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Scale-covariant theory of gravitation and astrophysical applications

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By associating the mathematical operation of scale transformation with the physics of using different dynamical systems to measure space-time distances, we formulate a scale-covariant theory of gravitation. Corresponding to each dynamical system of units is a gauge condition which determines the otherwise arbitrary gauge function. For gravitational units, the gauge condition is chosen so that the standard Einstein equations are recovered. Assuming the atomic units, derivable from atomic dynamics, to be distinct from the gravitational units, a different gauge condition must be imposed. It is suggested that Dirac's large-number hypothesis be used for the determination of this condition so that gravitational phenomena can be described in atomic units. The result allows a natural interpretation of the possible variation of the gravitational constant without compromising the validity of general relativity. A geometrical interpretation of the scale-covariant theory is possible if the covariant tensors in Riemannian space are replaced by cocovariant cotensors in an integrable Weyl space. A scale-invariant action principle is constructed from the metrical potentials of the integrable Weyl space. Application of the dynamical equations in atomic units to cosmology yields a family of homogeneous solutions characterized by $R \sim t$ for large cosmological times. Equations of motion in atomic units are solved for spherically symmetric gravitational fields. Expressions for perihelion shift and light deflection are derived. They do not differ from the predictions of general relativity except for secular variations, having the age of the universe as a time scale. Similar variations of periods and radii for planetary orbits are also derived. The generalized hydrodynamic equations derived for atomic units are studied. It is found that the stellar structure equations are formally unchanged, except that G and M can now be functions of the cosmological time. This in turn would imply secular variations of the stellar luminosities. The effects of these results on the past climatology of the earth and other geological effects are discussed. None of the consequences of the theory investigated so far is found to be in disagreement with observations.

I. INTRODUCTION

In recent years, owing to the scaling behavior exhibited in high-energy particle scattering experiments, there has been considerable interest in manifestly scale-invariant theories.[1] However, such theories are considered valid only in the limit of high energies or vanishing rest masses. This is due to the fact that in elementary particle theories, rest masses are considered constants, and it is well known that scale invariance is generally valid only when the constant-rest-mass condition is relaxed.[2] An alternative explanation is that, if a theory has an *a priori* given mass, a length scale intrinsic to the theory can be constructed. If the rest mass vanishes, no such intrinsic scale exists and the theory would thus be invariant under scale transformation. Implicit in the above consideration is the language of quantum fields. Not only is the space-time metric *a priori* given, it is furthermore assumed that distances can be measured independent of the dynamics of the field under consideration. The scale invariance referred to above states that the dynamic equations of the fields are covariant with respect to local variations of the units of measurement. While such local scale transformation is mathematically well defined, its physical significance is obscure unless one can prescribe physical processes which could reveal the local variations. To understand how this can be done, it is necessary to understand what constitutes a unit of measurement.

We note that in the early days of quantum mechanics, it was felt that atomic physics, since it is fundamental, provided the only natural system of units[3] whereby all physical quantities are measured, and that it was unphysical to consider general scale transformations away from the atomic units.[4] This point of view was modified when different kinds of interaction emerged and there was no apparent unification among them. To make measurements, a physical reference system is needed. Such systems must themselves obey physical laws. Thus if different reference systems are governed by dynamical laws corresponding to different interactions, independent systems of units can be obtained. For example, the astronomical unit of length is conventionally taken to be the sun-earth distance. Since this unit results from gravitational dynamics, insofar as a unified theory of gravitation and quantum electrodynamics does not exist, it is logically conceivable that the astronomical unit mentioned above and the atomic unit, e.g., the Bohr radius, are related generally by a scalar function of space-time. Thus, with the mathematical operation of local scale transformation we associate the physics of using different dynamical systems to measure space-time distances.

With this understanding of scaling, we wish to consider gravitational phenomena under such transformations. Since gravitation is of infinite range, it is often said to be mediated by a massless particle, the graviton. In view of our previous discussion, gravitational theories are expected to be scale invariant. While this is clear from the formalism of quantum fields, we wish to see what scale invariance implies for a classical gravitation theory. Einstein's general relativity was originally constructed as a theory of gravitation only. As such, the implied system of units, which corresponds to a nonvarying gravitational constant, must be gravitational.[5] We shall apply a general scale transformation to it so that a scale-covariant theory is obtained. The physical consequences of this theory will then be explored.

There has been a long history of similar, but more ambitious generalizations of Einstein's theory. We mention the two prominent ones. In an attempt to unify electromagnetism with gravitation, Weyl[6] generalized Riemannian geometry by allowing lengths to change under parallel displacement. Although the theory was soon rejected as being unphysical, a mathematical operation known as gauge transformation was introduced. It represents, as was pointed out by Eddington,[7] a change of units of measurement and hence gives a general scaling of the physical system being investigated. In a paper, Dirac[8] rebuilt Weyl's unified theory by introducing the notion of two metrics and an additional gauge function $\beta(x)$. A scale-invariant variational principle was proposed from which gravitational and electromagnetic field equations can be derived. As will be seen later, an arbitrary gauge function is necessary in all scale-covariant theories. The concept of two metrics, derived from the Milne hypothesis, was introduced by Dirac decades ago[9] to make compatible his large-number hypothesis (LNH, a brief summary can be found in Appendix B) and Einstein's general relativity. The idea that there exist two metrics, one corresponding to gravitational (or Einstein) units, the other, to atomic units, is identical to our notion oi local scale transformation from gravitational dynamics to quantum electrodynamics.

To investigate the physical significance of a scale-covariant theory of gravitation, we shall first introduce Dirac's theory and define its range of applicability in accordance with our own interpretation. In the process, minor modifications will be introduced. But we believe that the fundamental concepts are consistent with Dirac's ideas.

A scale-covariant theory provides the necessary theoretical framework in which it becomes sensible to discuss the possible variation of the gravitational constant G. Letting G be a constant amounts to the adoption of a particular system of units, the gravitational (or Einstein) units. Conversely, G can be found to vary if, and only if, measurements are made with respect to units other than the gravitational ones. Furthermore, it is incorrect to search for effects of varying G by fitting data with Einstein's equations and simply allowing G to be variable, for in this case the equations are inconsistent with the conservation laws that are commonly assumed. Sensible analysis of the effects of varying G can be given only through a system of modified field equations and modified conservation laws.

We contend also that Dirac's LNH[10-13] can be incorporated in the present theoretical framework. The statement that the gravitational constant is inversely proportional to the age of the universe when measured in atomic units amounts to a statement concerning the relation between gravitational and atomic units. This we believe was the intent of Dirac's LNH for the last forty years. With the scale-covariant theory of gravitation, the necessary mathematical relations can be given for observational tests of the LNH to be carried out. It should be noted, however, that the LNH is not an essential ingredient of the present theory. As will be seen below, we shall use it to fix a gauge condition, which in principle could be determined by actual observation or by some fundamental principle, once a unified theory has been constructed.

In Sec. II, we shall derive the generalized gravitational field equations in three different, but equivalent ways: (1) by performing a direct scale transformation, (2) by extending Riemannian geometry to Weyl geometry, through the introduction of the notion of cotensors, and (3) from a variational principle. Modified conservation laws will be presented. Having obtained a set of dynamical equations, we shall derive astrophysical consequences in analogy with the standard theory, viz., dynamical equations will be solved for specific problems at hand. In particular, we consider homogeneous cosmological solutions in Sec. III. In Sec. IV, we study the geodesic equations and derive expressions for the perihelion shifts, light deflections, and secular variations of planetary orbital elements. We also derive the stellar structure equations for a star in quasistatic equilibrium. A short discussion of the effects of the above results on the past thermal history of the earth as well as of other geophysical effects is also given. Finally, we shall speculate on the possible relation of the present theory with gauge field theories and their predictions of cosmological constants.

The appearance of a scalar gauge function as well as the metric tensor in the present theory can give the erroneous impression that we are dealing with a special case of scalar-tensor theories[14,15]. We shall try to point out the essential differences here. Firstly, scalar-tensor theories are intended to be complete dynamical theories. Not only gravitation, but the dynamics of atomic systems (at least in the classical description) are also included. We, on the other hand, are only concerned with gravitation and do not consider at this point atomic dynamics or its coupling to gravitation. The scalar function, having no independent field equation for it, will not be introduced as a field variable. Its physical significance will be unambiguously defined by Eq. (2.4) below to be the relation between the measuring instrument and a gravitationally constructed clock. As such, the scalar does not participate in the dynamics of gravitational interaction. Thus Einstein's theory of gravitation is not modified. We have simply allowed for different measuring procedures.

We note also that since in scalar-tensor theories, the scalar field was introduced explicitly to modify gravitational dynamics, the theories necessarily give different predictions from Einstein's theory regardless of measuring instruments, and hence can be ruled out by improved experimental confirmation of Einstein's theory. The scale-covariant theory which we shall present does not run into the same difficulty. More details on this point will be given in Sec. IV.

II. SCALE-COVARIANT THEORY OF GRAVITATION

A. Transformation of units–Einstein field equation

Given an atomic system, one could use it to provide a unit for the measurement of space-time intervals. A collection of such measurements then provides an operationally defined metric tensor $g_{\mu\nu}$. To the extent that the atomic system need not be considered a source of the gravitational fields, i.e., in the approximation that the measurements do not disturb the gravitational field present, $g_{\mu\nu}$ need not be identified with $\bar{g}_{\mu\nu}$, the metric tensor given by measurements using gravitational units, and which is intrinsic in the theory of geometrodynamics. Thus we start with the Einstein equations in Einstein units

$$\bar{G}_{\mu\nu} = -8\pi\bar{\mathfrak{J}}_{\mu\nu} + \bar{\Lambda}\bar{g}_{\mu\nu}\,, \tag{2.1}$$

where

$$\bar{G}_{\mu\nu} = \bar{R}_{\mu\nu} - \tfrac{1}{2}\bar{g}_{\mu\nu}\bar{R} \tag{2.2}$$

is the Einstein tensor. The bars indicate that Einstein units are being used. The line element $d\bar{s}$ is given by

$$d\bar{s} = \bar{g}_{\mu\nu}dx^{\mu}dx^{\nu}\,, \tag{2.3}$$

where the coordinate interval is dimensionless. $\bar{\mathfrak{J}}_{\mu\nu}$ is the matter energy-momentum tensor expressed in geometric units, i.e., $(\text{length})^{-2}$, with lengths in Einstein units. Under a transformation

$$d\bar{s} \to ds = \beta^{-1}(x)d\bar{s}\,, \tag{2.4}$$

it is easily seen that since

$$ds^2 = g_{\mu\nu}dx^{\mu}dx^{\nu} \tag{2.5}$$

then

$$\bar{g}_{\mu\nu} = \beta^2 g_{\mu\nu}\,. \tag{2.6}$$

Equation (2.6) represents a conformal transformation from a geometry described by $\bar{g}_{\mu\nu}$ to one described $g_{\mu\nu}$. The corresponding transformation of the Ricci tensors and therefore of the Einstein tensor is well known.[16] We have[17]

$$\bar{G}_{\mu\nu} = G_{\mu\nu} + \frac{2\beta_{\mu;\nu}}{\beta} - \frac{4\beta_{\mu}\beta_{\nu}}{\beta^2} - g_{\mu\nu}\left(2\frac{\beta^{\lambda}{}_{;\lambda}}{\beta} - \frac{\beta^{\lambda}\beta_{\lambda}}{\beta^2}\right)\,, \tag{2.7}$$

where on the right-hand side, covariant differentiation as well as index raising and lowering operations are carried out with respect to $g_{\mu\nu}$.

The cosmological term can be written as

$$\bar{\Lambda}\bar{g}_{\mu\nu} = \bar{\Lambda}\beta^2 g_{\mu\nu} = \Lambda g_{\mu\nu} \tag{2.8}$$

with

$$\Lambda = \beta^2\bar{\Lambda}\,. \tag{2.9}$$

This does not complete the transformation of units on the Einstein equations since the consideration of conformal transformation of geometries does not tell us how $\bar{\mathfrak{J}}_{\mu\nu}$ transforms. To find out, we consider a further transformation of (2.7). Let

$$g_{\mu\nu} = \varphi^2 g'_{\mu\nu}\,. \tag{2.6$'$}$$

Denoting covariant differentiation with respect to $g'_{\mu\nu}$ by a colon, the expression on the right-hand side of (2.7) can be written as

$$G'_{\mu\nu} + 2\frac{(\beta\varphi)_{\mu:\nu}}{\beta\varphi} - 4\frac{(\beta\varphi)_{\mu}(\beta\varphi)_{\nu}}{(\beta\varphi)^2} - g'_{\mu\nu}\left(2\frac{(\beta\varphi)^{\lambda}{}_{:\lambda}}{\beta\varphi} - \frac{(\beta\varphi)^{\lambda}(\beta\varphi)_{\lambda}}{(\beta\varphi)^2}\right)\,.$$

Noting that

$$\bar{g}_{\mu\nu} = \beta^2 g_{\mu\nu} = \beta^2\varphi^2 g'_{\mu\nu} = \beta'^2 g'_{\mu\nu}\,,$$

we see that the above exercise demonstrates the form invariance of the right-hand side of (2.7).

