Natural suppression of symmetry violation in gauge theories: Muon- and electron-lepton-number nonconservation

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We analyze the circumstances under which the violations of an approximate symmetry in a unified gauge theory of weak interactions are naturally suppressed; in particular, we investigate approximate muon- and electron-type lepton-number conservation as an example of such a symmetry. Extending earlier work, we propose a unified treatment of this symmetry together with strangeness conservation by the weak neutral current and CP invariance. The rate for the decay $\mu \to e\gamma$ is calculated for a general $SU(2) \times U(1)$ gauge model. From this and a similar study of the decay $\mu \to ee\bar{e}$ we derive a set of conditions which guarantees that the violation of muon- and electron-type lepton-number conservation is naturally strongly suppressed. As part of this, we compute the nondiagonal electromagnetic vertex to one-loop order for an arbitrary $SU(2) \times U(1)$ gauge theory. We then focus on the phenomenological predictions of a particular gauge model with three left-handed doublets of leptons and quarks. These include the existence of charged and neutral heavy leptons and of small violations of μ-e universality and the relation $G_F^\beta \sec\theta_C = G_F^\mu$. Other muon- and electron-number-violating effects include nonvanishing rates for the decays $K^\pm \to \pi^\pm e\bar{\mu}$ and $K_L \to e\bar{\mu}$, and for the reactions $\mu + N \to e + N$ and $\nu_\mu + N \to e^- + X$.

I. INTRODUCTION

There are a number of interesting examples in weak interactions of approximate conservation laws which hold to a very high degree of accuracy. These include the conservation of strangeness by the weak neutral current, separate muon- and electron-type lepton-number conservation, and finally CP invariance. The conservation of μ- and e-type lepton number may well be exact; we shall, however, take the view here of regarding it as an approximate symmetry, the validity of which has been experimentally demonstrated to a given level of precision.[1]

These three cases share an important common feature, which we should like to focus upon in this paper, with special emphasis on μ- and e-type lepton-number nonconservation. It is not difficult to construct a model in which approximate conservation laws hold in lowest order in the weak-coupling constant, G_F. The striking feature of the examples cited above is that the violation which, in principle, could occur in second order, i.e., in order $G_F\alpha$, is in fact further suppressed. We propose, extending earlier work,[2,3] a unified approach to these three problems based on the idea of a mechanism which "naturally" suppresses violation of the given symmetry, with μ- and e-type lepton-number nonconservation an an example.

Historically, the smallness of the K_L-K_S mass difference, of the rate for $K_L \to \mu\bar{\mu}$, and of other strangeness-changing weak neutral-current processes necessitated the incorporation of the Glashow-Iliopoulos-Maiani (GIM) cancellation mechanism in gauge theories of weak interactions. This mechanism precludes an s-d transition through the Z coupling in lowest order in the weak Lagrangian. But it does more than this; as has been discussed elsewhere,[2] it also works at the one-loop level to suppress $\Delta S \neq 0$ induced neutral-current effects to the level $G_F\alpha(\Delta m_q^2/m_W^2)\epsilon_C$, where Δm_q^2 refers to the difference of certain quark masses squared, and $\epsilon_C = \sin\theta_C \cos\theta_C$, in the minimal Weinberg-Salam (WS) theory.[5] Similarly, in gauge theories of microweak CP violation,[3] the magnitude of CP violation in $\Delta S = 1$ processes is not of order $G_F\epsilon$, but, instead, of order $G_F(\Delta m_q^2/m_W^2)\epsilon$, where $\epsilon \sim 10^{-3}$ is the conventional measure of CP violation in the neutral K system. Furthermore, in this class of theories the electric dipole moment of a quark arises only in two-loop order, and is of order 10^{-30} cm.

It is important to distinguish two aspects of this mechanism in a natural theory in which parameters of the theory are arbitrary. The first aspect is the appearance of certain mixing parameters such as ϵ_C and ϵ, whose moduli are constrained to be less than unity, but whose magnitudes are otherwise arbitrary. For the present, these must be regarded as empirical quantities to be taken from experiment. The second aspect, which is the crux of the matter, is the occurrence of the mass ratio $(\Delta m_q^2/m_W^2)$ for any values of the parameters; this depends only on the representation content of the model. Since m_W is presumably much larger than m_q in any model so far contemplated, the above mass ratio represents a substantial suppression of the violation of the given approximate conservation law. This second aspect is what we mean by a natural suppression mechanism.

The suppression factor $(\Delta m_q{}^2/m_W{}^2)$ reflects the fact that two different quark transitions contribute with opposite sign and, in the absence of any mass difference, with equal magnitude. Because of this, the leading term of various amplitudes cancels, and the remainder is of the order of the above mass ratio, in the absence of infrared singularities which might arise in the limit $m_q \to 0$.

The theoretical basis for the natural suppression mechanism has recently been studied systematically in $SU(2)\times U(1)$ gauge theories in which quark mass terms are arbitrary (i.e., in those models in which there is no zeroth-order natural relation among masses and mixing parameters; see below).[6] Glashow and Weinberg have established general necessary and sufficient conditions which guarantee that the weak neutral current naturally conserves fermion flavors in orders G_F and $G_F\alpha$. The conditions are (1) that quarks of a given charge and chirality have the same weak T and T_3, and (2) that quarks of a given charge receive their mass either from a gauge-invariant bare mass term or from their couplings with a single neutral Higgs field (but not from both). One of us determined in the same framework the conditions which guarantee that $|\Delta S| = 2$ transitions, both CP conserving and violating, are naturally suppressed.[3] They are (1) the Glashow-Weinberg conditions (but without the bare-mass option, since the possibility of such a coupling is precluded by the next condition), and (2) the requirement that quarks of charge q and quarks of charge $q\pm 1$ do not belong to the same isomultiplets for at least one chirality.

Let us now shift our attention to the leptonic sector and consider the role of the natural suppression mechanism with regard to the conservation laws of separate electron- and muon-type lepton number. In unified gauge theories, these laws have occupied a rather special position. In contrast to the conservation of electric charge, the conservation of muon-type and electron-type lepton numbers is not associated with the gauge-invariant coupling of a conserved μ- or e-type lepton current to a massless gauge vector boson. That is, these conservation laws, if exact, are not realized in nature as gauge symmetries. Indeed, since the weak gauge symmetry is spontaneously broken, the eigenstates of the weak gauge group are in general not eigenstates of the mass matrix. Therefore, there will in general be mixing between fermions of the same chirality and charge, which will prevent the existence of a conserved quantum number assigned to the particles in a particular weak multiplet.

As has been remarked upon before,[7] in the minimal Weinberg-Salam model with just two left-handed doublets[8] $(\nu_e, e)_L$ and $(\nu_\mu, \mu)_L$, the exact degeneracy of the neutrinos (guaranteed by their masslessness) implies separate μ- and e-type lepton-number conservation. More generally, consider a model with n left-handed doublets $(N_1, e)_L$, $(N_2, \mu)_L$, $(N_3, L_3^-)_L, \ldots, (N_n, L_n^-)_L$, where the neutral leptons $N_1, \ldots, N_n$ are degenerate. Certainly the natural way to guarantee this degeneracy is to make these neutral leptons massless, but this is not necessary. Now one can define $\nu_e \equiv N_1$, $\nu_\mu \equiv N_2$, $\nu_i = N_i$ as *both* the mass eigenstates *and* the weak gauge group eigenstates. Assign to the electron multiplet the electron-type lepton number $\mathfrak{N}_e$, and similarly for $\mathfrak{N}_\mu, \mathfrak{N}_3, \ldots, \mathfrak{N}_n$. Then in such a theory these n quantum numbers $\mathfrak{N}_e, \mathfrak{N}_\mu, \ldots, \mathfrak{N}_n$ will be exactly conserved.[9] For simplicity let us return to the minimal Weinberg-Salam model. If one (1) allows ν_e and ν_μ to be nondegenerate in mass, or (2) adds either left- or right-handed doublets with neutral or doubly charged heavy leptons (called generically L^0, L^{--}) which can communicate with muons and electrons or (3) enlarges the original doublets to triplets or higher-dimensional representations of weak SU(2), then μ- and e-type lepton numbers will not be separately conserved. The mixing of ν_e and ν_μ in case (1) is quite analogous to the situation which obtains in the hadronic sector of the minimal model, where there is no separate conservation of $(u, d_\theta)_L$-quark number and $(c, s_\theta)_L$-quark number because of the mixing of d and s, or equivalently u and c. It is the purpose of this paper to analyze the conditions which guarantee that the nonconversation of e- and μ-type lepton number is naturally suppressed and to discuss their consequences.[9] In doing this we shall also present a compendium of expressions for electromagnetic vertices—both parity-conserving and parity-violating, and both on-diagonal and off-diagonal—of fermions in any $SU(2)\times U(1)$ gauge theory. We shall briefly comment on special circumstances in which the fermion mass matrix is not arbitrary due to the representation content of the theory, so that the effects of particle mixing are naturally suppressed. In such models, the model of Cheng and Li being a prime example,[10] the general conditions need not apply. With our suppression mechanism one obtains predictions for such processes as $\mu \to e\gamma$ and $\mu \to ee\bar{e}$ which, for a wide range of parameters (heavy lepton masses and mixing angles) are in accord with, but not extremely small compared to, present experimental limits.

The remainder of this paper is organized as follows. In Sec. II, we review our general matrix formalism and discuss the one-loop calculation of the general fermion electromagnetic vertex. Section III contains a description of the various models to be considered. In Sec. IV we present our calculation of the rate for the decays $\mu \to e\gamma$ and $\mu \to ee\bar{e}$ in these models. In Sec. V we focus on the Kobayashi-Maskawa (KM) model[11,12] involving three left-handed doublets of leptons and quarks. Certain interesting phenomenological implications of the model are analyzed. Our concluding remarks are given in Sec. VI. Appendix A contains a discussion of the self-energy graphs.

II. THE GENERAL FERMION ELECTROMAGNETIC VERTEX TO ONE-LOOP ORDER

In this section we shall compute the general electromagnetic vertex to one-loop order for fermions in an $SU(2)\times U(1)$ gauge theory. For the special case of a diagonal, real-photon vertex, in models where our approximations are valid, our results can be used to determine the anomalous magnetic dipole moment and electric dipole moment of an arbitrary fermion.[13] The main applications of the nondiagonal, real-photon amplitude are to radiative hyperon decays, $Y \to N+\gamma$, where $Y=\Lambda$, Σ, or Ξ,[14] and radiative lepton decays, such as $L^- \to e+\gamma$ (where L^- is a heavy lepton) and $\mu \to e+\gamma$. The nondiagonal virtual-photon amplitude is used in the calculation of rare K decays such as $K_L \to \mu\bar{\mu}$, $K^+ \to \pi^+ e\bar{e}$, $K_S^0 \to \pi^0 e\bar{e}$, and the leptonic decay $\mu \to ee\bar{e}$. In the present paper, as mentioned before, we shall concentrate on the application to μ- and e-type lepton-number nonconservation.

In all these experimentally interesting hadronic and leptonic applications, there are two simplifying features. First, the main contribution to the amplitude comes from diagrams in which the mass of the internal virtual fermion (a charmed quark or heavy lepton) is considerably larger than the masses of the external fermion (d or s quark, or μ, e, ν lepton). This fact justifies an approximation which we shall use in evaluating the Feynman parameter integrals; namely, we shall keep only the non-negligible fermion masses. Second, in the decay amplitude for $f_1 \to f_2+\gamma_{\text{virtual}}$, the momentum q of the photon satisfies $q^2 < m_1^2$, where m_1 is the mass of the initial fermion. Consequently, q^2 is also small compared to the charmed-quark or heavy-lepton mass squared and can be neglected in the parametric integrals. With these approximations the parametric integrals will be of the form

$$a_0 + b_0 \ln\frac{m_W^2}{m_F^2} + \left(\frac{m_F^2}{m_W^2}\right)\left(a_1 + b_1 \ln\frac{m_W^2}{m_F^2}\right)$$

(where m_F is the mass of the relevant virtual fermion) rather than some complicated and, for our purposes, not very useful, expression involving dilogarithms. Moreover, since q^2 is small, it is not necessary to compute the full off-shell vector and axial-vector form factors but rather only the V and A parts of the charge radius. This expedites the computation.

We shall perform the calculation using the ξ-limiting procedure as formulated for spontaneously broken non-Abelian gauge theories by Fujikawa.[15] In this formulation, there is no interaction term of the type $em_W(A_\mu W^{-\mu}\phi^+ + A_\mu W^{+\mu}\phi^-)$, where A_μ is the photon field, and $\phi^\pm$ are unphysical scalar fields, in contrast to the regular R_ξ gauge,[16] in which this term is present. Furthermore, the WWA vertex differs from the form in the R_ξ gauge by the addition of a term linear in ξ [see Eq. (2.27)]. Finally, as in the R_ξ gauge, the W propagator is

$$i\Delta_{\alpha\beta}(k) = -i\left(g_{\alpha\beta} - \frac{k_\alpha k_\beta(1-1/\xi)}{k^2 - m_W^2/\xi}\right) \times \frac{1}{k^2 - m_W^2 + i\epsilon} \tag{2.1}$$

and similarly, with appropriate changes, for the Z propagator. The advantages of using this ξ-limiting procedure are first, that there are no diagrams involving $\phi^\mp W^\pm A$ vertices. Second, for the physical quantities which we calculate at the one-loop level, diagrams such as those of Figs. 1(a) and 1(b), but with $W^\pm$ replaced by $\phi^\pm$, both vanish in the limit $\xi \to 0$. (The limit is taken after all integrations are performed.) This is useful because our general matrix formalism (see below) is simplest in diagrams which do not involve un-

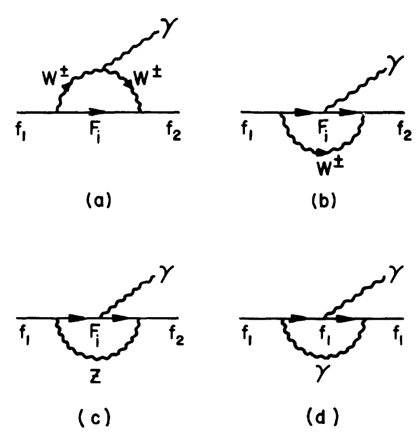

FIG. 1. Diagrams contributing in a general $SU(2) \times U(1)$ gauge model to the process $f_1 \to f_2+\gamma$, where $f_{1,2}$ are (identical or different) fermions. The symbol F_i denotes any fermion which can couple in these graphs. For the decay $\mu \to e\gamma$, $f_1=\mu$, $f_2=e$, and in graphs (a) and (b), $F_i = L^{--}$ or L^0, in accord with the upper or lower sign choices for $W^\pm$. It is shown in the text that graph (c) can only occur for the diagonal amplitude; hence in this graph $f_i = F_i = f_2$.

physical scalars.[17]

Before presenting the results of the calculations, we must introduce our general notation.[3] The crucial virtue of this notation is that it allows one to treat in a unified way all SU(2) × U(1) gauge models. Let us denote by ξ_L and ξ_R the left and right chiral lepton fields. The components of $\xi_{L,R}$ are labeled by T, T_3, Y (or equivalently, Q), and α, where α distinguishes different multiplets with the same T and Y. The coupling of the gauge bosons to fermions is given by

$$\mathcal{L}_{GF} = gJ^\mu_W W^+_\mu + \text{H.c.} + (g^2+g'^2)^{1/2} J^\mu_Z Z_\mu \ , \tag{2.2}$$

where

$$J^\mu_W = \bar{\xi}_L \gamma^\mu T^L_+ \xi_L + \bar{\xi}_R \gamma^\mu T^R_+ \xi_R \tag{2.3}$$

and

$$J^\mu_Z = \bar{\xi}_L \gamma^\mu T^L_Z \xi_L + \bar{\xi}_R \gamma^\mu T^R_Z \xi_R \ . \tag{2.4}$$

In Eq. (2.3), $T^{L,R}_i$, $i=1,2,3$, are the representations of the weak-isospin generators for the various left and right fermion multiplets and $Q = T_3 + Y/2$ is the electric charge operator. The $T^{L,R}_i$ satisfy the commutation relation

$$[T^{L,R}_i, T^{L,R}_j] = i\epsilon_{ijk} T^{L,R}_k \ . \tag{2.5}$$

For a doublet representation of SU(2), for example, $T^{L,R}_i = \frac{1}{2}\tau_i$. The weak-isospin raising and lowering operators are defined as

$$T^{L,R}_\pm = \frac{T^{L,R}_1 \pm i T^{L,R}_2}{\sqrt{2}} \ , \tag{2.6}$$

so that

$$[T^{L,R}_+, T^{L,R}_-] = T^{L,R}_3 \ . \tag{2.7}$$

In Eq. (2.4),

$$T^{L,R}_Z = T^{L,R}_3 - \sin^2\theta_W Q^{L,R} \ , \tag{2.8}$$

where

$$Q = Q^L L + Q^R R \tag{2.9}$$

and

$$\left.\begin{matrix} L \\ R \end{matrix}\right\} = \left(\frac{1 \mp \gamma_5}{2}\right) \ . \tag{2.10}$$

In order to express this coupling in terms of physical fermion fields, we must diagonalize the mass matrix, which is of the form

$$\bar{\xi}_L M \xi_R + \text{H.c.}$$

Here M is a general matrix which is constrained only to commute with Q:

$$Q^L M = M Q^R \ . \tag{2.11}$$

Since in general $[M, M^\dagger] \neq 0$, there does not exist a unitary matrix U such that $UMU^\dagger$ is diagonal. Consider, however, $MM^\dagger$ and $M^\dagger M$; these are Hermitian and have the same positive-semidefinite eigenvalues, so that there exist unitary matrices U_L and U_R such that

$$\begin{aligned} U_L M M^\dagger U_L^\dagger &= M_D^2 \ , \\ U_R M^\dagger M U_R^\dagger &= M_D^2 \ , \end{aligned} \tag{2.12}$$

where M_D^2 is diagonal. Then

$$U_L M U_R^\dagger = M_D \ . \tag{2.13}$$

From Eq. (2.11), it follows that

$$[Q^L, U_L] = [Q^R, U_R] = 0 \ . \tag{2.14}$$

The fermion mass eigenstates are then defined as

$$\begin{aligned} &\psi_L = U_L \xi_L, \quad \psi_R = U_R \xi_R \ , \\ &\psi = L\psi_L + R\psi_R \ . \end{aligned} \tag{2.15}$$

In terms of the mass eigenstates, the fermion–gauge-boson coupling takes the form of Eq. (2.2) with

$$J^\mu_W = \bar{\psi}_L \gamma^\mu \mathcal{T}^L_+ \psi_L + \bar{\psi}_R \gamma^\mu \mathcal{T}^R_+ \psi_R \tag{2.16}$$

and

$$J^\mu_Z = \bar{\psi}_L \gamma^\mu \mathcal{T}^L_Z \psi_L + \bar{\psi}_R \gamma^\mu \mathcal{T}^R_Z \psi_R \ , \tag{2.17}$$

where

$$\mathcal{T}^{L,R}_i = U_{L,R} T^{L,R}_i U^\dagger_{L,R} \tag{2.18}$$

and

$$\mathcal{T}^{L,R}_Z = \mathcal{T}^{L,R}_3 - \sin^2\theta_W Q^{L,R} \ . \tag{2.19}$$

Let us proceed with the calculation. The diagrams which can, in general, contribute in one-loop order to the electromagnetic transition amplitude from an initial fermion f_1 to a final fermion f_2 are shown in Fig. 1. In the nondiagonal case where $f_1 \neq f_2$, there are also the self-energy graphs shown (for $f_1 = \mu$, $f_2 = e$) in Fig. 3. For the real-photon amplitude, these give a zero contribution and for the virtual-photon amplitude, they contribute only to the renormalization of the (nondiagonal) vector and axial-vector form factors. The treatment of these self-energy graphs is discussed in greater detail in the Appendix. In the nondiagonal case, the graph involving a Z in Fig. 1(c) will only contribute if there are appropriate lowest-order nondiagonal Z-fermion coupling terms in the Lagrangian. Stating one of our results in advance, it will in fact be shown that if this Z graph could contribute to the decay $\mu \to e\gamma$ it would give much too large a rate. The photon graph 1(d) is, of course, present only for the diagonal electromagnetic vertex; it gives [cf. Eqs. (2.20) and (2.64)] $\bar{F}^V_2(0)_{ab} = -(\alpha/2\pi) Q_a{}^2 \delta_{ab}$, where $\bar{F}^V_2(0)_{aa} \equiv F^V_2(0)_{aa}/Q_a$ is the anomalous magnetic moment.

The invariant amplitude for $f_1 \to f_2 + \gamma$ (virtual or real) has the general Lorentz and Dirac structure (with $p_1 = p_2 + q$)

$$\mathfrak{M}_\mu(f_1(p_1) \to f_2(p_2) + \gamma(q)) = -i\bar{u}_2(p_2)\Big[\gamma_\mu(F_1^V(q^2) + F_1^A(q^2)\gamma_5) + \frac{i\sigma_{\mu\nu}q^\nu}{(m_1+m_2)}(F_2^V(q^2)+F_2^A(q^2)\gamma_5) + q_\mu(F_3^V(q^2)+F_3^A(q^2)\gamma_5)\Big]u_1(p_1) \,. \tag{2.20}$$

In Eq. (2.20) $u_1(p_1)$ is to be regarded as a tensor product of a Dirac four-spinor and an n-dimensional vector, where n denotes the number of hadronic or leptonic flavors, i.e., mass eigenstates. The form factors $F_i(q^2)$ are $n \times n$ matrices in the space of physical quark or lepton fields. We have normalized the $F_2^{V,A}$ matrices so that the diagonal elements are equal to the anomalous magnetic moment (times the charge) and electric dipole moment of the corresponding fermions.

Electromagnetic current conservation requires that

$$q^\mu \mathfrak{M}_\mu = 0 \,. \tag{2.21}$$

which implies the two relations

$$(m_1 - m_2)F_1^V(q^2) + q^2 F_3^V(q^2) = 0 \,, \tag{2.22a}$$

$$-(m_2 + m_1)F_1^A(q^2) + q^2 F_3^A(q^2) = 0 \,. \tag{2.22b}$$

For $q^2 = 0$, since $F_3^{V,A}$ are analytic at this point,

$$F_1^V(0) = Q_{f_1}\delta_{f_1 f_2} \,, \tag{2.23a}$$

$$F_1^A(0) = 0 \,, \tag{2.23b}$$

where f_1 and f_2 label the initial and final fermions. In particular, for the decays $\mu \to e\gamma$ or (for nondegenerate neutrinos) $\nu_2 \to \nu_1\gamma$, $F_1^V(0) = 0$. Furthermore, for a real photon, the full amplitude is

$$\mathfrak{M} = \epsilon^\mu(q)\mathfrak{M}_\mu \,, \tag{2.24}$$

and $\epsilon \cdot q = 0$, so that the $F_3^{V,A}$ terms make zero contribution to $f_1 \to f_2 + \gamma(q^2 = 0)$. Thus in order to determine the rate $\Gamma(\mu \to e\gamma)$, we need only calculate the quantities $F_2^{V,A}(0)$.

