being subjected to any further conditions other than the finiteness of ρ at the center—has of itself turned out to describe a distribution of matter which is also globally balanced, i.e., is spatially bounded. In this respect, the present solution is of the same character as that describing the structure of a neutron star.[1]

To illustrate the cutoff in ρ, we have in Fig. 1

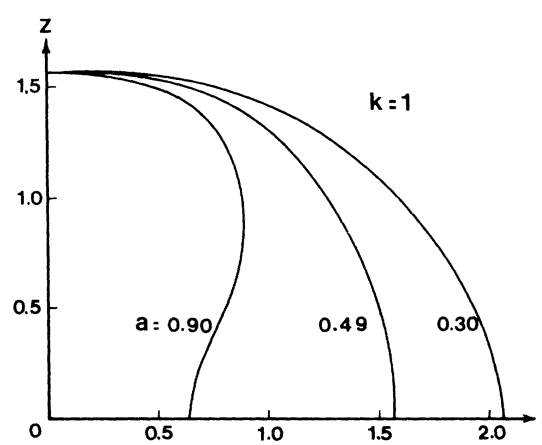

FIG. 1. The curves $\rho(r,z)=0$ for $k=1$ and $a=0.30$, 0.49, 0.90; r and z are in units of r_0. The surface of rotation obtained from each of these curves represents the upper boundary of the corresponding system. The polar radii of the shown systems are equal since all three correspond to the same value of k.

plotted the boundaries of the objects under consideration, i.e., the curves $\rho(r,z)=0$, for a single value of k and several different values of a; note that, according to Eq. (20), all quantities possess equatorial as well as axial symmetry. Of course since the boundary of a system is not a coordinate-independent concept, no direct physical significance should be attached to the shapes shown in Fig. 1. On the other hand, that the equatorial radii in this figure increase with a decreasing value of a reflects the general-relativistic nature of the density cutoff. As we shall see in the following subsection, the Newtonian limit of solution (20) corresponds to $a \ll 1$. Therefore, in this limit, the second term (which constitutes the contribution of magnetic energy density) will be absent from the right-hand side of Eq. (23a), and hence ρ will no longer reduce to zero at a finite radius: Eqs. (13) and (19) ensure that $\nabla^2\lambda>0$ for all r.

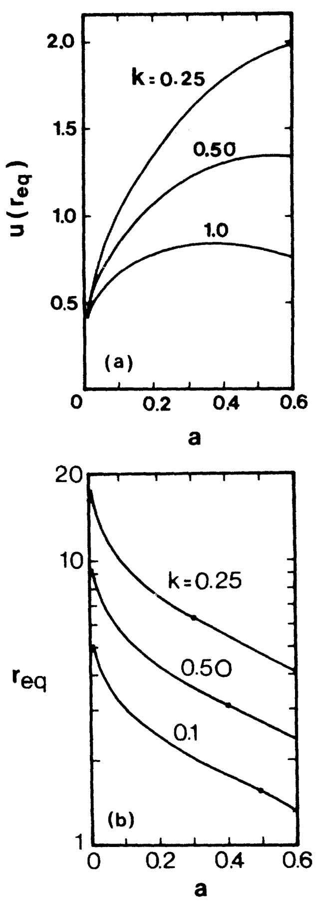

FIG. 2. (a) Dependence of $u(r_{eq})=\lambda(r_{eq},0)$ on the parameters a and k. The dimensionless quantity $u(r_{eq})$ represents the maximum value attained by the metric function λ in the interior of the system. (b) Dependence of the equatorial radius, r_{eq}, of the system on the parameters a and k. Here, r_{eq} is in units of r_0 and the dot on each curve specifies those values of a and k for which the polar and the equatorial radii of the system are equal.

If we now hold a constant and vary k, we find that not only the polar radius (which is always equal to $\pi/2k$), but also the equatorial radius, r_{eq}, is a monotonically decreasing function of k [see Fig. 2(b)]. This numerical result can easily be explained by noting that $\rho(r,z=0)$ vanishes at a point where

$$\frac{1+r^2}{2r}\frac{du}{dr}=\frac{du}{d\xi}=1 \quad (r=r_{eq}). \tag{26}$$

Since $du/d\xi$ starts out with the value a (<1) at the origin and increases monotonically (see Sec. III A), it is clear that the sharper the rise in $du/d\xi$, the sooner will the point r_{eq} be reached. On the other hand, $du/d\xi$ will rise more steeply the higher the value of k [see Eq. (19a)]. Furthermore, since $du/d\xi$ is proportional to a, one can now see why r_{eq} should also be a decreasing function of a [Fig. 2(b)].

It is worth noting that the value of the polar radius, $\pi/2k$, can here be endowed with a physical significance which is lacking in that of the equatorial radius: since the frame of reference in which we are working is locally inertial for all interior points along the z axis (see Sec. III A), for making measurements along this line segment we can proceed as in a Euclidean space. The world line $r=0$, $-\pi/2k\le z\le \pi/2k$, is a null geodesic of metric (4); and hence, if an observer located at $r=0$, $z=\pi/2k$ were to measure the radius of the system by sending a light signal along the axis of symmetry and receiving back its reflection from the center, then he would in fact obtain the same value as $\pi/2k$ (see Ref. 13).