Similar invariance can be trivially ascertained for the cosmological term in the Einstein equation. We conclude therefore that the matter source term must also be form invariant and that the field equations in general units can be written as

$$G_{\mu\nu}+2\frac{\beta_{\mu;\nu}}{\beta}-4\frac{\beta_\mu\beta_\nu}{\beta^2}-g_{\mu\nu}\left(2\frac{\beta^\lambda{}_{;\lambda}}{\beta}-\frac{\beta^\lambda\beta_\lambda}{\beta^2}\right)$$
$$=-8\pi\,\mathfrak{T}_{\mu\nu}+\Lambda g_{\mu\nu}\,. \quad (2.10)$$

Here, by definition,

$$\overline{\mathfrak{T}}_{\mu\nu}\equiv\mathfrak{T}_{\mu\nu}(\beta), \quad (2.11a)$$

or explicitly

$$\overline{G}\,\overline{T}_{\mu\nu}=G(\beta)T_{\mu\nu}(\beta) \quad (2.11b)$$

with

$$\overline{G}\equiv G(\beta=1), \quad \overline{T}_{\mu\nu}\equiv T_{\mu\nu}(\beta=1)\,,$$

where $G(\beta)$ and $T_{\mu\nu}(\beta)$ are the gravitational constant and the energy-momentum tensor in general units. Even though the specific functional dependence on β is unknown at this point, we can anticipate some of the future results by pointing out that in the restricted case of a pressureless perfect fluid, (2.11b) reduces to

$$\overline{G}\,\overline{\rho}\,\overline{u}_\mu\overline{u}_\nu=G(\beta)\,\rho(\beta)u_\mu u_\nu\,.$$

However,

$$\overline{u}^\mu=\frac{dz^\mu}{d\overline{s}}=\beta^{-1}\frac{dz^\mu}{ds}=\beta^{-1}u^\mu$$

and

$$\overline{u}_\mu=\overline{g}_{\mu\nu}\overline{u}^\nu=\beta^2 g_{\mu\nu}\beta^{-1}u^\nu=\beta u_\mu\,,$$

we finally obtain

$$\beta^2\overline{G}\,\overline{\rho}=G(\beta)\rho(\beta)\,. \quad (2.11c)$$

i.e., the product $G(\beta)\rho(\beta)$ must transform like β^2, no matter how $G(\beta)$ and $\rho(\beta)$ transform individually.

B. Co-covariant equations–geodesic equations

Having arrived at the gravitational field equations (2.10) in general units we seek to characterize the nature of the space-time manifold underlying such equations. In general relativity, space-time is taken to be Riemannian and any needed equation can be found by taking the pertinent equation from special relativity and writing it so that it is form invariant under arbitrary coordinate transformations. We will see that the space-time underlying Eq. (2.10) is an integrable Weyl (IW) manifold, and that pertinent equations of macroscopic gravitational phenomena can be found by taking the corresponding equations in general relativity and writing them so that they are form invariant under both arbitrary coordinate and arbitrary scale transformations. Whereas equations form-invariant under arbitrary coordinate transformations are called covariant, equations form-invariant under both arbitrary coordinate and arbitrary scale transformations will be called co-covariant after Dirac.[8] Use of the term covariant is reserved for properties related exclusively to the metric tensor $g_{\mu\nu}$ as in Riemannian theory.

In Riemannian geometry, if a displacement vector δx^μ is parallel transported, its length does not change along the path. Thus

$$d(g_{\mu\nu}\delta x^\mu\delta x^\nu)=0\,. \quad (2.12)$$

However, under a general scale transformation

$$ds\to ds'=\varphi ds\,, \quad (2.13)$$

the metric tensor becomes

$$g'_{\mu\nu}=\varphi^2 g_{\mu\nu}\,. \quad (2.14)$$

The length of the displacement vector in this new system of units will generally change under parallel transport. In fact,

$$d(g'_{\mu\nu}\delta x^\mu\delta x^\nu)=2g'_{\mu\nu}\delta x^\mu\delta x^\nu d(\ln\varphi)\,. \quad (2.15)$$

Consequently, a generalization of Riemannian geometry is called for. Such a generalization was provided by Weyl,[6] and we shall use the mathematics developed for this generalized geometry to describe our scale-covariant theory of gravitation. We have given a concise summary of the essential features of Weyl's geometry in Appendix A. More details can be found in the books by Eddington[7] or Weyl[6] himself.

It should be pointed out that Einstein[18] had objected to the use of Weyl geometry to describe the physics of electromagnetic as well as gravitational phenomena. The essence of his objection[19] rests in the fact that sharp spectral lines are observed even in the presence of electromagnetic field, whereas in Weyl's theory, the electromagnetic field would imply a nonintegrable length which in turn implies that different atoms, having very different past world lines, should not be emitting radiation at the same frequency. The same objection still applies even though a different system of units can be set up, since transformation of units such as given by (2.13) does not alter the gauge-invariant integrability condition (see Appendix A)

$$k_{\mu;\nu}-k_{\nu;\mu}=0\,. \quad (2.16)$$

However, in order to include scale covariance considerations of gravitational phenomena, we do not need the fully generalized Weyl space. Indeed, comparing (2.15) with (A3) we need generalize the

Riemannian geometry to the extent that Weyl's metric vector k_μ can be expressed as a gradient

$$k_\mu = \Phi_{,\mu}\,, \tag{2.17}$$

in which case (2.16) is satisfied and Einstein's objection does not affect our use of such an integrable Weyl geometry (IW geometry). In the literature, one often finds statements to the effect that whenever (2.16) is satisfied, the geometry is Riemannian. It is true that when (2.17) holds, the space is conformally equivalent to a Riemannian space. However, to identify the two is to assert that k_μ is unobservable and is completely irrelevant to the description of the physical world. We do believe that an "absolute" k_μ has no physical significance and hence is unobservable. In fact, this is the reason for imposing scale invariance.[20] But the relative k_μ, which describes the difference between two systems of units, such as those provided by gravitational theory and atomic theory, does have physical significance. The nonmeasurability of the "absolute" metric vector allows one to stipulate that k_μ is identically zero in one system of units which we choose to be Einstein units.

Using general units of measure, the natural description of gravitational phenomena is given by the IW space whose metrical properties are given by the metric tensor $g_{\mu\nu}$ and a scalar potential Φ. But to make use of the mathematics developed for Weyl geometry, it is convenient to retain the scale vector k_μ with the understanding that it is a gradient vector field.

Having determined the mathematical space for our description of physical phenomena, it is easy to infer that in a gauge-covariant theory, the physical equations must involve tensors in Weyl space, called cotensors.

The notion of a cotensor and its power, and the concept of co-covariant differentiation which brings a cotensor into a cotensor of the same power, are described in Appendix A. We observe here that $\beta(x)$, defined in (2.4) as the scale factor between Einstein units and any general units, can be easily demonstrated to be a coscalar of power -1.

From considerations of the transformation properties of the Einstein equation in the preceding section, we expect that the field equation in a gauge-covariant theory can be written as in-tensor equation having the form

$$^*G_{\mu\nu} = -8\pi\, ^*\mathfrak{J}_{\mu\nu} + {}^*\Lambda_{\mu\nu}\,. \tag{2.18}$$

$^*G_{\mu\nu}$ is given by (A20) and $^*\Lambda_{\mu\nu}$ is the cosmological term. Here each term is an in-tensor and we consider them separately. Since (2.18) is the generalization of (2.1), the in-tensors must be the generalizations of the tensors in (2.1). Since the metric tensor is a cotensor of power $+2$, we can write

$$^*\Lambda_{\mu\nu} = \Lambda g_{\mu\nu}\,, \tag{2.19}$$

where

$$\Lambda = \beta^2 \bar{\Lambda} \tag{2.20}$$

is a coscalar of power -2.

Since we have already argued that $^*\mathfrak{J}_{\mu\nu}$ is scale invariant, we shall write

$$^*\mathfrak{J}_{\mu\nu} \equiv \mathfrak{J}_{\mu\nu} \equiv \bar{\mathfrak{J}}_{\mu\nu}\,, \tag{2.21}$$

so that finally (2.18) can be written as

$$G_{\mu\nu} - k_{\mu;\nu} - k_{\nu;\mu} + 2g_{\mu\nu}k^\lambda{}_{;\lambda} - g_{\mu\nu}k^\lambda k_\lambda - 2k_\mu k_\nu = -8\pi\mathfrak{J}_{\mu\nu} + \Lambda g_{\mu\nu}\,, \tag{2.22}$$

where the term $k_{\mu;\nu} - k_{\nu;\mu}$ has been dropped because (2.16) has been assumed. Equation (2.22) is identical to (2.10) if

$$k_\mu = -\partial_\mu(\ln\beta) = -\frac{\beta_{,\mu}}{\beta}\,. \tag{2.23}$$

This amounts to prescribing the gauge potential of IW space as follows: In Einstein units, the natural gauge $k_\mu = 0; \Phi = \text{constant}$ is used. For any other system of units, the gauge must be changed, and the gauge induced by such a change of units is precisely (2.23). Thus, in general the metric potential Φ must be written as

$$\Phi = -\ln\beta\,, \tag{2.24}$$

where β is the scale factor between the units being used and the Einstein units.

In addition to the field equations (2.22), one can easily generalize other equations in relativity to the scale-covariant theory. Thus, the conservation law

$$\bar{\mathfrak{J}}^{\mu\nu}{}_{;\nu} = \bar{G}\,\bar{T}^{\mu\nu}{}_{;\nu} = 0\,, \tag{2.25}$$

which follows from the Einstein equations, must now be written as

$$\mathfrak{J}^{\mu\nu}{}_{*\nu} = (G(\beta)\,T^{\mu\nu})_{*\nu} = 0\,. \tag{2.26}$$

In a similar manner the geodesic equation in general relativity, (GR),

$$\bar{u}^\mu{}_{;\nu}\bar{u}^\nu = 0\,,$$

gets generalized to

$$u^\mu_{*\nu}u^\nu = 0\,, \tag{2.27}$$

where u^μ and $\bar{u}^\mu$ are particle four-velocities normalized to unity.

To ascertain that our prescription for generalizing the relativistic equations is indeed correct, we note that if one starts with the geodesic equation in Einstein units

$$\frac{d^2x^\mu}{d\bar{\lambda}^2}+\bar{\Gamma}^\mu{}_{\nu\rho}\frac{dx^\nu}{d\bar{\lambda}}\frac{dx^\rho}{d\bar{\lambda}}=0$$

and lets

$$d\bar{\lambda}=\beta d\lambda,\ \ \bar{g}_{\mu\nu}=\beta^2 g_{\mu\nu}$$

and uses procedures similar to that of Sec. II A, it can be shown that the new in-geodesic equation becomes

$$\frac{d\xi^\mu}{d\lambda}+\Gamma^\mu{}_{\alpha\beta}\xi^\alpha\xi^\beta=-k_\nu(\epsilon g^{\mu\nu}-\xi^\mu\xi^\nu)\,, \tag{2.28}$$

where

$$\epsilon=g_{\mu\lambda}\xi^\mu\xi^\lambda,\quad \xi^\lambda=\frac{dx^\mu}{d\lambda}\,. \tag{2.29}$$

Using the definition (A16) and remembering that u^μ is a covector of power -1, it is straightforward to demonstrate that (2.27) and (2.28) are equivalent. Depending on whether we are dealing with massive or massless particles, ϵ in (2.28) can be set equal to 1 or 0, respectively.

In summary, to obtain the generalized equations in the scale-covariant theory, we take the general relativistic equations, write the tensors in cotensor form, and use co-covariant differentiations. It should be noted that in addition to the variables that exist in the general relativistic equation, we now have also β, whose functional form is not specified. We shall return to this subject after a scale-invariant variational principle, from which (2.22) can be derived, has been introduced. The physical interpretation of Eqs. (2.26) and (2.28) as well as a possible way to determine β for the above equations will be discussed after the formal development of the theory is completed.

C. Scale-invariant action principle

The ideas that led to the generalized field equation (2.10) can be used to construct a generalized variational principle as well. Here we shall make connection with Dirac's paper[8] alluded to at the beginning, and spell out the minor modifications we have introduced. Since, as we have seen, the equations are scale as well as coordinate covariant, the Lagrangian density must be an in-scalar. As we have also determined that the relevant geometrical space is an integrable Weyl space, the gravitational field Lagrangian must be constructed from the elements of IW space, the metric tensor $g_{\mu\nu}$ and metric potential Φ, which in accordance with our discussion in the preceding section is replaced by β. The simplest way to proceed is to generalize Einstein's action in the same manner we constructed co-covariant equations: replacing the covariant quantities by the corresponding co-covariant ones. Thus we can write the action principle as

$$\delta\int\beta^2\,{}^*R\sqrt{g}\,d^4x=0\,, \tag{2.30a}$$

where

$$g=\left|\det(g_{\mu\nu})\right|\,.$$

The in-invariant character of (2.30a) is ensured by the multiplication factor β^2. In principle, one can add to (2.30a) terms involving co- covariant derivatives of β, and a term quartic in β, so that (c, c_1 = constants)

$$I=\int dx^4\sqrt{g}\,(-\beta^2\,{}^*R+c_1\beta^{*\mu}\beta_{*\mu}+c\beta^4)\,. \tag{2.30b}$$

The in-invariance requirement dictates that only a quartic term can appear. The middle term, while having the correct invariance properties, has no contribution in our theory because

$$\beta_{*\mu}=\beta_{,\mu}-\Pi k_\mu\beta=0\,,$$

where Π is the power of β. The first equality follows from the definition (A14a). The second equality follows from (2.23) and the fact that $\Pi=-1$ for β. Including a matter Lagrangian, we can then state our action principle as follows:

$$\begin{aligned}\delta I&=\delta\int d^4x\sqrt{g}\,(-\beta^2\,{}^*R+c\beta^4+16\pi\mathcal{L})\\&=\delta\int d^4x\sqrt{g}\,(-\beta^2R+6\beta^\mu\beta_\mu+c\beta^4+16\pi\mathcal{L})\\&=0\,.\end{aligned} \tag{2.31}$$

At this point, we do not specify the matter Lagrangian $\mathcal{L}$ aside from stressing that by definition it is a coscalar with $\Pi=-4$ and that

$$\mathfrak{I}^{\mu\nu}=g^{\mu\lambda}g^{\nu\rho}\mathfrak{I}_{\lambda\rho}=\beta^{-2}\frac{2}{\sqrt{g}}\frac{\delta}{\delta g_{\mu\nu}}(\sqrt{g}\,\mathcal{L})\,, \tag{2.32}$$

which is the natural generalization of

$$\bar{\mathfrak{I}}^{\mu\nu}=\frac{2}{(\bar{g})^{1/2}}\frac{\delta}{\delta\bar{g}_{\mu\nu}}[(\bar{g})^{1/2}\bar{\mathcal{L}}]$$

in general relativity. The factor β^{-2} is necessary for the cotensor power of both sides of (2.32) to be -4. We note that the first line of (2.31) is manifestly scale invariant. The second line has been written out because it is easier to derive the field equations from it and will serve as a basis for our comparison with other theories.