For the related decay $\mu \to ee\bar{e}$, we shall need the $F_1^{V,A}$ terms. Since in this decay $q^2 \lesssim m_\mu^2 \ll m_L^2 \ll m_W^2$ it is useful to expand the vector and axial-vector form factors, keeping only the largest terms. From Eqs. (2.22a) and (2.22b) we obtain the following relations linking the vector and axial-vector charge radii $dF_1^{V,A}(0)/dq^2$ with the scalar and pseudoscalar form factors:

$$\frac{dF_1^V}{dq^2}(0) = \frac{-1}{m_1 - m_2} F_3^V(0) \tag{2.25a}$$

(for $m_1 \neq m_2$) and

$$\frac{dF_1^A}{dq^2}(0) = \frac{1}{m_1 + m_2} F_3^A(0) \,. \tag{2.25b}$$

We shall use Eqs. (2.25a) and (2.25b) to determine $dF_1^{V,A}/dq^2$, since this method is easier than a direct computation of $F_1^{V,A}(q^2)$. Once having utilized $F_3^{V,A}$ in this manner, we will drop them because they make a zero contribution to the amplitude for $\mu \to ee\bar{e}$.

Diagram 1(a) gives a contribution

$$eg^2 Q_W \int \frac{d^4k}{(2\pi)^4}\gamma^\beta(\mathcal{T}_\pm^L L + \mathcal{T}_\pm^R R)\frac{(\not{p} - \not{k} + M_D)}{(p_1 - k)^2 - M_D^2}\gamma^\alpha(\mathcal{T}_\mp^L L + \mathcal{T}_\mp^R R)\Delta_{\alpha\rho}(k)\Delta_{\beta\sigma}(k-q)\Gamma_{WW\gamma}^{\rho\mu\sigma}(k, -q, q-k) \,, \tag{2.26}$$

where the chiral projection operators L and R are given in Eq. (2.10) and a sum over both upper and lower signs is understood. The $WW\gamma$ vertex is given by

$$ie\Gamma_{W^+W^-\gamma}^{\alpha_1\alpha_2\alpha_3}(k_1,k_2,k_3) = ie[(k_1-k_2)_{\alpha_3}g_{\alpha_1\alpha_2} + (k_2-k_3)_{\alpha_1}g_{\alpha_2\alpha_3} + (k_3-k_1)_{\alpha_2}g_{\alpha_3\alpha_1}] + ie\xi[(k_1)_{\alpha_1}g_{\alpha_2\alpha_3} - (k_2)_{\alpha_2}g_{\alpha_1\alpha_3}] \tag{2.27}$$

(with the momenta defined as going into the vertex). In Eq. (2.26), $Q_W = \pm 1$, corresponding to the upper and lower choices of signs in the $\mathcal{T}_\pm$ operators. Similarly, diagram 1(b) gives a contribution

$$eg^2\int\frac{d^4k}{(2\pi)^4}\gamma^\beta(\mathcal{T}_\pm^L L + \mathcal{T}_\pm^R R)\frac{(\not{p}_2 - \not{k} + M_D)}{(p_2-k)^2 - M_D^2}\gamma^\mu Q\frac{(\not{p}_1 - \not{k} + M_D)}{(p_1-k)^2 - M_D^2}\gamma^\alpha(\mathcal{T}_\mp^L L + \mathcal{T}_\mp^R R)\Delta_{\alpha\beta}(k) \,. \tag{2.28}$$

The contribution of diagram 1(c) is given by Eq. (2.28) with the replacements $g^2 \to g^2 + g'^2$, $\mathcal{T}_\pm^{L,R} \to \mathcal{T}_Z^{L,R}$, $m_W \to m_Z$, and $\xi \to \eta$, where η is the gauge parameter for the Z field.

It is convenient to separate the LL, RR and LR, RL parts of the form factors arising from the W and Z graphs 1(a)–1(c):

$$F_i^{V,A} \equiv (F_i^{V,A})_{LL,RR} + (F_i^{V,A})_{LR,RL}, \quad i = 1,2,3 \,. \tag{2.29}$$

Performing some Dirac algebra using the chiral projection operators L and R in Eqs. (2.26) and (2.28), we determine the general structure of the LL, RR and LR, RL parts of the form factors to be as listed below. In these equations a sum over the indices $a(\bar{a}) = +(-), -(+), Z(Z)$ is understood corresponding to the contributions of W^-, W^+, and Z graphs, respectively:

$$F_1^V(q^2)_{LL,RR} = -q^2[\mathcal{T}_a^L C_3^{LL,RR}\mathcal{T}_{\bar{a}}^L + \mathcal{T}_a^R C_{3,a}^{LL,RR}\mathcal{T}_{\bar{a}}^R] , \tag{2.30a}$$

$$F_1^A(q^2)_{LL,RR} = q^2[\mathcal{T}_a^L C_{3,a}^{LL,RR}\mathcal{T}_{\bar{a}}^L - \mathcal{T}_a^R C_{3,a}^{LL,RR}\mathcal{T}_{\bar{a}}^R] , \tag{2.30b}$$

$$F_2^V(q^2)_{LL,RR} = (m_1+m_2)^2[\mathcal{T}_a^L C_{2,a}^{LL,RR}\mathcal{T}_{\bar{a}}^L + \mathcal{T}_a^R C_{2,a}^{LL,RR}\mathcal{T}_{\bar{a}}^R] , \tag{2.31a}$$

$$F_2^A(q^2)_{LL,RR} = (m_1^2-m_2^2)[\mathcal{T}_a^L C_{2,a}^{LL,RR}\mathcal{T}_{\bar{a}}^L - \mathcal{T}_a^R C_{3,a}^{LL,RR}\mathcal{T}_{\bar{a}}^R] , \tag{2.31b}$$

$$F_3^V(q^2)_{LL,RR} = (m_1-m_2)[\mathcal{T}_a^L C_{3,a}^{LL,RR}\mathcal{T}_{\bar{a}}^L + \mathcal{T}_a^R C_{3,a}^{LL,RR}\mathcal{T}_{\bar{a}}^R] , \tag{2.32a}$$

$$F_3^A(q^2)_{LL,RR} = (m_1+m_2)[\mathcal{T}_a^L C_{3,a}^{LL,RR}\mathcal{T}_{\bar{a}}^L - \mathcal{T}_a^R C_{3,a}^{LL,RR}\mathcal{T}_{\bar{a}}^R] , \tag{2.32b}$$

$$F_2^V(q^2)_{LR,RL} = (m_1+m_2)[\mathcal{T}_a^L C_{2,a}^{LR,RL}M_D\mathcal{T}_{\bar{a}}^R + \mathcal{T}_a^R C_{2,a}^{LR,RL}M_D\mathcal{T}_{\bar{a}}^L] , \tag{2.33a}$$

$$F_2^A(q^2)_{LR,RL} = (m_1+m_2)[\mathcal{T}_a^L C_{2,a}^{LR,RL}M_D\mathcal{T}_{\bar{a}}^R - \mathcal{T}_a^R C_{2,a}^{LR,RL}M_D\mathcal{T}_{\bar{a}}^L] , \tag{2.33b}$$

$$F_1^{V,A}(q^2)_{LR,RL} \simeq 0 , \tag{2.34}$$

$$F_3^{V,A}(q^2)_{LR,RL} \simeq 0 . \tag{2.35}$$

In Eqs. (2.30)–(2.33) $C_{2,a}$ and $C_{3,a}$ are real $n\times n$ diagonal matrices, the values of which will be given below. In the approximation in which the external fermion masses are much smaller than the internal virtual fermion masses, the C matrices are independent of the external lepton masses. Thus the symmetry under the interchange $m_1 \leftrightarrow m_2$ is manifest in the above equations. One can observe that, under a parity transformation, the vector form factors are symmetric and the axial-vector form factors are antisymmetric, as they must be.

In Eq. (2.35), $F_3^{V,A}(q^2)_{LR,RL}$ actually has the form, suppressing $\mathcal{T}_\pm^{L,R}$ matrices,

$$F_3^{V,A}(q^2)_{LR,RL} \sim \frac{eG_F m_F}{8\pi^2}\left[0+\mathcal{O}\left(\frac{m_{1,2}^{\ 2}}{m_V^{\ 2}}\right)\right] , \tag{2.36}$$

where m_F denotes the mass of a virtual fermion in Fig. 1, and $m_V = m_W$ or m_Z. In contrast, from Eqs. (2.33a) and (2.33b), (2.44)–(2.47), (2.50), (2.54), and (2.57), it follows that

$$F_2^{V,A}(q^2)_{LR,RL} \sim \frac{eG_F m_F(m_1+m_2)}{8\pi^2}\left[a+b\left(\frac{m_F^{\ 2}}{m_V^{\ 2}}\right)\right] , \tag{2.37}$$

where a is of order unity (and b is either of order unity or $\ln(m_W^2/m_F^2)$, depending on the model). Hence the contribution of $F_3^{V,A}(q^2)q_\mu \sim (m_1+m_2) \times F_3^{V,A}(q^2)$ is negligible for both diagonal and nondiagonal transitions, being smaller than that of $F_2^{V,A}(q^2)$ by the factor $(m_{1,2}^{\ 2}/m_V^2)$. A similar remark applies to Eq. (2.34).

With the weak matrix structure of the form factors thus determined, we next consider the restrictions on the amplitude for $f_1 \to f_2+\gamma$ arising from the Hermiticity of the Lagrangian and from time-reversal invariance. This amplitude constitutes an effective proper $f_1 f_2 \gamma$ vertex

$$\mathcal{L}_{\text{eff}} = \bar{\psi}\Gamma_\mu \psi A^\mu \tag{2.38}$$

(where ψ is a tensor product of a Dirac field and an n-dimensional vector of all flavors of quark or lepton fields).

In momentum space this vertex is

$$\bar{u}(p_2)\Gamma_\mu(p_1,p_2;q)u(p_1)$$

and the Hermiticity of the Lagrangian implies that

$$\Gamma_\mu(p_1,p_2;q) = \gamma_0\Gamma^\dagger(p_2,p_1;-q)\gamma_0 . \tag{2.39}$$

Equation (2.39) requires that the total form factors satisfy the following relations:

$$F_1^{V,A}(m_1,m_2) = F_1^{V,A}(m_2,m_1)^\dagger , \tag{2.40}$$

$$F_2^V(m_1,m_2) = F_2^V(m_2,m_1)^\dagger , \tag{2.41}$$

$$F_2^A(m_1,m_2) = -F_2^A(m_2,m_1)^\dagger , \tag{2.42}$$

$$F_3^V(m_1,m_2) = -F_3^V(m_2,m_1)^\dagger , \tag{2.43}$$

$$F_3^A(m_1,m_2) = F_3^A(m_2,m_1)^\dagger . \tag{2.44}$$

One can easily check that these conditions are satisfied by Eqs. (2.25) and (2.30)–(2.35), given that with our approximations the C matrices are independent of m_1 and m_2. Note that the current conservation equations, (2.22a) and (2.22b) are consistent with these Hermiticity relations. In the (CP-invariant) diagonal case, Eq. (2.43) implies that $F_3^V(m_1=m_2)=0$ [also implied, for $q^2\neq 0$, by Eq. (2.22a)], and Eq. (2.42) yields

$$F_2^A(m_1=m_2) = 0 . \tag{2.45}$$

Finally, CP invariance implies that the F_j matrices satisfy the relations[18]

$$F_j^{V,A}(m_1,m_2) = F_j^{V,A}(m_2,m_1)^* , \quad j=1,2,3 . \tag{2.46}$$

The C matrices may be conveniently written in the form (suppressing nonmatrix indices)

$$C_{ij}=\frac{eg^2}{32\pi^2 m_W^2}C_i\delta_{ij} \tag{2.47}$$

(with no sum on i). For the $W^\pm$ graphs we use the relation, in the Weinberg-Salam model,[19]

$$\frac{g^2}{8m_W^2}=\frac{G_F}{\sqrt{2}}\ . \tag{2.48}$$

For the Z graph we make use of the relation

$$\frac{g^2+g'^2}{8m_Z^2}=\kappa\,\frac{g^2}{8m_W^2} \tag{2.49}$$

and define C_{ij} similarly to Eq. (2.47), with

$$\kappa=\frac{\frac{1}{2}\sum(T^2-T_3^2+T)\lambda_{T,T_3}^2}{\sum(T_3)^2\lambda_{T,T_3}^2}\ . \tag{2.50}$$

In these equations g and g' are the coupling constants for the SU(2) and U(1) subgroups of SU(2) $\times$ U(1), respectively; λ_{T,T_3} is the vacuum expectation value of (the neutral member of) a Higgs multiplet with weak isospin T, and the sums in Eq. (2.50) are over all Higgs multiplets. For the $W^\pm$ terms obviously

$$(C_{k,\pm})^{LL,RR}=(C_{k,\pm}^{1(a)+1(b)})^{LL,RR} \tag{2.51}$$

for $k=2,3$, and similarly for the LR,RL part. From our present and past calculations[2,20,21] we find that the diagram of Fig. 1(a) yields

$$(C_{2,\pm}^{1(a)})_i^{LL,RR}=(Q_1-Q_{F_i})(\tfrac{5}{6}-\tfrac{1}{4}\epsilon_i)\ , \tag{2.52}$$

$$(C_{2,\pm}^{1(a)})_i^{LR,RL}=(Q_1-Q_{F_i})(-2+\tfrac{3}{2}\epsilon_i)\ , \tag{2.53}$$

$$(C_{3,\pm}^{1(a)})_i^{LL,RR}=(Q_1-Q_{F_i})\left[-\tfrac{35}{12}-\tfrac{4}{3}\ln\xi+\frac{1}{4\xi}+\epsilon_i(-\tfrac{1}{2}+\tfrac{1}{4}\ln\xi)\right]\ , \tag{2.54}$$

while diagram 1(b) gives

$$(C_{2,\pm}^{1(b)})_i^{LL,RR}=Q_{F_i}(-\tfrac{2}{3}+\tfrac{1}{2}\epsilon_i)\ , \tag{2.55}$$

$$(C_{2,\pm}^{1(b)})_i^{LR,RL}=Q_{F_i}\left[2+\epsilon_i\left(-4\ \ln\frac{1}{\epsilon_i}+6\right)\right]\ , \tag{2.56}$$

$$(C_{3,\pm}^{1(b)})_i^{LL,RR}=Q_{F_i}\left(\tfrac{2}{3}\ln\frac{1}{\epsilon_i}+\tfrac{1}{9}+\tfrac{3}{2}\epsilon_i\right)\ . \tag{2.57}$$

In these equations Q_1 and Q_{F_i} are the charges of the initial and ith virtual fermion, and

$$\epsilon_i=\frac{m_{F_i}^2}{m_W^2}\ . \tag{2.58}$$

We recall that these results apply in the limit $\xi\to 0$. As is expected, $F_3(q^2)$ is a non-Abelian gauge-dependent quantity; in Sec. IV, it will be shown that it has precisely the correct ξ dependence to cancel the ξ dependence of the Z and W^+W^- exchange contribution to $\mu\to ee\bar{e}$ and yield a ξ-independent S-matrix element for this process.

The Z graph 1(c) contributions are obtained by simply multiplying those of graph 1(b) by κ and replacing m_W^2 by m_Z^2 in Eq. (2.58):

$$(C_{2,Z}^{1(c)})_i^{LL,RR}=\kappa Q_{F_i}(-\tfrac{2}{3}+\tfrac{1}{2}\delta_i) \tag{2.59}$$

$$(C_{2,Z}^{1(c)})_i^{LR,RL}=\kappa Q_{F_i}\left[2+\delta_i\left(-4\ \ln\frac{1}{\delta_i}+6\right)\right]\ , \tag{2.60}$$

$$(C_{3,Z}^{1(c)})_i^{LL,RR}=\kappa Q_{F_i}\left(\tfrac{2}{3}\ln\frac{1}{\delta_i}+\tfrac{1}{9}+\tfrac{3}{2}\delta_i\right)\ , \tag{2.61}$$

where

$$\delta_i=\frac{m_{F_i}^2}{m_Z^2}\ . \tag{2.62}$$

The invariant matrix element for the radiative decay $f_1\to f_2+\gamma$ is given by

$$i\mathfrak{M}(f_1(p_1)\to f_2(p_2)+\gamma(q))=\bar{u}_2(p_2)\,\frac{i\sigma_{\mu\nu}q^\nu\epsilon^\mu}{(m_2+m_1)}\,(F_2^V(0)_{21}+F_2^A(0)_{21}\gamma_5)u_1(p_1)\ , \tag{2.63}$$

where for notational convenience we have separated the Dirac and weak-gauge-group matrix structures, defining

$$F_i^{V,A}(q^2)_{ba}=\langle f_b|F_i^{V,A}(q^2)|f_a\rangle\ . \tag{2.64}$$

The rate is then

$$\Gamma(f_1\to f_2+\gamma)=\frac{m_1}{8\pi}\left(1-\frac{m_2}{m_1}\right)^2\left(1-\frac{m_2^2}{m_1^2}\right)\times[|F_2^V(0)_{21}|^2+|F_2^A(0)_{21}|^2]\ . \tag{2.65}$$

From these results we can immediately derive several constraints on models of weak interactions. We concentrate on the leptonic sector here, since the quark sector has been discussed before.[2,3] First, the existing experimental bound on the branching ratio for the decay $\mu\to ee\bar{e}$, viz., $\mathrm{BR}(\mu\to ee\bar{e})<6\times10^{-9}$, prohibits a direct lowest-order $Z\mu e$ coupling in the Lagrangian. For if there were such a coupling this decay would proceed at a rate comparable to that of the decay $\mu\to e\bar{\nu}_e\nu_\mu$ unless the mixing of the muon and electron were unnaturally small. But if $\mathcal{T}_Z^L$ and $\mathcal{T}_Z^R$ are to be diagonal in the μ-e subspace for an *arbitrary* mass matrix M and hence *arbitrary* U_L and U_R, they must in fact be multiples of the unit matrix in the entire $Q=-1$ subspace. That is, all left-handed leptons of charge $Q=-1$ must have the same value of weak $\mathcal{T}_3^L$ and similarly for right-handed leptons of charge $Q=-1$. As a consequence the Z graph 1(c) does not contribute to nondiagonal electromagnetic leptonic transitions. Parenthetically, it may be mentioned that, in the hadronic case, the smallness of the $K_L K_S$ mass difference and of the rate for the decay $K_L \to \mu\bar{\mu}$ similarly prohibits a lowest-order nondiagonal Zsd vertex. By the argument given above, the absence of such a vertex can be guaranteed naturally only if all quarks of a given charge and chirality have the same value of weak T_3.

Let us consider next the W graphs 1(a) and 1(b). For the $\mu \to e\gamma$ decay, the LL, RR part of these graphs will, in general, give a contribution to $F_2^{V,A}(0)$ of order $eG_F m_\mu{}^2/(8\pi^2)$, while the LR, RL part will give a contribution of order $eG_F m_L m_\mu/(8\pi^2)$, where again, m_L denotes the mass of a generic internal virtual lepton (specifically, the maximum mass in the case of widely disparate masses) in Figs. 1(a), and 1(b). Thus the LL, RR part will produce a $\mu \to e\gamma$ branching ratio of order (α/π), while the LR, RL part, if present, will yield an even larger rate of the order $(\alpha/\pi)(m_L/m_\mu)^2$. We must therefore require that the corresponding matrix elements $\langle e|\mathcal{T}_\pm^L C_{2,\pm}^{LL,RR}\mathcal{T}_\mp^L|\mu\rangle$, $\langle e|\mathcal{T}_\pm^R C_{2,\pm}^{LL,RR}\times\mathcal{T}_\mp^R|\mu\rangle$, $\langle e|\mathcal{T}_\pm^L C_{2,\pm}^{LR,RL} M_D \mathcal{T}_\mp^R|\mu\rangle$, and $\langle e|\mathcal{T}_\pm^R C_{2,\pm}^{LR,RL}\times M_D\mathcal{T}_\mp^L|\mu\rangle$ all vanish to leading order. Here, in contrast to Eqs. (2.30)–(2.33), there is no implied sum over both upper and lower signs, i.e., each matrix element must vanish for each sign choice. Now to leading order, the matrices $C_{2,\pm}^{LL,RR}$ are proportional to the identity matrix in each charge sector. Therefore, for upper and lower sign choices individually, the LL, RR matrix elements can be written

$$\langle e|\mathcal{T}_\pm^L C_{2,\pm}^{LL,RR}\mathcal{T}_\mp^L|\mu\rangle \propto \langle e|\mathcal{T}_\pm^L\mathcal{T}_\mp^L|\mu\rangle = \tfrac{1}{2}\langle e|[(\vec{\mathcal{T}}^L)^2 - (\mathcal{T}_3^L)^2 \pm \mathcal{T}_3^L]|\mu\rangle . \quad (2.66)$$

In order for this matrix element to vanish, in general, $(\vec{\mathcal{T}}^L)^2$ and $\mathcal{T}_3^L$ must be diagonal in the μ-e subspace. Again, if this is to be true for an arbitrary beginning mass matrix M, $(\vec{\mathcal{T}}^L)^2$ and $\mathcal{T}_3^L$ must actually be multiples of the identity in the $Q=-1$ charge sector. The same argument and conclusion apply to $(\vec{\mathcal{T}}^R)^2$ and $\mathcal{T}_3^R$. We thus find from the requirement that the LL, RR Wgraph contributions do not give too large a branching ratio for the decay $\mu \to e\gamma$ that leptons of charge $Q=-1$ and a given chirality must have the same value of weak T and T_3. As mentioned before, this is in fact precisely the condition on lepton representation content which guarantees natural conservation of $Q=-1$ leptonic flavors by the weak neutral current.

Proceeding to the analysis of the LR, RL matrix elements in the W graph contributions to $\mu \to e\gamma$, we note that, as in the LL, RR case, to leading order, $C_{2,\pm}^{LR,RL}$ are proportional to the identity matrix in each charge subspace. Consequently we require that for each sign choice individually

$$\langle e|\mathcal{T}_\pm^L M_D\mathcal{T}_\mp^R|\mu\rangle = \langle e|\mathcal{T}_\pm^R M_D \mathcal{T}_\mp^L|\mu\rangle = 0 . \quad (2.67)$$

This implies, by the same reasoning as was given before, that

$$\langle l_2(Q=-1)|\mathcal{T}_\pm^L M_D\mathcal{T}_\mp^R|l_1(Q=-1)\rangle = \langle l_2(Q=-1)|\mathcal{T}_\pm^R M_D\mathcal{T}_\mp^L|l_1(Q=-1)\rangle = 0 , \quad (2.68)$$

where $l_{1,2}$ are arbitrary leptons.

This is thus a special case of the second condition for microweak CP violation discussed elsewhere (for quarks).[3] In words, this condition, for leptons, is that, for at least one chirality, leptons of charge q and $q\pm 1$ do not belong to the same weak isomultiplet. In the specific case of the μe matrix element it is necessary that leptons of charge $Q=0$ and $Q=-2$ do not belong to the same weak isomultiplet as electrons and muons, for at least one chirality.