C. Electric current density and magnetic field strength

The inhomogeneous Maxwell equations together with Eqs. (1b) and (5) show that the only surviving component of the electric current density is the following (contravariant) φ component:

$$j^{\varphi}=-\frac{c}{4\pi r_0^{\,4}}\frac{e^{2(\lambda-\nu)}}{r}\left[\frac{\partial}{\partial r}\left(\frac{e^{2\lambda}}{r}A_r\right)+\frac{\partial}{\partial z}\left(\frac{e^{2\lambda}}{r}A_z\right)\right]. \tag{27}$$

With the aid of Eqs. (12) and (23), this can be alternatively written as

$$j^{\varphi}=\frac{c}{4\pi}\frac{B_0}{r_0^{\,2}}e^{\lambda}\frac{\rho}{\rho_0}, \tag{28a}$$

where

$$B_0=(4\pi\rho_0c^2)^{1/2}. \tag{28b}$$

Therefore, another simple feature of the present solution is the vanishing of the invariant quantity

$$(-j_\mu j^\mu)^{1/2} = \frac{c}{4\pi}\frac{B_0}{r_0} r \frac{\rho}{\rho_0} \tag{29}$$

along with rest-mass density at the boundary of the system. It should perhaps be mentioned here that, although j^r and j^z are equal to zero, from the point of view of inertial observers located within the system all three spatial components of the electric current density are in general nonvanishing; in this sense, the current distributions within the magnetically balanced systems considered here are in fact rather complicated.

The magnetic field strength, as measured in a locally inertial frame, is according to Eqs. (1b), (12), and (28b) given by

$$B \equiv (\tfrac{1}{2} F_{\mu\nu} F^{\mu\nu})^{1/2} = B_0 \frac{e^{\lambda-\nu}}{r} |\vec{\nabla}\lambda| . \tag{30}$$

In particular then, from Eqs. (30) and (23) we have

$$\left.\frac{B}{B_0}\right|_{r=0} = 2a \cos kz \tag{31a}$$

and

$$\frac{B}{B_0} = \frac{e^{\lambda-\nu}}{r}\left(\frac{2ru'}{1+r^2}\cos kz\right)^{1/2}\Bigg|_{\text{boundary}}, \tag{31b}$$

for the variations of B along the z axis and along the boundary of the system, respectively. Hence, the central value of the magnetic field strength is

$$B_c = 2aB_0 , \tag{32}$$

and the ratio B/B_c which is zero at the pole, assumes the value

$$\hat{B}_{\text{eq}} = \frac{B}{B_c} = \frac{1}{a}\frac{e^{\lambda-\nu}}{1+r^2} \quad (z=0,\ r=r_{\text{eq}}), \tag{33}$$

at the equator [see Eq. (26)]. The quantity $\hat{B}_{\text{eq}}$, which can be regarded as a representative value of the strength of the surface magnetic field relative to that of the central magnetic field, has here been plotted for a range of values of the parameters a and k in Fig. 3. In this as well as in Figs. 4 and 5, the curves have been drawn only up to those values of a for which the equation $z = z(r)$ describing the boundary of the system is single valued. The solutions thus excluded, which correspond to extreme values of the central magnetic field (see below), are in any case physically irrelevant insofar as stellar configurations are concerned.

From Eqs. (25) and (32), we can now infer the following relationship between the central values of ρ and B and the parameter a:

$$\frac{B_c^2}{4\pi\rho_c c^2} = \frac{a}{1-a} . \tag{34}$$

Clearly, departure of the value of the invariant $B_c^2/4\pi\rho_c c^2$ from zero specifies the degree to which

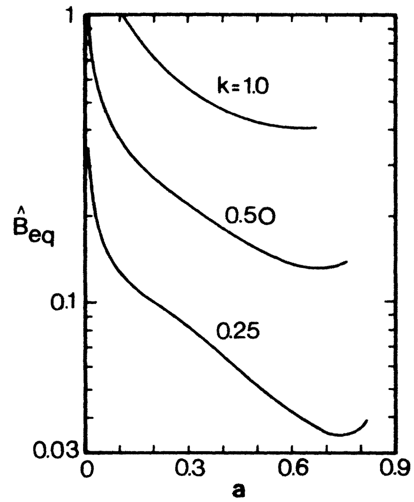

FIG. 3. Dependence of the magnitude of the surface magnetic field on the parameters a and k. Here, $\hat{B}_{\text{eq}}$ is in units of the central magnetic field strength B_c and represents the value of B at $r = r_{\text{eq}}$, $z=0$.

a given solution is relativistic. The limit $a \to 0$, for which we also have $\lambda \to 0$, corresponds to the Newtonian theory, $a = \frac{1}{2}$ represents the case in which, from the point of view of an inertial observer at the origin, the magnetic and the rest-mass energy densities are equal; and $a \to 1$ corresponds to an ultrarelativistic situation where the magnetic field, in addition to being the sole agent balancing gravity, is itself the dominant source of the existing gravitational field. In Fig. (2a), we have plotted $\lambda(r_{\text{eq}}, 0)$, i.e., $u(r_{\text{eq}})$ as a function of a for several values of k. Inasmuch as a light signal emitted at the center will be red-shifted by the amount $\Delta f/f = \exp[-u(r_{\text{eq}})] - 1$ when it arrives at the point $r = r_{\text{eq}}$, $z=0$ on the boundary, Fig. (2a) depicts the role that is also played by k in specifying the degree to which a given solution is relativistic: the smaller the value of k is, the smaller should the value of a be in order that the corresponding solution may approach the Newtonian limit.

IV. MASS AND MAGNETIC MOMENT

A. Mass

Since we are here dealing with time-independent gravitational and electromagnetic fields, the total energy of matter plus fields, i.e., the mass of the system, is given by[13]

$$M = \frac{c^2 r_0}{4\pi G}\int \sqrt{-g}\, R_0{}^0\, dV , \tag{35}$$

in which both the volume element $dV = r\,dr\,dz\,d\varphi$ and the integrand are dimensionless, and the integration extends over all space. For metric (4), this becomes

$$M = \frac{c^2 r_0}{4\pi G} \int \nabla^2 \lambda \, dV , \tag{36}$$

which can immediately be shown to reduce to an identity by converting the volume integral into a surface integral and using the following asymptotic form for λ at infinity:

$$\lambda \simeq -\frac{GM}{r_0 c^2} \frac{1}{(r^2+z^2)^{1/2}} + \text{const.} \tag{37}$$

Clearly, to evaluate the integral in Eq. (36) directly, we would need to know the exterior solution for λ as well as the interior solution given by Eq. (20). However, in the absence of an exterior solution, we can resort to the exterior field equations themselves to extract the necessary information for evaluating the above integral. As we shall see below, this is indeed a viable alternative in the present instance.