It should be remarked that when Dirac[8] introduced his action principles in the form of (2.31), β was considered a new scalar field in addition to Weyl's metric vector k_μ. Furthermore, since

Dirac's k_μ is not related to β by (2.23), $\beta_{*\mu}\beta^{*\mu}$ does not vanish. With this additonal term in the Lagrangian, Dirac could ensure that no independent equation is derived for the variable β, only if he puts $c_1=6$ in (2.30b). With our introduction of IW space and β as a metric potential, the indeterminacy of β becomes a natural consequence of the theory. A variational principle formally identical to the one given by the second line of (2.31) was also considered by Anderson[21] and Bicknell.[22] Independently varying $g_{\mu\nu}$ and β, and using (2.31) and (2.32) we find

$$\delta I=\int d^4x\sqrt{g}\,\Big\{\beta^2\Big[R^{\mu\nu}-\tfrac{1}{2}g^{\mu\nu}R+2\frac{\beta^{\mu;\nu}}{\beta}-4\frac{\beta^\mu\beta^\nu}{\beta^2}-g^{\mu\nu}\Big(2\frac{\beta^\lambda{}_{;\lambda}}{\beta}-\frac{\beta^\lambda\beta_\lambda}{\beta^2}\Big)+8\pi\mathfrak{I}^{\mu\nu}+\tfrac{1}{2}c\beta^2g^{\mu\nu}\Big]\delta g_{\mu\nu}$$
$$+\Big(4c\beta^3+16\pi\frac{\delta\mathfrak{L}}{\delta\beta}-2\beta R-12\beta^\mu{}_{;\mu}\Big)\delta\beta\Big\}\,. \tag{2.33}$$

Hence,

$$R^{\mu\nu}-\tfrac{1}{2}g^{\mu\nu}R+2\frac{\beta^{\mu;\nu}}{\beta}-4\frac{\beta^\mu\beta^\nu}{\beta^2}-g^{\mu\nu}\Big(2\frac{\beta^\lambda{}_{;\lambda}}{\beta}-\frac{\beta^\lambda\beta_\lambda}{\beta^2}\Big)=-8\pi\mathfrak{I}^{\mu\nu}+\Lambda g^{\mu\nu}\,, \tag{2.34}$$

$$\beta R+6\beta^\mu{}_{;\mu}=-4\Lambda\beta+8\pi\,\delta\mathfrak{L}/\delta\beta\,, \tag{2.35}$$

where we have set $\frac{1}{2}c\,\beta^2=-\Lambda$. Equation (2.34) is seen to be identical to the Einstein equation we have derived previously for general units. Although (2.35) appears to be an independent field equation, we shall show that this is not the case. In vacuum,

$$\mathfrak{L}=0,\quad \mathfrak{I}^{\mu\nu}=0,$$

and it can be easily seen that the trace of (2.34) is identical to (2.35). More generally, the trace of (2.34) can be written as

$$\beta R+6\beta^\mu{}_{;\mu}=-4\Lambda\beta+8\pi\beta\mathfrak{I}^\mu{}_\mu\,. \tag{2.34$'$}$$

Comparison with (2.35) gives

$$\beta\mathfrak{I}^\mu{}_\mu=\frac{\delta\mathfrak{L}}{\delta\beta}\,. \tag{2.36}$$

However, this relation must be an identity by construction if I is to be scale invariant. To see this, we consider an infinitesimal scale transformation,

$$ds\to ds'=(1+\lambda)ds$$

so that

$$\delta g_{\mu\nu}=2\lambda g_{\mu\nu}\,,$$
$$\delta\beta=-\lambda\beta\,.$$

When the above variations are put into (2.33) and δI is required to vanish under such transformations, we find exactly (2.36).

It should be pointed out that the matter Lagrangian considered here is phenomenological. It gives rise to a source term which is again a phenomenological energy-stress tensor, as in classical fluid dynamics. Within the context of the present gravitation theory, source strengths can only be measured through the effects the sources have on the geometry. We can thus scale the source term like geometrical quantities and impose scale invariance on the matter Lagrangian. Implicit is the stipulation that the scale-breaking part of the latter is neglected even though it may contribute to the gravitational field. As will be stressed below, it is precisely such terms which are required to complete the theory of coupled dynamics.

The fundamental assumptions in all the above methods of deriving the gravitational field equations are the same. We imposed scale covariance as well as general coordinate covariance. In the first two methods the indeterminacy of β is clear, since it was introduced as an arbitrary scale factor and no new equation could be derived for its determination. With the variational method, although a new equation was obtained, it has been shown not to be independent of the rest of the field equations. Such an under-determinacy is not new. It is well known that for any action invariant under certain continuous transformations, there exist Bianchi-type identities among the field variables. In GR, the under-determinacy is taken care of by imposing coordinate conditions. In the present theory, we shall impose gauge conditions so as to eliminate the arbitrariness in β. However, before we elaborate on how this can be done, we shall try to clarify certain conceptual differences between the two kinds of conditions. Coordinate systems are considered to be *a priori* devoid of dynamical significance. The measuring units, on the other hand, are generally considered to be dynamically determined. In fact, it has been suggested[5,23] that every proper theory should provide in and by itself its own means for defining quantities with which it deals. The gauge freedom that exists in the present theory is a result of our professed ignorance of the proper coupling between different kinds of dynamics, specifically the coupling between geometrodynamics and quantum electrodynamics. If the coupling is stipulated *a priori*, such as in the usual superposition of gravitational action and QED action, one no longer has gauge freedom. In fact, β must then be constant. We do not feel that such a simple superposition is the only possible coupling of the two kinds of dynamics, for within its framework, there is no possibility for the gravitational constant measured with respect to atomic units, to vary, an issue which ought to be resolved by accurate experimental verification, rather than by *a priori* conjecture. To the extent that the measuring system does not perturb the gravitational field being measured, the above given formulation of the gravitational theory in arbitrary units can be considered applicable. Thus, one could even take such measurements as an *a posteriori* determination of the unknown scale factor.

In view of the above discussion, we emphasize again that the dynamics of the measuring device is not included in our action principle. As Anderson[21] has pointed out, if one makes the gravitational source scale invariant, one cannot construct dynamical systems which measure differently from the intrinsic gravitational units. This is indeed quite clear from the first two derivations we have given for the generalized field equation: Since only gravitational dynamics is involved, only one dynamical clock can be constructed. In a unified dynamical theory, there would be a single unit corresponding to this dynamics, which would be neither gravitational nor atomic. In such a theory, one could conceive of limiting cases in which isolated atomic systems as passive measuring devices do not act as sources of fields. As such, the measuring device or clock loses dynamical meaning in the restricted dynamical problem being considered. The unified theory, if such existed, would be able to supply the relation between the atomic measurements and those made intrinsically with the restricted dynamics. That is, the scale invariance would be broken and Eq. (2.36) would become the field equation for β rather than a Bianchi identity.

One could of course turn the problem around and consider, as Dirac does, that the large-number coincidences are statements of relations between the units of different dynamics. Consequently, the gauge condition can be inferred using such relations and we shall show below how this can be done.

Thus we see that the LNH can be considered the observational input which determines the atomic gauge relative to the Einstein gauge, and therefore the function β. With this known functional form of β and hence known variation of G, the field equations derived earlier are complete and solutions can then be obtained to yield various cosmological models just as in general relativity.

It should be remarked that we have not used LNH in the general form (B5). Rather, we are considering (B4a) and (B4b) as separate hypotheses which can be adopted in conjunction or separately. Our purpose is to use relations of the type (B4a) and (B4b) to determine the gauge condition. Whether (B5) is consistent with the scale-covariant theory can then be subjected to tests using the dynamical equations. It is interesting to note that if both (B4a) and (B4b) are used in the determination of β, (B6) follows as will be shown in the following.

D. Conservation laws

In any action principle, corresponding to coordinate transformation (CT) and gauge transformation (GT) invariance, there are associated conservation laws. In the case of the vacuum, Dirac[8] has already given the details of the derivation of these laws. For CT invariance, one gets the generalized Bianchi identities and for GT invariance, one simply gets an expression which is identically zero.

When matter was present, the GT-invariant conservation equation was derived in Sec. II C. Aside from ensuring that the scalar field equation is not independent, it does not seem to have any sensible physical interpretation. For CT invariance, one can proceed formally as indicated by Dirac.[8] After some tedious algebra, one arrives at precisely the conservation equation (2.26). Instead of producing all the details of this derivation, we shall pursue (2.26) further by introducing the energy-momentum tensor of a perfect fluid,

$$T^{\mu\nu}=(\rho+p)u^{\mu}u^{\nu}-pg^{\mu\nu}\,. \tag{2.37}$$

Introducing (2.37) into (2.26), and using (A19), we obtain

$$\dot{\rho}+(\rho+p)u^{\mu}{}_{;\mu}=-\rho\left(\frac{\dot{G}}{G}+\frac{\dot{\beta}}{\beta}\right)-3p\frac{\dot{\beta}}{\beta} \tag{2.38}$$

and the Euler equation

$$(\rho+p)\dot{u}^{\mu}=(g^{\mu\nu}-u^{\mu}u^{\nu})\times\left[p_{,\nu}+p\frac{G_{,\nu}}{G}+(\rho-p)\frac{\beta_{,\nu}}{\beta}\right], \tag{2.39}$$

where for any A, $\dot{A}\equiv A_{;\mu}u^{\mu}$. For a comoving volume $\mathcal{V}\sim R^3$, an alternative form of (2.38) is

$$\frac{1}{\rho\mathcal{V}G\beta}\frac{D\rho\mathcal{V}G\beta}{Dt}=-3\frac{p}{\rho}\frac{1}{R\beta}\frac{DR\beta}{Dt}\,,\quad \frac{D}{Dt}\equiv u^{\mu}\partial_{\mu}\,. \tag{2.40}$$

We recall that the gravitational "constant" is now a function of space-time and its derivatives do not vanish in general. Equations (2.38) and (2.39) show explicitly how the variation of G and β modifies the energy and momentum conservation laws of general relativity when written in general units.

Next we consider yet another conservation equation whose physical content is not contained in the action principle. In hydrodynamic problems encountered in general relativity, it is necessary to have an equation for the number density of particles in order for the system of hydrodynamic equations to be closed.

Let us consider the mass conservation law in Einstein units, i.e.,

$$(\bar{\mathfrak{M}}\bar{u}^\mu)_{;\mu}=0\,, \tag{2.41}$$

where

$$\bar{\mathfrak{M}}=\bar{m}\,\bar{n} \tag{2.42}$$

is the rest-mass density.

Since Dirac's LNH raises the possibility of nonconservation of baryonic number, we shall seek a generalization of (2.41). According to our co-covariant considerations, (2.41) can be simply written as [see A16]

$$(\mathfrak{M}u^\mu)_{*\mu}=(\mathfrak{M}u^\mu)_{;\mu}-(\Pi+4)\mathfrak{M}u^\mu k_\mu =0. \tag{2.43}$$

The cotensor power Π of $\mathfrak{M}u^\mu$ can be deduced as follows. First, it is clear from its definition that u^μ has power -1. $\mathfrak{M}$ is the classical limit of ρ, the energy density, and hence has the same power as ρ, which we denote by $\Pi(\rho)$. Furthermore, we denote the coscalar power of G by $\Pi(G)$. From (2.11c) we can write

$$\Pi(\rho)+\Pi(G)=-2, \tag{2.44}$$

so that

$$\Pi\equiv\Pi(\mathfrak{M}u^\mu)=\Pi(\mathfrak{M})+\Pi(u^\mu) =\Pi(\rho)-1 =-\Pi(G)-3. \tag{2.45}$$

Consequently, Eq. (2.43) becomes

$$(\mathfrak{M}u^\mu)_{;\mu}-[\Pi(G)-1]\mathfrak{M}\frac{\dot\beta}{\beta}=0\,, \tag{2.46}$$

where Eq. (2.23) has been used in the above reduction. In atomic units, particle rest mass is constant and we obtain an equation for the particle number density.

$$(nu^\mu)_{;\mu}-[\Pi(G)-1]n\frac{\dot\beta}{\beta}=0. \tag{2.47}$$

$\Pi(G)$ cannot be specified independent of the gauge condition. Examples of its determination will be given in the next section. Finally, we note that the assumption of validity of (2.41) in Einstein units and its consequences in (2.46) and (2.47) is consistent with our previous treatment of the scale-covariant field equations. It is easy to see that in the classical limit, when $p=0$, (2.40) is equivalent to (2.46). But we emphasize that (2.46) is an independent equation since it is assumed to be valid even when matter pressure is nonvanishing.