In passing, we mention that the LR, RL contribution in the diagonal case is also larger than that of the LL, RR terms, at least in the experimentally interesting case in which $m_1{}^2 (=m_2{}^2) \ll m_L{}^2$. For example, in the Georgi-Glashow model such an LR, RL contribution is present and yields an anomalous magnetic moment proportional to a heavy lepton mass, as in Eqs. (2.33a) and (2.33b):

$$F_2(0)_{GG} \propto G_F m_{M^0} m_\mu . \quad (2.69)$$

In contrast, in the WS model, with only LL, RR contributions,

$$F_2(0)_{WS} \propto G_F m_\mu{}^2 . \quad (2.70)$$

The large LR contribution in the Georgi-Glashow model was in fact used to set a phenomenologically important upper bound on the heavy lepton mass m_{M^0}.[13]

Finally, consider the C_3 matrices, which do not contribute to $\mu \to e\gamma$ but do to $\mu \to ee\bar{e}$. The same conditions which guarantee that the $\mu \to e\gamma$ branching ratio is not too large also ensure that graph 1(a) gives a reasonable contribution. However, this is not true of diagram 1(b) becuase of the presence of the $\ln(1/\epsilon_i)$ in $(C_{3,+}^{1(b)})^{LL,RR}$, as given in Eq. (2.57).[22] Assuming that the other conditions are satisfied, in particular the condition that all leptons of charge $Q=-1$ have the same weak T and T_3, this $\ln(1/\epsilon_i)$ will produce a $\mathrm{BR}(\mu \to ee\bar{e}) \sim (\alpha/\pi)^2\epsilon \ln^2(m_{L_2}{}^2/m_{L_1}{}^2)$, where $L_{1,2}^{--}$ are generic doubly charged leptons, and ϵ is a product of mixing angles. Unless m_{L_2} and m_{L_1} are very close to being degenerate and/or the mixing angle(s) is very small, such a branching ratio would conflict with the present experimental limit. In view of this, it is therefore desirable to avoid the possibility of an LL, RR contribution from diagram 1(b). This requires, in particular, that there be no doubly negatively charged leptons which communicate with e and μ in such a way as to yield nonzero matrix elements $\langle e|\mathcal{T}_+^L C_{3,+}^{LL,RR}\mathcal{T}_-^L|\mu\rangle$ or $\langle e|\mathcal{T}_+^R C_{3,+}^{LL,RR}\mathcal{T}_-^R|\mu\rangle$.

III. GAUGE MODELS

In order to illustrate how certain features of μ- and e-type lepton-number nonconservation depend on characteristics of the underlying gauge theory, we shall apply the general formulas derived in the preceding section to a variety of $SU(2)\times U(1)$ gauge models. Subsequently, we shall focus upon the $V-A$ six-quark KM model. The leptonic representation content of these models is depicted in Table I. In this table,

$$R_2(\theta)=\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix} \tag{3.1}$$

and $\mathfrak{U}$ is a 3×3 unitary matrix (which depends on four parameters; see below).

Model (a) is the original Weinberg-Salam model of leptons,[5] generalized to allow for nonzero neutrino mass. Given that the neutrino mass eigenstates ν_1 and ν_2 are nondegenerate, they will mix by an arbitrary rotation R_2 through an angle θ to form the weak eigenstates ν_e and ν_μ. (The spinors (ν_e,ν_μ), (ν_1,ν_2) in Table I(a) are used solely for notational convenience and do not represent doublets of any group.) We include model (a) only to illustrate the fact that with the present experimental upper limits on the neutrino masses, it cannot account for a branching ratio for $\mu\to e\gamma$ larger than $\sim 10^{-45}$. Henceforth, we shall accordingly restrict two of the neutral leptons which couple to e_L and μ_L to be massless by decreeing that they have no right-handed components.

The rest of the models include heavy leptons. Model (b), the KM model,[11] is obtained by expanding the minimal model with the addition of neutral and singly charged heavy leptons L^0 and L^-, as shown in Table I. The left-handed component of L^0 is distributed among the left-handed neutral weak eigenstates via the unitary mapping $\mathfrak{U}$. One merit of this model is that it would incorporate in a natural way the heavy lepton discovered at SPEAR[23] and corroborated by DESY[23] if in fact this heavy lepton is found to decay via a V-A current. Lepton-quark universality and the cancellation of anomalies are easily achieved by postulating that the hadronic sector also consists of three left-handed doublets, $(u_d,d)_L$, $(u_s,s)_L$, and $(u_b,b)_L$, where

$$\begin{pmatrix}(u_d)_L\\(u_s)_L\\(u_b)_L\end{pmatrix}=\mathfrak{V}\begin{pmatrix}u_L\\c_L\\t_L\end{pmatrix} \tag{3.2}$$

with $\mathfrak{V}$ a unitary matrix. With such a quark sector, however, the KM model cannot account for the anomalous y distribution in inclusive antineutrino-nucleon scattering reported by the Harvard-Pennsylvania-Wisconsin-Fermilab experiment[24] and recently supported by the Caltech-Fermilab experiment.[25,26] (See, however, the note added in proof, Ref. 26.)

The fact that $\mathfrak{U}$ is unitary is equivalent to the fact

TABLE I. Lepton multiplets in the models considered. See the text for an explanation of the R_2 and $\mathfrak{U}$ matrices. The mixing of L_{1R}^{--} and L_{2R}^{--} to form L_{eR}^{--} and $L_{\mu R}^{--}$ in model (d) is determined by $R_2(\theta)$. Similarly, in model (e) the mixings of L_{1L}^{--} and L_{2L}^{--} to form L_{eL}^{--} and $L_{\mu L}^{--}$ and of L_{1R}^{--} and L_{2R}^{--} to form $L_{eR}'^{--}$ and $L_{\mu R}'^{--}$ are determined respectively by $R_2(\theta_L)$ and $R_2(\theta_R)$. These mixing formulas are omitted for brevity. See the text for the mixing in model (f).

(a) $\begin{pmatrix}\nu_e\\e\end{pmatrix}_L\begin{pmatrix}\nu_\mu\\\mu\end{pmatrix}_L,\quad \begin{matrix}\nu_{1R} & \nu_{2R}\\ e_R & \mu_R\end{matrix}$

where

$$\begin{pmatrix}\nu_e\\\nu_\mu\end{pmatrix}_L=R_2(\theta)\begin{pmatrix}\nu_1\\\nu_2\end{pmatrix}_L$$

(b) $\begin{pmatrix}N_e\\e\end{pmatrix}_L\begin{pmatrix}N_\mu\\\mu\end{pmatrix}_L\begin{pmatrix}N_L\\L^-\end{pmatrix}_L\ e_R\ \mu_R\ \begin{matrix}L_R^0\\L_R\end{matrix}$

where

$$\begin{pmatrix}(N_e)_L\\(N_\mu)_L\\(N_L)\end{pmatrix}=\mathfrak{U}\begin{pmatrix}(\nu_1)_L\\(\nu_2)_L\\(L^0)_L\end{pmatrix}.$$

(c) $\begin{pmatrix}\nu_e\\e\\L_e^{--}\end{pmatrix}_L\begin{pmatrix}\nu_\mu\\\mu\\L_\mu^{--}\end{pmatrix}_L\quad \begin{matrix}e_R & \mu_R\\ L_{1R}^{--} & L_{2R}^{--}\end{matrix}$

where

$$\begin{pmatrix}(L_e^{--})_L\\(L_\mu^{--})_L\end{pmatrix}=R_2(\theta)\begin{pmatrix}(L_1^{--})_L\\(L_2^{--})_L\end{pmatrix}.$$

(d) $\begin{matrix}\begin{pmatrix}\nu_e\\e\end{pmatrix}_L & \begin{pmatrix}\nu_\mu\\\mu\end{pmatrix}_L\\ L_{1L}^{--} & L_{2L}^{--}\end{matrix}\quad\begin{pmatrix}e\\L_e^{--}\end{pmatrix}_R\begin{pmatrix}\mu\\L_\mu^{--}\end{pmatrix}_R.$

(e) $\begin{pmatrix}\nu_e\\e\\L_e^{--}\end{pmatrix}_L\begin{pmatrix}\nu_\mu\\\mu\\L_\mu^{--}\end{pmatrix}_L\begin{pmatrix}e\\L_e'^{--}\end{pmatrix}_R\begin{pmatrix}\mu\\L_\mu'^{--}\end{pmatrix}_R.$

(f) $N_L'N_L''\ \begin{pmatrix}N_e'\\e\end{pmatrix}_L\begin{pmatrix}N_\mu'\\\mu\end{pmatrix}_L,\quad\begin{pmatrix}N_e\\e\end{pmatrix}_R\begin{pmatrix}N_\mu\\\mu\end{pmatrix}_R.$

that the model has natural flavor conservation by the neutral current. This is clear since the first condition for flavor conservation by J^{μ}_{Z} is true if and only if the matrices U_L and U_R in Eq. (2.9) are unitary not just *in toto*, but also when restricted to any (T, T_3, Q) subspace. But $\mathfrak{U}$ is precisely the restriction of the matrix $U_L^{\dagger}$ to the $Q=0$ subspace, and so the equivalence asserted above is proved. (A similar remark applies to the unitarity of $\mathfrak{V}$.) Allowing for redefinition of the phases of fermion fields and the extraction of a single overall unobservable phase, the matrix $\mathfrak{U}$ depends on four parameters. Since a 3×3 orthogonal matrix only depends on three parameters this leaves an additional one, which we can choose as a (*CP*-violating) phase. For generality we shall denote the elements of $\mathfrak{U}$ by $\mathfrak{U}_{ij}$ and avoid using any explicit trigonometric forms.

As in the WS model with massless quarks we shall denote the normalized linear combination of the zero-mass physical fields ν_1 and ν_2 which couple to e and μ by ν_e and ν_μ:

$$\nu_e=\frac{\mathfrak{U}_{11}\nu_1+\mathfrak{U}_{12}\nu_2}{(|\mathfrak{U}_{11}|^2+|\mathfrak{U}_{12}|^2)^{1/2}}, \tag{3.3a}$$

$$\nu_\mu=\frac{\mathfrak{U}_{21}\nu_1+\mathfrak{U}_{22}\nu_2}{(|\mathfrak{U}_{21}|^2+|\mathfrak{U}_{22}|^2)^{1/2}}. \tag{3.3b}$$

There are several important features to note here. First, in the WS model, with either massless or massive neutrinos, the $T=\frac{1}{2}$, $T_3=\frac{1}{2}$ weak gauge group eigenstates, which in that case are ν_e and ν_μ, are orthonormal. The analogous $T=\frac{1}{2}$, $T_3=\frac{1}{2}$ weak eigenstates N_e, N_μ, and N_L in the KM model are similarly orthonormal, so that $\mathfrak{U}$ is unitary. However, the zero-mass linear combinations $\mathfrak{U}_{j1}\nu_1+\mathfrak{U}_{j2}\nu_2$, $j=1,2$, which couple to e and μ, respectively, are neither of unit norm nor orthogonal. Of course there is no reason why they should be orthonormal, since they do not constitute complete weak-gauge-group eigenstates.[27] With our definitions (3.3a) and (3.3b) of ν_e and ν_μ the deviation from unit norm is rendered explicit; for example the $We\nu_e$ vertex is

$$\left(-\frac{ig}{\sqrt{2}}\right)(|U_{11}|^2+|U_{12}|^2)^{1/2}(\bar{e}\gamma_\alpha L\nu_e W^{-\alpha}+\text{H.c.}).$$

The inner product of ν_e and ν_μ is

$$\langle\nu_e|\nu_\mu\rangle=\frac{-\mathfrak{U}_{13}^{*}\mathfrak{U}_{23}}{(|\mathfrak{U}_{11}|^2+|\mathfrak{U}_{12}|^2)^{1/2}(|\mathfrak{U}_{21}|^2+|\mathfrak{U}_{22}|^2)^{1/2}}; \tag{3.3c}$$

where we have used the fact that $\sum_j \mathfrak{U}_{1j}^{*}\mathfrak{U}_{2j}=0$. Since the couplings of e to ν_e and μ to ν_μ are not in general equal, the KM model violates $\mu-e$ universality. Moreover, in consequence of the fact that ν_e and ν_μ are not orthogonal, there is neutrino mixing. However, since the neutrinos are degenerate this mixing is not time-dependent, i.e., there are no neutrino oscillations.[28] In Sec. V the phenomenological implications of the KM model will be examined in greater detail. Here we shall simply observe that there are three important experimental constraints on the amount of mixing which can be allowed in the KM model. First, one must respect $e\mu$ universality, i.e., the equality of $g_{We\nu_e}{}^2$ and $g_{W\mu\nu_\mu}{}^2$, where $g_{We\nu_e}$ and $g_{W\mu\nu_\mu}$ are the coupling constants for the $We\nu_e$ and $W\mu\nu_\mu$ vertices. The most accurate test of this comes from a comparison of the rates for the decays $\pi\to e\bar{\nu}_e$ and $\pi\to\mu\bar{\nu}_\mu$.[29] Second, there is the requirement that $g_{Wdu}{}^2\sec^2\theta_C=g_{W\mu\nu_\mu}{}^2$ as determined by measurement of the rate of β decay versus μ decay and by the measurement of the Cabibbo angle from hyperon decay or $K\to\mu\bar{\nu}_\mu$ versus $\pi\to\mu\bar{\nu}_\mu$. Finally, the amount of nonorthogonality between ν_e and ν_μ is restricted by the nonobservation of e^- from an incident beam of ν_μ scattering off nucleons.[30] As will be shown in Sec. V, these constraints allow mixings

$$|\mathfrak{U}_{13}|\sim|\mathfrak{U}_{23}|\lesssim 0.22\ .$$

The next model (c) proposed recently by Wilczek and Zee (WZ)[31] contains two left-handed triplets in which the $T_3=-1$ states L_{eL}^{--} and $L_{\mu L}^{--}$ are mixtures of the heavy leptons states L_{1L}^{--} and L_{2L}^{--}. For the cancellation of anomalies (between quark and lepton sectors) it is necessary to postulate new quarks. However, since these quarks are a peripheral issue here we shall not discuss them in detail. On another matter there is a more immediate requirement; the experimental fact that to the precision of ~one percent, the coupling constant for the $\bar{u}dW$ vertex, multiplied by $\sec\theta_C$, is equal to that for the $\bar{\nu}_\mu\mu W$ vertex. Because model (c) places the left-handed leptons in triplets rather than doublets, the $\bar{\nu}_e eW$ vertex increases in strength from g to $\sqrt{2}\,g$. Of course this just induces a trivial redefinition of g^2 in terms of $G_F m_W{}^2$, but it has the nontrivial effect of forcing the u_L and d_L quarks to be placed in a triplet to retain quark-lepton universality for the weak coupling constant. Furthermore, in order to maintain the naturality of the hadronic sector, this also forces the c and s quarks to be placed in a triplet.

The WZ model (d)[31] is similar to model (c) in that it adds two new L^{--} leptons to serve as intermediate states for the μ-e transition. However, in contrast with model (c), it places these in two right-handed doublets, with their weak hypercharge shifted down by two units in order to ensure naturalness. This is the leptonic analogue of a model for the hadron sector discussed recently in connection with CP violation[3,6] and nonscaling anomalies in (anti) neutrino scattering.[26] Model (e) is included to illustrate the effect of an unsuppressed μ-e transition of the form $\mu_L \to (L_{1,2}^{--})_L$; $(L_{1,2}^{--})_R \to e_R$ or $\mu_R \to (L_{1,2}^{--})_R$; $(L_{1,2}^{--})_L \to e_L$. We will find that such a left-right transition gives much too large a rate for the decay $\mu \to e\gamma$.

Model (f) is an interesting model proposed recently by Cheng and Li,[10] and further discussed by a number of authors,[32] involving two neutral heavy leptons. The right-handed components are arranged in two doublets with

$$\begin{pmatrix} (N_e)_R \\ (N_\mu)_R \end{pmatrix} = R_2(\theta) \begin{pmatrix} (N_1)_R \\ (N_2)_R \end{pmatrix}. \tag{3.4}$$

The left-handed components of N_1 and N_2 are distributed among two singlets N_L'' and N_L''', and the $T_3 = +\frac{1}{2}$ members of two doublets, N_{eL}' and $N_{\mu L}'$.

All of these models except (f) satisfy both conditions for natural leptonic-flavor conservation by the weak neutral current. The fact that models (a)–(c) meet the first condition is obvious from Table I; to show the validity of the second requires a description of the Higgs content in each model. Before doing this, we note that, as is again evident from Table I, all the models except (e) and (f) satisfy the condition that leptons of charge q and $q \pm 1$ do not belong to the same weak isomultiplet for at least one chirality. In model (a) one complex doublet $\binom{\phi^+}{\phi^0}$ suffices to give mass to e and μ. This doublet is sufficient in model (b) to give mass to e, μ, and L; another, $\binom{\phi'^0}{\phi'^-}$, is needed to produce a mass for L^0. It is economical, but not necessary, to take this second doublet as the SU(2)-transformed charge conjugate of the first:

$$\begin{pmatrix} \phi'^0 \\ \phi'^- \end{pmatrix} = i\sigma_2 \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}^* = \begin{pmatrix} \bar\phi^0 \\ -\phi^- \end{pmatrix}. \tag{3.5}$$

Model (c) requires two Higgs triplets,

$$\begin{pmatrix} \phi^+ \\ \phi^0 \\ \phi^- \end{pmatrix},$$

for the e and μ masses, and

$$\begin{pmatrix} \chi^{++} \\ \chi^+ \\ \chi^0 \end{pmatrix}$$

for the L_1^{--} and L_2^{--} masses. In model (d) we need a slightly more complicated Higgs structure, $\binom{\phi^+}{\phi^0}$, for the $L_{1,2}^{--}$ masses and a triplet, which can be written as in a traceless $(\frac{1}{2}, \frac{1}{2})$ matrix representation

$$\begin{bmatrix} H^+/\sqrt{2} & H^{++} \\ H^0 & -H^+/\sqrt{2} \end{bmatrix}$$

for the e and μ masses. The Higgs content of model (e) consists of a triplet and doublet; although there are neutral Higgs fields contained in both of these multiplets, they couple to leptons of different charge.

The Cheng-Li (CL) model (f) fails to satisfy the first condition for the neutral ($Q=0$) sector, the second condition for the lepton mass term, and the last condition; transitions $e, \mu \leftrightarrow N_e, N_\mu$ can take place through both chiralities. Nevertheless, this model, in the form presented, is adequate to suppress e-and μ-lepton-number nonconservation to the desired level, thanks to a special property of the Higgs system. Specifically, the neutral leptons receive their masses both from their couplings to a Higgs doublet $\binom{\phi^0}{\phi^-}$,

$$\frac{m_1}{v}\bar N_L''(\bar\phi^0, \phi^+)\left[\cos\theta \begin{pmatrix} N_e \\ e \end{pmatrix}_R - \sin\theta \begin{pmatrix} N_\mu \\ \mu \end{pmatrix}_R\right],$$

$$+\frac{m_2}{v}\bar N_L'''(\bar\phi^0, \phi^+)\left[+\sin\theta \begin{pmatrix} N_e \\ e \end{pmatrix}_R + \cos\theta \begin{pmatrix} N_\mu \\ \mu \end{pmatrix}_R\right],$$

and from the gauge-invariant bare mass term

$$\left[m_e(\bar N_e', \bar e)_L \begin{pmatrix} N_e \\ e \end{pmatrix}_R + m_\mu(\bar N_\mu', \bar\mu)_L \begin{pmatrix} N_\mu \\ \mu \end{pmatrix}_R\right].$$

However, the crucial point is that only *one* Higgs doublet is involved in giving masses to the leptons; as a consequence, there is a natural zeroth-order relation among the lepton masses and the mixing of the neutral leptons. This relation guarantees that the neutral current is diagonal in $Q=-1$ leptonic flavors down to the level $G_F\alpha$, as in a fully natural model. Note that if one started with more than one doublet of Higgs, one could always redefine the Higgs doublets so that one and only one picks up a vacuum expectation value and hence is involved with giving mass to the leptons. If, however, the Higgs representation content were expanded to include triplets or higher-dimensional multiplets, then the $Q=0$ leptons would indeed receive their masses from couplings to more than one neutral Higgs field, and the zeroth-order relation constraining the mixing of neutral leptons would no longer be valid.

In this theory the four left-handed weak eigenstates N_{eL}', $N_{\mu L}'$, N_L'', and N_L''' are unitary transforms of the mass eigenstates ν_{1L}, ν_{2L}, N_{1L}, and N_{2L}, while the two neutral right-handed weak eigenstates N_{eR} and $N_{\mu R}$ are unitary transforms of N_{1R} and N_{2R}, as prescribed in Table I and Eq. (3.1). When one diagonalizes the mass matrix and expresses the neutral weak eigenstates in terms of mass eigenstates, one finds that the mixing is suppressed by the small factors m_e/m_L or m_μ/m_L and is just such that the leading induced LR,RL contribution to $F_2^{V,A}(0)$ vanishes. The remaining, subdominant LR,RL contribution, which will be given in the next section, is of the same order as the LL,RR part. One can easily see this without actually having to diagonalize the mass matrix and compute the mixing explicitly, by using the original weak eigenbasis and nondiagonal matrix M. Writing out the mass terms listed above, we have for the neutral lepton mass terms in the CL model,

$$\mathcal{L}_M^{\rm CL} = -(\overline{N}'_{eL}, \overline{N}'_{\mu L}, \overline{N}''_L, \overline{N}'''_L) M \begin{pmatrix} N_{eR} \\ N_{\mu R} \end{pmatrix} + \text{H.c.}, \quad (3.6)$$

where (with $s = \sin\theta$, $c = \cos\theta$)

$$M = \begin{bmatrix} m_e & 0 \\ 0 & m_\mu \\ m_1 c & -m_1 s \\ m_2 s & m_2 c \end{bmatrix}. \quad (3.7)$$

As mentioned before, to leading order, $C_2^{LR,RL}$ is proportional to the identity in each charge subspace. Moreover, in Eqs. (2.31a) and (2.31b) only the lower sign choice, $a = -$, contributes to the $e\mu$ matrix element of $F_2^{V,A}(0)$, corresponding to the fact that the only available intermediate lepton state has charge $Q = 0$, not $Q = -2$. Accordingly, reexpressing Eqs. (2.31a) and (2.31b) in terms of $T_\pm^{L,R}$ and M, and using the fact that

$$U_{L,R} |l\rangle = |l_{L,R}\rangle, \quad l = e, \mu \quad (3.8)$$

we have, for the dominant part of $(F_2^V(0))_{e\mu}^{LR,RL}$,

$$\begin{aligned}(F_2^V(0))_{e\mu}^{LR,RL} &= (F_{2,-}^V(0))_{e\mu}^{LR,RL} \\ &= \langle e_L | T_-^L M T_+^R | \mu_R \rangle + \langle e_R | T_-^R M^\dagger T_+^L | \mu_L \rangle \\ &= \tfrac{1}{2} \langle N'_{eL} | M | N_{\mu R} \rangle + \tfrac{1}{2} \langle N_{eR} | M^\dagger | N'_{\mu L} \rangle . \end{aligned} \quad (3.9)$$

But with M as given in Eq. (3.7),

$$\begin{aligned}\langle N'_{eL} | M | N_{\mu R} \rangle &= \langle N_{eR} | M^\dagger | N'_{\mu L} \rangle \\ &= 0 . \end{aligned} \quad (3.10)$$

The same argument applies to $(F_2^A(0))_{e\mu}^{LR,RL}$, and thus the dominant LR,RL contributions to $(F_2^{V,A}(0))_{e\mu}$ vanish, as asserted. If the leading LR, RL term in $F_2^{V,A}(0)$ had not vanished, it would have given much too large a branching ratio for $\mu \to e\gamma$. In passing, it may be recalled that the LR,RL transitions do not give a leading contribution to the nondiagonal charge radius which enters into the $\mu \to ee\bar{e}$ amplitude, as is evident from Eqs. (2.34) and (2.35).