The exterior field equations may be inferred from Eqs. (6a) and (27) simply by setting $\rho = 0$ and $j^\varphi = 0$:

$$\nabla^2 \lambda - \frac{e^{2\lambda}}{r^2} |\vec{\nabla} \hat{A}|^2 = 0 , \tag{38a}$$

$$\frac{\partial}{\partial r}\left(\frac{e^{2\lambda}}{r} \hat{A}_r\right) + \frac{\partial}{\partial z}\left(\frac{e^{2\lambda}}{r} \hat{A}_z\right) = 0 , \tag{38b}$$

where, in accordance with the definition of α in Eq. (12),

$$\hat{A} = \frac{A}{\alpha} \tag{38c}$$

is a dimensionless quantity. It is well known[12] that, outside $r = 0$, Eqs. (38) are equivalent to the following two statements of continuity:

$$\vec{\nabla} \cdot \left(\vec{\nabla}\lambda - \frac{e^{2\lambda}}{r^2} \hat{A} \vec{\nabla} \hat{A}\right) = 0 \tag{39a}$$

and

$$\vec{\nabla} \cdot \left(\frac{e^{2\lambda}}{r^2} \vec{\nabla} \hat{A}\right) = 0 . \tag{39b}$$

If we now apply Gauss's theorem to integrate Eq. (39a) over an infinite volume from which the interior of the system and the z axis are excluded, then in the light of Eq. (36) we obtain

$$\frac{M}{M_0} = \int \left(\vec{\nabla}\lambda - \frac{e^{2\lambda}}{r^2} \hat{A} \vec{\nabla} \hat{A}\right) \cdot \hat{n} \, dS , \tag{40a}$$

where

$$M_0 = \rho_0 r_0^{\,3} , \tag{40b}$$

the surface of integration is the boundary of the interior, and $\hat{n}$ is the corresponding outward normal. In this expression, there is no contribution from the integral over the surface enclosing the z axis because, as expected on the basis of the symmetry of the problem, Eqs. (37) and (38) imply that $\partial A/\partial r$ vanishes at $r = 0$. This property of the exterior solution is also evident in the asymptotic form of the function $A(r, z)$ at infinity: to within an additive constant which we have chosen to be zero, this asymptotic form may be written (see Appendix C)

$$A \simeq \frac{\mu r^2}{(r^2+z^2)^{3/2}} , \tag{41}$$

where μ is the magnetic moment of the system.

In Appendix B, by employing the relevant junction conditions, we have shown that the functions λ, ν, and A are all continuous across the boundary of the system. Consequently, the integrand in Eq. (40a) is known to us, and with the aid of the interior solutions (12) and (20) we can in fact determine the mass directly from this expression. According to Eq. (12), in the interior of the system

$$\hat{A} = -e^{-\lambda} + 1 , \tag{42}$$

in which the choice of integration constant is dictated by the requirement that A should be continuous at the boundary point $r = 0$, $z = \pi/2k$ [see Eqs. (18), (20), and (41)]. Hence, with the notation of Eq. (40a),

$$\frac{M}{M_0} = \int \left[1 + \frac{1}{r^2}(1 - e^{\lambda})\right] \vec{\nabla}\lambda \cdot \hat{n} \, dS , \tag{43}$$

which, when partly integrated with the aid of Gauss's theorem and Eqs. (13) and (20), becomes

$$\frac{M}{M_0} = \frac{8\pi a}{k} - \int \frac{e^{\lambda}}{r^2} \vec{\nabla}\lambda \cdot \hat{n} \, dS . \tag{44}$$

Yet another expression for the mass can be found by transforming the right-hand side of Eq. (43) to a volume integral over the interior of the system, and using Eqs. (13) and (23a):

$$\frac{M}{M_0} = \int e^{2\nu - \lambda} \frac{\rho}{\rho_0} dV . \tag{45}$$

In this form, the total mass M can be readily compared with the rest-mass of the system which (assuming neutron matter) is given by the product of mass m and total number N of nucleons:

$$\frac{Nm}{M_0} = \int e^{2\nu - 3\lambda} \frac{\rho}{\rho_0} dV . \tag{46}$$

Here, $e^{2\nu - 3\lambda} dV$ is the 3-space volume element as measured by a local inertial observer.

In Fig. 4, we have plotted $\hat{M} = 2[a(1-a)]^{1/2} M/M_0$ as a function of the parameters a and k. Note that the dimensionless quantity $\hat{M}(a, k)$ is here defined such that, for given values of the central density ρ_c and the parameters a and k, the total mass can be

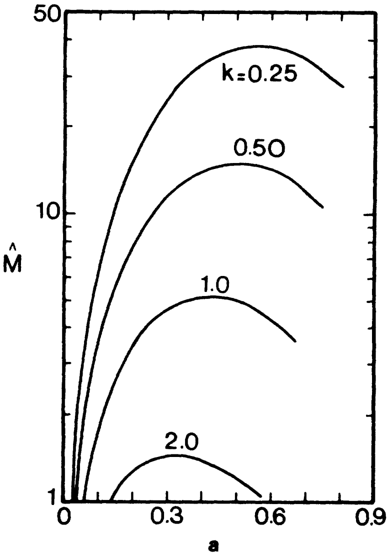

FIG. 4. Dependence of mass on the parameters a and k. The total mass of the system M may be deduced from the value of the dimensionless quantity $\hat{M}$ plotted here and Eq. (47).

deduced from

$$M=\left(\frac{c^2}{4\pi G}\right)^{3/2}\rho_c^{-1/2}\hat{M} \tag{47a}$$

$$=3.52\times 10^{40}\rho_c^{-1/2}\hat{M}\text{ g} \tag{47b}$$

[see Eqs. (40b), (25), (23b), and (14)]. Figure 4 therefore shows that for central densities of the order of 10^{14} g cm^{-3}, the value of M lies within the range of stellar masses if $10^{-2}\lesssim a\lesssim 10^{-1}$ and $10^{-1}\lesssim k\lesssim 1$.