E. LNH as a gauge condition

Since we do not yet know the functional form of G or β, we have only the formal structure of a theory. To be able to solve dynamical problems, we must specify β which corresponds to choosing a gauge. We shall now give an example of how the LNH can be used to specify β in cosmology.

G is a coscalar, and we assume it has power $\Pi(G)$. In Einstein units, it has a constant value $\bar{G}$. β has been shown to be a coscalar of power -1. In Einstein units it is a constant which we can set equal to unity. Thus, generally, we can write

$$G=\bar{G}\beta^{-\Pi(G)}\sim\frac{1}{t}\,, \tag{2.48}$$

where the second relation results from a consequence of the LNH, namely that the gravitational constant in atomic units is inversely proportional to the cosmological time. We next consider (2.46) in a cosmological context. Equation (2.46) implies for a comoving volume $\mathfrak{V}$

$$\frac{1}{\mathfrak{V}}\frac{D\mathfrak{M}\mathfrak{V}}{Dt}=[\Pi(G)-1]\mathfrak{M}\frac{\dot\beta}{\beta}$$

or

$$\frac{D\mathfrak{M}\mathfrak{V}G\beta}{Dt}=0\,, \tag{2.49}$$

a result to be expected as a particular case of (2.40) since when $p\to0$, $\rho\to\rho_0=\mathfrak{M}$ (rest-mass density). We therefore have

$$\mathfrak{M}\mathfrak{V}\sim G^{-1}\beta^{-1}\sim\beta^{[\Pi(G)-1]}\sim t^2\,, \tag{2.50}$$

where the second relation states that the mass in a comoving element increases like the square of cosmological time, which is another consequence of LNH. Combining (2.48) and (2.50), we find

$$\beta\sim\frac{1}{t} \tag{2.51}$$

and

$$\Pi(G)=-1\,, \tag{2.52}$$

$$G=\bar{G}\beta\,. \tag{2.53}$$

On the other hand, if we do not assume spontaneous mass creation and require

$$\mathfrak{M}\mathfrak{V}\sim\beta^{[\Pi(G)-1]}\sim t^0\,, \tag{2.54}$$

we obtain instead

$$\beta \sim t\,, \tag{2.55}$$

$$\Pi(G)=1\,, \tag{2.56}$$

$$G=\bar{G}\beta^{-1}\,. \tag{2.57}$$

III. COSMOLOGY

Having determined the atomic gauge and hence a specific functional form for β, the dynamical equations can be applied for cosmological considerations. We shall assume, as in the standard cosmological model, spatial homogeneity and isotropy. The line element can be written in the Robertson-Walker form

$$ds^2=dt^2-R^2(t)\left(\frac{dr^2}{1-kr^2}+r^2d\theta^2+r^2\sin^2\theta\, d\varphi^2\right), \tag{3.1}$$

where k is a parameter which can be normalized to ±1 or 0. We keep in mind that ds is in atomic units. When the occasion arises, quantities in Einstein units will be indicated by a bar over the symbol as has been done in Sec. II.

With (3.1) and (2.37) the field equations (2.10) become

$$\left(\frac{\dot R}{R}+\frac{\dot\beta}{\beta}\right)^2+\frac{k}{R^2}=\frac{8\pi}{3}G\rho+\tfrac{1}{3}\Lambda\,, \tag{3.2a}$$

$$\frac{\ddot R}{R}+\frac{\ddot\beta}{\beta}+\frac{\dot\beta}{\beta}\frac{\dot R}{R}-\frac{\dot\beta^2}{\beta^2}=-\frac{4\pi G}{3}(3p+\rho)+\tfrac{1}{3}\Lambda\,. \tag{3.2b}$$

As in ordinary cosmology, the dynamic equation (3.2a) must be supplemented by the "energy conservation" equation (2.38)

$$\dot\rho+3\frac{\dot R}{R}(\rho+p)=-\rho\frac{1}{G\beta}\frac{DG\beta}{Dt}-3p\frac{\dot\beta}{\beta}\,,$$

which, for any equation of state of the form

$$p=c_s^2\rho\,,$$

can be integrated to give

$$G(\beta)\rho(\beta)R^{3(1+c_s^2)}\sim\frac{1}{\beta^{1+3c_s^2}}\,. \tag{3.3}$$

Specifically [see (2.53) and (2.57)]

$$\rho R^{3(1+c_s^2)}\sim\begin{cases}\beta^{-2-3c_s^2} & \text{(A) matter creation,}\\ \beta^{-3c_s^2} & \text{(B) no matter creation.}\end{cases} \tag{3.4}$$

For dust ($c_s^2=0$) and radiation ($c_s^2=\frac{1}{3}$) the previous equations specialize to

$$\text{(A) } \rho_m(t)\sim\frac{1}{\beta^2(t)}R^{-3}(t),\ \text{(B) } \rho_m(t)\sim R^{-3}(t)\,, \tag{3.5a}$$

$$\text{(A) } \rho_\gamma(t)\sim\frac{1}{\beta^3(t)}R^{-4}(t),\ \text{(B) } \rho_\gamma(t)\sim\frac{1}{\beta(t)}R^{-4}(t). \tag{3.5b}$$

Equations (3.2) and (3.5) can also be obtained directly by scale transforming the corresponding results in Einstein units and using the fact that $\beta^2G\rho=\bar G\bar\rho$. Using (3.3) to eliminate $G(\beta)\rho(\beta)$, we can now solve Eq. (3.2a), which we shall write for the case when matter dominates over radiation, $c_s^2=0$,

$$\frac{dF}{\beta\,dt}=\frac{dF}{d\tau}=\left(\frac{a}{F}+\tfrac{1}{3}\bar\Lambda F^2-k\right)^{1/2},$$

where

$$F=R(t)\beta(t),\quad a\equiv\frac{8\pi\bar G}{3}\rho_{m0}R_0^3. \tag{3.6}$$

Upon integrating, we find

$$\tau=\int^{F/F_0}\frac{dF}{(a/F-k+\frac{1}{3}\bar\Lambda F^2)^{1/2}}\,. \tag{3.7}$$

For $k=\pm1,0$, the $\Lambda=0$ solutions are well known. Translating back into atomic units, the $R(t)$ vs t functions now read for the case (A), $\beta=t_0/t$ (matter creation: For $k=0$,

$$\beta(t)R(t)=[1+\tfrac{3}{2}t_0\bar H_0\ln(t/t_0)]^{2/3}; \tag{3.8}$$

for $k=-1$,

$$\beta(t)R(t)=\bar q_0(1-2\bar q_0)^{-1}(\cosh\psi-1)\,, \tag{3.9a}$$

$$t_0\bar H_0\ln(t/t_0)+A=\bar q_0(1-2\bar q_0)^{-3/2}(\sinh\psi-\psi)\,, \tag{3.9b}$$

$$1=\bar q_0(1-2\bar q_0)^{-1}(\cosh\psi_0-1)\,, \tag{3.9c}$$

$$A=\bar q_0(1-2\bar q_0)^{-3/2}(\sinh\psi_0-\psi_0)\,; \tag{3.9d}$$

for $k=+1$,

$$\beta(t)R(t)=\bar q_0(2\bar q_0-1)^{-1}(1-\cos\theta)\,, \tag{3.10a}$$

$$t_0\bar H_0\ln(t/t_0)+B=\bar q_0(2\bar q_0-1)^{-3/2}(\theta-\sin\theta)\,, \tag{3.10b}$$

$$1=\bar q_0(2\bar q_0-1)^{-1}(1-\cos\theta_0)\,, \tag{3.10c}$$

$$B=\bar q_0(2\bar q_0-1)^{-3/2}(\theta_0-\sin\theta_0)\,. \tag{3.10d}$$

For the case (B), $\beta=t/t_0$, i.e., without matter creation, one can derive an analogous set of equations, where, however, the terms $t_0\ln(t_0/t)$ must be substituted by

$$\int_t^{t_0}\beta(t)dt-\tfrac{1}{2}t_0\left[1-\left(\frac{t}{t_0}\right)^2\right]. \tag{3.11}$$

In particular, for large values of t, the cases $k=0$ and $k=-1$ yield

$$R(t)\sim t^{1/3}\ (k=0);\ R(t)\sim t\ (k=-1). \tag{3.12}$$

Analogously, for small θ's, the case $k=+1$ gives

$$(k=+1), \quad R(t)\sim t^{1/3}. \tag{3.13}$$

Finally, we must explain the relations between $\overline{H}_0$, $\overline{q}_0$ in Einstein's units and the corresponding H_0, q_0 in atomic units. In fact, Eqs. (3.8)–(3.10) still contain the Hubble constant and the deceleration parameter expressed in Einstein units.

Introducing the notations

$$q_0=-\left(\frac{R\ddot{R}}{\dot{R}^2}\right)_0, \quad H_0=\left(\frac{\dot{R}}{R}\right)_0,$$
$$Q_0=-\left(\frac{\beta\ddot{\beta}}{\dot{\beta}^2}\right)_0, \quad h_0=\left(\frac{\dot{\beta}}{\beta}\right)_0, \tag{3.14}$$

we can easily derive from (3.2) the relations ($\rho_0 \equiv \rho_{m0}$),

$$\frac{\rho_0}{\rho_c}=2q_0+\frac{2\Lambda}{3H_0^2}+2(1+Q_0)\frac{h_0^2}{H_0^2}-2\left(\frac{h_0}{H_0}\right),$$
$$\frac{k}{R_0^2}=(2q_0-1)H_0^2+\Lambda+(1+2Q_0)h_0^2-4h_0H_0, \tag{3.15}$$

or

$$\frac{k}{R_0^2}=\left[\frac{\rho_0}{\rho_c}-\left(1+\frac{h_0}{H_0}\right)^2\right]\left(\frac{H_0}{c}\right)^2+\frac{\Lambda}{3} \tag{3.16}$$

a generalization of the well-known relations in Einstein units,

$$\frac{\overline{\rho}_0}{\overline{\rho}_c}=2\overline{q}_0+\frac{2\overline{\Lambda}}{3\overline{H}_0^2}, \quad \frac{k}{R_0^2}=(2\overline{q}_0-1)\overline{H}_0^2+\overline{\Lambda},$$
$$\frac{k}{R_0^2}=\left(\frac{\overline{\rho}_0}{\overline{\rho}_c}-1\right)\left(\frac{\overline{H}_0}{c}\right)^2+\frac{\Lambda}{3}. \tag{3.17}$$

The relation between $\overline{q}_0$ and q_0 is derived to be

$$\overline{q}_0=q_0\left(\frac{H_0}{h_0+H_0}\right)^2+(1+Q_0)\left(\frac{h_0}{h_0+H_0}\right)^2-\frac{h_0H_0}{(h_0+H_0)^2}, \tag{3.18}$$

$$\overline{H}_0=H_0+h_0, \quad \overline{\rho}_c=\rho_c\left(\frac{\overline{H}_0}{H_0}\right)^2.$$

At this point we must discuss a very important point concerning q_0 and $\overline{q}_0$. In traditional cosmology the search for the value of the curvature has been pursued in the last 16 years by Sandage and his collaborators. Recently, however, the abundance of deuterium, an element very difficult to form but present in the early universe, has proved to be a more sensitive test than any of the ones used by Sandage so far. The conclusion based on the abundance of deuterium is that the universe is open and the value of the deceleration parameter is much less than unity. This value is often mistakenly identified with q_0, thus ruling out the $k=+1$, $=+1,0$ cases for $\beta=t/t_0$, since $q_0\approx 2$. However, this is not correct. The experimental value should be identified with $\overline{q}_0$, because the cosmological models employed in the nucleosynthesis computations done so far correspond to Einstein units with $\Lambda=0$.

The first case to be considered will be the one corresponding to no-matter creation. Since in the three cases the $R(t)$ function can be written approximately as $t^a(a=\frac{1}{3}$ or $1)$, it is easy to check that (3.18) becomes

$$\overline{q}_0=\frac{1-a}{1+a}, \quad q_0=\frac{1-a}{a}>0,$$
$$\overline{q}_0=q_0\frac{a}{1+a}. \tag{3.19}$$

The value of $\overline{q}_0$ is clearly less than one for any of the three curvatures $k=\pm1,0$; in particular, $\overline{q}_0=0.5$ for $k=+1,0$ and $\overline{q}_0=0$ for $k=-1$, whereas q_0 is 0 or 2. This clearly indicates how incorrect it is to compare q_0 instead of $\overline{q}_0$ with observations.