The crucial property of M used in Eq. (3.10) is the fact that $M_{12} = 0$. Note that even if the bare mass terms had originally included cross terms of the form

$$a(\overline{N}'_e, \bar{e})_L \begin{pmatrix} N_\mu \\ \mu \end{pmatrix}_R + b(\overline{N}'_\mu, \bar{\mu})_L \begin{pmatrix} N_e \\ e \end{pmatrix}_R + \text{H.c.},$$

which would lead to nonzero M_{12} and M_{21} (as well as direct $e\mu$ mixing in the $Q = -1$ sector of the mass matrix), it would always be possible to eliminate such cross terms by unitary transformations of the left-handed doublets and, separately, of the two right-handed doublets. In contrast, however, if there are one or more Higgs triplets then there will in general be mass terms arising from Higgs couplings of the form

$$\frac{1}{v}(\overline{N}'_e, \bar{e})_L \vec{\tau} \cdot \vec{\Phi} \left[\alpha \begin{pmatrix} N_e \\ e \end{pmatrix}_R + \beta \begin{pmatrix} N_\mu \\ \mu \end{pmatrix}_R \right] + \frac{1}{v}(\overline{N}'_\mu, \bar{\mu})_L \vec{\tau} \cdot \vec{\Phi} \left[\gamma \begin{pmatrix} N_e \\ e \end{pmatrix}_R + \delta \begin{pmatrix} N_\mu \\ \mu \end{pmatrix}_R \right] + \text{H.c.},$$

where the Higgs triplet is written as a (traceless) matrix

$$\Phi = \begin{bmatrix} \Phi^0/\sqrt{2} & \Phi^+ \\ \Phi^- & -\Phi^0/\sqrt{2} \end{bmatrix} \quad (3.11)$$

and

$$\langle \Phi \rangle_0 = \begin{bmatrix} v & 0 \\ 0 & -v \end{bmatrix}. \quad (3.12)$$

In this case it would not be possible to eliminate the cross terms proportional to β and γ by allowed redefinitions of fields. We have already observed that the presence of such Higgs triplet(s) in addition to the Higgs doublet(s) violates the second condition for natural flavor conservation by the neutral current. We see now that such triplet(s) would also yields a nonzero M_{12} and thereby lead to much too large a $\mu \to e\gamma$ rate in the CL model.

For reference, we list below the expressions for the weak eigenstates N'_{eL} and $N'_{\mu L}$ in terms of the mass eigenstates ν_1, ν_2, N_1, and N_2. The coefficients are accurate to lowest order in $m_\mu{}^2/m_L{}^2$:

$$\begin{aligned} N'_{eL} &= \nu_e + \frac{m_e}{m_1} \cos\theta\, N_1 + \frac{m_e}{m_2} \sin\theta\, N_2 , \\ N'_{\mu L} &= \nu_\mu - \frac{m_\mu}{m_1} \sin\theta\, N_1 + \frac{m_\mu}{m_2} \cos\theta\, N_2 , \end{aligned} \quad (3.13)$$

where ν_e and ν_μ are defined as the linear combinations of (the degenerate) ν_1 and ν_2 which couple to N'_{eL} and $N'_{\mu L}$, respectively.

IV. THE DECAYS $\mu \to e\gamma$ AND $\mu \to ee\bar{e}$

We shall now compare the rates predicted for the decays $\mu \to e\gamma$ and $\mu \to ee\bar{e}$ by the various models described in the preceding section. We begin with the WS theory, as modified to include nondegenerate neutrinos. In this model, $C_2=(C_2^{1(a)})^{LL,RR}$. The weak-isospin-raising operator is a 4×4 matrix which in the mass eigenbasis ordered as $(\nu_{1L},\nu_{2L},e_L,\mu_L)$ takes the form

$$\mathcal{T}_+^L=U_L T_+^L U_L^\dagger=\begin{pmatrix}0 & \frac{1}{\sqrt{2}}R_2^{-1}(\theta)\\ 0 & 0\end{pmatrix} \tag{4.1}$$

since $U_L=R_2^{-1}(\theta)$. Thus[33]

$$\langle e|\mathcal{T}_-^L C_{2,-}^{LL,RR}|\mathcal{T}_+^L\mu\rangle = -\sin\theta\cos\theta[(C_{2,-}^{LL,RR})_{11}-(C_{2,-}^{LL,RR})_{22}] \simeq\frac{-eG_F}{32\pi^2\sqrt{2}}\left(\frac{\Delta m_\nu^2}{m_W^2}\right)\sin\theta\cos\theta \tag{4.2}$$

and hence

$$F_2^V(0)_{e\mu}=F_2^A(0)_{e\mu} \simeq\frac{-eG_F m_\mu^2}{32\pi^2\sqrt{2}}\left(\frac{\Delta m_\nu^2}{m_W^2}\right)\sin\theta\cos\theta\,, \tag{4.3}$$

where $\Delta m_\nu^2=m_{\nu_1}^2-m_{\nu_2}^2$. Using Eq. (2.65), we obtain the rate

$$\Gamma(\mu\to e\gamma)\simeq\frac{3\alpha}{32\pi}\left(\frac{\Delta m_\nu^2}{m_W^2}\right)^2\sin^2\theta\cos^2\theta \times\Gamma(\mu\to e\bar\nu_e\nu_\mu)\,, \tag{4.4}$$

where

$$\Gamma(\mu\to e\bar\nu_e\nu_\mu)=\frac{G_F m_\mu^5}{192\pi^3}\,. \tag{4.5}$$

The present upper limit on Δm_ν^2, which comes from the nonobservation of neutrino oscillations, is[34]

$$\Delta m_\nu^2<25\text{ eV}^2\,. \tag{4.6}$$

Using this limit and the value $m_W\simeq 60$ GeV, we compute for the branching ratio (denoted B)

$$B(\mu\to e\gamma)_{\rm WS}\lesssim(2.6\times10^{-45})\left(\frac{\Delta m_\nu^2}{25\text{eV}^2}\right)^2\,, \tag{4.7}$$

which is far smaller than the level which could be detected by any foreseeable experiment.

The situation with the other models is quite different since they have heavy leptons which can serve as intermediate states for the $\mu\to e$ transition. In the KM model again only diagram 1(a) contributes, and we find

$$F_2^V(0)_{e\mu}=F_2^A(0)_{e\mu} =\frac{eG_F m_\mu^2}{32\pi^2\sqrt{2}}\left(\frac{m_{L^0}^2}{m_W^2}\right)\mathfrak{U}_{13}\mathfrak{U}_{23}^*\,. \tag{4.8}$$

Hence

$$B(\mu\to e\gamma)_{\rm KM}=\frac{3\alpha}{32\pi}\left(\frac{m_{L^0}^2}{m_W^2}\right)^2|\mathfrak{U}_{13}\mathfrak{U}_{23}^*|^2\,. \tag{4.9}$$

With $m_W=60$ GeV and $|\mathfrak{U}_{13}\mathfrak{U}_{23}^*|=0.5\times10^{-1}$–$0.5\times10^{-2}$ (see Sec. V)

$$B(\mu\to e\gamma)_{\rm KM}=(0.4\times10^{-9}\text{–}0.4\times10^{-11})\left(\frac{m_{L^0}}{10\text{ GeV}}\right)^4\,. \tag{4.10}$$

For $m_{L^0}=10$ GeV the range of branching ratios corresponding to the above range of $|\mathfrak{U}_{13}\mathfrak{U}_{23}^*|$ is

$$B(\mu\to e\gamma)_{\rm KM}=0.4\times10^{-9}\text{–}0.4\times10^{-11}\,.$$

These values are safely smaller than the present experimental limit,[35]

$$B(\mu\to e\gamma)_{\rm exp}<2.2\times10^{-8}\,, \tag{4.11}$$

but are not so small as to be without experimental interest.

The next two models, (c) and (d), give a comparable rate for the decay $\mu\to e\gamma$. In the first WZ model, (c),[31,36]

$$B(\mu\to e\gamma)_{\rm (c)}=\frac{75\alpha}{32\pi}\left(\frac{\Delta m_L^2}{m_W^2}\right)^2\sin^2\theta\cos^2\theta\,. \tag{4.12}$$

The branching ratio is the same in the second WZ model, (d), in terms of the mixing angle θ and heavy lepton masses of that model.

Model (e) provides an instructive illustration of the effect of unsuppressed left-right transitions of the form $\mu_L\to(L_{1,2}^{--})_L$; $(L_{1,2}^{--})_R\to e_R$, or $\mu_R\to(L_{1,2}^{--})_R$; $(L_{1,2}^{--})\to e_L$. When such LR,RL transitions occur, the chiral projection operators in the vertices do not clear the numerators of the Feynman integrals of the virtual heavy lepton masses as they do in the LL,RR case. Consequently, the leptonic GIM mechanism does not operate effectively. The rate is found to be

$$\Gamma(\mu\to e\gamma)_{\rm (e)}=\frac{3\alpha}{2\pi}\left[\frac{(m_{L_1}-m_{L_2})^2}{m_\mu^2}(c_Ls_R+c_Rs_L)^2+\frac{(m_{L_1}+m_{L_2})^2}{m_\mu^2}(c_Ls_R-c_Rs_L)^2\right]\Gamma(\mu\to e\nu_e\nu_\mu)\,, \tag{4.13}$$

where $c_L=\cos\theta_L$, $s_L=\sin\theta_L$, etc. Unless to high precision $c_Ls_R=c_Rs_L$ and $m_1=m_2$ this rate is in conflict with experiment. Hence, this theory, although it satisfies both conditions for natural flavor conservation by the neutral current, fails to satisfy condition (2.67) and hence is ruled out.

Finally, we consider the CL model, (f). For the LL,RR contribution to the $\mu\to e\gamma$ decay amplitude we find that only the RR part is present. This is again most easily seen by working with the weak, rather than mass, eigenstates and the nondiagonal matrix M; to leading order, i.e., order $(eG_F m_\mu/\pi^2)(\Delta m_L^2/m_W^2)$,

$$\langle e|\mathcal{T}_-^L C_{2,-}^{LL,RR}\mathcal{T}_+^L|\mu\rangle = \text{const}\times\langle N_{eL}|MM^\dagger|N_{\mu L}\rangle = 0\,. \quad (4.14)$$

From the RR part we obtain

$$(F_2^V(0))_{e\mu}^{LL,RR} = -(F_2^A(0))_{e\mu}^{LL,RR} = \left(-\frac{1}{4}\right)\frac{eG_F m_\mu^2}{8\pi^2\sqrt{2}}\left(\frac{\Delta m_L^2}{m_W^2}\right)\sin\theta\cos\theta \quad (4.15)$$

(where $\Delta m_L^2 = m_{L_1}^2 - m_{L_2}^2$). The LR,RL contribution (which was omitted in Ref. 10) is

$$(F_2^V(0))_{e\mu}^{LR,RL} = -(F_2^A(0))_{e\mu}^{LR,RL} = m_\mu\langle e|\mathcal{T}_-^R C_{2,-}^{LR,RL} M_D \mathcal{T}_+^L|\mu\rangle\,. \quad (4.16)$$

The $\langle e|\mathcal{T}_-^L C_{2,-}^{LR,RL} M_D \mathcal{T}_+^R|\mu\rangle$ term is smaller by the factor (m_e/m_μ) and is thus negligible. The right-hand side of Eq. (4.16) is

$$m_\mu\langle e|\mathcal{T}_-^R C_{2,-}^{LR,RL} M_D \mathcal{T}_+^L|\mu\rangle = \frac{eG_F m_\mu^2}{8\pi^2\sqrt{2}}\left(\frac{-3}{2m_W^2}\right)\langle N_{eR}|M^\dagger MM^\dagger|N'_{\mu L}\rangle = \left(\frac{3}{2}\right)\frac{eG_F m_\mu^2}{8\pi^2\sqrt{2}}\left(\frac{\Delta m_L^2}{m_W^2}\right)\sin\theta\cos\theta\,. \quad (4.17)$$

Thus the total transition form factors are

$$(F_2^V(0))_{e\mu} = -(F_2^A(0))_{e\mu} = \frac{5eG_F m_\mu^2}{32\pi^2\sqrt{2}}\left(\frac{\Delta m_L^2}{m_W^2}\right)\sin\theta\cos\theta$$

from which it follows that

$$B(\mu\to e\gamma)_{\text{CL}} = \frac{75\alpha}{32\pi}\left(\frac{\Delta m_L^2}{m_W^2}\right)^2\sin^2\theta\cos^2\theta\,, \quad (4.18)$$

which is a factor of 25 larger than the result originally given by Cheng and Li.[10] This branching ratio is, for $\Delta m_L^2\sim$ a few GeV^2, in agreement with experiment. Thus, taking $m_W = 60$ GeV, $\Delta m_L^2 = 1\ \text{GeV}^2$, and $\sin^2\theta\cos^2\theta = \frac{1}{4}$, we have $B(\mu\to e\gamma)_{\text{CL}} = 1.0\times10^{-10}$.

In addition to the total rate there are several interesting and informative correlation terms which can in principle be measured in the decay $\mu\to e\gamma$. First, consider an experiment in which a polarized muon decays in the $e\gamma$ mode. The decay angular distribution in the muon rest frame is given by

$$\frac{1}{\Gamma}\frac{d\Gamma}{d\Omega}(\theta)_{\mu\to e\gamma} = \frac{1}{4\pi}(1+\alpha P\cos\theta)\,, \quad (4.19)$$

where $\vec{P} = P\hat{n}$ is the polarization vector of the muon and $\cos\theta = \hat{n}\cdot\hat{p}_2$, with $\hat{p}_2$ being the three-momentum of the electron in the muon rest frame. The asymmetry parameter is

$$\alpha = \frac{2\,\text{Re}(F_2^V(0)_{e\mu}F_2^A(0)_{e\mu}^*)}{|F_2^V(0)_{e\mu}|^2 + |F_2^A(0)_{e\mu}|^2} \quad (4.20)$$

and serves to measure the amount of parity violation in the decay. For theories with purely left-handed charged currents such as the WS, KM, and (c) models, the decay amplitude is of the form $\bar{u}(p_2)i\sigma_{\mu\nu}q^\nu\epsilon^\mu Ru(p_1)$, where $R = \frac{1}{2}(1+\gamma_5)$, and hence $\alpha = +1$. The opposite is true if the decay $\mu\to e\gamma$ arises from an RR transition; accordingly, in model (d) $\alpha = -1$. In the CL model, interestingly, although there are both RR and LR contributions to the amplitude, the signs in Eqs. (2.33a) and (2.33b) relative to those in Eqs. (2.31a) and (2.31b) are such that $\alpha = -1$. Thus in the decay angular distribution from a polarized muon, the isotropic component determines $|F_2^V(0)_{e\mu}|^2 + |F_2^A(0)_{e\mu}|^2$, while the component proportional to $\cos\theta$ yields the relative phase between $F_2^V(0)_{e\mu}$ and $F_2^A(0)_{e\mu}$.

One could also consider measuring the polarization of the photon from the decaying muon. The expression for the angular distribution, assuming that the muon is polarized, and neglecting terms of order (m_e/m_μ), is

$$\frac{1}{\Gamma}\frac{d\Gamma}{d\Omega}(\theta,\epsilon)_{\mu\to e\gamma} = \frac{1}{8\pi}[\hat{\epsilon}\cdot\hat{\epsilon}^*(1+\alpha P\cos\theta) + i\hat{q}\cdot(\hat{\epsilon}\times\hat{\epsilon}^*)(P\cos\theta+\alpha)]\,. \quad (4.21)$$

In this equation, α, P, and $\cos\theta = \hat{n}\cdot\hat{p}_2$ are as in Eqs. (4.19) and (4.20) and $\vec{q} = -\hat{p}_2$ and $\hat{\epsilon}$ are the three-momentum and polarization vector of the photon in the muon rest frame. All four correlation terms are time reversal invariant; $\hat{\epsilon}\cdot\hat{\epsilon}^*$ and $i\hat{q}\cdot(\hat{\epsilon}\times\hat{\epsilon}^*)\hat{n}\cdot\hat{p}_2$ are even under parity, while $\hat{\epsilon}\cdot\hat{\epsilon}^*\hat{n}\cdot\hat{p}_2$ and $i\hat{q}\cdot(\hat{\epsilon}\times\hat{\epsilon}^*)$ are odd (and accordingly are multiplied by α). Observe that even if the decaying muon is unpolarized, one can still obtain $\text{Re}(F_2^V(0)_{e\mu}F_2^A(0)_{e\mu}^*)$ by measuring the $i\hat{q}\cdot(\hat{\epsilon}\times\hat{\epsilon}^*)$ term.

Finally, we note that to obtain information from the measurement of the electron spin would be quite difficult since the electron spin correlation terms are suppressed by the factor (m_e/m_μ) relative to the dominant angular correlation terms. For example, the angular distribution in the case where one measures the polarization of the muon and the electron, but not that of the photon, is

$$\frac{1}{\Gamma}\frac{d\Gamma}{d\Omega} = \frac{1}{8\pi}\left\{1 + 4\left(\frac{m_e}{m_\mu}\right)\left(\frac{p_1\cdot s_2 p_2\cdot s_1}{m_\mu^2}\right) + \frac{2\alpha}{m_\mu}\left[p_2\cdot s_1 + \left(\frac{m_e}{m_\mu}\right)p_1\cdot s_2\right]\right\}, \quad (4.22)$$

where we have dropped terms of higher order in (m_e/m_μ) and for symmetry reasons have left the

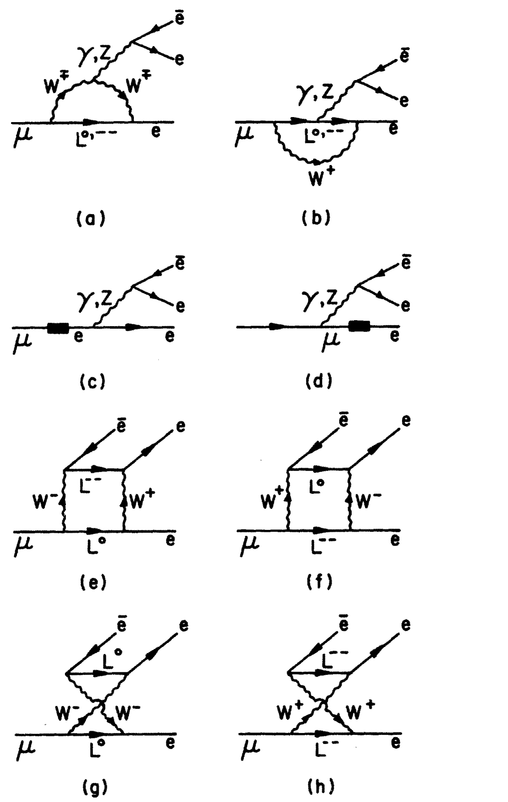

FIG. 2. Diagrams contributing to a general SU(2)×U(1) model to the decay $\mu \to ee\bar{e}$. In graph $(b)_\gamma$, the virtual lepton, can only be L^{--}. In graphs (c) and (d) the rectangular box denotes a nondiagonal lepton self-energy insertion; for the graphs contributing to this, see Fig. 3. Each graph represents a difference of two graphs, related to each other by interchange of final electron momenta.

expression in terms of the (pseudo) scalar products of the four-momenta p_1 and p_2 with s_1 and s_2, the spin vectors of the muon and electron, respectively.

We next proceed to consider the decay $\mu \to ee\bar{e}$, concentrating on the KM model. We exclude theories which gave hopelessly small $\mu \to e\gamma$ rates (e.g., the WS model) and theories which give $\mu \to e\gamma$ rates so large as to be in conflict with experiment [e.g., the illustrative model (e)]. There are three classes of graphs, as shown in Fig. 2,

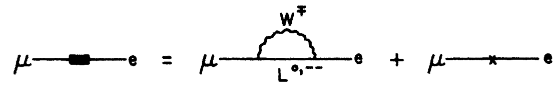

FIG. 3. Diagrams contributing in a general gauge model to the nondiagonal μ-e self-energy. The cross represents a counterterm.

which contribute to this decay. These include, first, diagrams $2(a)_\gamma$–$2(d)_\gamma$ with a photon as the virtual gauge boson which creates the $e\bar{e}$ pair. Graphs $2(a)_\gamma$ and $2(b)_\gamma$ are just the analogs of graphs 1(a) and 1(b) with a virtual rather than real photon. The nondiagonal lepton self-energy insertions in graphs $(c)_\gamma$ and $(d)_\gamma$ are depicted in Fig. 3; actually (see below) they do not contribute to $\mu \to ee\bar{e}$ decay. The second class of diagrams consists of Figs. $2(a)_Z$–$2(d)_Z$ with Z as the virtual gauge boson which creates the $e\bar{e}$ pair. Here the nondiagonal self-energy graphs do contribute. Finally, the third class consists of the W^+W^- exchange diagrams shown in Figs. 2(e)–2(h). In Figs. 2(a), 2(b), and 2(e)–2(h) the generic symbols L^0 and L^- are used to refer to all neutral and doubly negatively charged leptons which can couple in these graphs. Of course in theories without doubly charged leptons which couple to both e and μ only graphs 2(a), $2(b)_Z$, 2(c), 2(d), and 2(g) are present. On the other hand, in models such as (c) and (d) in which the μ-e transition proceeds only via L^{--} leptons, graphs 2(e), 2(f), and 2(g) are absent. Each diagram in Fig. 2 is understood to represent the difference of two graphs; the second is related to the first by interchange of final electron lines, in accord with the requirement of antisymmetry for identical fermions.

Let us begin with the class of virtual-photon graphs $2(a)_\gamma$–$2(d)_\gamma$. As is shown in the Appendix, the self-energy graphs only contribute to the renormalization of $F_1^{V,A}(0)$. Since $\langle e|F_1^{V,A}(0)|\mu\rangle = 0$ they therefore do not affect the $\mu \to ee\bar{e}$ amplitude. Now, taking account of the fact that $q^2 \ll m_L^{\,2}$, we expand $F_1^{V,A}(q^2)$ in a Taylor series and use Eqs. (2.23a), (2.23b), (2.25a), and (2.25b), the latter two of which enable us to calculate the vector and axial-vector charge radii $dF_1^{V,A}(0)/dq^2$ in terms of $F_3^{V,A}(0)$, as discussed in Sec. II. The virtual-photon amplitude is thus

$$\mathfrak{M}^{(\gamma)}(\mu \to ee\bar{e}) = i\bar{u}_e(p_2)E_\alpha^{(\gamma)}(p_2,p_1)u_\mu(p_1)\left(\frac{-ig^{\alpha\beta}}{q^2}\right)[ie\bar{u}(l_2)\gamma_\beta v(l_1)] - (p_2 \leftrightarrow l_2)\,, \tag{4.23}$$

where, in analogy with the notation of Ref. 2, $iE_\alpha^\gamma(p_2,p_1)$ denotes the full one-loop effective $\gamma\mu e$ vertex:

$$E_\alpha^{(\gamma)}(p_2,p_1) = \left[\frac{q^2}{m_\mu}\gamma_\alpha(-F_3^V(0)_{e\mu} + F_3^A(0)_{e\mu}\gamma_5) + \frac{i\sigma_{\alpha\beta}q^\beta}{m_\mu}(F_2^V(0)_{e\mu} + F_2^A(0)_{e\mu}\gamma_5)\right]. \tag{4.24}$$

Since $q^2 \lesssim m_\mu^{\,2}$, $E_\alpha^{(\gamma)}$ is of the same order as $F_2^{V,A}(0)$; in the KM model,

$$E_\alpha^{(\gamma)} \sim F_2^{V,A}(0) \sim m_\mu F_3^{V,A}(0) \sim \frac{eG_F m_\mu^2}{8\pi^2}\left(\frac{m_{L_0}^2}{m_W^2}\right). \tag{4.25}$$

Evaluating Eq. (4.24) for the KM model we find

$$\mathfrak{M}^{(\gamma)}(\mu \to ee\bar{e})_{\rm KM} = i\left(\frac{e^2 G_F}{4\pi^2\sqrt{2}}\right)\left(\frac{m_{L_0}^2}{m_W^2}\right)\mathfrak{U}_{13}\mathfrak{U}^*_{23}\bar{u}_e(p_2)\left[\gamma_\alpha L(\tfrac{1}{2} - \tfrac{1}{4}\ln\xi) + \frac{i\sigma_{\alpha\beta}q^\beta m_\mu}{q^2}R(-\tfrac{1}{4})\right]u_\mu(p_1)\bar{u}_e(l_2)\gamma^\alpha \bar{v}_e(l_1) - (p_2 \leftrightarrow l_2). \tag{4.26}$$

For the Z and W^+W^- exchange graphs, we can simply use, with appropriate changes, the results of our previous general R_ξ gauge calculations,[37] letting $\xi \to 0$ match the ξ-limiting procedure used for the photon graphs. Note that the only W^+W^- graph is Fig. 2(g). In contrast to the case with the photon graphs, in the Z and W^+W^- graphs there is no reason why the leading γ_α and $\gamma_\alpha\gamma_5$ terms [the analogs of $F_1^{V,A}(0)_{e\mu}$] must vanish, and indeed they do not. They dominate over the $i\sigma_{\alpha\beta}q^\beta$ and $i\sigma_{\alpha\beta}q^\beta\gamma_5$ terms by a power of (m_W^2/m_μ^2), since in order to form the latter terms one loses one, and hence by symmetric integration, two powers of loop momenta in the Feynman integrals. As in the photon graphs, the q_μ and $q_\mu\gamma_5$ terms give zero contribution.

