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Low-frequency limit of gravitational scattering*

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We consider the low-frequency limit of cross sections for massless scalar, electromagnetic, and gravitational waves scattering on a Schwarzschild black hole. Explicit results are found for the scattering amplitudes, which differ from the Born approximation results.

I. PLANE WAVES

For any process, the calculation of a scattering amplitude via the time-independent formalism requires a statement of a fiducial plane wave. For scattering on a Schwarzschild black hole, a decomposition into spherical harmonics is essential. Any expression written for a "plane" wave must be "distorted" owing to the long-range field.[1,2] The asymptotic scalar "plane" wave is[1]

$$\phi_{\text{plane}} \sim \frac{(4\pi)^{1/2}}{2i\omega r} \sum_{l=0}^{\infty} (2l+1)^{1/2} {}_0Y_l^0 \times [e^{i\omega r^*} - (-1)^l e^{-i\omega r^*}] e^{-i\omega t} \ , \quad (1.1)$$

where ${}_0Y_l^0$ is a spin-weight-zero spherical harmonic,

$${}_0Y_l^0 \equiv Y_l^0 = \left(\frac{2l+1}{4\pi}\right)^{1/2} P_l \ ,$$

in the usual notation.[3] The appearance of

$$r^* = r + 2M \ln(r/2M - 1) + \text{const} \ . \quad (1.2)$$

in the exponent describes the distortion of the wave owing to the long-range field (M is the mass of the hole, ω is the frequency of the wave, and r and t are the usual Schwarzschild coordinates).

For the electromagnetic and gravitational cases one needs to either work with gauge-invariant quantities or to fix a gauge. In these cases we choose to present the ingoing and the outgoing pieces of the wave in two different gauges, both of which are asymptotically transverse (and traceless). We have for the vector potential component $A_\mu m^\mu$ and the metric-tensor perturbation component $h_{\mu\nu} m^\mu m^\nu$ (see Ref. 2)

$$A^{\text{down}}_{m\ \text{plane}} \sim \int d\bar\omega \sum_{lmP} \tilde K^{\text{down}}_{lm\bar\omega P} \frac{e^{-i\bar\omega r^*} e^{-i\bar\omega t}}{r} {}_1Y_l^m \ , \quad (1.3a)$$

$$A^{\text{up}}_{m^*\ \text{plane}} \sim \int d\bar\omega \sum_{lmP} \tilde K^{\text{up}}_{lm\bar\omega P} \frac{e^{i\bar\omega r^*} e^{-i\bar\omega t}}{r} {}_{-1}Y_l^m \ , \quad (1.3b)$$

$$h^{\text{down}}_{mm\ \text{plane}} \sim \int d\bar\omega \sum_{lmP} K^{\text{down}}_{lm\bar\omega P} \frac{e^{-i\bar\omega r^*} e^{-i\bar\omega t}}{r} {}_2Y_l^m \ , \quad (1.4a)$$

and

$$h^{\text{up}}_{m^* m^*\ \text{plane}} \sim \int d\bar\omega \sum_{lmP} K^{\text{up}}_{lm\bar\omega P} \frac{e^{i\bar\omega r^*} e^{-i\bar\omega t}}{r} {}_{-2}Y_l^m \ . \quad (1.4b)$$

Here the constant mode amplitudes K, $\tilde K$ are, for circularly polarized incident waves,

$$\tilde K^{\text{down}}_{lm\bar\omega P} = -\frac{A\pi}{\sqrt{2}\, i\omega} (-1)^l \left(\frac{2l+1}{4\pi}\right)^{1/2} \times [\delta_{m1}\delta(\omega - \bar\omega) + P\delta_{m-1}\delta(\omega + \bar\omega)] \ , \quad (1.5a)$$

$$\tilde K^{\text{up}}_{lm\bar\omega P} = (-1)^{l+m} \tilde K^{\text{down}}_{lm\bar\omega P} \ , \quad (1.5b)$$

$$K^{\text{down}}_{lm\bar\omega P} = -\frac{h\pi}{i\omega} (-1)^l \left(\frac{2l+1}{4\pi}\right)^{1/2} \times [\delta_{m2}\delta(\bar\omega - \omega) - P\delta_{m-2}\delta(\bar\omega + \omega)] \ , \quad (1.6a)$$