The case (A) can also be treated. By writing approximately $R=t(\ln t)^b$ $(b=\frac{2}{3}$ or 1), it is easy to derive that

$$\overline{q}_0=\frac{1-b}{b},$$
$$-q_0=\frac{b\ln t+b(b-1)}{(b+\ln t)^2}. \tag{3.20}$$

Here again, as before, $\overline{q}_0<1$ and in particular, $\overline{q}_0=0.5$ for $k=+1$ and 0 and $\overline{q}_0=0$ for $k=-1$. This completes our exposition of the cosmological consequences of the gauge-covariant theory of gravitation. As will be shown later, case (A), corresponding to matter creation, seems at present favorable over case (B). In this case we would suggest that the $k=-1$ curvature case, with $R\sim t\ln t$, is more likely to be the model that best fits the cosmological data in that it yields the smallest value of $\overline{q}_0$, namely zero, as the growing evidence from the abundance of deuterium seems to indicate. At the level of numerical coincidences and Mach's principle

$$\frac{MG}{Rc^2}\sim\text{const}, \tag{3.21}$$

Dirac has suggested that $R\sim t$, for the matter creation case. In fact, if $M\sim t^2$, $G\sim t^{-1}$, R must go like t. Clearly such a behavior is only reproduced by the $k=-1$ curvature case, without matter creation, however. Within the matter creation case it is clear that a pure $R\sim t$ is not an admissible solution for the $\Lambda=0$ case. However, it ought to be remembered that the Mach principle (3.21) is actually not incorporated into the set of Einstein equations (3.2), and so its use corresponds to an extra boundary condition. This is most clearly seen if we write (3.21) as

$$\rho GR^2\sim\text{const}. \tag{3.22}$$

The product ρG is a coscalar of power −2 and so

$$\bar{\rho}\bar{G}(\beta R)^2 = \text{const.} \tag{3.23}$$

This implies $\beta R = \mathfrak{R} = \text{const}$, i.e., we must have a static Einstein universe in Einstein units. Case (B), with $\beta \sim t$, is evidently excluded, since it would imply $R \sim 1/t$, i.e., a contracting universe, a fact against all existing evidence. For case (A) with matter creation, $\beta \sim 1/t$, we have $R \sim t$, i.e., the universe expands, an admissible solution. In this case the cosmological constant Λ must be different from zero.

Mach's principle as expressed by (3.21) is imposed in addition to the field equations and is not a natural result of the latter. However, we prefer to stick to the exact solutions represented by (3.8), (3.9), and (3.10) without postulating any additional external boundary condition.

Finally we would like to comment on the existence of the large number [see (B3)]

$$N_3 = \frac{4\pi}{3}\frac{\rho}{m_p}\left(\frac{c}{H_0}\right)^3 \approx 10^{78} \sim t^2 . \tag{3.24}$$

By asserting that (3.24) should hold for all cosmological times, Dirac concluded that one must require matter creation. But in the construction of the large number N_3, the present expansion parameter H_0 was used to define the visible universe, whose coordinate boundaries may change with time. Hence the variation of N_3 with time need not imply matter creation. In fact, by using $\rho_m(t)$ from case (B), (3.5a), corresponding to nonmatter creation, $\rho_m \sim 1/R^3(t)$ and (3.12) and (3.13) for either $k=0$ or $+1$, the quantity

$$N_3 \sim \frac{\rho(t)}{H^3(t)} \simeq \frac{1}{R^3(t)}\left[\frac{R(t)}{\dot{R}(t)}\right]^3 \sim \frac{1}{\dot{R}^3(t)} \sim t^2$$

goes exactly like t^2 and no matter creation is needed. Granting that the LNH can be meaningfully used to fix the gauge funtion $\beta(t)$, we must emphasize that the cosmological solutions presented here are valid only for large cosmological times and cannot be extrapolated to early times. If one does so,[24,25] one finds that the mean free time for nuclear interactions as well as the mean free time for photon Compton scattering are greater than the expansion time of the universe itself and therefore no nucleosynthesis could have taken place.

As repeatedly stressed by Dirac, the LNH is an asymptotic condition and it cannot be used to fix the value of $\beta(t)$ at times when nucleosynthesis occurred. A new condition must be found. For exactly the same reason we cannot at this moment make any sensible comment on the existence of a horizon, since that again implies the knowledge of the function $R(t)$ and therefore $\beta(t)$ for any t.

IV. APPLICATION TO LOCAL GRAVITATIONAL PHENOMENA

A. Equations of motion

In this section, we shall consider three classical tests of general relativity where effects of a nonconstant β may be observable. Since, as we have emphasized, we have introduced a scale function β to put Einstein's theory of gravitation in a scale-covariant form, any purely gravitational experiment is not expected to produce results different from the predictions of GR. In particular, the perihelium advances per revolution and the deflection of light rays by a spherically symmetric gravitational field, being measured in radians, are necessarily scale invariant. Likewise, the equations of motion for planetary orbits, expressed in terms of coordinates, must be identical to those of GR, since coordinates are also invariant under scale transformations. It is only when radial distances and orbital periods in atomic units are considered that the present theory yields predictions different from those of GR. Specifically, both distances and time will be scaled with respect to those of GR by a factor β^{-1}. With this scaling, it is easy to obtain predictions from the present theory once those of GR are known.

However, in discussions of planetary orbits with a varying gravitational constant there has been in the literature some confusion as to what the correct equations of motion ought to be. We shall therefore outline below the set of new equations of motion that are relevant for the class of problems under consideration and indicate how integrals of these equations can be obtained. Discussions of specific changes in measured quantities required by the present theory will then follow.

We consider motion of particles (massive or massless) under the influence of a spherically symmetric gravitational field. In GR, the metric is given by the Schwarzchild solution of the source-free Einstein equation, and the line element is written as (*in this section t and r are taken here to be coordinates*)

$$d\bar{s}^2 = \bar{g}_{\mu\nu}dx^\mu dx^\nu = (\bar{G}\bar{M})^2[(1-2/r)dt^2 - (1-2/r)^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)] . \tag{4.1}$$

With arbitrary β, an exact solution of the source-free (2.34) is given by

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = \beta^{-2}d\bar{s}^2 = \beta^{-2}(\bar{G}\bar{M})^2[(1-2/r)dt^2 - (1-2/r)^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)] . \tag{4.2}$$

The Christoffel symbols for the two metrics are related by

$$\Gamma_{\nu\lambda}{}^{\mu}=\overline{\Gamma}_{\nu\lambda}{}^{\mu}-\delta_{\nu}{}^{\mu}\frac{\beta_{\lambda}}{\beta}-\delta_{\lambda}{}^{\mu}\frac{\beta_{\nu}}{\beta}+g_{\nu\lambda}\frac{\beta^{\mu}}{\beta}. \tag{4.3}$$

The components of the in-geodesic equation (2.28) can now be written explicity:

$$\ddot{t}-\frac{A}{B}\frac{\beta_{,t}}{\beta}\dot{r}^{2}+\left(\frac{B_{,r}}{B}-\frac{\beta_{,r}}{\beta}\right)\dot{t}\dot{r}-\frac{r^{2}}{B}\frac{\beta_{,t}}{\beta}\dot{\varphi}^{2}-\beta\beta_{,t}\frac{\epsilon}{B}=0, \tag{4.4a}$$

$$\ddot{r}+\left(\frac{B_{,r}}{2A}-\frac{B}{A}\frac{\beta_{,r}}{\beta}\right)\dot{t}^{2}-\frac{\beta_{,t}}{\beta}\dot{r}\dot{t}+\frac{1}{2}\frac{A_{,r}}{A}\dot{r}^{2}-\frac{r}{A}\left(1-r\frac{\beta_{,r}}{\beta}\right)\dot{\varphi}^{2}+\frac{\epsilon}{A}\beta\beta_{,r}=0, \tag{4.4b}$$

$$\ddot{\varphi}-\frac{\beta_{,t}}{\beta}\dot{t}\dot{\varphi}+\left(\frac{2}{r}-\frac{\beta_{,r}}{\beta}\right)\dot{\varphi}\dot{r}=0. \tag{4.4c}$$

For ease of notation we have taken $\overline{GM}=1$; a dot indicates differentiation with respect to the parameter λ, and we have used the notation

$$B=A^{-1}=1-2/r. \tag{4.5}$$

For simplicity we have assumed $\beta=\beta(t,r)$ and furthermore, restricted the motion to the $\theta=\pi/2$ plane. The normalization condition (2.29) can be shown to be an integral of (2.28) and can now be written explicitly,

$$B\dot{t}^{2}-A\dot{r}^{2}-r^{2}\dot{\varphi}^{2}=\epsilon\beta^{2}. \tag{4.6}$$

Dividing through by $\dot{r}$, (4.4c) can be integrated to yield

$$\frac{d}{d\lambda}(\ln\dot{\varphi}+2\ln r-\ln\beta)=0$$

or

$$\beta^{-1}r^{2}\dot{\varphi}=\text{const}=J, \tag{4.7}$$

which is the generalized angular momentum conservation equation. Using (4.6), it can be shown that (4.4a) reduces to

$$\frac{d}{d\lambda}\ln(\beta^{-1}\dot{t}B)=0$$

or

$$\beta^{-1}\dot{t}B=\text{const}=E, \tag{4.8}$$

which is the generalized energy conservation equation.

Finally, with the aid of (4.6)–(4.8), (4.4b) yields

$$\frac{d}{d\lambda}\left(\beta^{-2}A\dot{r}^{2}+\frac{J^{2}}{r^{2}}-\frac{E^{2}}{B}\right)=0.$$

It can be seen that this is another way of writing (4.6):

$$\beta^{-2}A\dot{r}^{2}+\frac{J^{2}}{r^{2}}-\frac{E^{2}}{B}=\text{const}=\epsilon. \tag{4.9a}$$

One can eliminate the parameter λ in favor of φ or t in (4.9a), using (4.7) and (4.8) to give

$$\frac{A}{r^{4}}J^{2}\left(\frac{dr}{d\varphi}\right)^{2}+\frac{J^{2}}{r^{2}}-\frac{E^{2}}{B}=\epsilon \tag{4.9b}$$

or

$$\frac{A}{B^{2}}\left(\frac{dr}{dt}\right)^{2}+\frac{J^{2}}{r^{2}}-\frac{E^{2}}{B}=\epsilon. \tag{4.9c}$$

These are seen to be identical to the equations used in the study of perihelion precession, light deflection, and radar echo delay in GR. Thus, without further mathematical analysis, we can immediately write down the following[26]:

(*a*) *Perihelion shift*:

$$(\Delta\varphi)_{ps}=\frac{6\pi}{L}, \tag{4.10}$$

where

$$2L=\frac{1}{r_{+}}+\frac{1}{r_{-}}; \tag{4.11}$$

$r_{\pm}$ are the coordinates of aphelia and perihelia.

(*b*) *Deflection of light*:

$$(\Delta\varphi)_{ld}=\frac{4}{r_{0}}, \tag{4.12a}$$

where r_0 is the radial coordinate of closest approach of the light path. If this coordinate is taken to be that of the solar limb, we then write

$$(\Delta\varphi)_{ld}=\frac{4GM_{s}}{R_{s}}, \tag{4.12b}$$

where we have restored units (atomic) to the right-hand side of (4.12a) and where R_s is the solar radius. It should be noted that even though GM_s scales like a geometric length, R does not since it is not a quantity determined by gravitation alone [see Eq. (4.17)].

(*c*) *Radar echo delay*: The coordinate time delay is given by

$$\Delta t=4\left(1+\ln\frac{4r_{1}r_{2}}{r_{0}^{2}}\right), \tag{4.13a}$$

where r_1 and r_2 are the radial coordinates of the reflector and emitter respectively and have been taken to be constants. r_0 is the closest radial approach by the radar signal characterized by $(dr/d\varphi)_{r=r_0}=0$. A maximum delay is obtained if r_0 corresponds to the coordinate of the solar limb. We can finally write for the *atomic proper-time delay*

$$(\Delta t_A)_{\max}=g_{00}{}^{1/2}\,\Delta t=\beta^{-1}\overline{G}\overline{M}\,B^{1/2}(r)\,\Delta t\simeq GM\,\Delta t$$
$$=4GM\left(1+\ln\frac{4R_1R_2}{R_s{}^2}\right), \tag{4.13b}$$

where we have used (2.50), i.e., $GM\beta=\text{const}$.

Since R_s is not gravitationally determined, the expression (4.13b) does not scale simply like a time. We have

$$\beta(\Delta t_A)_{\max}=(\Delta t_A)^0_{\max}-8G_0M_{s0}\ln\left(\beta\frac{R_s}{R_{s0}}\right), \tag{4.13c}$$

where the index zero corresponds to the value today.

Arguments based on homological transformations[13] give the following relation for the radius of the sun:

$$R_s\sim G^{g_1}M^{m_1}, \tag{4.14}$$

where

$$g_1=\frac{n+k_2-4}{n+3+3k_1+k_2},\qquad m_1=\frac{n-1+k_1+k_2}{n+3+3k_1+k_2}, \tag{4.15}$$

where n, k_1, and k_2 are the indices in the nuclear source term and the opacity, i.e.,

$$\epsilon=\epsilon_0\rho T^n,\quad \tilde{k}=\tilde{k}_0\rho^{k_1}T^{k_2}. \tag{4.16}$$

For the p-p chain, $n=4.5$, and for the case of Kramer's opacity, $k_1=1$ and $k_2=-3.5$. With these values, we find

$$\frac{R_s}{R_{s0}}=\left(\frac{M_s}{M_{s0}}\right)^{1/7}\left(\frac{G}{G_0}\right)^{-3/7}, \tag{4.17}$$

which can be used in (4.13c) to show explicitly the time dependence of $(\Delta t_A)_{\max}$.

B. Planetary orbits

Instrumentation technology has permitted a high-accuracy measurement of planetary distances and orbital periods in atomic units. It has been suggested that such measurements could in the near future reveal deviations from predictions of the standard gravitational theory, such as the secular variation of the orbital period P of two gravitating bodies. In this section, we shall derive some predictions of the scale-covariant theory relevant to such measurements.