Denoting the full one-loop effective $Z\mu e$ vertex by $iE_\alpha^{(Z)}(p_2,p_1)$, we calculate the Z-exchange amplitude to be

$$\mathfrak{M}^{(Z)} = i\bar{u}_e(p_2)E_\alpha^{(Z)}(p_2,p_1)u_\mu(p_1)\left(\frac{-ig^{\alpha\beta}}{q^2 - m_Z^2}\right)[i(g^2+g'^2)^{1/2}]\,\bar{u}_e(l_2)(-\tfrac{1}{2}\gamma_\beta L + \sin^2\theta_W\gamma_\beta)v_e(l_1) - (p_2 \leftrightarrow l_2), \tag{4.27}$$

where

$$E_\alpha^{(Z)} = \frac{g^2(g^2+g'^2)^{1/2}}{32\pi^2}\,\frac{m_{L_0}^2}{m_W^2}\mathfrak{U}_{13}\mathfrak{U}_{23}^*\left(\ln\frac{m_W^2}{m_{L_0}^2} - \tfrac{3}{2} - \tfrac{1}{4}\ln\xi\right)\gamma_\alpha L. \tag{4.28}$$

Since the leading γ_α term does not vanish, $E_\alpha^{(Z)}$ is larger than $E_\alpha^{(\gamma)}$ by the factor (m_W^2/m_μ^2). However, the actual amplitudes $\mathfrak{M}^{(\gamma)}$ and $\mathfrak{M}^{(Z)}$ are of the same order (up to logarithms) because the Z propagator is smaller than the photon propagator by the factor (m_μ^2/m_W^2). Observe also that $E_\alpha^{(Z)}$ arises, to leading order, only from the weak isospin current in $J_Z^\alpha = J_3^\alpha - \sin^2\theta_W J_{\rm em}^\alpha$, as is evident from the absence of any term proportional to $\sin^2\theta_W$ in Eq. (4.28). Indeed, the contribution of the electromagnetic current to $E_\alpha^{(Z)}$ is just $\sin^2\theta_W E_\alpha^{(\gamma)}$, which is negligible in comparison.

The W^+W^- exchange amplitude arises in the KM model from graph 2(g). We find that

$$\mathfrak{M}^{(WW)} = i\left(\frac{g^2 G_F}{4\pi^2\sqrt{2}}\right)\left(\frac{m_{L_0}^2}{m_W^2}\right)\mathfrak{U}_{13}\mathfrak{U}_{23}^* \times \bar{u}_e(p_2)\gamma_\alpha L u_\mu(p_1)\left(-\ln\frac{m_W^2}{m_{L_0}^2} - \tfrac{1}{2} + \tfrac{1}{4}\ln\xi\right)\bar{u}_e(l_2)\tfrac{1}{2}\gamma^\alpha L v_e(l_1) - (p_2 \leftrightarrow l_2). \tag{4.29}$$

The total amplitude is then

$$\begin{aligned}\mathfrak{M}(\mu \to ee\bar{e})_{\rm KM} &= \mathfrak{M}^{(\gamma)} + \mathfrak{M}^{(Z)} + \mathfrak{M}^{(WW)}\\ &= i\left(\frac{e^2 G_F}{4\pi^2\sqrt{2}}\right)\left(\frac{m_{L_0}^2}{m_W^2}\right)\mathfrak{U}_{13}\mathfrak{U}_{23}^*\bar{u}_e(p_2)\left[\gamma_\alpha L\left(2 - \ln\frac{m_W^2}{m_{L_0}^2}\right) + \frac{i\sigma_{\alpha\beta}q^\beta m_\mu}{q^2}R(-\tfrac{1}{4})\right]u_\mu(p_1)[\bar{u}_e(l_2)\gamma^\alpha v_e(l_1)]\\ &\quad - i\left(\frac{g^2 G_F}{4\pi^2\sqrt{2}}\right)\left(\frac{m_{L_0}^2}{m_W^2}\right)\mathfrak{U}_{13}\mathfrak{U}_{23}^*[\bar{u}_e(p_2)\gamma_\alpha L u_e(p_1)][\bar{u}_e(l_2)\gamma^\alpha L v_e(l_1)] - (p_2 \leftrightarrow l_2).\end{aligned} \tag{4.30}$$

This amplitude is independent of the non-Abelian gauge parameter ξ, as it must be. Observe that the cancellation of the $\ln\xi$ term occurs between the Z and W^+W^- graphs for the part proportional to g^2 and between the Z and γ graphs for the part proportional to $e^2(=g^2\sin^2\theta_W)$.

In order to calculate the decay rate from the amplitude (4.30), we shall make the approximation of retaining only the logarithm. This gives for the $\mu \to ee\bar{e}$ rate in the KM model

$$B(\mu \to ee\bar{e})_{\rm KM} \simeq \frac{3\alpha^2}{16\pi^2}\left(\frac{m_{L_0}^2}{m_W^2}\right)^2\ln^2\left(\frac{m_W^2}{m_{L_0}^2}\right)|\mathfrak{U}_{13}\mathfrak{U}_{23}^*|^2. \tag{4.31}$$

For $m_{L_0} = 10$ GeV, $m_W = 60$ GeV, and $|\mathfrak{U}_{13}\mathfrak{U}_{23}|$ in the range 0.05–0.005 this branching ratio is in the range 2×10^{-11}–2×10^{-13}. This is safely below the present experimental upper bound[38]

$$B(\mu \to ee\bar{e})_{\rm exp} < 6\times10^{-9}. \tag{4.32}$$

It is of interest to form the ratio of the $\mu \to ee\bar{e}$ and $\mu \to e\gamma$ decay rates; in the KM model we have

$$\frac{\Gamma(\mu \to ee\bar{e})_{\rm KM}}{\Gamma(\mu \to e\gamma)_{\rm KM}} \simeq \frac{2\alpha}{\pi} \ln^2\left(\frac{m_W^2}{m_{L_0}^2}\right), \tag{4.33}$$

which is independent of the product of mixing parameters $|\mathfrak{U}_{13}\mathfrak{U}_{23}^*|$. For the values of m_{L_0} and m_W selected above the ratio (4.33) is equal to ~0.06. This ratio, which is formally of order (α/π), is actually somewhat enhanced by the $\ln^2(m_W^2/m_{L_0}^2)$ factor. The origin of this log term can be traced to the Z-exchange amplitude (4.27) and (4.28).

In the WZ models the $\mu \to e + \gamma_{\rm virtual}$ transition proceeds by way of doubly negatively charged heavy leptons. Thus diagram $2(\mathrm{b})_\gamma$ contributes, and gives an unsuppressed $\ln(m_W^2/m_{L_j}^2)$ term, where $j=1,2$, in $(C_{3,+}^{LL,RR})_j$ [cf. Eq. (2.47)]. This means that the leptonic GIM mechanism does not operate nearly as effectively as it would if the leading term were independent of m_L. The V and A charge radius part of the virtual-photon amplitude $\mathfrak{M}^{(\gamma)}$ is larger than in a fully GIM-suppressed model such as the KM model by roughly the factor $(m_W^2/\Delta m_L^2)$.[39] The $F_2^{V,A}$ terms are still adequately suppressed to the level $eG_F m_\mu^2(\Delta m_L^2/m_W^2)$, so that $\mathfrak{M}^{(\gamma)}$ is completely dominated by the charge radius terms. Moreover, $\mathfrak{M}^{(Z)}$ and $\mathfrak{M}^{(WW)}$ are not enhanced, and consequently $\mathfrak{M}(\mu \to ee\bar{e}) \simeq \mathfrak{M}^{(\gamma)}$. We calculate a branching ratio in model (c),

$$B(\mu \to ee\bar{e})_{(\mathrm{c})} \simeq \frac{\alpha^2}{3\pi^2} \ln^2\left(\frac{m_{L_2}^2}{m_{L_1}^2}\right) \sin^2\theta \cos^2\theta\,. \tag{4.34}$$

In order for this branching ratio to be smaller than the experimental limit (4.32), m_{L_1} and m_{L_2} must be very close to being degenerate and/or the mixing angle θ must be very small, both rather special conditions. Specifically, it is necessary that

$$\sin^2\theta \cos^2\theta \ln^2\left(\frac{m_{L_2}^2}{m_{L_1}^2}\right) < 3.3 \times 10^{-3}\,. \tag{4.35}$$

Equation (4.34) also applies to model (d), where L_1^{--} and L_2^{--} are the corresponding heavy leptons and θ is the mixing angle in that model.[36] Because of the fact that the natural suppression mechanism does work effectively for the $\mu \to e\gamma$ decay in the WZ models, the ratio of the $\mu \to ee\bar{e}$ and $u \to e\gamma$ decay rates is much larger than the value (α/π):

$$\frac{\Gamma(\mu \to ee\bar{e})_{(\mathrm{c}),(\mathrm{d})}}{\Gamma(\mu \to e\gamma)_{(\mathrm{c}),(\mathrm{d})}} = \frac{32}{225}\left(\frac{\alpha}{\pi}\right)\left(\frac{m_W^2}{\Delta m_L^2}\right)^2 \ln^2\left(\frac{m_W^2}{|\Delta m_L^2|}\right). \tag{4.36}$$

For $m_{L_1} \simeq 2$ GeV, $m_{L_2} \simeq 4$ GeV, and $m_W = 60$ GeV the ratio (4.35) is $\simeq 10^3$.

The decay $\mu \to ee\bar{e}$ in the CL model has been calculated approximately in Ref. 32 and we shall not duplicate this work here. For reference it is found, from a leading logarithm approximation, that for $\ln(m_W/\bar{m}_L) \simeq 3$, where $\bar{m}_L = \frac{1}{2}(m_{L_1} + m_{L_2})$, and $\sin^2\theta_W \simeq \frac{1}{3}$,

$$\frac{\Gamma(\mu \to ee\bar{e})_{\rm CL}}{\Gamma(\mu \to e\gamma)_{\rm CL}} \simeq 0.07\,.$$

V. PHENOMENOLOGY OF THE KM MODEL

A. Constraints on mixing angles

In this section we shall briefly discuss a number of phenomenological aspects of the KM model.

The representation content of the quark sector in this theory is analogous to that of the leptonic sector, shown in Table I(b) (except for the absence of ν_{1R} and ν_{2R} in the leptonic case):

$$\begin{pmatrix} u_d \\ d \end{pmatrix}_L, \quad \begin{pmatrix} u_s \\ s \end{pmatrix}_L, \quad \begin{pmatrix} u_b \\ b \end{pmatrix}_L, \quad \begin{matrix} u_R, c_R, t_R\,, \\ d_R, s_R, b_R\,. \end{matrix} \tag{5.1}$$

In a natural model such as this it is a convention whether one takes the $T_3 = \frac{1}{2}$ or $T_3 = -\frac{1}{2}$ fermions to be both mass eigenstates and weak eigenstates. (With no loss of generality in a natural model one can always choose one or the other option.) In order to maintain the formal similarity between quark and lepton sectors we have chosen to consider the $T_3 = \frac{1}{2}$ quarks as mixtures of mass eigenstates. This mixing is prescribed by Eq. (3.2), where the 3×3 unitary matrix $\mathfrak{V}$, like its leptonic analog $\mathfrak{U}$, is in general a function of four angles, one of which violates CP. The CP-violating phase angle can of course be set equal to zero by decree; in both the leptonic and hadronic cases we prefer to retain the most general form of $\mathfrak{U}$ and $\mathfrak{V}$, allowing for the possibility of CP violation.

It is useful to consider the experimental constraints on the mixing angles in this theory. One may observe first that there are really several ways to define the Fermi constant, all of which coincide in the minimal WS model, but are in general different in the KM theory. From nuclear beta decay one can define

$$\frac{G_F^\beta}{\sqrt{2}} \cos\theta_C = \frac{g^2}{8m_W^2} \mathfrak{V}_{11}(|\mathfrak{U}_{11}|^2 + |\mathfrak{U}_{12}|^2)^{1/2}\,. \tag{5.2}$$

A symmetrized version of this constant which includes the u-s coupling as well as the u-d coupling is

$$\frac{G_F^u}{\sqrt{2}} = \frac{g^2}{8m_W^2} (|\mathfrak{V}_{11}|^2 + |\mathfrak{V}_{21}|^2)^{1/2} (|\mathfrak{U}_{11}|^2 + |\mathfrak{U}_{12}|^2).^{1/2} \tag{5.3}$$

Alternatively, the Fermi constant can be determined from a measurement of the muon decay rate; in this case what is measured is

$$\frac{G_F^\mu}{\sqrt{2}} = \frac{g^2}{8m_W^2}(|\mathcal{U}_{11}|^2+|\mathcal{U}_{12}|^2)^{1/2}(|\mathcal{U}_{21}|^2+|\mathcal{U}_{22}|^2)^{1/2}. \tag{5.4}$$

Consider the special case in which the u and c quarks mix only among themselves, and similarly with ν_1 and ν_2, so that

$$\mathcal{V} = \begin{bmatrix} \cos\theta_C & -\sin\theta_C & 0 \\ \sin\theta_C & \cos\theta_C & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{5.5}$$

and similarly with $\mathcal{U}$ in terms of the corresponding leptonic angle θ. In this case, where the "old" quark and old lepton sectors of the KM model coincide with those of the WS theory, $G_F^\beta = G_F^u = G_F^\mu = G_F$, with G_F as defined in Eq. (2.48).

The Cabibbo angle is given by

$$\tan\theta_C = \frac{\mathcal{V}_{21}}{\mathcal{V}_{11}}. \tag{5.6}$$

Accordingly, a measurement of nuclear beta decay alone only yields the combination $\mathcal{V}_{11} = \cos\theta_C(|\mathcal{V}_{11}|^2 + |\mathcal{V}_{21}|^2)^{1/2}$, which is not sufficient by itself to determine θ_C. Similar comments apply to semileptonic hyperon decays and the meson decays $K\to l\bar{\nu}_l$, $\pi\to l\bar{\nu}_l$, taken individually. This situation contrasts with that in the minimal model where any of these measurements could in principle (modulo different radiative corrections, etc.) be used to determine θ_C. It is convenient to define another weak coupling constant G_F^c and another angle θ_C' in analogy with Eqs. (5.3) and (5.5) but based on the coupling of the c quark to the d and s quarks:

$$\frac{G_F^c}{\sqrt{2}} = \frac{g^2}{8m_W^2}(|\mathcal{V}_{12}|^2+|\mathcal{V}_{22}|^2)^{1/2}(|\mathcal{U}_{11}|^2+|\mathcal{U}_{12}|^2)^{1/2}, \tag{5.7}$$

$$\tan\theta_C' = -\mathcal{V}_{12}/\mathcal{V}_{22}. \tag{5.8}$$

With this notation established, we can now state the experimental constraints on the leptonic and hadronic mixing matrices $\mathcal{U}$ and $\mathcal{V}$. First, there is the observed μ-e universality, which is best tested by a comparison of $\pi\to e\bar{\nu}_e$ and $\pi\to\mu\bar{\nu}_\mu$ decays. The ratio of the decay rates in the KM model is, in the absence of radiative corrections,

$$\frac{\Gamma(\pi\to e\bar{\nu}_e)}{\Gamma(\pi\to\mu\bar{\nu}_\mu)} = R_{e/\mu}{}^2\left(\frac{m_e^2}{m_\mu^2}\right)\frac{(1-m_e^2/m_\pi^2)^2}{(1-m_\mu^2/m_\pi^2)^2}, \tag{5.9}$$

where

$$R_{e/\mu} = \left|\frac{\langle N_e|\nu_e\rangle}{\langle N_\mu|\nu_\mu\rangle}\right| = \frac{(|\mathcal{U}_{11}|^2+|\mathcal{U}_{12}|^2)^{1/2}}{(|\mathcal{U}_{21}|^2+|\mathcal{U}_{22}|^2)^{1/2}}. \tag{5.10}$$

The radiative corrections[40] reduce the theoretical prediction for the ratio by ~ −3.9%. From the experimentally measured ratio[41] of $(1.247\pm0.028)\times10^{-4}$ one infers that at the one standard deviation level,

$$0.983 < R_{e/\mu} < 1.005. \tag{5.11}$$

Note that this constraint, taken alone, is compatible with large mixing parameters $|\mathcal{U}_{13}|$ and $|\mathcal{U}_{23}|$ as long as they are sufficiently close to being equal.

The second experimental constraint delimits the size of (the product of) these two parameters. This input is provided by neutrino experiments which search for electrons produced from the scattering of an incident beam of ν_μ off a nucleon target. Several such searches have recently been made, primarily with the motivation of looking for neutrino oscillations.[28] These oscillations cause an original beam which is composed purely of ν_μ to develop a nonvanishing component of ν_e if the corresponding mass eigenstates are not degenerate and the mixing angles are nonzero. As was mentioned before, in the KM model there are no neutrino oscillations since by assumption $m_{\nu_1} = m_{\nu_2} = 0$. However, the neutrino states ν_e and ν_μ are nonorthogonal; the matrix element $\langle\nu_e|\nu_\mu\rangle$ is proportional to $\mathcal{U}_{11}^*\mathcal{U}_{21} + \mathcal{U}_{12}^*\mathcal{U}_{22} = -\mathcal{U}_{13}^*\mathcal{U}_{23}$, as given in Eq. (3.3c).[30] The relative cross section for an incident ν_μ beam to produce electrons rather than muons is thus given by

$$\Sigma_{e/\mu} = \frac{\sigma(\nu_\mu N\to e^- X)}{\sigma(\nu_\mu N\to\mu^- X)} = \left|\frac{\langle N_e|\nu_\mu\rangle}{\langle N_\mu|\nu_\mu\rangle}\right|^2 = R_{e/\mu}^{-2}|\langle\nu_e|\nu_\mu\rangle|^2 \simeq |\langle\nu_e|\nu_\mu\rangle|^2. \tag{5.12}$$

The main experimental background consists of the small ν_e contamination in the ν_μ beam. The Gargamelle bubble-chamber experiment finds electron events at the level (0.50 + 0.08)%, while the estimated background is (0.46 + 0.10)%, and concludes that at the 90% confidence level, $\Sigma_{e/\mu} < 0.3\%$.[42] We can therefore infer the bound

$$\langle\nu_e|\nu_\mu\rangle = \frac{|\mathcal{U}_{13}\mathcal{U}_{23}^*|}{(|\mathcal{U}_{11}|^2+|\mathcal{U}_{12}|^2)^{1/2}(|\mathcal{U}_{21}|^2+|\mathcal{U}_{22}|^2)^{1/2}} \lesssim 0.055. \tag{5.13}$$

The third constraint is that of the "universality" of the weak-interaction coupling strength among quarks and leptons. Of course at a fundamental level this universality is automatically built into a unified gauge theory since weak quark eigenstates and weak lepton eigenstates in multiplets of equal dimensionality couple to $W^\pm$ with the same strength. As conventionally stated, this principle is the equality

$$\frac{G_\mu}{G_\beta \sec\theta_C} = 1 . \tag{5.14}$$

This equality is realized automatically in the Cabibbo-WS theory; in the KM model, it must be explicitly enforced. In terms of $\mathfrak{U}$ and $\mathfrak{V}$ this Cabibbo universality is the condition that

$$R_{\mu/\beta} = \frac{(|\mathfrak{U}_{21}|^2 + |\mathfrak{U}_{22}|^2)^{1/2}}{(|\mathfrak{V}_{21}|^2 + |\mathfrak{V}_{22}|^2)^{1/2}} = 1 . \tag{5.15a}$$

Experimentally,[43]

$$R_{\mu/\beta} \simeq 1.01 , \tag{5.15b}$$

where the model-dependent nuclear radiative corrections are of order 1%. Note that the conventional form of quark-lepton universality in Eq. (5.14) leaves much of $\mathfrak{U}$ and $\mathfrak{V}$ unconstrained. In particular, it is consistent with large values of $|\mathfrak{U}_{13}|$ and, independently, $|\mathfrak{V}_{13}|$. Furthermore, one should note that even large values of $|\mathfrak{U}_{23}|$ and $|\mathfrak{V}_{23}|$ are allowed, as long as $|\mathfrak{U}_{23}| \simeq |\mathfrak{V}_{23}|$ so that Eq. (5.15) remains satisfied to the level of accuracy demonstrated by experiment. Parenthetically, we mention that one might postulate a more sweeping form of universality such as $\mathfrak{U} = \mathfrak{V}$.[44] There is no experimental evidence for this; however, it seems not to be inconsistent with present data.

The fourth constraint is that the relative strength and phase of the u-d and u-s couplings agree with the successful Cabibbo theory, i.e., that the angle defined by Eq. (5.6) be in fact equal to the experimentally measured Cabibbo angle. This angle can be measured from a combination of $\Delta S = 0$ and $|\Delta S| = 1$ semileptonic baryon decays, such as nuclear beta decay and semileptonic hyperon decays, or from the relative rates for $K \to l\bar{\nu}_l$ and $\pi \to l\bar{\nu}_l$. Note that this is true in the KM model as well as in the minimal model since the factor $(|\mathfrak{U}_{11}|^2 + |\mathfrak{U}_{12}|^2)^{1/2}$ at the $We\bar{\nu}_e$ vertex [or in the meson case also $(|\mathfrak{U}_{21}|^2 + |\mathfrak{U}_{22}|^2)^{1/2}$ at the $W\mu\bar{\nu}_\mu$ vertex] cancels out in ratios. A recent comparison of hyperon and nuclear beta decay data yields[45]

$$\sin\theta_C = 0.230 \pm 0.003 . \tag{5.16}$$

An analogous condition would be that θ_C' defined by Eq. (5.8) be equal to θ_C, i.e., $(\mathfrak{V}_{21}/\mathfrak{V}_{11}) = -(\mathfrak{V}_{12}/\mathfrak{V}_{22})$. Experiments on decays of charmed particles are not nearly accurate enough yet to test this seriously. From a comparison of the invariant mass plots for decays of the $D^0(1865)$ into $\pi^\pm K^\mp$ channels versus the $\pi^+\pi^-$ channel one might crudely estimate[46] that $\tan^2\theta_C' \lesssim 0.1$. An even stronger universality assumption would be that $\mathfrak{V}_{11} = \mathfrak{V}_{22}$, which, together with the equation $\theta_C = \theta_C'$, implies $\mathfrak{V}_{21} = -\mathfrak{V}_{12}$. Again it should be stressed that there is no theoretical reason in the context of the KM model for these equalities or the $\theta_C' = \theta_C$ relation. In contrast, in the special case (5.5) for $\mathfrak{V}$ (and similarly for $\mathfrak{U}$), they are all automatically valid.