$$K^{\text{up}}_{lm\bar\omega P} = (-1)^{l+m} K^{\text{down}}_{lm\bar\omega P} \ , \quad (1.6b)$$

and A and h are the electromagnetic and metric perturbation amplitudes of the plane wave. Also, for the purposes of this discussion the label down (up) means incoming (outgoing). In Eqs. (1.3)–(1.6) the symbol $P = \pm 1$ is the parity. This quantum number does not enter scalar and electromagnetic scattering partial waves. Their scattering is independent of the parity. The gravitational scattering, however, is not. However, in all cases, the plane wave is a sum of positive- and negative-parity pieces. The expressions (1.3)–(1.6) are for left circularly polarized waves if $\omega > 0$. The other handedness is obtained by reversing the sign of ω everywhere.

The variables $Ke^{\pm i\bar\omega r^*}$, $\tilde K e^{\pm i\bar\omega r^*}$ described in (1.3) and (1.4) are simply related to the asymptotic form of the gauge-independent wave variables (here collectively denoted X) described by Moncrief.[4] These wave variables X all satisfy a wave equation,

$$\left\{\frac{d^2}{dr^{*2}}+\omega^2-\left(1-\frac{2M}{r}\right)\left[\frac{l(l+1)}{r^2}+V(r)\right]\right\}X=0\ , \tag{1.7}$$

where $V(r)$ is a function of order r^{-3}. [$V(r)$ is different for the scalar, the electromagnetic, and the two ($P=\pm1$) gravitational cases.]

One calculates a scattering solution by imposing the boundary condition that the wave be pure ingoing at the horizon. At $r=\infty$ this solution can be asymptotically split into its up (outgoing) and down (incoming) parts. The normalization of the wave is then adjusted so that the down part is identical to the down part of the plane wave, via (1.1), (1.3), or (1.4). The difference between the up part of the now normalized solution and the plane-wave value of the up part is the scattered wave amplitude.

II. LOW-FREQUENCY SCATTERING

For many purposes it is desirable to have simple analytical closed forms for limiting regimes of the scattering problem. Here we concentrate on the limit $\omega\to0$. This is the appropriate limit, for instance, for the cross sections needed in "test particle" virtual quantum calculations. As we now show, it is actually an approximation in ω/l and the analytical results it produces for large l complement numerical calculations for small l.

The appearance of the term $l(l+1)/r^2$ in Eq. (1.7) shows that for $l\neq0$, and for the $\omega\to0$ limit considered here, the s-wave scalar scattering is different from the scattering of the other modes. For any other mode, the classical turning point is $r_{\rm TP}\simeq l/\omega$ for small ω. This makes the influence of the $(r)^{-3}$ terms of Eq. (1.7) negligible compared to the angular momentum term $l(l+1)/r^2$, and it means that the peak value of the potential barrier in Eq. (1.7) is very large compared to the energy, ω^2, of the wave. Hence the scattering becomes elastic for small $M\omega/l$, and a description in terms of (real) phase shifts becomes satisfactory. (We shall denote such phase shifts as γ_l.) The substitution

$$X=Y(1-2M/r)^{-1/2} \tag{2.1}$$

gives a wave equation of the form

$$\left[\frac{d^2}{dr^2}+\omega^2+\frac{4M\omega^2}{r}+\frac{12M^2\omega^2-l(l+1)}{r^2}+O(r^{-3})\right]Y=0\ . \tag{2.2}$$

For $M\omega\ll l$, (2.2) has the form of the Schrödinger equation for a Coulomb scattering problem with $\hbar^2/2\mu=1$, where μ is the mass of the scattered particle, and with attractive charges of magnitude $Ze^2=4M\omega^2$. The asymptotic solutions of (1.7) which have a phase $e^{\pm i\omega r^*}$ go over asymptotically to phases $e^{\pm i\omega r_c}$, with

$$r_c\equiv r+2M\ln(2\omega r)\ , \tag{2.3}$$

the combination which appears in the well-known[1,2,5] solutions to the Coulomb problem for the parameters described above. Obviously the asymptotic relation