As was noted before, the equations for the coordinates of the orbit are identical with those of GR. In particular, we can derive from (4.9b) the following:

$$\frac{d^2u}{d\varphi^2}+u=\left(\frac{\overline{G}\overline{M}}{h}\right)^2+3u^2, \tag{4.18}$$

where

$$h=J(\overline{G}\overline{M})^2.$$

For simplicity, we limit ourselves to circular orbits whose radial coordinate is

$$r=h^2/(\overline{G}\overline{M})^2.$$

The orbital radius, given in units of atomic length, is then

$$R=\beta^{-1}(\overline{G}\overline{M})r=h^2/(\beta\overline{G}\overline{M})=h^2/(\beta^2GM). \tag{4.19}$$

Defining $n=\dot{\varphi}$ and eliminating h from (4.19) and (4.7), we recover Kepler's law in it usual form

$$n^2R^3=GM. \tag{4.20}$$

From (4.19) and (4.20) one easily obtains for the time variation of n and R,

$$\frac{\dot n}{n}=\frac{2}{GM}\frac{DGM}{Dt}+3\frac{\dot\beta}{\beta}, \tag{4.21}$$

$$\frac{\dot R}{R}=-\frac{1}{GM}\frac{DGM}{Dt}-2\frac{\dot\beta}{\beta}. \tag{4.22}$$

Hence we have the following:

(a) primitive theory[27,28]:

$$\beta=1,\quad G\sim t^{-1},\quad M=\text{const}, \tag{4.23}$$

$$\frac{\dot n}{n}=2\frac{\dot G}{G},\quad \frac{\dot R}{R}=-\frac{\dot G}{G}; \tag{4.24}$$

(b) scale-covariant theory ($GM\beta=\text{const}$), matter creation

$$\beta\sim t^{-1},\quad G\sim t^{-1},\quad M\sim t^2, \tag{4.25}$$

$$\frac{\dot n}{n}=\frac{\dot G}{G},\quad \frac{\dot R}{R}=-\frac{\dot G}{G}; \tag{4.26}$$

(c) scale-covariant theory ($GM\beta=\text{const}$), no matter creation

$$\beta\sim t,\quad G\sim t^{-1},\quad M=\text{const}, \tag{4.27}$$

$$\frac{\dot n}{n}=-\frac{\dot G}{G},\quad \frac{\dot R}{R}=\frac{\dot G}{G}. \tag{4.28}$$

The preceding equations can alternatively be obtained remembering that $n=\overline{n}\beta$, $\overline{R}=R\beta$, and $GM\beta=\text{const}$.

We have presented a dynamical derivation to emphasize consistency of our reasoning and caution against the *ad hoc* introduction of dynamical equations without a theoretical framework.[29,30]

To conform to the notation most widely used in the literature regarding the earth-moon motion, we shall cast cases (a), (b), and (c) in the form

$$\frac{\dot{G}}{G}=f\frac{\dot{n}_a-\dot{n}_t}{n} \tag{4.29}$$

with $f=\frac{1}{2}$ for (a), $f=+1$ (matter creation) and $f=-1$ (no matter creation). Here $\dot{n}_t$ is the contribution due to the tidal motion and $\dot{n}_a$ is the atomic contribution. The most recent data indicate that[31]

$$\begin{aligned}\dot{n}_t&=(-26.0\pm 2.0)''/\text{century}^2 ,\\ \dot{n}_a&=(-36.0\pm 5.0)''/\text{century}^2 ,\end{aligned} \tag{4.30}$$

Since $\dot{n}_a-\dot{n}_t$ turns out to be negative and $\dot{G}/G$ is by definition negative in any of the previous theories, it follows that

$$\frac{\dot{G}}{G}=f(-5.8\pm 3.1)10^{-11}/\text{yr} \tag{4.31}$$

with f either $\frac{1}{2}$ or 1. The case without matter creation ($f=-1$) seems indeed to be excluded. Can one decide between $f=\frac{1}{2}$ and $f=1$?

In the cosmological context of the present theory, Eqs. (3.8)–(3.10) indicate that R is almost a linear function of t, so that with good approximation, $-\dot{G}/G$ can be written as

$$-\frac{\dot{G}}{G}=\frac{1}{t_0}\simeq\frac{\dot{R}_0}{R_0}=H_0 . \tag{4.32}$$

The value of H_0 is uncertain and present estimates range anywhere between 50 and 100 km $\text{sec}^{-1}\,\text{Mpc}^{-1}$ However, even with the smallest value of H_0 ($\approx 5\times10^{-11}/\text{yr}$), Eq. (4.31) does not admit $f=\frac{1}{2}$. The present-day indication is therefore in favor of the version of the scale-covariant theory with matter creation, and not of the primitive theory.

C. Stellar structure equations

In the scale-covariant theory of gravitation, one accepts the possibilities of a gradual weakening of the gravitational field and continuous matter creation. Thus, a star in hydrodynamic equilibrium may undergo secular variations induced by the variations of gravitational field strength and the total mass of the star. In this section, we apply the field equations (2.34) to the problem of stellar structure. Assuming spherical symmetry as usual, the line element and the velocity field can be written as

$$ds^2=e^{2\phi(t,r)}dt^2-e^{2\psi(t,r)}dr^2-r^2(d\theta^2+\sin^2\theta\, d\varphi^2) , \tag{4.33}$$

$$u^\mu=(\gamma e^{-\phi}, (\gamma^2-1)^{1/2}e^{-\psi}, 0, 0) , \tag{4.34}$$

where γ is a function of r and t.

After some lengthy algebra, the nontrivial field equations are, dropping v^2 terms and letting $\gamma\to 1$,

$$\frac{\partial}{\partial r}[r(1-e^{-2\psi})]=2G(4\pi r^2\rho)-\left(3\frac{\dot{\beta}^2}{\beta^2}+2\frac{\dot{\beta}}{\beta}\dot{\psi}\right)r^2e^{-2\phi} , \tag{4.35}$$

$$\dot{\psi}=-\frac{\dot{\beta}}{\beta}r\phi' , \tag{4.36}$$

$$\frac{2\phi'}{r}e^{-2\psi}-\frac{1-e^{-2\psi}}{r^2}=8\pi Gp+\left(2\frac{\ddot{\beta}}{\beta}-2\frac{\dot{\beta}}{\beta}\dot{\phi}-\frac{\dot{\beta}^2}{\beta^2}\right)e^{-2\phi} . \tag{4.37}$$

Using (4.36), (4.35) can be integrated to yield

$$e^{-2\psi}=1-2\frac{GM}{r}+r^2e^{-2\phi}\left(\frac{\dot{\beta}}{\beta}\right)^2 , \tag{4.38}$$

where

$$M(t,r)=4\pi\int_0^r \rho(t,r')r'^2dr' . \tag{4.39}$$

Since we are considering cosmologically induced variations of stellar structure, $\dot{\beta}/\beta$ is of order $1/t_0$, where t_0 is the age of the universe. For a local system, $(r/t_0)^2\ll 1$ (we have put the velocity of light $c=1$), so

$$e^{-2\psi}\simeq 1-2\frac{GM}{r} , \tag{4.40}$$

which is formally analogous to the standard stellar equilibrium solution in general relativity. But in the present context both G and M are functions of time.

In the same approximation, (4.37) and the radial component of (2.39) can be written as

$$\begin{aligned}\phi'&\cong\frac{G}{r}\left(\frac{M+4\pi r^3p}{r-2GM}\right) ,\\ \frac{dp}{dr}&\cong-\phi'(\rho+p) ,\end{aligned} \tag{4.41}$$

where (4.40) has been used. We finally arrive at the stellar structure equation

$$\frac{dp}{dr}\cong-\frac{G}{r}\frac{(\rho+p)(M+4\pi r^3p)}{(r-2GM)} . \tag{4.42}$$

This equation indicates again that any cosmologically induced variation of stellar structure is, to an accuracy of (r/t_0), implicitly contained in the variation of G and M. Consequently, classical results such as the luminosity of a star[13]

$$L\sim G^7M^5\mu^8 \tag{4.43}$$

and the polytrope relation[13] ($p\sim\rho^\Gamma$, $\Gamma=1+1/n$, n = polytropic index),

$$R^{3\Gamma-4}GM^{2-\Gamma}=\text{const} , \tag{4.44}$$

remain valid up to the same accuracy.

If we apply (4.44) to a galaxy, for which the polytropic index n is of the order of 5 or 6 and so in good approximation $\Gamma \rightarrow 1$, (4.44) yields for the size of the galaxy

$$R \propto GM \propto 1/\beta \tag{4.45}$$

since, as we know, $GM\beta = \text{const}$. We therefore conclude that the size of a galaxy scales like β^{-1}, i.e., exactly like the orbit of one of its peripheral stars [Eq. (4.26)] as expected.

D. Surface temperature of the earth–geological effects

Cosmological ideas such as the ones presented in this paper are sometimes tested using arguments based on the acceptable temperature of the earth in the past few billion years. Several arguments against and in favor of a time varying G have been published over the years but no firm conclusion can yet be reached.

The absolute luminosity of the sun is known to vary as[13]

$$L(t) = L_{\rm ev}(t) G^{\gamma}(t) M^{\delta}(t) , \tag{4.46}$$

where $L_{\rm ev}(t)$ corresponds to the change due to the evolution of the chemical composition, i.e., the molecular weight μ. For Kramer's opacity, $\gamma \sim 7$, $\delta \sim 5$. Defining an effective temperature as

$$\sigma T_e^4 = S(t) = \frac{L}{4\pi R^2} , \tag{4.47}$$

where R is the sun–earth distance, it is easy to see that even if G and M are constant, the temperature was lower in the past since μ was smaller. Using (4.19) for R, it is easy to derive the time variation of the solar constant $S(t)$ as

$$S(t) = S_0 \left(\frac{L_{\rm ev}(t)}{L_0} \right) \beta^4(t) \left(\frac{G(t)}{G_0} \right)^{\gamma+2} \left(\frac{M(t)}{M_0} \right)^{\delta+2} , \tag{4.48}$$

where $S_0 = 1.9885$ cal cm^{-2} min^{-1}. For the cases under consideration, we obtain the following:

(1) stellar evolution only,

$$S_1(t) = S_0 \left(\frac{L_{\rm ev}(t)}{L_0} \right);$$

(2) primitive theory,

$$S_2(t) = S_1(t_0/t)^{\gamma+2};$$

(3) scale-covariant theory, no matter creation,

$$S_3(t) = S_1(t_0/t)^{\gamma-2};$$

(4) scale-covariant theory, matter creation

$$S_4(t) = S_1(t/t_0)^{2\delta-\gamma-2}.$$

The effective temperature T_e is then obtained as $210.69\, S^{1/4}(t)$ °K. Using standard stellar model computations to evaluate $L_{\rm ev}(t)$ one can evaluate $S_1(t)$ and the remaining ones. It turns out that 1.2 eons ago, $T_e = 244$ °K, 285 °K, 266 °K, and 240 °K for $\gamma = 7$ and $\delta = 5$. Analogously, 2.5 eons ago, the results are 238 °K, 334 °K, 288 °K, and 230 °K.

We should point out, however, that the above estimates of T_e, evaluated from (4.47), cannot be directly compared with temperatures derived from geological data since the greenhouse effect and possibly other geothermal effects have been ignored, so that T_e does not represent the physical temperature at the surface of the earth. Adjusting the chemical composition of the atmosphere, by introducing a small amount of ammonia, Sagan and Mullen[32] were able to get such a large greenhouse effect that the lower luminosity in the past was amply compensated for, and a higher "surface" temperature was obtained. Consequently, the past thermal history of the earth cannot be used to argue conclusively for or against a given cosmology by estimating the variation of the solar constant alone. A more thorough analysis of the problem, including the varying greenhouse effect with a varying solar constant, is now being attempted, and the results will be published elsewhere.

Other geophysical effects of a varying G cosmology are often discussed and we shall limit our discussion here to showing that the present theory does not contradict any well-established fact.

An update survey of implications for geophysics as arising from nonstandard cosmologies can be found in a paper by Wesson.[33] We shall discuss here two major effects: the expansion of the earth radius and the spin down. Thorough discussion and pertinent references can be found in the paper by Wesson.

Having shown that in the present theory the hydrostatic equations governing the stability of a star are unaffected by the scale function $\beta(t)$, we can write down the expression to be satisfied by R, G, and M [Eq. (4.44), as

$$R \sim M^{(\Gamma-2)/(3\Gamma-4)} G^{-1/(3\Gamma-4)} , \tag{4.49}$$

which yields the desired results for the time variation of the earth's radius, namely

$$\begin{aligned} &\text{matter creation, } \frac{\dot{R}}{R} = \frac{2\Gamma-3}{3\Gamma-4}\left(\frac{1}{t}\right), \\ &\text{no matter creation, } \frac{\dot{R}}{R} = \frac{1}{3\Gamma-4}\left(\frac{1}{t}\right), \end{aligned} \tag{4.50}$$

where R_0/t_0 is 0.425, 0.354, and 0.319 mm/yr for $t_0 = 15$, 18, and 20 billion years, respectively. Several independent estimates (Wesson quotes 21 of them) lead to the result

$$\dot{R}_0 = (0.5-0.6) \text{ mm/yr} \tag{4.51}$$

for the last 500 million years.