Taken together, these four experimental constraints significantly limit the amount by which the quark sector can differ from the Cabibbo-WS model and the way in which ν_1, ν_2, and L^0 can mix to form N_e, N_μ, and N_L. For the purpose of our present calculations of μ- and e-number nonconserving processes, perhaps the most important constraint is the one on $|\mathfrak{U}_{13}\mathfrak{U}_{23}^*|$, Eq. (5.13), since this factor controls the rate of decays such as $\mu \to e\gamma$, $\mu \to ee\bar{e}$, $K_L \to \mu\bar{e}$, and $K^- \to \pi^- e\bar{e}$.

B. Rare K decays

Let us recall first that, as was shown by Gaillard and Lee,[2] the GIM mechanism in the WS model works not only at the tree level to ensure the absence of direct lowest-order nondiagonal Z-quark couplings; it also operates adequately at the one-loop level to suppress processes such as $K_L \to \mu\bar{\mu}$ and $K^0 \leftrightarrow \bar{K}^0$ (which gives rise to the $K_L K_S$ mass difference). Indeed it succeeds in doing this while at the same time allowing other, nonsuppressed decays such as $K_{L,S} \to \gamma\gamma$, $K_S \to \pi^0\gamma\gamma$, and $K^- \to \pi^- e\bar{e}$ to proceed at their experimentally observed rates. Since the KM model, like the minimal WS theory, satisfies the two conditions for natural flavor conservation by the neutral current and also the condition (2.67), it also succeeds in accounting for the relative rates of the various K decays. The free quark approximation used in Ref. 2 will again be satisfactory for our purposes here.

We shall compute first the decay rate for $K_L \to \mu\bar{e}$ and compare it with that for the analogous μ- and e-lepton-number-conserving rare decay $K_L \to \mu\bar{\mu}$. The graph for the elementary process $s\bar{d} \to e\bar{\mu}$ in the KM model is shown in Fig. 4; as in Figs. 2 and 3 the symbol L^0 refers to all the neutral leptons which can couple, namely, ν_1, ν_2, and the particular heavy lepton L^0. An interesting aspect of the calculation is that there are GIM mechanisms operating on both the quark and lepton sides of the graph. However, as will be evident, the graph is not suppressed any more than it would be if there were only a GIM cancellation on one side, as is the case for $K_L \to \mu\bar{\mu}$, for example. This is quite analogous to the situation regarding the

$K_L K_S$ mass difference[2] ($s\bar{d}\to\bar{s}d$ transition), where a double GIM mechanism only suppresses the amplitude by a single power of $(m_c^2-m_u^2)/m_W^2$. The explanation lies in the infrared behavior of the relevant Feynman integral, which has a pole rather than just a logarithmic divergence for zero fermion mass (see below).

We calculate the amplitude for $s\bar{d}\to e\bar{\mu}$ to be

$$\mathfrak{M}(s\bar{d}\to e\bar{\mu})=\frac{ig^2}{8\pi^2}\frac{G_F}{\sqrt{2}}\,\epsilon_L(\bar{d}\gamma_\alpha Ls)(\bar{e}\gamma^\alpha L\mu)\mathfrak{U}_{13}\mathfrak{U}_{23}^*\sum_{j=1}^{3}\mathfrak{V}_{1j}\mathfrak{V}_{2j}^*I(\epsilon_L,\epsilon_{q_j})\,, \tag{5.17}$$

where $\epsilon_L=m_{L_0}{}^2/m_W{}^2$, $\epsilon_{q_j}=m_{q_j}{}^2/m_W{}^2$, $q_{1,2,3}=u,\ c,\ t$, and

$$I(\epsilon_L,\epsilon_{q_j})=\left(\frac{1}{\epsilon_L-\epsilon_{q_j}}\right)\left[\frac{(\epsilon_{q_j}\ln\epsilon_{q_j}+1-\epsilon_{q_j})}{(1-\epsilon_{q_j})^2}-\frac{(\epsilon_L\ln\epsilon_L+1-\epsilon_L)}{(1-\epsilon_L)^2}\right]. \tag{5.18}$$

By using Eqs. (5.3) and (5.6)–(5.8), and the unitarity of $\mathfrak{V}$, one can recast this amplitude in the form

$$\mathfrak{M}(s\bar{d}\to e\bar{\mu})=\frac{i}{2\pi^2}m_{L_0}{}^2(\bar{d}\gamma_\alpha Ls)(\bar{e}\gamma^\alpha L\mu)\left(\frac{\mathfrak{U}_{13}\mathfrak{U}_{23}^*}{|\mathfrak{U}_{11}|^2+|\mathfrak{U}_{12}|^2}\right)$$
$$\times[(G_F^u)^2(\cos\theta_C\sin\theta_C^*)\{I(\epsilon_L,\epsilon_u)-I(\epsilon_L,\epsilon_t)\}-(G_F^c)^2(\sin\theta_C'\cos\theta_C'^*)\{I(\epsilon_L,\epsilon_c)-I(\epsilon_L,\epsilon_t)\}]\,, \tag{5.19}$$

which exhibits more explicitly the GIM cancellation operating among the u, c, and t quark contributions. In order to illustrate the statement about the double GIM mechanism, it is convenient to consider the special case of the KM model where $\mathfrak{V}$ has the form (5.5) while $\mathfrak{U}$ is still arbitrary, subject to the various experimental constraints discussed previously. In this case, neglecting m_u^2 relative to m_c^2, we have

$$\mathfrak{M}(s\bar{d}\to e\bar{\mu})=\frac{ig^2G_F}{8\pi^2\sqrt{2}}\mathfrak{U}_{13}\mathfrak{U}_{23}^*\sin\theta_C\cos\theta_C(\bar{d}\gamma_\alpha Ls)(\bar{e}\gamma^\alpha L\mu)\frac{\epsilon_L\epsilon_c}{\epsilon_L-\epsilon_c}\ln\left(\frac{m_L{}^2}{m_c{}^2}\right). \tag{5.20}$$

Thus the double GIM mechanism has indeed produced the product $\epsilon_L\epsilon_c$ in the numerator; however, one power of ϵ is essentially canceled by the infrared pole of the Feynman integral, $1/(\epsilon_L-\epsilon_c)$. [By infrared pole, we mean a pole in λ as one scales ϵ_L and ϵ_c down by the factor λ; the total expression in Eq. (5.20) is, of course, regular as $\epsilon_L\to\epsilon_c$.]

Reverting to the general form of the amplitude again, we use the fact that

$$\langle 0|\bar{s}\gamma_\alpha Ld|K^0\rangle=\tfrac{1}{2}f_K(p_K)_\alpha \tag{5.21}$$

to obtain the result

$$\mathfrak{M}(K_L\to e\bar{\mu})=\frac{ig^2G_Ff_Km_\mu}{8\pi^2}\epsilon_L\mathfrak{U}_{13}\mathfrak{U}_{23}^*\sum_{j=1}^{3}\mathrm{Re}(\mathfrak{V}_{1j}\mathfrak{V}_{2j}^*)I(\epsilon_L,\epsilon_{q_j})\bar{e}R\mu\,. \tag{5.22}$$

It is interesting to compare this amplitude for $K_L\to e\bar{\mu}$ with the analogous $|\Delta S|=1$ neutral K decay which conserves e- and μ-lepton number, $K_L\to\mu\bar{\mu}$. In order to get an estimate of the relative rates, it suffices to take the special case of the KM model in which the mixing in the hadronic sector is prescribed by Eq. (5.5). Then the free quark approximation to the decay amplitude is (again dropping m_u^2)

$$\mathfrak{M}(K_L\to\mu\bar{\mu})=\frac{ig^2}{16\pi^2}G_Ff_Km_\mu\epsilon_c\sin\theta_C\cos\theta_C[\mathfrak{U}_{13}\mathfrak{U}_{23}^*\epsilon_LI(\epsilon_L,\epsilon_c)-2]\bar{\mu}\gamma_5\mu\,. \tag{5.23}$$

Making the plausible assumption that $I(\epsilon_L,\epsilon_c)$ is of order unity, we therefore have for the ratio of rates in the free quark approximation

$$\frac{\Gamma(K_L\to e\bar{\mu})}{\Gamma(K_L\to\mu\bar{\mu})}\simeq|\mathfrak{U}_{13}\mathfrak{U}_{23}^*|^2\,. \tag{5.24}$$

However, as was discussed in Ref. 2, the main contribution to the $K_L\to\mu\bar{\mu}$ amplitude comes not from the short-distance free quark graphs but rather from the conventional long-distance $K_L\to\gamma\gamma\to\mu\bar{\mu}$ process. Indeed, if one calculates the part of the $K_L\to\mu\bar{\mu}$ rate due to the elementary quark contribution in the KM model, neglecting the term proportional to $\mathfrak{U}_{13}\mathfrak{U}_{23}^*$ in Eq. (5.23), or equivalently, in the WS model, one finds (normalizing to a typical K decay)

$$\frac{\Gamma(K_L\to\mu\bar{\mu})}{\Gamma(K^-\to\mu^-\bar{\nu})}\simeq\frac{(G_Fm_c{}^2)^2}{2\pi^4}\cos^2\theta_C\frac{(1-4m_\mu{}^2/m_K{}^2)^2}{(1-m_\mu{}^2/m_K{}^2)^2}\,, \tag{5.25}$$

from which it follows that

$$B(K_L\to\mu\bar{\mu})_{\text{free quark}}\simeq0.7\times10^{-11}\,. \tag{5.26}$$

The unitarity bound on the $K_L\to\mu\bar{\mu}$ rate is determined by computing the contribution of the 2γ in-

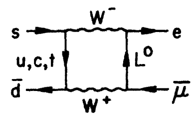

FIG. 4. Graph for the transition $s\bar{d}\to e\bar{\mu}$ in the KM model.

termediate state to the imaginary part of the amplitude. In fact, the 2γ contribution dominates this imaginary part; it can be calculated from the measured rate for the decay $K_L\to\gamma\gamma$ and yields the bound[47]

$$\begin{aligned}\Gamma(K_L\to\mu\bar{\mu}) > \Gamma(K_L\to\mu\bar{\mu})_{\text{absorptive}} \\ \simeq (1.2\times10^{-5})\Gamma(K_L\to\gamma\gamma) \\ \simeq (6\times10^{-9})\Gamma(K_L\to\text{all}). \quad (5.27)\end{aligned}$$

The actual rate is comparable to this bound[48]:

$$\Gamma(K_L\to\mu\bar{\mu})_{\text{exp}} = (1.0\pm0.3)\times10^{-8}. \quad (5.28)$$

Thus evidently the short-distance free-quark processes contribute only about a fraction 10^{-3} of the total rate for $K_L\to\mu\bar{\mu}$.

In view of this fact, if one wishes to estimate the actual decay rate for $K_L\to e\bar{\mu}$, it is of some importance to ascertain whether the short-distance, free-quark contribution to this decay is similarly only a small part of the total rate, and to attempt to compute a unitarity bound from whatever is the dominant intermediate state contribution. We will see that in fact the relative sizes of the direct $s\bar{d}\to e\bar{\mu}$ quark process and the phenomenological $K_L\to\gamma\gamma\to e\bar{\mu}$ process are reversed, compared to the order of importance of $s\bar{d}\to\mu\bar{\mu}$ and $K_L\to\gamma\gamma\to\mu\bar{\mu}$ in the decay $K_L\to\mu\bar{\mu}$. Consequently, modulo short-distance strong-interaction corrections, the elementary reaction $s\bar{d}\to e\bar{\mu}$ probably provides a lower bound not too far below the actual rate for $K_L\to e\bar{\mu}$. Thus for the total rates we estimate (fq denotes free quark contribution)

$$\begin{aligned}\frac{\Gamma(K_L\to e\bar{\mu})_{\text{tot}}}{\Gamma(K_L\to\mu\bar{\mu})_{\text{tot}}} \simeq \frac{\Gamma(K_L\to e\bar{\mu})_{\text{fq}}}{10^3\Gamma(K_L\to\mu\bar{\mu})_{\text{fq}}} \\ \simeq 10^{-3}|\mathfrak{U}_{13}\mathfrak{U}_{23}^*|^2 \quad (5.29)\end{aligned}$$

from which it follows that, for $|\mathfrak{U}_{13}\mathfrak{U}_{23}|^2\sim10^{-2}\text{–}10^{-4}$,

$$B(K_L\to e\bar{\mu})_{\text{tot}} \simeq 10^{-13}\text{–}10^{-15}. \quad (5.30)$$

Unfortunately, if these estimates are reliable, the $K_L\to e\bar{\mu}$ decay is beyond experimental reach at present.

The statement that $\Gamma(K_L\simeq e\bar{\mu})_{\text{tot}}\simeq\Gamma(K_L\to e\bar{\mu})_{\text{fq}}$ is demonstrated by estimating the contribution of the dominant conventional decay chain $K_L\to2\gamma\to e\bar{\mu}$ to the imaginary part of the decay amplitude. Since the photons are on the mass shell, the hadronic side of the graph is then calculable, just as it was for $K_L\to\mu\bar{\mu}$, in terms of the rate for $K_L\to\gamma\gamma$. However, the amplitude for $\gamma\gamma\to e\bar{\mu}$ is far smaller then the one for $\gamma\gamma\to\mu\bar{\mu}$. The Feynman diagrams for the equivalent process $\mu\bar{e}\to\gamma\gamma$ are shown in Fig. 5. There are three general classes of graphs; the first consists of graphs 5(a) and 5(b) in which one photon is emitted via the transition magnetic and electric dipole moment coupling (2.63) and the other is emitted directly. The second set is comprised of the graphs 5(c), 5(d), and 5(e) in which both photons are emitted directly and the μ-e transition proceeds by way of a nondiagonal self-energy insertion. Finally, there are irreducible graphs symbolized by Fig. 5(f) and depicted explicitly in Fig. 6. For the reaction $\mu\bar{e}\to\gamma\gamma$, each of the graphs in Figs. 5 and 6 represents a sum of two separate graphs related to each other by interchange of final photon momenta. The quark analogs of these graphs have been analyzed previously[2] in order to determine an effective Lagrangian for the process $s\bar{d}\to\gamma\gamma$, of relevance to $K_L\to\gamma\gamma$ decay. It was pointed out that the quark counterpart of graph 6(a) (with L^{--} replaced by u and c) dominates over all the other graphs and yields an amplitude of order $\sim(\alpha/\pi)G_F m_K f_K\sin\theta_C\cos\theta_C$. In the present case, however, graphs 6(a) and 6(b) are absent, and hence the amplitude is more severely suppressed by the leptonic GIM mechanism. A detailed calculation of the graphs of Figs. 5, 6(c), and 6(d)

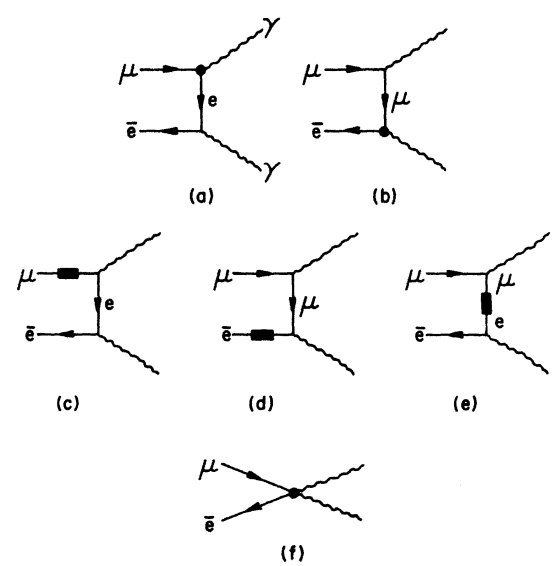

FIG. 5. Graphs contributing in a general SU(2) × U(1) model to the $\mu\bar{e}\to\gamma\gamma$ amplitude. The heavy dots in (a) and (b) represent insertions of the $\mu\to e+\gamma$ one-loop amplitude. The rectangular boxes in (c)–(e) denote nondiagonal μ-e self-energy insertions (see Fig. 3). The heavy dot in Fig. (f) is the sum of graphs in Fig. 6.

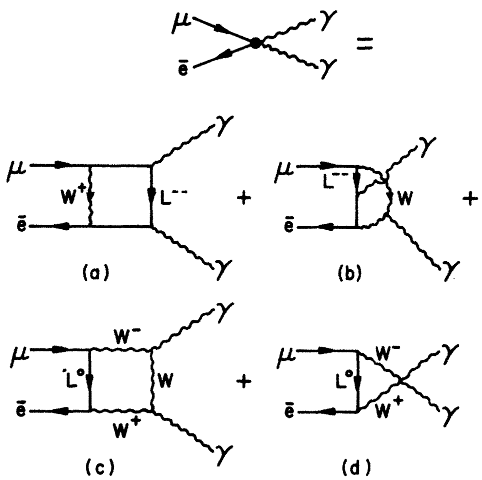

FIG. 6. Graphs contributing in a general SU(2)×U(1) model to the irreducible part of the $\mu\bar{e}\to\gamma\gamma$ amplitude.

would be rather involved; from considerations of (electromagnetic) gauge invariance, we estimate that the total amplitude is of order $(\alpha/\pi)G_F m_K f_K \times(m_L^2/m_W^2)\mathcal{U}_{13}\mathcal{U}_{23}^*$. In contrast, the $\mu\bar{\mu}\to\gamma\gamma$ amplitude is simply of order e^2. Accordingly, we estimate that the ratio of the squares of the imaginary parts of the $K_L\to e\bar{\mu}$ and $K_L\to\mu\bar{\mu}$ amplitudes arising from the two-photon intermediate state is

$$\frac{\Gamma(K_L\to e\bar{\mu})_{\gamma\gamma,\,\text{absorptive}}}{\Gamma(K_L\to\mu\bar{\mu})_{\gamma\gamma,\,\text{absorptive}}}\sim\frac{1}{\pi^4}(G_F m_K f_K)^2\left(\frac{m_{L^0}{}^2}{m_W{}^2}\right)^2 \times|\mathcal{U}_{13}\mathcal{U}_{23}^*|^2\,. \tag{5.31}$$

Using $(m_{L^0}/m_W)\simeq\frac{1}{6}$, $|\mathcal{U}_{13}\mathcal{U}_{23}^*|\lesssim .05$, and Eq. (5.27), we find that

$$[\Gamma(K_L\to e\bar{\mu})_{\gamma\gamma,\,\text{absorptive}}]/\Gamma(K_L\to\text{all})\lesssim 10^{-30}\,.$$

Indeed, even without the GIM suppression factor $(m_{L^0}{}^2/m_W{}^2)|\mathcal{U}_{13}\mathcal{U}_{23}^*|^2$ this ratio would be of the order 10^{-21}. These are admittedly very rough estimates, but they show that because of the μ- and e-type lepton-number-violating nature of the decay $K_L\to e\bar{\mu}$ the elementary free quark process $s\bar{d}\to e\bar{\mu}$ probably dominates the total amplitude, as was claimed.

We next turn to K decays of the form $K\to\pi e\bar{\mu}$ and compare them with the corresponding muon- and electron-number conserving decays $K\to\pi e\bar{e}$. Since our primary interest is in μ- and e-number nonconservation effects, we again choose $\mathcal{V}$ to have the form given in Eq. (5.5). The fundamental quark transitions involved are respectively $s\to de\bar{\mu}$ and $s\to de\bar{e}$. The graph for the former process is just Fig. 4 with the d quark line crossed, and the resulting amplitude in the KM model is given by the crossed version of Eq. (5.20) for $\mathfrak{M}(s\bar{d}\to e\bar{\mu})$. The diagrams contributing to $s\to de\bar{e}$ are the counterparts of Figs. 2(a)–2(d) and 2(g), with appropriate replacements of leptons by quarks. The calculation is similar to the one performed previously.[2] The dominant contribution to the amplitude comes from the unsuppressed logarithm in the vector and axial-vector charge radius arising from the quark analog of diagram $2(\text{b})_\gamma$. This can be calculated via a Fierz transform in terms of the quantum electrodynamics vacuum polarization integral; the result is the same as in the WS model and yields an amplitude

$$\mathfrak{M}(s\to d+e+\bar{e})\simeq\frac{2i}{3}\,Q_c\frac{\alpha}{\pi}\frac{G_F}{\sqrt{2}}\ln\left(\frac{m_c{}^2}{m_K{}^2}\right)\sin\theta_C\cos\theta_C \times(\bar{d}\gamma_\alpha Ls)(\bar{e}\gamma^\alpha e)\,, \tag{5.32}$$

where $Q_c=\frac{2}{3}$.

Using Eqs. (5.20) and (5.32), we find

$$\frac{\Gamma(K^-\to\pi^-e\bar{\mu})}{\Gamma(K^-\to\pi^-e\bar{e})}=\frac{\Gamma(K_S^0\to\pi^0e\bar{\mu})}{\Gamma(K_S^0\to\pi^0e\bar{e})} \simeq 2\left(\frac{3\csc^2\theta_W}{8Q_c}\right)^2\left[\frac{\left(\frac{\epsilon_L\epsilon_c}{\epsilon_L-\epsilon_c}\right)\ln\left(\frac{m_L{}^2}{m_c{}^2}\right)}{\ln\left(\frac{m_L{}^2}{m_K{}^2}\right)}\right]^2|\mathcal{U}_{13}\mathcal{U}_{23}^*|^2\,. \tag{5.33}$$

For $m_c=1.5$ GeV, $m_{L^0}=10$ GeV, $m_W=60$ GeV, $\sin^2\theta_W\sim\frac{1}{3}$, and $|\mathcal{U}_{13}\mathcal{U}_{23}^*|^2\lesssim .05$ the ratio (5.33) is $\sim 10^{-4}$. The free-quark approximation for $B(K^-\to\pi^-e\bar{e})$ calculated[2] from Eq. (5.32) is

$$B(K^-\to\pi^-e\bar{e})_{\text{fq}}\simeq 5\times10^{-7}\,. \tag{5.34}$$

In contrast to the case with the short-distance, free-quark contribution to $K_L\to\mu\bar{\mu}$, this is comparable to the measured value

$$B(K^-\to\pi^-e\bar{e})_{\text{exp}}\simeq(2.6\pm0.5)\times10^{-7}\,. \tag{5.35}$$

Therefore, from Eqs. (5.33) and (5.34) we estimate using the same values of parameters that

$$B(K^-\to\pi^-e\bar{\mu})\simeq 10^{-11}\,, \tag{5.36}$$

which again is too small to be of much experimental interest. A similar comment applies to the decay $K_S\to\pi^0e\bar{\mu}$.