$$r^*\sim r_c \tag{2.4}$$

will be satisfied if the so far unspecified constant in r^* is adjusted so that

$$r^*=r+2M\ln(r/2M-1)+2M\ln(4M\omega)\ . \tag{2.5}$$

We shall call the limit $M\omega\ll1$ of Eq. (2.2), i.e.,

$$\left[\frac{d^2}{dr^2}+\omega^2-\frac{l(l+1)}{r^2}+\frac{4M\omega^2}{r}\right]Y=0\ . \tag{2.6}$$

the comparison Newtonian problem. Solving Eq. (2.6) exactly is equivalent to solving (1.7) or (2.2) in the $\omega\to0$ limit. Since (2.6) is exactly a Coulomb radial wave equation for $\hbar^2/2\mu=1$ and $Ze^2=4M\omega^2$, we may immediately take over results from the quantum-mechanical Coulomb problem. In particular, the phase shifts η_l found for (2.6) will be those of the Coulomb scattering problem.

In the low-frequency limit considered here the two parities for gravitational wave scattering approach each other, a result which is straightforward from the form of the wave equation (2.6) in this limit. In general, for finite ω, the two gravitational wave parities scatter differently. This may be seen by referring to the metric formulation of Zerilli[6] and Moncrief[4], where a wave equation of the form of Eq. (1.7) is obtained, with explicitly different $V(r)$ in the two parity cases. An alternative viewpoint on this problem is afforded by a formulation[2] in terms of the Teukolsky[7] equation. This equation deals with Riemann tensor components and is explicitly independent of the parity P. When dealing with the Riemann tensor formalism, however, the expression for the phase of the plane wave *is parity dependent*, with a parity dependence which disappears only at $\omega\to0$ as considered here and at $\omega\to\infty$ (the WKB limit)[2]; this parity dependence of the plane wave then manifests itself in a parity dependence of the scattering. The work of Chandrasekhar[8] and of Chandrasekhar and Detweiler[9] showing the relation between the two parity solutions to the metric perturbation problem is closely related to the Teukolsky equation approach of Ref. 2.

III. SCATTERING AMPLITUDE

We have seen that the phase shifts $\gamma_l^{(a)}$ (for $l \neq 0$) tend to the Coulomb phase shifts η_l for $M\omega/l \to 0$ (here a labels the spin: 0 for scalar, 1 for electromagnetic, and 2 for gravitational). Furthermore, the two parities $P = \pm 1$ behave identically in this limit, even for scattering of gravitational waves. We use the notation ${}_sY_l^m(\theta, \varphi) = e^{im\varphi}{}_sS_l^m(\theta)$. In the three cases scalar, electromagnetic, and gravitational, there is a scattering amplitude for circularly polarized incident waves given in the $\omega \to 0$ limit by the following[10]:

$$\text{scalar: } f(\theta) = \frac{(4\pi)^{1/2}}{2i\omega} \sum_{l=0}^{\infty} (2l+1)^{1/2} {}_0Y_l^0 (e^{2i\gamma_l^{(0)}} - 1) \tag{3.1a}$$

$$\underset{\omega \to 0}{\sim} \frac{(4\pi)^{1/2}}{2i\omega} {}_0S_0^0 e^{2i\gamma_0^{(0)}} + \frac{(4\pi)^{1/2}}{2i\omega} \sum_{l=1}^{\infty} (2l+1)^{1/2} {}_0S_l^0 e^{2i\eta_l} \;; \tag{3.1b}$$

$$\text{electromagnetic[11]: } a(\theta) = \frac{(4\pi)^{1/2}}{2i\omega} \sum_{l=1}^{\infty} (2l+1)^{1/2} {}_{-1}S_l^1 (e^{2i\gamma_l^{(1)}} - 1) \tag{3.2a}$$

$$\underset{\omega \to 0}{\sim} \frac{(4\pi)^{1/2}}{2i\omega} \sum_{l=1}^{\infty} (2l+1)^{1/2} {}_{-1}S_l^1 e^{2i\eta_l} \;; \tag{3.2b}$$

$$\text{gravitational: } g(\theta) = \frac{(4\pi)^{1/2}}{2i\omega} \sum_{l=2}^{\infty} (2l+1)^{1/2} {}_{-2}S_l^2 (e^{2i\gamma_l^{(2)}} - 1) \tag{3.3a}$$

$$\underset{\omega \to 0}{\sim} \frac{(4\pi)^{1/2}}{2i\omega} \sum_{l=2}^{\infty} (2l+1)^{1/2} {}_{-2}S_l^2 e^{2i\eta_l} \;. \tag{3.3b}$$