Let us now look at the spin-down effect. It seems to be an accepted fact that the earth is not only expanding but also slowing down at a rate of

$$\dot{P}_0 = 1.6 \text{ msec/century}. \tag{4.52}$$

We have already proved that within the scale-covariant theory, the conserved angular momentum is given by Eq. (4.7), from which we deduce that the time variation of the period is given by (using 4.49)

$$\frac{\dot{P}}{P} = -\frac{2}{3\Gamma - 4}\frac{\dot{G}}{G} + \frac{2\Gamma - 4}{3\Gamma - 4}\frac{\dot{M}}{M} + \frac{\dot{\beta}}{\beta}. \tag{4.53}$$

Hence we have the following:

(a) primitive theory,

$$\frac{\dot{P}}{P} = \frac{2}{3\Gamma - 4}\frac{1}{t}; \tag{4.54a}$$

(b) scale-covariant theory, matter creation,

$$\frac{\dot{P}}{P} = \frac{\Gamma - 2}{3\Gamma - 4}\frac{1}{t}; \tag{4.54b}$$

(c) scale-covariant theory, no matter creation,

$$\frac{\dot{P}}{P} = \frac{3\Gamma - 3}{3\Gamma - 4}\frac{1}{t}, \tag{4.54c}$$

where P_0/t_0 is 0.576, 0.480, and 0.432 msec century for $t_0 = 15$, 18, and 20 billion years, respectively.

By fitting an expression of the type $p = a\rho^{\Gamma}$ to the numerical values of p and ρ for the earth,[34] we concluded that $4.5 \le \Gamma \le 7$. We shall take $\Gamma = 6$, so that

$$\text{matter creation, } \dot{R}_0 = \begin{cases} 0.273 \\ 0.227, \\ 0.205 \end{cases}$$
$$\text{no matter creation, } \dot{R}_0 = \begin{cases} 0.03 \\ 0.025 \\ 0.023 \end{cases} \tag{4.55}$$

in mm/yr. Analogously

$$\text{(a) } \dot{P}_0 = \begin{cases} 0.082 \\ 0.068, \\ 0.061 \end{cases} \quad \text{(b) } \dot{P}_0 = \begin{cases} 0.164 \\ 0.137, \\ 0.123 \end{cases} \quad \text{(c) } \dot{P}_0 = \begin{cases} 0.617 \\ 0.514 \\ 0.463 \end{cases} \tag{4.56}$$

in msec/century. None of these figures is in contradiction with the observed values.

V. CONNECTION WITH GAUGE FIELDS FINAL REMARKS

In this paper we have presented a scale-covariant theory of gravitation, characterized by a set of equations which are complete only after a choice is made of the scale function $\beta(t)$. Among an *a priori* infinite number of choices, two seem particularly appropriate: Einstein gauge ($\beta = \text{const}$) and atomic gauge.

Since no general principle has yet been given as to how to choose $\beta(t)$ in atomic units, we have suggested the use of the large dimensionless numbers relating atomic and gravitational constants. Several results, ranging from cosmology, planetary orbits, stellar structure, and earth's geology are then derived and shown to be consistent with a variety of well-known facts.

Even though such proofs of consistency must be given, they constitute a necessary but not sufficient *raison d'être* for such a new theory. Other more fundamental reasons exist which justify the study of a covariant theory of gravitation. The generalization is being pursued; we have in mind the relation between gravitational and atomic phenomena, a relation that in spite of having been discussed in the scientific literature with increasing frequency has not yet led to a satisfactory picture. Gravity is recently being considered in a much broader light and its hoped-for relation to the structure of matter is more closely investigated; the ultimate goal is the unification of all types of interactions, an endeavor that has been recently crowned by encouraging success.

From the theoretical point of view, Weinberg and Salam have convincingly conjectured that electromagnetism and weak interactions can be combined into a unique non-Abelian gauge theory. Experimental evidence is so far in favor of such a theory. (Einstein theory of gravity is also non-Abelian.) From the experimental point of view, strong interactions have recently been shown to exhibit scale invariance, a property so far possessed only by electromagnetic interactions.

Seemingly dividing barriers have either fallen or become more brittle upon close inspection and the gate seems to have finally opened to a flood of new interesting though still unrelated proposals.

In this paper we have focused our attention on a direction so far unexplored, namely scale invariance. We do not claim to have shown that gravity must be scale invariant, but only that a gravitational theory endowed with such a property leads to no contradictions with well-established facts ranging from geology to cosmology.

Since local gravitational phenomena have been historically the major cause of the high rate of casualties for other generalizations of Einstein equations, we have given a detailed presentation of the three classical tests, with the result that at any given instant of time the present theory yields the same results as ordinary standard theory.

Having passed that hurdle, we have indicated how the present theory can enlarge our interpretation of several phenomena, not ultimately being the only consistent theoretical framework which can accommodate a possible variation of the gravitational constant with cosmological time, a possibility entirely excluded by ordinary Einstein equations.

Besides passing several crucial tests, a theory must also be able to make predictions. In this respect we believe that the present theory can solve what has been a major difficulty concerning the cosmological constant Λ, within the framework of gauge fields and broken symmetries. Although it is not known whether Λ is needed to explain cosmological facts such as the magnitude vs red-shift relations, it is unquestionably true that the stability of galactic clusters put limits on its magnitude. In fact, $|\Lambda|$ must be less than 10^{-57} cm^{-2}.

Since the cosmological constant Λ can physically be interpreted as the vacuum contribution to the energy-momentum tensor of matter,[35] it is possible to derive the following expression[36-38] within the framework of the gauge fields:

$$\Lambda = -\frac{\pi}{\sqrt{2}}\frac{Gm_\varphi^{\,2}}{G_F} < 10^{-6}\ \text{cm}^{-2}$$

some 50 orders of magnitude larger than the previous value. This large discrepancy, which has even been considered as undermining the credibility of the Higgs mechanism,[36] can be drastically reduced if not totally accounted for in the present theory. In fact, on the basis of (2.20) and (2.51), Λ must have a time dependence of the form

$$\Lambda(t) = \Lambda_0(t_0/t)^2 \,.$$

If $\Lambda_0 < 10^{-57}$ today, $\Lambda(t) < 10^{-6}$ was achieved at $t \approx 10^{-8}$ sec, a time not drastically different from the quoted $t \approx 10^{-14}$ sec, i.e., $T \approx 300$ GeV at which the computation is usually performed. The computations can be improved further once we have a better understanding of the behavior of $\beta(t)$ at early cosmological times. In fact, we have reason to believe that $\beta(t)$ scales faster than t^{-1}, thus moving 10^{-8} sec to earlier times.

The analogy with gauge fields and broken symmetries is probably even deeper. In fact, we have learned from the work of Linde and Weinberg that a possible phase transition existed at very high temperatures or equivalently at very early cosmological times. Above a critical temperature $T_c \approx 300$ GeV, the differences in strength among weak, electromagnetic, and nuclear interactions disappeared, all the forces becoming unified.

Our proposal here is complementary in spirit since it deals with gravitation, an interaction not covered by the gauge fields theory of Weinberg and Salam. The present work indicates the possibility of an increase of G at early times. Supposing that G increases always like t^{-1}, gravitational interaction would equal in strength electric forces at $t = 10^{-22}$ sec.

Since, however, we do not know the exact form of $\beta(t)$ at early times, the previous values serve only as orientative. Should the phase transition become definitely established, one could use it to fix the behavior of $\beta(t)$ for early times.

In conclusion, a great deal of future work remains to be done both from the standpoint of internal consistency, comparison and relation with theories of fundamental interactions as well as direct comparison with observations.

It is our feeling, however, that the preceding analysis has shown how a scale-covariant theory can enlarge the possibilities of taking one step further toward a unified theory of the various kinds of interactions, without contradicting any well-accepted facts.

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APPENDIX A: COTENSOR ANALYSIS

In this section, we shall first review the essential features of Weyl's geometry. Cotensors are then defined in Weyl space. Some mathematical relations in cotensor analysis pertinent to the main text of this paper will be derived here.

The fundamental postulates of Weyl geometry are as follows:

(A) There exist affine connections ${}^*\Gamma^\mu{}_{\nu\lambda}$ such that parallel transport of a vector ξ^μ can be defined as

$$d\xi^\mu = -{}^*\Gamma^\mu{}_{\nu\lambda}\xi^\nu dx^\lambda \,, \tag{A1}$$

where

$${}^*\Gamma^\mu{}_{\nu\lambda} = {}^*\Gamma^\mu{}_{\lambda\nu} \,. \tag{A2}$$

(B) The change of length of a vector by parallel transport is given by

$$d(g_{\mu\nu}\xi^\mu\xi^\nu) = 2g_{\mu\nu}\xi^\mu\xi^\nu k_\lambda dx^\lambda \,. \tag{A3}$$

Note that the metrical properties of Weyl space are specified by both $g_{\mu\nu}$ and k_λ. Since lengths are not assumed to be preserved, the scale vector k_λ gives their variation under parallel transport.

Let

$$^*\Gamma_{\mu,\nu\lambda} = g_{\mu\rho}\, {}^*\Gamma^\rho{}_{\nu\lambda}\,. \tag{A4}$$

It can be easily shown that the affine connections are related to the metric and scale tensors by

$$^*\Gamma_{\mu,\nu\lambda} = \tfrac{1}{2}(g_{\mu\nu,\lambda} + g_{\mu\lambda,\nu} - g_{\nu\lambda,\mu}) - (g_{\mu\nu}k_\lambda + g_{\mu\lambda}k_\nu - g_{\nu\lambda}k_\mu)\,. \tag{A5a}$$

Hence

$$\begin{aligned} ^*\Gamma^\mu{}_{\nu\lambda} &= \tfrac{1}{2}g^{\mu\rho}(g_{\rho\nu,\lambda} + g_{\rho\lambda,\nu} - g_{\nu\lambda,\rho}) \\ &\quad - (g^\mu{}_\nu k_\lambda + g^\mu{}_\lambda k_\nu - g_{\nu\lambda}k^\mu) \\ &= \Gamma^\mu{}_{\nu\lambda} - (g^\mu{}_\nu k_\lambda + g^\mu{}_\lambda k_\nu - g_{\nu\lambda}k^\mu)\,, \end{aligned} \tag{A5b}$$

where $\Gamma^\mu{}_{\nu\lambda}$ are the Christoffel symbols defined in terms of $g_{\mu\nu}$ as in Riemannian geometry.

If we define a curvature tensor in Weyl space by means of parallel displacement of a vector along a closed curve, we get analogous to the Riemannian case

$$^*R^\mu{}_{\nu\lambda\rho} = \frac{\partial\, {}^*\Gamma^\mu{}_{\nu\lambda}}{\partial x_\rho} - \frac{\partial\, {}^*\Gamma^\mu{}_{\nu\rho}}{\partial x^\lambda} + {}^*\Gamma^\eta{}_{\nu\lambda}\, {}^*\Gamma^\mu{}_{\eta\rho} - {}^*\Gamma^\eta{}_{\nu\rho}\, {}^*\Gamma^\mu{}_{\lambda\eta}\,. \tag{A6}$$

The associated contracted tensors $^*R_{\mu\nu}$ and $^*R$ can be written as

$$\begin{aligned} ^*R_{\mu\nu} &= {}^*R^\lambda{}_{\mu\lambda\nu} \\ &= R_{\mu\nu} - 2(k_{\mu;\nu} - k_{\nu;\mu}) - (k_{\mu;\nu} + k_{\nu;\mu}) \\ &\quad - g_{\mu\nu}k^\lambda{}_{;\lambda} - 2k_\mu k_\nu + 2g_{\mu\nu}k^\lambda k_\lambda\,, \end{aligned} \tag{A7}$$

$$^*R = g^{\mu\nu}\, {}^*R_{\mu\nu} = R - 6k^\lambda{}_{;\lambda} + 6k^\lambda k_\lambda\,, \tag{A8}$$

where $R_{\mu\nu}$ and R are the Ricci tensor and scalar curvature defined in terms of $g_{\mu\nu}$. Clearly, if $k_\mu = 0$, the affine connections as well as the curvature tensors reduce to the Riemannian case, and Weyl space in this limit becomes Riemannian space. We note also that a semicolon is used in this paper to denote the normal covariant differentiation defined using $\Gamma^\mu{}_{\nu\lambda}$ rather than $^*\Gamma^\mu{}_{\nu\lambda}$.

Next consider a general scale transformation of the form

$$ds \to ds' = l(x)ds\,. \tag{A9}$$

Since

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu \tag{A10}$$

and dx^μ, since it is a coordinate differential, does not change under scaling, we have

$$g_{\mu\nu} \to g'_{\mu\nu} = l^2(x) g_{\mu\nu}\,. \tag{A11}$$

Equation (A11) can be recognized as a conformal transformation. We remark that given (A10) as the definition of the line element, conformal transformation and scale transformation imply each other. The latter is also called a gauge transformation and we shall be using these terminologies interchangeably in this paper.

From (A3) it can be shown that under the scale transformation (A9), k_μ transform as follows:

$$k_\mu \to k'_\mu = k_\mu + (\ln l)_{,\mu}\,. \tag{A12}$$

It is easy to show using (A11) and (A12) that $^*\Gamma^\mu{}_{\nu\lambda}$ is invariant under gauge transformation. It is of course not a tensor. But the tensor properties of $^*R^\mu{}_{\nu\lambda\rho}$, $^*R_{\mu\nu}$, $^*R$ can be easily established. Furthermore, since $^*\Gamma^\mu{}_{\nu\lambda}$ is gauge invariant, inspection of (A6) and (A7) shows that $^*R^\mu{}_{\nu\lambda\rho}$ and $^*R_{\mu\nu}$ are also gauge invariant.