In the case of the decay $K_L\to\pi^0e\bar{e}$, CP conservation forbids the occurrence of the quark analogs of the γ- and Z-exchange diagrams, 2(a) and $2(\mathrm{b})_\gamma$ and $2(\mathrm{a})_Z$–$2(\mathrm{d})_Z$. (This decay is CP conserving even if $\mathfrak{U}$ contains a CP-violating phase, as long as $\mathfrak{V}$ itself is CP conserving.) Thus, only the W^+W^- exchange diagram contributes, and in contrast to (the charge radius term in) graph $2(\mathrm{b})_\gamma$, it is fully suppressed by the GIM mechanism. Indeed, the local form of the interaction (i.e., the form in which external momenta are neglected relative to m_W) due to W^+W^- exchange is also of the current × current type, and consequently CP invariance implies that its hadronic matrix element vanishes. This is true for both of the decays $K_L \to \pi^0 e\bar{e}$ and $K_L \to \pi^0 e\bar{\mu}$ (or $K_L \to \pi^0 \bar{e}\mu$). Nonlocal effects are present at the level $(m_K{}^2/m_W{}^2)$ and they do in principle allow these processes. We expect that

$$\Gamma(K_L \to \pi^0 e\bar{\mu}) \simeq |\mathfrak{U}_{13}\mathfrak{U}_{23}^*|^2 \Gamma(K_L \to \pi^0 e\bar{e})\,;$$

both rates however, are extremely small. For the larger one, normalizing to a dominant semileptonic decay mode, we have, for the free quark approximation,

$$\Gamma(K_L \to \pi^0 e\bar{e})/\Gamma(K_L \to \pi^0 e\bar{\nu}_e) \sim (\alpha/\pi)^2 (m_c{}^2/m_W{}^2)^2 (m_K{}^2/m_W{}^2)^2 \sim 10^{-20}\,.$$

Even if the estimated branching ratio were nonnegligible, the π^0 in the final state would render it experimentally difficult to set a very stringent upper bound on the decay $K_L \to \pi^0 e\bar{\mu}$.

Thus, the effects of μ- and e-type lepton-number nonconservation in K meson decays may well be too small to measure. We shall proceed to consider certain lepton-hadron reactions and decays of hevy leptons, where such effects are probably more easily observable.

C. Production and decay of L^0 and L^-

If the decay $\mu \to e\gamma$ is indeed detected at the 10^{-9}–10^{-10} level in branching ratio it would be reasonable to consider the heavy lepton L^0 in the KM model to have a mass $m_{L^0} \sim 10$ GeV. For example, if $B(\mu \to e\gamma) = 10^{-9}$, $m_W = 60$ GeV, and $|\mathfrak{U}_{13}\mathfrak{U}_{23}^*|^2 = 0.3 \times 10^{-2}$, then $m_{L^0} \simeq 12$ GeV. Moreover, as was mentioned earlier, it is plausible, although not necessary, to entertain the possibility that L^- is the heavy lepton observed at SPEAR and DESY.[23] If one chooses to make this identification then $m_{L^-} \simeq 1.9$ GeV, substantially lighter than the L^0. A crucial test of this hypothesis regarding L^- is to determine whether the experimentally observed heavy lepton decays via a $V-A$ coupling. Of course it is entirely possible that (a) the decay $\mu \to e\gamma$, if it exists at all, proceeds with a branching ratio much smaller than 10^{-9}–10^{-10} and/or (b) the KM model represents only a part of the presumed complete gauge model, and L^- is not the SPEAR-DESY heavy lepton. Accordingly, there is no strong theoretical or experimental reason why the L^- cannot have a greater mass than the L^0. Indeed some of the trimuon events recently observed in the Caltech-Fermilab[49] and Harvard-Pennsylvania-Wisconsin-Fermilab[50] neutrino experiments may be due to the sequential production and decay chain

$$\begin{aligned}\nu_\mu + N \to\ & L^- + X \\ & \searrow L^0 + \mu^- + \bar{\nu}_\mu \\ & \qquad \searrow \mu^- + \mu^+ + \nu_\mu\,.\end{aligned} \tag{5.37}$$

If one identifies the two heavy leptons in this process with the L^- and L^0 of the KM model, then of course it is necessary that $m_{L^-} > m_{L^0}$. In a phenomenological spirit we shall consider both orderings of L^- and L^0 masses.

Let us first consider production mechanisms for the L^- and L^0.[51] The L^- can be produced in the reaction $e^+e^- \to L^+L^-$ with a well-known cross section. The $e\mu$ events seen at SPEAR would arise from the process

$$\begin{aligned}e^+e^- \to\ & L^+L^- \\ & \searrow e^-\bar{\nu}_e\nu_L \\ & \searrow \mu^+\nu_\mu\bar{\nu}_L\ ,\end{aligned} \tag{5.38}$$

where we define ν_L to be the unit-normalized linear combination of ν_1 and ν_2 which couples to L^-,

$$\nu_L = \frac{\mathfrak{U}_{31}\nu_1 + \mathfrak{U}_{32}\nu_2}{(|\mathfrak{U}_{31}|^2 + |\mathfrak{U}_{32}|^2)^{1/2}} \tag{5.39}$$

in analogy with Eqs. (3.2a) and (3.2b) for ν_e and ν_μ. The $L^\pm$ can also be produced in the high-energy neutrino-nucleon reactions

$$\begin{aligned}\nu_\mu N \to\ & L^- + X \\ & \searrow \mu^-\bar{\nu}_\mu\nu_L \\ & \quad\ e^-\bar{\nu}_e\nu_L \\ & \quad\ \nu_L + \text{hadrons}\end{aligned} \tag{5.40}$$

and the corresponding reaction with incident $\bar{\nu}_\mu$. The cross section is suppressed by the mixing parameters; well above threshold for L^- production the relative rate is given by

$$\begin{aligned}\frac{\sigma(\nu_\mu N \to L^- + X)}{\sigma(\nu_\mu N \to \mu^- + X)} &= \frac{\sigma(\bar{\nu}_\mu N \to L^+ + X)}{\sigma(\bar{\nu}_\mu N \to \mu^+ + X)} \\ &= \frac{|\mathfrak{U}_{31}^*\mathfrak{U}_{21} + \mathfrak{U}_{32}^*\mathfrak{U}_{22}|^2}{|\mathfrak{U}_{21}|^2 + |\mathfrak{U}_{22}|^2} \\ &\simeq |\mathfrak{U}_{23}|^2\,.\end{aligned} \tag{5.41}$$

Thus the L^- production cross section is $\lesssim 5\times10^{-2}$ of the corresponding ordinary charged-current cross section. Furthermore, the leptonic decay modes are expected to comprise $\lesssim 40\%$ of the total, so that the signal will be present at the 10^{-2}–10^{-3} level. Reaction (5.40) will be difficult to observe in counter experiments because the μ^- decay mode simulates an ordinary charged-current reaction, the e^- is not detectable with present counter apparatus, and the semileptonic decay simulates a neutral-current event. The same statements apply to the reaction $\bar{\nu}_\mu N \to L^+ + X$. Because both the leptonic and hadronic charged currents in the KM model are purely $V-A$, in the valence-quark model, which serves as a reasonable approximation, $\sigma(\nu_\mu N \to L^- + X) = 3\sigma(\bar{\nu}_\mu N \to L^+ + X)$. In heavy liquid bubble-chamber experiments it is feasible to search for the electron decay mode of the L^- produced in reaction (5.40). Indeed if the L^- is sufficiently light and/or the time dilation factor is sufficiently large its tracks may be visible in the bubble chamber (see below). Characteristic kinematic features of heavy lepton production via reaction (5.40) include, first, a threshold behavior as a function of incident neutrino energy E and, for a fixed E, as a function of W, the invariant mass of the hadrons produced by the W boson-nucleon interaction at the "lower" vertex. Second, one would measure large values of $y_{\text{eff}} = 1-(E_{\mu^-}/E)$ since a considerable portion of the incident energy is carried off by unobserved neutrinos. Moreover, the semileptonic decay of the outgoing L^- will yield hadrons with significant fractions of the initial beam energy. If $m_{L^-} > m_{L^0}$ the sequential decay scheme of Eq. (5.37) will yield trimuon events, as was mentioned. Finally, note that since the neutral current is diagonal in flavors in the KM model, it is not possible to produce the L^- in a reaction such as $\mu^- + N \to L^- + X$ at a non-negligible rate.

The neutral heavy lepton L^0 is more difficult to produce than the L^-. Among e^+e^- reactions the one with the lowest threshold is $e^+e^- \to L^0\bar{\nu}_e$, which proceeds via W exchange in the t channel. Unfortunately, in addition to being a weak process, its cross section is further suppressed by the factor $|\mathfrak{U}_{13}|^2$. There is also a neutral-current reaction $e^+e^- \to L^0\bar{L}^0$ which is not suppressed by any small mixing angles. Of course until the energy is reasonably far above the respective thresholds these reactions will be suppressed by small phase-space factors. Since the cross sections in both cases are of the order $G_F^2 s$, in order for them to be significant one needs center-of-mass energies equal at least to the value $\sqrt{s} \sim 36$ GeV to be attained at PEP and PETRA. The L^0 cannot be produced via the neutrino reaction $\nu_\mu N \to L^0 + X$ for the reason given previously. It can be produced in the charged-current reaction $\mu^- + N \to L^0 + X$; however, the electromagnetic background from $\mu^- + N \to \mu^- + X$ is severe, and again the cross section is proportional to the mixing parameter eter $|\mathfrak{U}_{23}|^2$, which may be rather small.

The branching ratios for the various decay modes of the L^- and L^0 depend on their relative masses and on the mixing parameters in the matrix $\mathfrak{U}$. We consider first the leptonic decays of L^- and assume that it is lighter than the L^0. Then for the decay mode $L^- \to e\bar{\nu}_e \nu_L$ we have

$$\frac{\Gamma(L^- \to e\bar{\nu}_e\nu_L)}{\Gamma(\mu \to e\bar{\nu}_e\nu_\mu)} = \left(\frac{m_{L^-}}{m_\mu}\right)^5 \frac{(|\mathfrak{U}_{31}|^2 + |\mathfrak{U}_{32}|^2)}{(|\mathfrak{U}_{21}|^2 + |\mathfrak{U}_{22}|^2)} . \quad (5.42)$$

For small mixing this ratio is approximately equal to $(m_{L^-}/m_\mu)^5 (|\mathfrak{U}_{13}|^2 + |\mathfrak{U}_{23}|^2)$, i.e., $(|\mathfrak{U}_{21}|^2 + |\mathfrak{U}_{22}|^2) \simeq 1$. The same formula applies for the decay modes $L^- \to \mu\bar{\nu}_\mu\nu_L$ with the replacement of $(|\mathfrak{U}_{21}|^2 + |\mathfrak{U}_{22}|^2)$ by $(|\mathfrak{U}_{11}|^2 + |\mathfrak{U}_{12}|^2)$. If $m_{L^-} > m_{L^0}$ then the decay $L^- \to e\bar{\nu}_e L^0$ occurs, at a rate given by

$$\frac{\Gamma(L^- \to e\bar{\nu}_e L^0)}{\Gamma(\mu \to e\bar{\nu}_e\nu_\mu)} = \left(\frac{m_{L^-}}{m_\mu}\right)^5 \frac{|\mathfrak{U}_{33}|^2}{(|\mathfrak{U}_{21}|^2 + |\mathfrak{U}_{22}|^2)} f\left(\frac{m_{L^0}}{m_{L^-}}\right), \quad (5.43)$$

where

$$f(x) = (1-x^4)(x^4 - 8x^2 + 1) + 24x^4 \ln(1/x), \quad (5.44)$$

with $x = (m_{L^0}/m_{L^-})$. The rate for $L^- \to e\bar{\nu}_e L^0$ is in one way enhanced relative to that for $L^- \to e\nu_e\nu_L$ because $|\mathfrak{U}_{33}|^2 \sim 1$, whereas $(|\mathfrak{U}_{13}|^2 + |\mathfrak{U}_{23}|^2) \sim 10^{-2}$. However, in another way it is suppressed by the phase-space factor $f(m_{L^0}/m_{L^-})$; for example, $f(\frac{1}{2}) \simeq 0.16$. Equation (5.43) also applies to the decay mode $L^- \to \mu\bar{\nu}_\mu L^0$ with the replacement given above. In order to estimate the semileptonic modes we shall use the free-quark model or, equivalently to our order of approximation, the SPEAR results on $R = \sigma(e\bar{e} \to \text{hadrons})/\sigma(e\bar{e} \to \mu\bar{\mu})$. If $m_{L^-} \lesssim 2$ GeV then the decay channels which involve the t or b quarks are not open, and the $\bar{c}(s,d)$ channels are either below or only slightly above threshold. Consequently,

$$\frac{\Gamma(L^- \to \nu_L + \text{hadrons})}{\Gamma(\mu \to e\bar{\nu}_e\nu_\mu)} \simeq \frac{\sum_{q=d,s} \Gamma(L^- \to \nu_L \bar{u} q)}{\Gamma(\mu \to e\bar{\nu}_e\nu_\mu)} = 3\left(\frac{m_{L^-}}{m_\mu}\right)^5 \frac{(|\mathfrak{V}_{11}|^2 + |\mathfrak{V}_{21}|^2)(|\mathfrak{U}_{31}|^2 + |\mathfrak{U}_{32}|^2)}{(|\mathfrak{U}_{11}|^2 + |\mathfrak{U}_{12}|^2)(|\mathfrak{U}_{21}|^2 + |\mathfrak{U}_{22}|^2)}, \quad (5.45)$$

where the factor of 3 comes from the sum over quark colors. The ratio (5.45) is approximately equal to

$3(m_{L^-}/m_\mu)^5(|\mathfrak{U}_{31}|^2+|\mathfrak{U}_{32}|^2)$. Next consider the case of $m_{L^-}>m_c,m_t$. Then the $L^-\to\nu_L\bar{c}q,\nu_L\bar{t}q$, decay channels are open (where $q=d,s,b$). In the simple case where all quark masses are small compared to m_{L^-} the dependence upon $\mathfrak{V}$ disappears and we find

$$\frac{\sum_{\text{quarks}}\Gamma(L^-\to\nu_L\bar{q}_i q_j)}{\Gamma(\mu\to e\bar{\nu}_e\nu_\mu)}=9\left(\frac{m_{L^-}}{m_\mu}\right)^5\frac{(|\mathfrak{U}_{31}|^2+|\mathfrak{U}_{32}|^2)}{(|\mathfrak{U}_{11}|^2+|\mathfrak{U}_{12}|^2)(|\mathfrak{U}_{21}|^2+|\mathfrak{U}_{22}|^2)}. \tag{5.46}$$

If also $m_{L^-}\gg m_{L^0}$ the dependence upon $\mathfrak{U}$ in the numerator of the ratio (5.46) disappears and this ratio becomes approximately $9(m_{L^-}/m_\mu)^5$. Thus if $m_{L^-}\simeq 2$ GeV, with $m_{L^-}<m_{L^0},m_{t,b}$ the ratios of leptonic to semileptonic decays are

$$\frac{\Gamma(L^-\to\nu_L e^-\bar{\nu}_e)}{\Gamma(L^-\to\text{all})}=\frac{\Gamma(L^-\to\nu_L\mu^-\bar{\nu}_\mu)}{\Gamma(L^-\to\text{all})}\sim\tfrac{1}{5}, \tag{5.47}$$

whereas if $m_{L^-}\gg m_{L^0},m_q$, this ratio is approximately $\frac{1}{11}$.

For the low-mass case, summing the leptonic and semileptonic decay modes, we find

$$\frac{\Gamma(L^-\to\text{all})}{\Gamma(\mu^-\to e\bar{\nu}_e\nu_\mu)}\simeq 5\left(\frac{m_{L^-}}{m_\mu}\right)^5(|\mathfrak{U}_{31}|^2+|\mathfrak{U}_{32}|^2), \tag{5.48}$$

where we have used the approximation that $(|\mathfrak{U}_{j1}|^2+|\mathfrak{U}_{j2}|^2)\simeq 1$ for $j=1,2$. At the other extreme of very large L^- mass we have with the same approximation that

$$\frac{\Gamma(L^-\to\text{all})}{\Gamma(\mu^-\to e\bar{\nu}_e\nu_\mu)}\simeq 11\left(\frac{m_{L^-}}{m_\mu}\right)^5. \tag{5.49}$$

This relative decay rate is larger by at least a factor of 10^2 than the rate for small L^- mass, Eq. (5.48). The interesting thing to observe is that in the case of small m_{L^-} the rate is substantially suppressed by small mixing angles and consequently the L^- is longer lived by a factor of $(|\mathfrak{U}_{31}|^2+|\mathfrak{U}_{32}|^2)^{-1}\sim 10^2$ than a naive scaling from $m_\mu{}^5$ to $m_{L^-}{}^5$ would indicate. Numerically, Eq. (5.48) gives a lifetime for the small mass (sm) case of

$$\tau_{L^-}^{(\text{sm})}\simeq 6\times 10^{-10}\left(\frac{1\text{ GeV}}{m_{L^-}}\right)^5\left(\frac{10^{-2}}{|\mathfrak{U}_{31}|^2+|\mathfrak{U}_{32}|^2}\right)\text{sec}. \tag{5.50}$$

The lifetime in the large-mass case is

$$\tau_{L^-}^{(\text{lm})}\simeq 2.7\times 10^{-12}\left(\frac{1\text{ GeV}}{m_{L^-}}\right)^5. \tag{5.51}$$

The major factor in determining the lifetime is whether the $L^-\to L^0+\cdots$ transitions can occur, and if so, how much they are suppressed by phase space.

The analysis of the L^0 decay modes is quite similar and for brevity the details will be omitted. Note that if m_{L^0} is sufficiently large the decay modes $L^0\to L^-L^+\nu_L$ can occur. However, since the sum of the rates for these modes is proportional to $(|\mathfrak{U}_{31}|^2+|\mathfrak{U}_{32}|^2)$ they will not make a very large contribution to the decay rate even if $m_{L^0}\gg 2m_{L^-}$. The lifetimes in the low- and high-mass cases are thus given byEqs. (5.50) and (5.51) with the replacement $L^-\to L^0$.

In addition to these dominant decay modes there will be other, rarer decay modes. We shall not consider these here since they will be extremely difficult to observe. Yet another way in which the lepton L^0, if light enough, may make its existence manifest is in the leptonic decays of charmed pseudoscalar mesons, D^+, $F^+\to L^0 l^+$, where $l=e,\mu$.[52]

D. Other processes

We shall mention here two other interesting processes involving μ- and e-type lepton-number nonconservation. The first is the reaction $\mu+N\to e+N$, in which a slow muon is absorbed in matter and decays to an electron in the Coulomb field of a nucleus. This reaction was proposed and studied long ago as a means of testing for muon- and electron-number violation.[53] The signal is a high-energy electron emitted with $|\vec{p}_e|_{\text{lab}}\sim m_\mu$. The rate can be calculated from the general formulas of Ref. 53 together with our expression for the $\mu\to e+\gamma_{\text{virtual}}$ amplitude. A recent analysis yields an interestingly large ratio of rates in the KM model,[54] $R(\mu N\to eN)/R(\mu\to e\gamma)\sim 26$.

Second, there is the precise analog of K^0-$\bar{K}^0$ mixing ($s\bar{d}\leftrightarrow\bar{s}d$) which gives rise to the K_L-K_S mass difference, for the $\mu\bar{e}$ bound state, muonium, namely the transition $\mu\bar{e}\leftrightarrow\bar{\mu}e$.[55] We find that in contrast to the case with K^0-$\bar{K}^0$ mixing, a $\mu\bar{e}$ bound state will decay long before it has a significant probability to make a transition to $\bar{\mu}e$. This is easily seen as follows. The diagonal elements of the mass matrix $M-i\Gamma/2$ are dominated by the contribution to $i\Gamma$ arising from ordinary μ decay: $\Gamma\sim G_F{}^2m_\mu{}^5/(192\pi^3)$. The main contribution to the off-diagonal matrix elements arises from the W^+W^- exchange diagram shown in Fig. 7, which is the leptonic counterpart of the graph for $s\bar{d}\leftrightarrow\bar{s}d$. Other contributions to the off-diagonal matrix elements such as $\mu\bar{e}\to(\gamma\gamma)_{\text{virtual}}\to\bar{\mu}e$ are negligible in comparison. Diagram (7) gives a mass difference between the CP eigenstates $(\mu\bar{e}\pm\bar{\mu}e)/\sqrt{2}$ of

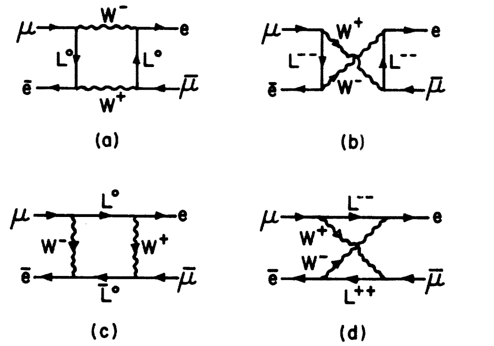

FIG. 7. Diagrams contributing in a general SU(2) × U(1) model to the transition $\mu\bar{e}\to\bar{\mu}e$.

$$\Delta m \sim \frac{\alpha}{\pi} G_F\left(\frac{m_{L^0}{}^2}{m_W{}^2}\right)|\mathfrak{U}_{13}\mathfrak{U}_{23}^*|^2|\psi(0)|^2, \qquad (5.52)$$

where $\psi(0)$ is the muonium wave function in the ground state. Using $|\mathfrak{U}_{13}\mathfrak{U}_{23}|^2\sim 10^{-3}$, $m_{L^0}/m_W \sim \frac{1}{6}$, and $|\psi(0)|^2 \simeq \alpha^3 m_e{}^3/\pi$, we find that $\Delta m/\Gamma \sim 10^{-18}$.

VI. CONCLUSIONS

In this work we have analyzed μ- and e-type lepton-number conservation, viewed an approximate symmetry such as strangeness conservation by neutral currents, or CP invariance. We have pointed out the special set of circumstances which guarantees exact μ- and e-lepton-number conservation in the minimal Weinberg-Salam model and have investigated the ways in which this invariance is violated, as it usually is, when one generalizes the minimal model. It has been stressed that although, *a priori*, one would expect violations of the three symmetries mentioned above to occur in order $G_F\alpha$, in fact experimentally such violations are further suppressed. Extending earlier work,[2,3] we have proposed a unified approach to these three approximate symmetries based on a mechanism which naturally suppresses the violation of a particular symmetry.

By calculating the $\mu\to e\gamma$ and $\mu\to ee\bar{e}$ decay rates in a general formalism applicable to any SU(2) × U(1) gauge theory, we have derived a set of conditions which ensures that, for arbitrary values of the parameters of the theory and thus a completely general mass matrix M, the nonconservation of μ- and e-type lepton number is naturally suppressed. These conditions are that (1) leptons of a given charge and chirality have the same weak T and T_3; (2) leptons of a given charge receive their masses from their couplings to a single neutral Higgs field, and (3) leptons of charge q and leptons of charge $q\pm 1$ do not belong to the same weak isomultiplet for at least one chirality. The first two conditions are essentially the Glashow-Weinberg criteria[6] for natural leptonic-flavor conservation by the neutral current. The last condition is the leptonic analog of the criterion for microweak CP violation derived previously.[3] Furthermore, in order for a model[9] to have a branching ratio for $\mu\to e\gamma$ of order 10^{-10} or larger, it must include at least one neutral or doubly negatively charged heavy lepton which is coupled to both e and μ.

We have considered several models which meet these conditions, focusing on the $V-A$ three-doublet model of Kobayashi and Maskawa.[11] In the KM model, for a wide range of parameters (mixing angles and heavy-lepton masses) the branching ratio for the decay $\mu\to e\gamma$ is in accord with, but not extremely small compared to, the present experimental limit; specifically, assuming that $|\mathfrak{U}_{13}\mathfrak{U}_{23}^*|^2=10^{-3}$, $B(\mu\to e\gamma)\simeq 1.7\times 10^{-10}(m_{L^0}/10\ \text{GeV})^4$. In addition to measuring the total decay rate, it will be very useful to determine the angular distribution when the muon is polarized and also the photon polarization, the latter even if the muon is unpolarized. This will give information on the relative sizes and signs of $F_2^V(0)_{e\mu}$ and $F_2^A(0)_{e\mu}$.