In all these cases the second line, following the symbol ~, is obtained from the first by noticing that

$$\sum_{l=|s|}^{\infty} (2l+1)^{1/2} {}_sS_l^m$$

is proportional to a δ function in the forward direction. Since we anticipate the same forward divergence as in the Coulomb case, owing to the long-range Newtonian force, dropping this term which arises from the (−1) in Eqs. (3.1a), (3.2a), and (3.3a) is a permissible transformation.[12] The scattering amplitudes then are undefined up to a phase which amounts to adding a constant to each of the phase shifts, which in turn is equivalent to adjusting the constant in Eq. (2.5). Hence the constant in r^* is not relevant for calculating the cross section.[1] Our choice in (2.5) has made the phases for $l > 0$ equal to those in the usual treatment of the comparison Newtonian problem.

We use the formula[13] with $x = \cos\theta$:

$$\int_{-1}^{1} [\sin^2(\theta/2)]^\sigma P_l(\cos\theta) d\cos\theta = \int_{-1}^{1} 2^{-\sigma}(1-x)^\sigma P_l(x) dx = \frac{2(-1)^l \Gamma^2(\sigma+1)}{\Gamma(\sigma+l+2)\Gamma(1+\sigma-l)} \quad (\mathrm{Re}\sigma > -1) \;. \tag{3.4}$$

This is the expression which is used to "explain" the summation of the expression which is the scattering amplitude for the scalar version of the comparison Newtonian problem (2.6). A small amount of manipulation using the properties of $\Gamma(Z)$ gives

$$F(\theta) = M[\sin^2(\theta/2)]^{-1+2iM\omega} e^{2i\eta_0} \tag{3.5a}$$

$$= \frac{(4\pi)^{1/2}}{2i\omega} \sum_{l=0}^{\infty} (2l+1)^{1/2} {}_0S_l^0 e^{2i\eta_l} \;, \tag{3.5b}$$

where

$$e^{2i\eta_l} = \frac{\Gamma(l+1-2iM\omega)}{\Gamma(l+1+2iM\omega)} \;, \tag{3.6}$$

and where (3.5a) is recognized as the scattering amplitude for the scalar limiting Newtonian problem, Eq. (2.6). This explanation has to be taken with some reservation since the term $[\sin^2(\theta/2)]^{-1}$ in (3.5a) means that $\mathrm{Re}\sigma = -1$ in Eq. (3.4). However, we shall continue by viewing this integral as the limit as $\epsilon \to 0_+$ for $\mathrm{Re}\sigma = -1 + \epsilon$.

From Eqs. (3.1) and (3.5), we see

$$f(\theta) = F(\theta) + (e^{2i\gamma_0^{(0)}} - e^{2i\eta_0}) \frac{(4\pi)^{1/2}}{2i\omega} {}_0S_0^0 \;, \tag{3.7}$$

which is just the addition of an angle-independent complex constant to $F(\theta)$ which gives the Newtonian case.

Starobinskii[14] has shown that the $l = 0$ scalar absorption cross section is not zero at zero frequency. This implies that $\gamma_0^{(0)}$ is complex. From Starobinskii:

$$\mathrm{Im}\gamma_0^{(0)} = kM\omega \;,$$

where k is a positive real constant of order unity. In addition, Matzner[1] has shown that the low-frequency behavior of $\mathrm{Re}\gamma_0^{(0)}$ is

$$\mathrm{Re}\gamma_0^{(0)} \approx \omega(dM - \tfrac{1}{3}M\ln 2M\omega)$$

with d another constant of order unity.

Hence the additional constant term which modifies the Newtonian result is dominated by a term proportional to $\ln 2M\omega$, and diverges for $\omega \to 0$. Hence the low-frequency scalar scattering cross section for the gravitational case diverges at all angles as $\omega \to 0$.

We are left now with evaluating $a(\theta), g(\theta)$ according to (3.2) and (3.3). Since the two cases are so similar, the electromagnetic case is left to the Appendix.