B. Magnetic moment

As in the preceding subsection, we can apply the flux conservation laws to infer also the magnetic moment of a given equilibrium configuration directly from its interior solution. With the intention of deriving an equation of continuity suited to this purpose, we may start by rewriting Eq. (39b) outside $r=0$ as

$$\vec{\nabla}\cdot\left(\vec{\nabla}\hat{A}-\frac{2}{r}\hat{A}\hat{e}_r\right)+2\vec{\nabla}\lambda\cdot\vec{\nabla}\hat{A}=0\,. \tag{48}$$

Next, if we eliminate the last term from this equation by means of the identity

$$\vec{\nabla}\hat{A}\cdot\vec{\nabla}\lambda=\vec{\nabla}\cdot(\hat{A}\vec{\nabla}\lambda)-\hat{A}\nabla^2\lambda\,, \tag{49}$$

and make use of the following alternative form of Eq. (39a),

$$2\hat{A}\nabla^2\lambda=\vec{\nabla}\cdot\left(\frac{e^{2\lambda}}{r^2}\hat{A}^2\vec{\nabla}\hat{A}\right), \tag{50}$$

we will arrive at

$$\vec{\nabla}\cdot\left[\left(1-\frac{e^{2\lambda}}{r^2}\hat{A}^2\right)\vec{\nabla}\hat{A}+2\hat{A}\vec{\nabla}\lambda-\frac{2}{r}\hat{A}\hat{e}_r\right]=0\,. \tag{51}$$

Integration of this equation over the infinite-volume exterior to the system and to the z axis then yields

$$\frac{\mu}{\mu_0}=-\frac{1}{8\pi}\int\left[\left(1-\frac{e^{2\lambda}}{r^2}\hat{A}^2\right)\vec{\nabla}\hat{A}+2\hat{A}\vec{\nabla}\lambda-\frac{2}{r}\hat{A}\hat{e}_r\right]\cdot\hat{n}\,dS\,, \tag{52a}$$

where

$$\mu_0=r_0^{\,3}B_0\,, \tag{52b}$$

and contributions to the integral from the surfaces located at infinity and at the z axis have been found by using Eqs. (37) and (41) and by recalling that $A\sim r^2$ near $r=0$. Hence, from Eqs. (52) and (42) it follows that to evaluate the magnetic moment of the system,

$$\frac{\mu}{\mu_0}=\frac{1}{8\pi}\int\left\{\left[\frac{e^{\lambda}}{r^2}(1-e^{-\lambda})^2+e^{-\lambda}-2\right]\vec{\nabla}\lambda+\frac{2}{r}(1-e^{-\lambda})\hat{e}_r\right\}\cdot\hat{n}\,dS\,, \tag{53}$$

nothing more than the interior solution (20) for λ is required. As in the Newtonian case, it is the distribution of electric current in the interior of the system which determines the magnetic moment. Indeed, the volume-integral representation of Eq.

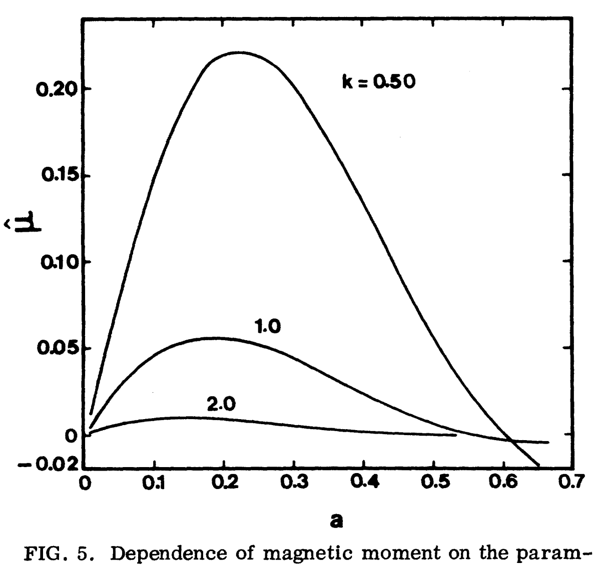

FIG. 5. Dependence of magnetic moment on the parameters a and k. The magnetic moment μ of the system may be deduced from the value of the dimensionless quantity $\hat{\mu}$ and Eq. (55). Note that for any given k there is a value of a for which $\hat{\mu}$ vanishes.

(53) is

$$\frac{\mu}{\mu_0}=\frac{1}{2cB_0}\int\left[1+\frac{1}{r^2}(1-e^{2\lambda})\right]e^{2(\nu-\lambda)}j_\varphi dV\,, \qquad (54)$$

where j_φ stands for the covariant component of the electric current density given in Eq. (27).

In Fig. 5, we have plotted $\hat{\mu}=4a(1-a)\mu/\mu_0$ as a function of the parameters a and k. Here, the quantity $\hat{\mu}(a,k)$ is again defined in such a way that, for given values of the central density ρ_c and the parameters a and k, the magnetic moment can be deduced from

$$\mu=\frac{c^4}{4\pi G^{3/2}}\rho_c^{-1}\hat{\mu} \qquad (55a)$$

$$=3.74\times10^{51}\rho_c^{-1}\hat{\mu}\ \mathrm{G\,cm^3} \qquad (55b)$$

[see Eqs. (52b), (28b), (23b), and (14)]. Figure 5 shows that, as suggested by Eq. (54), for certain values of a and k, the plotted quantity $\hat{\mu}$ equals zero; in other words, it shows that there can exist magnetically balanced equilibrium configurations whose fields at infinity correspond to multipoles higher than the dipole.