Now we introduce the notion of a cotensor. Let A denote a tensor of arbitrary rank, i.e., is under coordinate transformations, A has tensor properties. If in addition, under gauge transformation (A9),

$$A \to A' = l^\Pi A \tag{A13}$$

then A is called a cotensor of power Π. In particular, if $\Pi = 0$, A is called an in-tensor. Thus, we see that $^*R^\mu{}_{\nu\lambda\rho}$, $^*R_{\mu\nu}$ are in-tensors. From (A11), $g_{\mu\nu}$ is a cotensor of power 2. Since $g^{\mu\nu}$ is the inverse of $g_{\mu\nu}$, it is a cotensor of power -2.

Clearly, products of cotensors are again cotensors. In particular, let A_1, A_2 be cotensors of powers Π_1 and Π_2; thus

$$A = A_1 A_2$$

is a cotensor of power $\Pi = \Pi_1 + \Pi_2$. Consequently, $^*R$ is a coscalar of power -2. (In the present terminology, scalar and vector are special cases of tensors.) We mention the obvious fact that not all tensors are cotensors. For example, $R_{\mu\nu}$ and R do not transform like (A13), although they have tensor properties under coordinate transformations.

The extension of the concept of tensor to that of cotensor requires a corresponding extension of covariant differentiation. It is clear that the covariant derivative of a cotensor is in general not a cotensor. Let S, V, T be cotensors of power Π having ranks 0, 1, and 2, respectively. We define the co-covariant differentiation of these objects as follows:

$$S_{*\mu} \equiv S_{,\mu} - \Pi k_\mu S\,, \tag{A14a}$$

$$V^\mu{}_{*\nu} \equiv V^\mu{}_{,\nu} + {}^*\Gamma^\mu{}_{\nu\lambda}V^\lambda - \Pi k_\nu V^\mu\,, \tag{A14b}$$

$$V_{\mu*\nu} \equiv V_{\mu,\nu} - {}^*\Gamma^\lambda{}_{\mu\nu}V_\lambda - \Pi k_\nu V_\mu\,, \tag{A14c}$$

$$A^{\mu\nu}{}_{*\lambda} \equiv A^{\mu\nu}{}_{,\lambda} + {}^*\Gamma^\mu{}_{\lambda\rho}A^{\rho\nu} + {}^*\Gamma^\nu{}_{\lambda\rho}A^{\mu\rho} - \Pi k_\lambda A^{\mu\nu}\,, \tag{A14d}$$

$$A_{\mu\nu*\lambda} \equiv A_{\mu\nu,\lambda} - {}^*\Gamma^\rho{}_{\mu\lambda}A_{\rho\nu} - {}^*\Gamma^\rho{}_{\nu\lambda}A_{\mu\rho} - \Pi\, k_\lambda A_{\mu\nu}\,. \tag{A14e}$$

Generalization to higher-rank cotensors is immediate. It can be easily seen from expressions (A14) that the co-covariant derivative of a cotensor of power Π is again a cotensor of the same power.

The following relations will be found to be useful:

$$V^{\mu}{}_{*\nu} = V^{\mu}{}_{;\nu} - (\Pi+1)k_\nu V^\mu + k^\mu V_\nu - g^\mu{}_\nu k_\lambda V^\lambda , \quad \text{(A15)}$$

$$V^{\mu}{}_{*\mu} = V^{\mu}{}_{;\mu} - (\Pi+4)k_\mu V^\mu , \quad \text{(A16)}$$

$$A^{\mu\nu}{}_{*\lambda} = A^{\mu\nu}{}_{;\lambda} - (\Pi+2)k_\lambda A^{\mu\nu} - g^\mu{}_\lambda k_\rho A^{\rho\nu} - g^\nu{}_\lambda k_\rho A^{\mu\rho} + k^\mu A^\nu{}_\lambda + k^\nu A^\mu{}_\lambda , \quad \text{(A17)}$$

$$A^{\mu\nu}{}_{*\nu} = A^{\mu\nu}{}_{;\nu} - (\Pi+5)k_\nu A^{\mu\nu} - k_\rho A^{\rho\mu} + k^\mu A^\nu{}_\nu . \quad \text{(A18)}$$

If $A^{\mu\nu} = A^{\nu\mu}$, we have

$$A^{\mu\nu}{}_{*\nu} = A^{\mu\nu}{}_{;\nu} - (\Pi+6)k_\nu A^{\mu\nu} + k^\mu A^\nu{}_\nu . \quad \text{(A19)}$$

The metric tensor $g_{\mu\nu}$ satisfies the relations

$$g_{\mu\nu*\lambda} = 0 ,$$
$$g^{\mu\nu}{}_{*\lambda} = 0 .$$

The analog of the Einstein tensor $G_{\mu\nu}$ is

$$*G_{\mu\nu} = *R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} *R . \quad \text{(A20)}$$

APPENDIX B: DIRAC'S LARGE-NUMBER HYPOTHESIS

In this appendix, we briefly outline Dirac's large-number hypothesis (LNH) and some of its consequences. More detailed discussions can be found in Dirac's papers.[8-12] A comprehensive summary has also been given by Canuto and Lodenquai.[13]

The motivation of Dirac's hypothesis has been the coincidences among certain large dimensionless numbers first noted by Eddington, which have been known as the Eddington numbers. One of these is the ratio of electrostatic and gravitational forces between a proton and an electron:

$$N_1 = \frac{e^2}{G m_e m_p} = 2 \times 10^{39} . \quad \text{(B1)}$$

A second number arises when the age of the universe, approximated by the reciprocal of the Hubble expansion parameter, is divided by an atomic unit of time:

$$N_2 = \frac{m_e c^3}{H_0 e^2} = 7 \times 10^{40} . \quad \text{(B2)}$$

If the present average density of matter in the universe is taken to be $\rho = 10^{-30}$ g cm^{-3}, the total mass within the visible universe defined by this Hubble radius c/H_0 is given by $\frac{4}{3}\pi\rho(c/H_0)^3$. A third large number can thus be derived:

$$N_3 = \frac{4\pi}{3} \frac{\rho}{m_p} \left(\frac{c}{H_0}\right)^3 \simeq 10^{78} . \quad \text{(B3)}$$

The coincidences mentioned above refer to the fact that the following relations hold:

$$N_1 = a_2 N_2 , \quad \text{(B4a)}$$

$$N_3^{1/2} = a_3^{1/2} N_2 , \quad \text{(B4b)}$$

where a_1, a_3 are of order close to unity. Many theorists believe that the dimensionless constants in physics, such as $e^2/\hbar c$ or m_p/m_e, can in principle be explained theoretically. Likewise there have been numerous speculations about the coincidences of the Eddington numbers. Dirac pointed out that the ratios $N_1 : N_2 : N_3^{1/2}$ are of order unity. They are expected to be derivable theoretically as one would expect for the fine-structure constant. Accepting this point of view, and noting that N_2 corresponds to the cosmological epoch, he came to the conclusion that the gravitational constant measured in atomic units and the number of baryons in the visible universe must be a function of the epoch. Furthermore, he formulated the hypothesis that given any large dimensionless number N, it can be expressed as

$$N = a N_2^k , \quad \text{(B5)}$$

where a and k are constants of order unity. Clearly, Eqs. (B4) are special cases of (B5). It should be noted that (B5) is now taken to be a functional relation: As time passes, N_2 necessarily changes and N would change accordingly.

The immediate consequence of the large-number hypothesis is that the gravitational constant is inversely proportional to the epoch, and the number of baryons in this visible universe increases like the square of the epoch. When Dirac[12] applied the LNH to R, the radius of the universe measured in atomic units, he concluded that the exponent k in (B5) must be 1 and hence

$$R \sim t , \quad \text{(B6)}$$

where we have written t for N_2, which is the epoch in atomic units.

It should be emphasized that the large numbers considered thus far have been derived as ratios of macroscopic gravitational units and microscopic atomic units. In fact, this prompted Dirac[9] in his original article on the subject to suggest that the proper way to understand the LNH is by the consideration of two metrics. But this line of reasoning had not been taken up until recently.

Other astrophysical consequences of the LNH have been considered by various authors. The conclusions do not follow as simply from the LNH as do Eqs. (B4) and (B6), and various dynamical relations had to be used implicitly or explicitly. Hence, instead of summarizing these results here, we shall consider them anew in the main text as consequences of the modified dynamics of the gauge-covariant theory of gravitation.

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[1]C. G. Callan, S. Coleman, and R. Jackiw, Ann. Phys. (N.Y.) 59, 42 (1970).
[2]T. Fulton, F. Rohrlich, and L. Witten, Rev. Mod. Phys. 34, 442 (1962).
[3]In this paper, by different systems of units, we do not mean the trivial difference of, e.g., mks and cgs, or centimeter and angstrom. As will be explained, we associate a system of units with the governing dynamical laws.
[4]W. Pauli, *Theory of Relativity* (Pergamon, New York, 1958), p. 224.
[5]This point was first stressed by R. F. Marzke and J. A. Wheeler, in *Gravitation and Relativity*, edited by H. Y. Chiu and W. F. Hoffmann (Benjamin, New York, 1964), Chap. 3.
[6]H. Weyl, *Space, Time and Matter*, translated from the 4th ed. (Methuen, London, 1922) (reprinted by Dover, New York).
[7]A. S. Eddington, *Mathematical Theory of Relativity* (Cambridge Univ. Press, New York, 1924).
[8]P. A. M. Dirac, Proc. R. Soc. London A333, 403 (1973).
[9]P. A. M. Dirac, Nature 139, 323 (1937).
[10]P. A. M. Dirac, Proc. R. Soc. London A338, 439 (1974).
[11]P. A. M. Dirac, in *Theories and Experiments in High Energy Physics*, edited by B. Kurşunoglu, A. Perlmutter, and S. M. Widmayer (Plenum, New York, 1975).
[12]P. A. M. Dirac, in *Physicist's Conception of Nature*, edited by J. Mehra (Reidel, Dordrecht, The Netherlands, 1973).
[13]V. Canuto and J. Lodenquai, Astrophys. J. 211, 342 (1977).
[14]C. Brans and R. H. Dicke, Phys. Rev. 124, 925 (1961).
[15]R. V. Wagoner, Phys. Rev. D 1, 3209 (1970).
[16]L. P. Eisenhart, *Riemannian Geometry* (Princeton Univ. Press, Princeton, New Jersey, 1926), pp. 89–91.
[17]For any scalar field α, we use the notation $\alpha_\mu \equiv \alpha_{;\mu}$.
[18]A. Einstein, S. B. Preuss. Akad. Wiss. 478 (1918).
[19]See for example Pauli in Ref. 4, pp. 195 and 196.
[20]See e.g. T. D. Lee, in *Cargèse Lectures in Physics, 1966*, edited by M. Lévy (Gordon and Breach, New York, 1967), Vol. 1, p. 55 for a discussion of invariance principles and nonmeasurability.
[21]J. L. Anderson, Phys. Rev. D 3, 1689 (1971).
[22]G. V. Bicknell, J. Phys. A 9, 1077 (1976).
[23]N. Bohr and L. Rosenfeld, K. Dan. Vidensk. Selsk, Mat.—Fys. Medd. 12, No. 8 (1933).
[24]G. Steigman, Nature 261, 479 (1976). For a rebuttal see Ref. 25.
[25]V. Canuto, P. J. Adams, S. H. Hsieh, and E. Tsiang, Nature 264, 485 (1976).
[26]S. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1972).
[27]I. I. Shapiro, Science 162, 352 (1968).
[28]T. C. Van Flandern, in Seventh Texas Symposium on Relativistic Astrophysics, New York Academy of Sciences 262, 494 (1975).
[29]I. W. Roxburg, Nature 261, 301 (1976). For a rebuttal see Ref. 30.
[30]V. Canuto, P. J. Adams, S. H. Hsieh, and E. Tsiang, Nature, 265, 763 (1977).
[31]T. C. Van Flandern, Naval Observatory report, 1977 (unpublished).
[32]C. Sagan and G. Mullen, Science 52, 777 (1972).
[33]P. Q. Wesson, J. R. Astrophys. Soc. 14, 9 (1973).
[34]F. D. Stacey, *Physics of the Earth* (Wiley, New York, 1969), Table E1, pp. 281 and 282.
[35]Ya. B. Zeldovich, Usp. Fiz. Nauk 95, 209 (1968) [Sov. Phys.—Usp. 11, 381 (1968)].
[36]A. D. Linde, Zh. Eksp. Teor. Fiz. Pis'ma Red. 19, 320 (1974) [JETP Lett. 19, 183 (1974)]; J. Dreitlein, Phys. Rev. Lett. 33, 1243 (1974); M. Veltman, *ibid.* 34, 777 (1975); S. A. Bludman and M. A. Ruderman, Phys. Rev. Lett. 38, 255 (1977).
[37]D. Kohler, B. A. Watson, and J. A. Becker, Phys. Rev. Lett. 33, 1628 (1974). These authors show that m_φ cannot be in the range 1.03–18.2 MeV.
[38]K. Sato and H. Sato, Prog. Theor. Phys. 54, 1564 (1975). These authors show that $m_\varphi < 0.7 m_e$ is ruled out, otherwise the emission of Higgs mesons would affect the evolution of stars drastically.