We introduce the raising operator L^+, which raises the z component of angular momentum, m, by one unit[3,15]:

$$L^+ = \left(\partial_\theta - m\cot\theta - \frac{s}{\sin\theta}\right) \quad . \tag{3.8}$$

We have

$$L^+ {}_sS_l^m = [(l-m)(l+m+1)]^{1/2}\, {}_sS_l^{m+1} \ . \tag{3.9}$$

We also use the spin-weight-raising operator $\eth$ (see Ref. 3):

$$\eth = -(\partial_\theta - m/\sin\theta - s\cot\theta) : , \tag{3.10}$$

$$\eth\, {}_sS_l^m = [(l-s)(l+s+1)]^{1/2}\, {}_{s+1}S_l^m \ . \tag{3.11}$$

Now consider Eq. (3.5b). Since each term in the sum has $m=0$ and $s=0$, the operator L^+L^+ may be applied as a differential operator to the summed expression. We have

$$L^+L^+F(\theta) \equiv (\partial_\theta - \cot\theta)\partial_\theta F(\theta) \tag{3.12a}$$

$$= \frac{(4\pi)^{1/2}}{2i\omega}\sum_{l=0}^{\infty}(2l+1)^{1/2} L^+L^+\, {}_0S_l^0 e^{2i\eta_l} \tag{3.12b}$$

$$= \frac{(4\pi)^{1/2}}{2i\omega}\sum_{l=0}^{\infty}(2l+1)^{1/2}[(l-1)l(l+1)(l+2)]^{1/2} \times {}_0S_l^2 e^{2i\eta_l} \ , \tag{3.12c}$$

a sum which is even more divergent than (3.5b), but which may be evaluated by explicitly applying the differential operators to the expression $F(\theta)$ of (3.5a). Notice that the $l=0$, $l=1$ terms of (3.12c) vanish.

Similarly, since each term in the sum (3.3b) has the same values $s=-2$, $m=2$, we have

$$\eth\eth g(\theta) = \left(\partial_\theta - \frac{2}{\sin\theta} + \cot\theta\right)\left(\partial_\theta - \frac{2}{\sin\theta} + 2\cot\theta\right) g(\theta) \tag{3.13a}$$

$$= \frac{(4\pi)^{1/2}}{2i\omega}\sum_{l=2}^{\infty}(2l+1)^{1/2}\eth\eth\, {}_{-2}S_l^2 e^{2i\eta_l} \tag{3.13b}$$

$$= \frac{(4\pi)^{1/2}}{2i\omega}\sum_{l=2}^{\infty}(2l+1)^{1/2}[(l-1)l(l+1)(l+2)]^{1/2} \times {}_0S_l^2 e^{2i\eta_l} \ . \tag{3.13c}$$

The result (3.13c) is a sum identical to that of (3.12c).

We shall not concern ourselves with the divergent nature of these sums, since they obviously sum to the quantity

$$H = \sin\theta\, \partial_\theta\left(\frac{\partial_\theta F}{\sin\theta}\right) \ , \tag{3.14}$$

which diverges at $\theta = 0$. Instead we regard Eqs. (3.12c) and (3.13c) as giving us a linear second-order inhomogeneous differential equation to solve for $g(\theta)$:

$$\left(\partial_\theta - \frac{2}{\sin\theta} + \cot\theta\right)\left(\partial_\theta - \frac{2}{\sin\theta} + 2\cot\theta\right) g(\theta) = H \ . \tag{3.15}$$

Since the second-order operator is presented in a factored form, the integration of (3.15) is straightforward. Using the notation $y = \sin^2(\theta/2)$ we obtain

$$g(\theta) = \frac{B}{\cos^4(\theta/2)} + \frac{\tilde{B}y}{\cos^4(\theta/2)} + g_I \ , \tag{3.16}$$

$$\cos^4(\theta/2) g_I = Me^{2i\eta_0}\left\{y^{-1+2iM\omega} + \left(\frac{2iM\omega - 2}{2iM\omega}\right)\left[\left(\frac{2iM\omega - 1}{2iM\omega + 1}\right)y^{1+2iM\omega} - 2y^{2iM\omega}\right]\right\} \ , \tag{3.17}$$

where the constants B, B are the two constants of integration associated with the two solutions to the homogeneous version of (3.15) and g_I is the inhomogeneous solution. The constants B, $\tilde{B}$ can be determined by obtaining the moment of (3.16) against any two convenient ${}_{-2}S_l^2$. Since