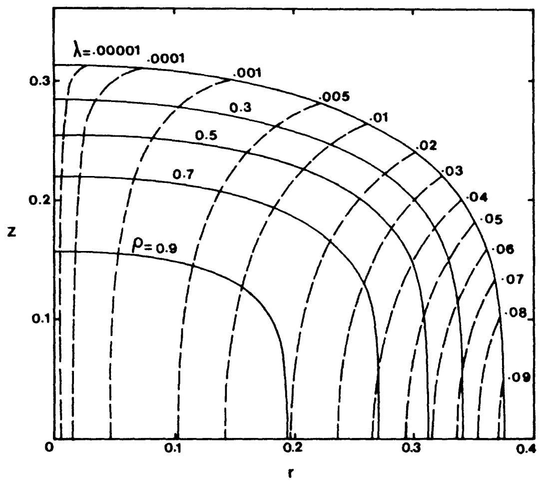

FIG. 6. Contours of ρ = const (solid lines) and λ = const (broken lines) for $a=0.47$, $k=5$. Here, ρ is in units of the central density ρ_c, and r and z are in units of r_0. In this particular case, the broken curves and the surfaces of rotation obtained from them may approximately be interpreted as the magnetic lines of force and the equipotential surfaces of the gravitational field, respectively.

V. DISCUSSION

Our treatment of the problem posed in Sec. I has so far been rather formal. To bring out the physical content of the analyses presented in Secs. II to IV, we shall start here by considering the topologies of the gravitational and the magnetic fields for a particular case where the curvature of space-time is negligibly small throughout the interior of the system. Such a case would either correspond to the Newtonian limit $a\ll1$, or to a relativistic situation in which the length scales of variation of the metric functions λ and ν are much larger than the extent of the system. The choice $a=0.47$, $k=5$, made here (see Figs. 6 and 7) corresponds to a system with the radial extent $r_{eq}=0.37$ for which everywhere $\lambda\le0.093$ and $|\lambda-\nu|\le0.045$. Hence, although dealing with a relativistic case, we can to within a good approximation interpret the quantities λ and B in Figs. 6 and 7 as the gravitational potential and the magnitude of the magnetic field, respectively. Furthermore, insofar as the quantity $\vec{\nabla}\times(A\hat{e}_\varphi/r)$ can in the Newtonian limit be regarded as the magnetic field vector (see Appendix A), from Eq. (9) it follows that the curves λ = const, φ = const, in Fig. 6 also approximately represent the magnetic lines of force. As the electric current in this case flows in the azimuthal direction [see Eq. (27)], the topology of both the magnetic and the gravitational forces can therefore be directly inferred from the equipotential surfaces λ = const. Note that at the boundary of the system and along the axis of symmetry there are no forces: where either $\rho=0$ or $r=0$ we also have $j^\mu j_\mu=0$ [see Eq. (29)], and $\vec{\nabla}\lambda\ (r=0)=0$. In Figs. 6 and 7 we have in addition plotted the contours of ρ = const and B = const; it can be seen that ρ decreases monotonically in all directions away from the center. The spatial distribution of B, on the other hand, is not always as shown in Fig. 7; for different choices of the parameters a and k, the behavior of B can be significantly different (also see Fig. 3).

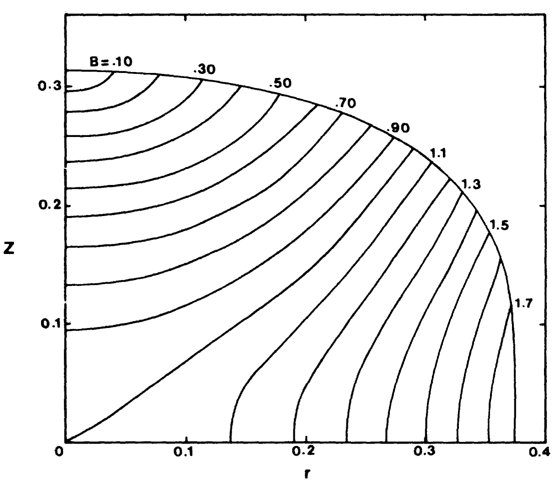

FIG. 7. Contours of B = const for $a=0.47$ and $k=5$. Here, B is in units of B_c and r and z are in units of r_0. In this particular case, B approximately represents the magnitude of the magnetic field vector.

Clearly, in an advanced stage of gravitational collapse the objects we have been considering here are primiarily composed of neutron matter. To ensure that the electric current in such objects can in principle be sufficiently strong to maintain the required magnetic fields, it is therefore necessary to estimate the degree of "charge separation" implied by Eq. (29). The orders of magnitude of the matter and the electric current densities appearing in this equation can be found from

$$\rho \simeq n_n m_n , \quad (-j^\mu j_\mu)^{1/2} \simeq n_e e v , \tag{55}$$

where m_n and n_n are the mass and the number density of neutrons, e and n_e are the charge and the number density of electrons, and v is an average velocity of electrons relative to protons (say). If we now insert these in Eq. (29) and make use of Eqs. (28b), (23b) and (14), we will obtain

$$\frac{n_e}{n_n} \simeq \frac{G^{1/2} m_n}{e} \left(\frac{c}{v}\right) r . \tag{56}$$

Note that $G^{1/2} m_n/e$, which equals the square root of the ratio of gravitational to electric force between two nucleons, has the value 0.90×10^{-18}. Therefore, in the relativistic regime where r is of the order of unity [see Fig. 2(b)], even for a population of low-energy electrons the ratio n_e/n_n required by the present solution is exceedingly small. According to current theories of collapsed stellar matter,[14] n_e/n_n at nuclear densities is by many orders of magnitude greater than what is here needed to maintain the required magnetic fields.

We have already pointed out that for a wide range of values of the parameters a and k the total masses of the equilibrium configurations considered here are comparable to stellar masses. Here, for one such configuration corresponding to $a = 0.01$, $k = 0.5$ we shall also compare the other physical parameters of the system with those of known astrophysical objects. This is a configuration which judging by $u(r_{eq}) = 0.472$ is only mildly relativistic. With a central density of 10^{11} g cm^{-3}, the mass of this system

$$M = 1.79 \times 10^{34} (\rho_c/10^{11})^{-1/2} \text{ g} , \tag{57}$$

equals 9 solar masses, and its magnetic field strength at the center,

$$B_c = 3.38 \times 10^{15} (\rho_c/10^{11})^{1/2} \text{ G} \tag{58}$$

is about three orders of magnitude greater than the surface field of a pulsar. Furthermore, since the extent of this system, which may be inferred[15] from the value of the polar radius

$$\frac{\pi r_0}{2k} = 6.48 \times 10^7 (\rho_c/10^{11})^{-1/2} \text{ cm} , \tag{59}$$

is appreciably greater than that of a neutron star, the magnetic moment in this case

$$\mu = 4.98 \times 10^{38} (\rho_c/10^{11})^{-1} \text{ G cm}^3 \tag{60}$$

turns out to be correspondingly large. However, this last feature is not characteristic of the present objects; as shown in Fig. 4, by a suitable choice of a and k one can also construct models which have vanishingly small magnetic moments.

In the context of the above particular model, we may now also examine the validity of having ignored pressure in our analysis. According to most equations of state,[16] the pressure p of neutron matter at the central densities envisaged here is of the order of $10^{-3} \rho_c c^2$. Since, on the other hand, the central value of the magnetic energy density for small values of a is $B_c^2/4\pi \simeq a\rho_c c^2$ [see Eq. (34)], we can see that $B_c^2/4\pi p \simeq 10^3 a$. In other words, the contribution of magnetic energy density to the energy-momentum tensor is for the present model approximately ten times greater than that of pressure. For models which are more relativistic ($a > 0.01$) this factor is of course larger.

In the absence of an exterior solution, we have not been able to extend the equipotential surfaces (λ = const) shown in Fig. 6 beyond the boundary of the system. However, since we also know that these surfaces assume a spherical shape at infinity, it is not difficult to infer the essential features of their global topology from what is already shown in Fig. 6. If we visualize the equipotential surfaces at infinity as concentric spherical balloons, then by gradually constricting these balloons along their equatorial planes as we approach the object, we would obtain a sequence of dumbbell-like closed surfaces which coincide with the equipotentials shown in Fig. 6 whenever they partly enter the interior of the object. Whether or not this sequence does approximately represent the equipotential surfaces depends of course on the validity of the assumption made here that an exterior solution exists and is nonsingular. The question of the existence and regularity of the exterior solution has been discussed in Appendix C on the basis of its asymptotic expansion. In this connection we should add that, if an exterior solution exists, then it can neither possess an event horizon nor be singular. It is well known[17] that all static (nonrotating) exterior solutions which possess an event horizon are characterized by the Reissner-Nordström metric. In the present case, not only is the metric different from that of Reissner-Nordström (see Appendix C), but also the equation describing an axisymmetric time-independent null hypersurface has no other solution than $g^{11} = 0$. Therefore, such a surface is absent in the present case, and since g^{11} is for that reason nowhere equal to zero, the exterior solutions for λ and ν are nonsingular.

In addition to the question of the existence of an exterior solution which has not been fully answered here, any definitive statements concerning the stability of the present objects must also await further studies. However, in that the magnetic field is in the present situation employed as a disruptive agent, it is clear that the stability problem we have to deal with here is precisely the opposite of that encountered in the confinement of plasmas by magnetic force. The mass of evidence accumulated on the instability of such plasmas may in fact be invoked as an indication that the present gravitationally bound equilibrium configurations should—at least for certain values of the parameters a and k—be stable.

Without having examined their stability, it may perhaps be considered rather premature to attempt to relate the properties of the objects discussed in this paper to astrophysical observations. However, it is interesting to note that none of the characteristic features of these objects is inconsistent with the observed properties of the collapsed members of binary x-ray sources. Indeed, since the present analysis allows of these objects attaining masses well above the maximum mass of a neutron star, we can find no reason to exclude the possibility that the collapsed member of Cygnus X-1 may in fact be a magnetically balanced equilibrium configuration rather than a black hole.

ACKNOWLEDGMENTS

We would like to thank Dr. B. Mashhoon for his interest and helpful correspondence and Professor H. Bondi for several constructive discussions.

APPENDIX A

In this appendix we will derive the counterpart of Eq. (11) directly from the nonrelativistic field equations and show that it is also possible to have magnetically balanced equilibrium configurations within the framework of the Newtonian theory.

The nonrelativistic condition of force balance and its corresponding gravitational field equation are

$$\frac{1}{4\pi}(\vec{\nabla}\times\vec{B})\times\vec{B}-\rho\vec{\nabla}\phi=0\,, \tag{A1}$$

$$\nabla^2\phi=4\pi G\rho\,, \tag{A2}$$

where $\vec{B}$ is the magnetic field and ϕ is the gravitational potential. It is well known that in the case of axial symmetry the divergenceless vector $\vec{B}$ can without any loss of generality be written as

$$\vec{B}=\frac{1}{r}\vec{\nabla}A\times\hat{e}_\varphi+\frac{1}{r}F\hat{e}_\varphi\,, \tag{A3}$$

in which A and F are independent of the coordinate φ. Therefore, with the intention of resolving Eq. (A1) into its components, we may express the magnetic field $\vec{B}$ everywhere in this equation in terms of the stream functions A and F to arrive at

$$\vec{\nabla}F\times\vec{\nabla}A=0 \tag{A4a}$$

and

$$\left(\nabla^2A-\frac{2}{r}A_r\right)\vec{\nabla}A+F\vec{\nabla}F+4\pi\rho r^2\vec{\nabla}\phi=0 \tag{A4b}$$

for the conditions of force balance in the toroidal and the poloidal directions, respectively. Equation (A4a) and the following consequence of Eq. (A4b):

$$\vec{\nabla}\phi\times\vec{\nabla}A=0\,, \tag{A5}$$

simply state that ϕ and F are functions of A, i.e., that the gravitational potential and the product of r with the toroidal component of magnetic field both remain constant along the magnetic lines of force. Since A/r is the φ component of the magnetic vector potential, Eq. (A5) is here clearly recognizable as the nonrelativistic counterpart of Eq. (9).