$$2\pi\int {}_sS_l^m\, {}_sS_{l'}^m\, dx = \delta_{ll'} \ , \quad (x = \cos\theta) \ , \tag{3.18}$$

$$\frac{2}{2i\omega}\left(\frac{2l+1}{4\pi}\right)^{1/2} e^{2i\eta_l} = \int g(\theta)\, {}_{-2}S_l^2 dx$$
$$= B\int_{-1}^{1} {}_{-2}S_l^2 \frac{dx}{\cos^4(\theta/2)} + \tilde{B}\int_{-1}^{1} {}_{-2}S_l^2 \frac{y}{\cos^4(\theta/2)}\, dx + \int_{-1}^{1} g_I\, {}_{-2}S_l^2 dx\ , \tag{3.19}$$

so evaluation of the integrals for any two l values uniquely determines B and $\tilde{B}$. Because the polynomials' expressions become more complicated for larger l, the sensible procedure is to use ${}_{-2}S_2^2$ and ${}_{-2}S_3^2$ to determine the values of B and $\tilde{B}$.

Now[3,15]

$${}_{-2}S_2^2 = \left(\frac{5}{4\pi}\right)^{1/2} \cos^4(\theta/2)\ , \tag{3.20}$$

while

$${}_{-2}S_3^2 = \left(\frac{7}{4\pi}\right)^{1/2} \cos^4(\theta/2)[1 - 6\sin^2(\theta/2)]. \tag{3.21}$$

The integrals are tedious but straightforward. We obtain

$$B = -\frac{2e^{2i\eta_0}}{2i\omega}\left(\frac{2-2iM\omega}{1+2iM\omega}\right)\ , \tag{3.22}$$

$$\tilde{B} = -\frac{2e^{2i\eta_0}}{2i\omega}\ , \tag{3.23}$$

which gives

$$g(\theta) = Me^{2i\eta_0}\frac{y^{2iM\omega}}{y} + \frac{2ye^{2i\eta_0}}{2i\omega}\left(y^{2iM\omega}\frac{1-4iM\omega}{1+2iM\omega} - 1\right)\Big/\cos^4(\theta/2) + \frac{4e^{2i\eta_0}}{2i\omega}\left(y^{2iM\omega} - \frac{1-iM\omega}{1+2iM\omega}\right)\Big/\cos^4(\theta/2)\ . \tag{3.24}$$

The first term in this sum is identical to the $F(\theta)$ for the comparison scalar Newtonian problem.

The last two terms in the expression (3.24) have an apparent divergence in the backward direction, $\theta \to \pi$. However, this divergence is not real, as we now show. Now

$$y^{2iM\omega} \equiv e^{2iM\omega \ln[1-\cos^2(\theta/2)]}\ . \tag{3.25}$$

As $\cos^2(\theta/2) \to 0$,

$$y^{2iM\omega} = 1 - 2iM\omega\cos^2(\theta/2) + \tfrac{1}{2}\cos^4(\theta/2)(2iM\omega)(2iM\omega - 1) + O(\cos^6(\theta/2))\ . \tag{3.26}$$

In this limit, the trigonometric factors in the last two terms combine to cancel the $[\cos^4(\theta/2)]^{-1}$ denominator. We find

$$g(\theta\to\pi) \to Me^{2i\eta_0}\left(\frac{1}{y} - 1 + O(\cos^2(\theta/2))\right)\ , \tag{3.27}$$

which vanishes at $\theta = \pi$. There is thus a remarkable cancellation of the backward divergence arising from the homogeneous terms B, $\tilde{B}$ in Eq. (3.16).

As discussed in Sec. II, the Newtonian problem calculated here approaches the relativistic scattering problem when the frequency becomes small, $M\omega \to 0$. The large parentheses in the last two terms in (3.24) contain the difference of quantities equal to $1 + O(M\omega)$ as $M\omega \to 0$ so the *a priori* low-frequency divergence in the last two terms is avoided.