If we now insert the above results in the forms $A=A(\phi)$ and $F=F(\phi)$ into Eq. (A4b) and express ρ with the aid of Eq. (A2) in terms of ϕ, then upon factoring out $\vec{\nabla}\phi$ we find

$$\left(1+\frac{r^2}{GA'^2}\right)\nabla^2\phi-\frac{2}{r}\phi_r+\frac{A''}{A'}|\vec{\nabla}\phi|^2+\frac{F'F}{A'^2}=0\,, \tag{A6}$$

where the primes denote differentiation with respect to ϕ. Apart from the last term, which would in any case be absent for $F=0$, this equation is of precisely the same mathematical structure as Eq. (11). Inasmuch as the metric function λ can be interpreted as the gravitational potential in the Newtonian limit, that Eqs. (10) and (11) reduce to Eq. (A6) when $\lambda\ll 1$ is of course expected. In the present case, however, the relativistic and the nonrelativistic equations happen in addition to be mathematically identical. The absence in Eq. (11) of a term corresponding to FF'/A'^2 can obviously be traced to the assumption embodied in the first member of Eq. (5); by relaxing this assumption in the nonrelativistic analysis, we have here allowed the azimuthal component of the magnetic field to be nonzero.

That for $A''=F'=0$ there exist physically tenable solutions to Eq. (A6) has already been shown in Sec. III. However, since the source term in the Newtonian field equation (A2) does not contain any contribution from the magnetic energy density, the particular solutions adopted in Sec. III no longer predict a cutoff of density in all directions away from the origin. Here, in order to obtain an equilibrium configuration which is spatially bounded, we must instead adopt a superposition of such solutions [see Eq. (17)], and besides the regularity condition at the center, we must also impose $\rho = 0$ (which corresponds to the condition $\phi_r = 0$) at the prescribed boundary of the system. It is nevertheless clear that as far as the feasibility of balancing self-gravitating systems by magnetic force is concerned, predictions of the relativistic and the Newtonian theories differ not in principle but merely in detail. That in the present case the relativistic theory must be used is dictated by properties of matter: our neglect of pressure is not permissible unless the magnetic and the rest-mass energy densities are sufficiently large (see Sec. V).

APPENDIX B

The purpose of this appendix is to show that the junction conditions for joining two solutions of the Einstein-Maxwell equations require in the present instance that the functions λ, ν, and A should be continuous across the boundary of the system.

With the symbol $[q]$ for the change $q - q'$ of a given quantity q across the boundary, the conditions of continuity for the metric and for the fluxes of energy, momentum, and magnetic field can be written as[18, 19]

$$[g_{\mu\nu}] = 0, \tag{B1}$$

$$[T^{\mu\nu}]n_\nu = 0, \quad [F^{*\mu\nu}]n_\nu = 0, \tag{B2}$$

where $F^{*\mu\nu}$ is the tensor dual to $F^{\mu\nu}$, and n_ν is the spacelike vector

$$n_\nu = (0, \vec{\nabla}\rho\,|_{\rho=0}) \tag{B3}$$

normal to the boundary of the system. Both $[\lambda] = 0$ and $[\nu] = 0$ are therefore immediate consequences of Eq. (B1). When these are taken into account, then the second member of Eq. (B2) becomes

$$[\vec{\nabla}A \times \vec{n}] = 0,$$

which simply states that from the point of view of local inertial observers at the boundary the normal component of the magnetic field is also continuous. (Here, $\vec{n}$ stands for the spatial part of the four-vector n_ν.)

Furthermore, the first member of Eq. (B2) together with the energy-momentum tensor appearing on the right of Eq. (1a) yields

$$[A_r^2 - A_z^2]n_1 + [2A_r A_z]n_2 = 0, \tag{B4a}$$

$$-[2A_r A_z]n_1 + [A_r^2 - A_z^2]n_2 = 0, \tag{B4b}$$

where n_1 and n_2 designate the r and the z components, respectively, of the poloidal vector $\vec{n}$. [Note that T^{11}, T^{12}, and T^{22} which are the only components of the energy-momentum tensor entering Eq. (B4) are the same for the regions interior and exterior to the system.] For Eqs. (B4a) and (B4b) to be rendered consistent, therefore, we should set the determinant of the coefficients of n_1 and n_2 equal to zero:

$$[A_r^2 - A_z^2]^2 + [2A_r A_z]^2 = 0, \tag{B5}$$

i.e., we should have

$$[A_r^2 - A_z^2] = 0, \quad [A_r A_z] = 0. \tag{B6}$$

Now, it is straightforward to show that Eqs. (B1) and (B6) together imply $[\vec{\nabla}A] = 0$. On the other hand, since we may add an arbitrary constant to the exterior function $A(r, z)$ without affecting either of the two invariant quantities given by Eqs. (40) and (52), from $[\vec{\nabla}A] = 0$ we can construe that $[A] = 0$. It should be noted that, as derived here, the continuity of A would remain valid even if we were to join the interior and exterior solutions across a surface of nonzero density. Since in the present case $j^\mu = 0$ at the boundary of the system, however, we could have alternatively used $[F^{\mu\nu}]n_\nu = 0$ instead of $[T^{\mu\nu}]n_\nu = 0$ to arrive at the same result. In fact, here, as a consequence of $[\rho] = 0$, the tensors $F^{\mu\nu}$ and $T^{\mu\nu}$ are themselves continuous across the boundary of the system.