As[16] $M\omega \to 0$

$$y^{2iM\omega} \equiv e^{2iM\omega \ln\sin^2(\theta/2)} \tag{3.28}$$

$$\simeq 1 + 2iM\omega \ln\sin^2(\theta/2)\ . \tag{3.29}$$

Hence, in the relevant limit $M\omega \to 0$

$$g(\theta) \to Me^{2i\eta_0}y^{2iM\omega}/y + \frac{2Me^{2i\eta_0}}{\cos^4(\theta/2)}\left\{y[\ln\sin^2(\theta/2) - 3] + 2\ln\sin^2(\theta/2) + 3\right\}\ . \tag{3.30}$$

Although qualitatively similar, the cross section calculated from the result (3.30) differs in detail from that found for the Born approximation by Westervelt[17] and Peters[18]:

$$\sigma(\theta) = M^2\frac{\sin^8(\theta/2) + \cos^8(\theta/2)}{\sin^4(\theta/2)}\ . \tag{3.31}$$

This difference is even more remarkable since the Born result for the scalar case of the Newtonian problem is well known to give the exact cross section for that problem. As shown in the Appendix, for the electromagnetic case the Born approximation agrees with the calculation here in the backward direction, and the cross section for the electromagnetic case is close to the Born result for essentially all angles.

APPENDIX: THE ELECTROMAGNETIC AMPLITUDE $a(\theta)$

To evaluate $a(\theta)$, we write

$$\eth a = L^+ F \ , \tag{A1}$$

$$\left(\partial_\theta - \frac{1}{\sin\theta} + \cot\theta\right) a = -M\partial_\theta[\sin^2(\theta/2)]^{-1+2iM\omega} e^{2i\eta_0} \ , \tag{A2}$$

$$a = \frac{\tilde{A}}{\cos^2(\theta/2)} + a_I \ , \tag{A3}$$

$$a_I = -\frac{Me^{2i\eta_0}}{\cos^2(\theta/2)} \left\{ [\sin^2(\theta/2)]^{-1+2iM\omega} - \frac{(2iM\omega - 1)}{2iM\omega} [\sin^2(\theta/2)]^{2iM\omega} \right\} \ . \tag{A4}$$

Here there is only the single constant $\tilde{A}$ to evaluate. We take, for simplicity, the moment of a against ${}_{-1}S_1^1$ (see Refs. 3 and 15). Now

$${}_{-1}S_1^1 = -\left(\frac{3}{4\pi}\right)^{1/2} \cos^2(\theta/2) \ . \tag{A5}$$

Hence

$$\int a \, {}_{-1}S_1^1 dx = \frac{(4\pi)^{1/2}\sqrt{3}\, e^{2i\eta_1}}{2\pi 2i\omega} = 2\left(\frac{3}{4\pi}\right)^{1/2} \tilde{A} + \int a_I \, {}_{-1}S_1^1 dx \ . \tag{A6}$$

We obtain

$$\tilde{A} = (2i\omega)^{-1} e^{2i\eta_0} \ , \tag{A7}$$

which gives

$$e^{-2i\eta_0} a \cos^2(\theta/2) = -M\{(\sin^2(\theta/2)]^{-1+2iM\omega} - [\sin^2(\theta/2)]^{2iM\omega}\} - \frac{1}{2i\omega} \{[\sin^2(\theta/2)]^{2iM\omega} - 1\} \ . \tag{A8}$$

Using Eqs. (3.28) and (3.29) for small $M\omega$, the scattering amplitude except near the forward direction is

$$a = -Me^{2i\eta_0} \left\{ \frac{[\sin^2(\theta/2)]^{2iM\omega}}{\sin^2(\theta/2)} + \frac{\ln \sin^2(\theta/2)}{\cos^2(\theta/2)} \right\} \ . \tag{A9}$$

Finally, we may investigate the $\theta \to \pi$ (backward) limit of this result using (3.26):

$$a \underset{\theta\to\pi}{\simeq} -Me^{2i\eta_0}\left[\frac{1}{\sin^2(\theta/2)} - 1\right] \cong -Me^{2i\eta_0}\left[\frac{\cos^2(\theta/2)}{\sin^2(\theta/2)}\right] . \tag{A10}$$

Only in this limit does the scattering amplitude agree with the form

$$\sigma(\theta) = M^2 \left[\frac{\cos^4(\theta/2)}{\sin^4(\theta/2)}\right] \tag{A11}$$

found by Westervelt[17] and Peters[18] in Born-approximation calculations.

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