APPENDIX C

The task of this appendix is to show that, at least to within a few leading orders, there exist power-series solutions to the exterior field equations (39) which possess the appropriate asymptotic behavior at infinity. The solutions which will be presented do not merely correspond to the multipole expansion of the magnetic field in a given space-time metric; here, the series expansions are carried to terms which in turn represent the effect of the electromagnetic field on the metric. That it has been at all possible to carry out this procedure without encountering any singularities is to be regarded as one of the main results of this appendix.

A well-known[11, 12] alternative form of the set of exterior field equations (39) which is more suited to the purposes of the present analysis is

$$\nabla^2\lambda = e^{-2(\lambda-\lambda_\infty)}|\vec{\nabla}\psi|^2, \tag{C1}$$

$$\nabla^2\psi = 2\vec{\nabla}\lambda \cdot \vec{\nabla}\psi, \tag{C2}$$

in which

$$\vec{\nabla}\psi = \frac{1}{r}e^{2(\lambda-\lambda_\infty)}\hat{e}_\varphi \times \vec{\nabla}A, \tag{C3}$$

and λ_∞ stands for the constant value of λ at infinity (see Sec. III A). As already indicated in Eqs. (37) and (41), those solutions of these field equations which are relevant to the present case should asymptotically reduce to

$$\lambda \simeq \lambda_\infty - \frac{m}{R}, \quad \psi \simeq \frac{\mu\cos\theta}{R^2} \quad (R \to \infty), \tag{C4}$$

where R and θ are the usual spherical coordinates, and the coefficients m and μ characterize the mass and the magnetic moment of the system, respectively. We shall now represent the expansions of λ and ψ in inverse powers of R by

$$\lambda = \lambda_\infty - \frac{m}{R} + \sum_{n=2}^{\infty} \frac{a_n(\theta)}{R^n}, \tag{C5}$$

$$\psi = \frac{\mu \cos\theta}{R^2} + \sum_{n=3}^{\infty} \frac{b_n(\theta)}{R^n}, \tag{C6}$$

and proceed to determine the unknown functions a_n and b_n by solving Eqs. (C1) and (C2) in steps corresponding to increasing orders of $1/R$. We shall also bear in mind that the equatorial symmetry of the problem in the present case requires λ and ψ to be, respectively, even and odd functions of $\cos\theta$.

Since the right-hand side of Eq. (C1) is $O(R^{-6})$, up to this order the metric function λ is governed by the empty-space field equation

$$\nabla^2\left(\lambda_\infty - \frac{m}{R} + \frac{a_2}{R^2} + \frac{a_3}{R^3}\right) = 0, \tag{C7}$$

whose solutions for a_2 and a_3 are proportional to Legendre functions of orders 1 and 2, respectively. However, as the requirement of symmetry would subsequently eliminate a_2, these solutions may be written

$$a_2 = 0, \quad a_3 = \alpha_3 P_2(\cos\theta), \tag{C8}$$

where the constant α_3 characterizes the quadrupole moment of the mass-energy distribution of the system. Hence, in order to satisfy Eq. (C2) to within $O(R^{-6})$ we should have

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{db_3}{d\theta}\right) + 6b_3 = -4m\mu \cos\theta, \tag{C9}$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{db_4}{d\theta}\right) + 12b_4 = -6mb_3, \tag{C10}$$

which yield

$$b_3 = -m\mu \cos\theta, \quad b_4 = \tfrac{3}{5} m^2\mu[\cos\theta + \beta_4 P_3(\cos\theta)]. \tag{C11}$$

Next, inserting this solution for b_3 in the expansion of Eq. (C1) and setting the coefficients of R^{-6} and R^{-7} in the resulting equation equal to zero, we obtain

$$\frac{1}{\sin\theta}\frac{d}{d\theta}(\sin\theta\, a_4) + 12a_4 = \mu^2(1 + 3\cos^2\theta), \tag{C12}$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}(\sin\theta\, a_5) + 20a_5 = -4m\mu^2\cos^2\theta, \tag{C13}$$

whose solutions are

$$\begin{aligned} a_4 &= \tfrac{1}{2}\mu^2\cos^2\theta, \\ a_5 &= \tfrac{1}{35} m\mu^2[(1 - 10\cos^2\theta) + \alpha_5 P_4(\cos\theta)]. \end{aligned} \tag{C14}$$

It is to be emphasized that there are no *a priori* reasons why the inhomogeneous Legendre equations emerging from the present analysis should all possess such well-behaved solutions as those appearing in Eqs. (C11) and (C14). The well-defined nature of the coefficients a_4 and a_5 which represent the contribution of the magnetic energy density to the curvature of space-time is clearly of physical significance. Furthermore, the following solutions, which were obtained above,

$$\lambda = \lambda_\infty - \frac{m}{R} + \alpha_3 \frac{P_2(\cos\theta)}{R^3} + \frac{\mu^2}{2}\frac{\cos^2\theta}{R^4} + \frac{m\mu^2}{35}\frac{\alpha_5 P_4(\cos\theta) + (1 - 10\cos^2\theta)}{R^5} + O(R^{-6}), \tag{C15}$$

$$\psi = \frac{\mu\cos\theta}{R^2} - m\mu\frac{\cos\theta}{R^3} + \tfrac{3}{5}m^2\mu\frac{\beta_4 P_3(\cos\theta) + \cos\theta}{R^4} + O(R^{-5}), \tag{C16}$$

already contain constants ($\alpha_3, \alpha_5, \beta_4$) that are unspecified. Since the complete solutions (C5) and (C6) are accordingly expected to contain an infinite number of such unspecified constants, it should in principle be possible to join these to the interior solutions (20) and (42) with any given degree of accuracy by extending the present analysis to higher orders.

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