The decay $\mu\to ee\bar{e}$ is estimated to proceed with a rate $\Gamma(\mu\to ee\bar{e})_{\rm KM}\simeq 0.06\Gamma(\mu\to e\gamma)_{\rm KM}$. In contrast, in models with doubly negatively charged heavy leptons which couple to both e and μ, such as the Wilczek-Zee models,[31] while the decay $\mu\to e\gamma$ is still fully suppressed by the leptonic GIM mechanism, the decay $\mu\to ee\bar{e}$ is not. Consequently in such models $\Gamma(\mu\to ee\bar{e})$ is considerably larger than $\Gamma(\mu\to e\gamma)$ [see e.g. Eq. (4.36)]. In order to keep the $\mu\to ee\bar{e}$ decay rate below the experimental upper limit, it is necessary that the heavy-lepton masses be very close to each other and/or their mixing angles be small. Our analysis also shows that in models such as that of Cheng and Li,[10] in which the mass matrix M is not completely arbitrary, owing to restrictions on the representation content of the Higgs bosons, the general conditions for naturally suppressed μ- and e-lepton-number violation can be weakened.

The mixing of mass eigenstates to form weak eigenstates in the KM model will in general cause not just muon- and electron-lepton-number violation but also small violations of μ-e universality, the hadron-lepton weak universality equality $G_F^\beta \sec\theta_C = G_F^\mu$, and the Cabibbo theory of weak decays of baryons. It is thus of continuing interest to improve the experimental limits on such violations. Furthermore, in this model ν_e and ν_μ are not orthogonal, so that one may expect to observe, at some level, the reaction $\nu_\mu + N\to e^- + X$, where the electron comes from the leptonic vertex. The heavy leptons in the KM model could be produced in e^+e^- reactions such as $e^+e^- \to L^+L^-$ and $e^+e^- \to L^0\bar{\nu}_e$, or in neutrino reactions such as $\nu_\mu + N \to L^- + X$; in the last case the cascade decay $L^- \to L^0\mu^-\bar{\nu}_\mu \to \mu^+\mu^-\mu^-\nu_\mu\bar{\nu}_\mu$ would produce trimuon events.

Thus the present experimental limits on decays such as $\mu \to e\gamma$, $\mu \to ee\bar{e}$, and other μ- and e-type lepton-number-violating processes can already be used to constrain models of weak interactions. Further experimental searches with improved sensitivity, whether they yield null or positive results, promise to contribute substantially to the understanding of the structure of weak interactions.

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APPENDIX

We shall give here a proof that the self-energy terms in Figs. 2 and 3 contribute only to the renormalization of $F_1^{V,A}(0)$ and therefore have no effect on the $\mu \to e\gamma$ amplitude or on $\mathfrak{M}^{(\gamma)}(\mu \to ee\bar{e})$, the virtual-photon contribution to the $\mu \to ee\bar{e}$ amplitude. We start with the electromagnetic Ward-Takahashi identity

$$q^\lambda V_\lambda^{V,A}(p_1,p_2;q) = Q[\Sigma^{s,p}(\not{p}_1) - \Sigma^{s,p}(\not{p}_2)], \qquad \text{(A1)}$$

where iV_λ is the proper (single-particle-irreducible) photon-fermion vertex between an initial and a final fermion of momentum p_1, and p_2, respectively, and $q = p_1 - p_2$, as in the text. Equation (A1) holds separately for the parity-conserving parts V_λ^V and Σ^s, and for the parity-violating parts V_λ^A and Σ^p.

The effective vertex, including off-diagonal fermion self-energy insertions, is

$$iE_\lambda^{V,A} = iV_\lambda^{V,A} + iQ\gamma_\lambda \frac{i}{\not{p}_1 - m_2} i\Sigma^{s,p}(\not{p}_1) + i\Sigma^{s,p}(\not{p}_2)\frac{i}{\not{p}_2 - m_1} iQ\gamma_\lambda.$$

We shall concentrate on the parity-conserving part iE_λ^V; the same argument applies for the parity-violating part. We note that

$$\bar{u}_2(p_2)iE_\lambda^V u_1(p_1)\big|_{p_2^2=m_2^2,\,p_1^2=m_1^2} = \bar{u}_2(p_2)i\left[V_\lambda^V(q) - Q\gamma_\lambda \frac{\Sigma^s(m_1) - \Sigma^s(m_2)}{m_1 - m_2}\right]u_1(p_1), \qquad \text{(A2)}$$

where

$$\bar{u}_2 V_\lambda^V(q)u_1 = \bar{u}_2 V_\lambda^V(p_1,p_2;q)u_1\big|_{p_1^2=m_1^2,\,p_2^2=m_2^2}. \qquad \text{(A3)}$$

Note first that the contribution of off-diagonal self-energy insertions is q^2 independent. The form of $V_\lambda^V(q)$ is

$$V_\lambda^V(q) = \gamma_\lambda V_1(q) + i\sigma_{\lambda\mu}q^\mu V_2(q^2) + q_\lambda V_3(q^2). \qquad \text{(A4)}$$

By taking the matrix element of Eq. (A1) between the spinors $\bar{u}_2(p_2)$ and $u_1(p_1)$, and making use of Eqs. (3, 4) we find that

$$V_1(0) = Q\frac{\Sigma^s(m_1) - \Sigma^s(m_2)}{m_1 - m_2}. \qquad \text{(A5)}$$

Combining now Eqs. (A2), (A4), and (A5)

$$\bar{u}_2(p_2)iE_\mu^V u_1(p_1) = \bar{u}_2(p_2)i\{\gamma_\mu[V_1(q^2) - V_1(0)] + i\sigma_{\mu\nu}q^\nu V_2(q^2) + q_\mu V_3(q^2)\}u_1(p_1). \qquad \text{(A6)}$$

Equations (A5) and (A6) prove our assertions at the beginning of this Appendix; note further that $V_1(q^2) - V_1(0)$ is finite whether or not V_λ^V and Σ^s have been renormalized.

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†With deepest regret, the tragic death on June 16, 1977 of Benjamin W. Lee must be recorded here.

*Operated by Universities Research Association Inc. under contract with the Energy Research and Development Administration.

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[12]For a brief discussion of muon- and electron-number nonconservation, see B. W. Lee, S. Pakvasa, R. E. Shrock, and H. Sugawara, Phys. Rev. Lett. 38, 937 (1977); 38, 1230(E) (1977). *Note added in proof.* The question of how much fermion mixing is allowed in the KM model discussed in this paper, and thus how large the muon-number nonconservation can be, has been the subject of some recent confusion in the literature. It has, for example, been claimed that quark-lepton Cabibbo universality and e-μ universality together imply that $|\mathfrak{U}_{j3}| < 0.1$ for $j = 1, 2$, and hence that in order to obtain a branching ratio $B(\mu \to e\gamma) \simeq 10^{-9}$ the neutral heavy lepton L^0 must be quite heavy: $m_{L^0} \gtrsim 20$ GeV. Both of these claims were somewhat overstated. In fact, as Eq. (5.15a) shows, quark-lepton Cabibbo universality is consistent with large values of $|\mathfrak{U}_{13}|$ (less than unity since it is an element of a unitary matrix) and only implies that $|\mathfrak{U}_{23}| - |\mathfrak{V}_{23}|$ is small, not that $|\mathfrak{U}_{23}|$ and $|\mathfrak{V}_{23}|$ must individually be small. Stated in words, quark-lepton Cabibbo universality places no restriction on the amount of mixing between different quarks; it only implies that these mixings must be sufficiently similar that in Eq. (5.15a) $R_{\mu/\beta}$ is equal to unity, to within ~1%. Furthermore, as Eq. (5.10) shows, e-μ universality places no restriction on the size of $|\mathfrak{U}_{13}|$ $|\mathfrak{U}_{23}|$ (less than unity); it simply requires that $|\mathfrak{U}_{13}| - |\mathfrak{U}_{23}|$ be sufficiently small that $R_{e/\mu}$ be as close to unity as Eq. (5.11) dictates. Again, in words, e-μ universality places no limit on how much e and μ are mixed with L^-; it only requires that they be mixed by the same or almost the same amount, as determined by the experimental limit of $|R_{e/\mu} - 1|$. The actual bound on $|\mathfrak{U}_{13}\mathfrak{U}_{23}^*|$ (which is the combination of these two mixing parameters relevant for μ- and e-number-violating processes) comes neither from the constraint of quark-lepton universality nor from the constraint of e-μ universality taken alone, but rather from the limit on $\sigma(\nu_\mu N \to e^- X)$, where the electron originates at the leptonic vertex. This limit implies that $|\mathfrak{U}_{13}\mathfrak{U}_{23}^*|^2 \lesssim 3\times 10^{-3}$, not 10^{-4}, as some have claimed. Thus for $B(\mu \to e\gamma) \simeq 10^{-9}$, $m_{L^0} \simeq 12$ GeV, not $m_{L^0} \gtrsim 20$ GeV.

[13]For earlier work on the diagonal electromagnetic vertex (g-2 calculation) see K. Fujikawa, B. W. Lee, and A. I. Sanda, Phys. Rev. D 6, 2923 (1972). There the calculation is carried out for the Weinberg-Salam, Georgi-Glashow, and Lee-Prentki-Zumino models in specific renormalizable R_ξ gauges and the gauge invariance of the results is demonstrated. A more detailed discussion of these calculations for continuous R_ξ gauges can be found in E. Abers and B. W. Lee, Phys. Rep. 9C, 1 (1973), and references cited therein. For calculations of the anomalous magnetic moment for the Weinberg-Salam model in the (singular) U gauge, see R. Jackiw and S. Weinberg, Phys. Rev. D 5, 2396 (1972); I. Bars and M. Yoshimura, *ibid.* 6, 374 (1972); W. Bardeen, R. Gastmans, and B. Lautrup, Nucl. Phys. B46, (1972); J. Primack and H. Quinn, Phys. Rev. D 6, 3171 (1972).

[14]For discussions of radiative hyperon decays, see, e.g. M. A. Ahmed and G. G. Ross, Phys. Lett. 59B, 293 (1975) and N. Vasanti, Phys. Rev. D 13, 1889 (1976).

[15]K. Fujikawa, Phys. Rev. D 7, 393 (1973). There are certain subtleties in the renormalization of the ghost fields in gauge theories with nonlinear gauge-fixing terms such as the one used by Fujikawa. For a discussion of this point, see J. Zinn-Justin, Bonn lectures, 1973 (unpublished). The complications in the ghost counterterms do not, however, affect the electromagnetic form-factor calculation to one-loop order. As pointed out by Fujikawa, there are exceptional cases in which the unphysical scalars give nonvanishing contributions even in the limit $\xi \to 0$. This does not happen in our calculations.

[16]The R_ξ gauge formalism was presented in K. Fujikawa, B. W. Lee, and A. I. Sanda, Ref. 13. For a proof of the gauge invariance of the renormalized S matrix in spontaneously broken gauge theories, see B. W. Lee and J. Zinn-Justin, Phys. Rev. D 5, 3121 (1972); 5, 3137 (1972); 5, 3155 (1972); B. W. Lee, Phys. Lett. 46B, 214 (1974); Phys. Rev. D 9, 933 (1974) and Les Houches lectures, 1975 (unpublished). See also G. 't Hooft, Nucl. Phys. B35, 167 (1971); G. 't Hooft and M. Veltman, *ibid.* B50, 318 (1972); A. Slavnov, Theor. Math. Phys. 10, 152 (1972); J. C. Taylor, Nucl. Phys. B33, 436 (1971).

[17]Thus, for example, the derivation of the conditions for natural flavor conservation by the weak neutral current is much simpler with our ξ-limiting procedure than in the general R_ξ gauge. The reason is that for a purely left-handed $W^\pm$ fermion vertex $\mathcal{T}_\pm^L \gamma_\mu L$ there are both left- and right-handed parts to the corresponding $\phi^\pm$-fermion vertex (where $\phi^\pm$ are unphysical scalars), and similarly with a purely right-handed $W^\pm$-fermion vertex. The same comment applies for the derivation of the conditions for microweak *CP* violation in Ref. 3. For an explicit calculation showing how a general rule which depends on the chiral structure of vertices and is manifestly obvious in U gauge also works, although not manifestly, in general R_ξ gauge, see R. Shrock, Ref. 20.

[18]See B. W. Lee, Ref. 3, in particular, Appendix B. The precise statement of *CP* invariance is

$$F_j^{V,A}(m_1, m_2; q^2 + i\epsilon) = F_j^{V,A}(m_2, m_1; q^2 - i\epsilon)^*,$$
$$j = 1, 2, 3.$$

With the restriction to spacelike q^2 made in Eq. (5.5) of Ref. 3, it is unnecessary to consider the charge in the sign of $i\epsilon$.

[19]For the WS model the Fermi constant in Eq. (2.48) is the physical quantity measured in μ decay, and (multiplied by $\cos\theta_C$) in nuclear beta decay. However, in the KM and CL models Eq. (2.48) should be regarded by a definition of a theoretical quantity G_F. In the KM model, for example, the "Fermi constant" measured in μ decay differs from that measured in nuclear beta decay, and they both differ from G_F, by factors which are equal to unity in the WS model. See Sec. V A. Equation (2.49) remains true in any model with natural flavor conservation by the neutral current (see also Ref. 36).

[20]R. Shrock, Phys. Rev. D 9, 743 (1974). This paper contains a calculation in general R_ξ gauge of the amplitude for $L^0 \to \nu_l \gamma$, from which one can extract the leading contribution to $C_2^{LR,RL}$. For a similar calculation in 't Hooft–Feynman gauge, and in addition a dispersive approach, see S.-Y. Pi and J. Smith, Phys. Rev. D 9, 1498 (1974).

[21]We note the following changes in the paper of M. K. Gaillard and B. W. Lee, Ref. 2. First, the Z-exchange amplitude for $K_L \to \mu\bar{\mu}$ contains an error in a subdominant non-log term so that although the log terms cancel between the Z and W^+W^- contributions, the resulting amplitude for $K_L \to \mu\bar{\mu}$ does not vanish. The correct Z-exchange amplitude is given in the second paper of Ref. 2. The correctness of the original conclusion, that the dominant contribution to the $K_L \to \mu\bar{\mu}$ amplitude comes from the conventional process $K_L \to \gamma\gamma \to \mu\bar{\mu}$, was not affected by this error. Second, in Appendix D on the effective $\mathfrak{N}\lambda\gamma$ vertex there is in fact no logarithm in the transition magnetic moment; the log term arising from graph 13(a) is canceled by another logarithm from graph 13(b). Again, this does not affect the correctness of the calculations which used the results of Appendix D, namely the $K^+ \to \pi^+ e\bar{e}$, and $K_S \to \pi^0 e\bar{e}$ decay rates. The reason for this is that the dominant contribution to the $\mathfrak{N}\lambda\gamma$ vertex is from the (correctly calculated) transition charge radius, which is larger than the magnetic moment term in the amplitude by the factor $\sim (m_W^2/m_c^2)\ln(m_c^2/m_K^2)$.

[22]The importance of the $\ln(1/\epsilon_i)$ term has also been noted by W. Marciano and A. I. Sanda, Phys. Lett. 67B, 303 (1977). See also Refs. 2, 31, and 54.

[23]M. Perl *et al*., Phys. Rev. Lett. 35, 1489 (1975); G. Feldman *et al*., *ibid*. 38, 117 (1977); R. Felst, invited talk at the Chicago APS meeting, 1977 (unpublished).

[24]A. Benvenuti *et al*., Phys. Rev. Lett. 36, 1478 (1976); 37, 189 (1976); 37, 1095 (1976) and references cited therein.

[25]B. C. Barish *et al*., Phys. Rev. Lett. 38, 314 (1977).

[26]For a recent comparison of the predictions of several gauge models with the data of Refs. 23 and 24 and with bubble-chamber experiments, see C. H. Albright and R. E. Shrock, Phys. Rev. D 16, 575 (1977). This paper contains further references to relevant experimental and theoretical work. [*Note added in proof*. Recent high-statistics data from the Caltech-Fermilab (CF) and CERN-Dortmund-Heidelberg-Saclay (CDHS) experiments indicates the absence of the high-y anomaly and associated substantial rise in $\langle y\rangle^{\bar{\nu}N}$ and $\sigma_T^{\bar{\nu}N}/\sigma^{\nu N}$ reported by the HPWF group (Ref. 24) and supported by the early CF data (Ref. 25). See B. C. Barish *et al*., Caltech Reports Nos. CALT 68-605, 68-606, 68-607 (unpublished); J. Steinberger, invited talk given at the European Conference on Particle Physics, Budapest, 1977 (unpublished). The predictions of the KM model are in agreement with this recent CF and CDHS data. However, the KM model's prediction for the amount of parity violation by the weak neutral current in heavy atoms, which is the same as that of the WS model, appears to be too large to agree with the recent, more accurate data from the Washington and Oxford experiments on ^{209}Bi. See E. N. Fortson and P. G. H. Sandars, talks given at the Washington APS Meeting, 1977 (unpublished).]

[27]The weak gauge group eigenstates $N_{eL}, N_{\mu L}$, etc., are necessarily orthonormal. Note that the linear combinations $\mathfrak{U}_{j1}\nu_1 + \mathfrak{U}_{j2}\nu_2$, $j = 1,2,3$, are mass eigenstates because of the special circumstance that ν_1 and ν_2 are degenerate; however, these linear combinations are not, in general, orthonormal.

[28]On the subject of neutrino oscillations, see B. Pontecorvo and C. Gribov and B. Pontecorvo, Ref. 1; J. Bahcall and S. Frautschi, Phys. Lett. 29B, 623 (1969); S. Eliezer and D. Ross, Phys. Rev. D 10, 3088 (1974) [we differ with their result for $\Gamma(\mu \to e\gamma)$]; S. Eliezer and A. Swift, Nucl. Phys. B105, 45 (1976); A. Mann and H. Primakoff, Phys. Rev. D 15, 655 (1977); H. Fritzsch and P. Minkowski, Phys. Lett. 62B, 72 (1976); S. Bilenky, S. Petcov, and B. Pontecorvo, Dubna Report No. JINR, E2-10374, 1977 (unpublished); E. Bellotti *et al*., Lett. Nuovo Cimento 17, 553 (1976); V. Barnes, *et al*., Argonne-Purdue report, 1977 (unpublished).

[29]Cf. D. Bailin and N. Dombey, Phys. Lett. 64B, 304 (1976).

[30]Experiments which test for neutrino oscillations by searching for electrons produced from a ν_μ beam also set a limit on the type of nonorthogonality of degenerate ν_e and ν_μ present in the KM model. See in this context Ref. 27.

[31]Models (c) and (d) were proposed in the context of muon-number nonconservation by F. Wilczek and A. Zee, Phys. Rev. Lett. 38, 531 (1977).

[32]The original Cheng-Li model and/or generalizations thereof have recently been discussed by S. B. Treiman, F. A. Wilczek, and A. Zee, Phys. Rev. D 16, 152 (1977); H. T. Nieh, SUNY Report No. ITP-58-77-12 (unpublished); and V. Barger and D. V. Nanopoulos, Wisconsin Report No. COO-583, 1977 (unpublished). We have been informed by J. D. Bjorken and S. Weinberg that they and K. Lane are also analyzing this model [Phys. Rev. D 16, 1474 (1977)]. For other calculations of μ- and e-number-violating processes, see W. K. Tung, Phys. Lett. 67B, 52 (1977); H. Fritzsch, *ibid*. 67B, 451 (1977); W. Marciano and A. I. Sanda, Ref. 22; S. Petcov, Dubna Report No. JINR-E2-10374, 1977 (unpublished); and S. Bilenky, S. Petcov, and B. Pontecorvo, in Ref. 28.

[33]In the WS model, in contrast to the other models considered, there are no large virtual lepton masses in the denominators of the Feynman parametric integrals. Thus, if for any of the quantities which we calculate, these integrals were to have infrared logarithms for zero lepton mass, we would have to retain the muon mass and would get terms of the form $\ln(m_W^2/m_\mu^2)$. However, the relevant integrals are in fact regular for zero lepton mass, as is evident from the absence of a

logarithm in Eqs. (4.2)–(4.4). (See the discussion at the beginning of Sec. II.)

[34]A. K. Mann and H. Primakoff, Ref. 28. For reference, the Particle Data Group value is $\Delta m_\nu^2 < (0.65\text{ MeV})^2$; the use of this limit gives $B(\mu \to e\gamma)_{WS} < 0.75 \times 10^{-24}$ for $m_W = 60$ GeV.

[35]This limit was set by S. Parker, H. L. Anderson, and C. Rey, Phys. Rev. 133, B768 (1964). See also the earlier result of S. Frankel *et al.*, Nuovo Cimento 27, 294 (1963), that $B(\mu \to e\gamma) < 4.3 \times 10^{-8}$, and the more recent result of M. Korenchenko *et al.*, Yad. Fiz. 13, 341 (1971) [Sov. J. Nucl. Phys. 13, 190 (1971)], that $B(\mu \to e\gamma) < 2.9 \times 10^{-8}$. A general discussion of these experiments is given by S. Frankel, Ref. 1.

[36]The increase in the strength of the $We\nu_e$ and $W\mu\nu_\mu$ vertices by the factor $\sqrt{2}$ in model (c) simply induces a redefinition of g^2 in terms of $G_F m_W^2$, as was noted earlier.

[37]M. K. Gaillard, B. W. Lee, and R. E. Shrock, Ref. 2. Note that in the $K_L \to \mu\bar{\mu}$ decay amplitude the vector part of the muon current gives zero contribution since it is contracted with $\langle 0|\bar{s}\gamma_\alpha(1-\gamma_5)d|K_L\rangle \propto (p_K)_\alpha$, and is therefore dropped.

[38]M. Korenchenko *et al.*, Yad. Fiz. 13, 341 (1971) [Sov. J. Nucl. Phys. 13, 728 (1971)].

[39]Cf. M. K. Gaillard and B. W. Lee, Ref. 2, Appendix D and Ref. 21 above.

[40]T. Kinoshita, Phys. Rev. Lett. 2, 477 (1959). See also S. Berman, *ibid.* 1, 468 (1958); A. Sirlin, Phys. Rev. D 5, 436 (1972); D. Bailin and N. Dombey, Ref. 29.

[41]E. di Capua *et al.*, Phys. Rev. 133, B1333 (1964).

[42]E. Bellotti *et al.*, Ref. 28. A similar bound for $\Sigma_{e/\mu}$ is found in the Columbia-Fermilab ν_μ-heavy-neon bubble-chamber experiment; C. Baltay (private communication).

[43]M. M. Nagels *et al.*, Nucl. Phys. B109, 1 (1976); K. Kleinknecht, in *Proceedings of the XVII International Conference on High Energy Physics*, edited by J. R. Smith (Rutherford Laboratory, Chilton, Didcot, Berkshire, England, 1974), p. III-23. The model-dependent (but nuclear-independent) radiative correction to the beta decay coupling constant G_F^β is expected to be of order 1% and invloves terms such as $\ln(m_Z^2/m_p^2)$ and $\ln(m_Z^2/m_A^2)$, where m_A is the mass parameter entering into the axial-vector form factor. This has been calculated by A. Sirlin, Nucl. Phys. B71, 29 (1974).

[44]Such a conjecture has recently been made by S. L. Glashow, Harvard Report No. HUTP-77/A008 (unpublished).

[45]K. Kleinknecht, Ref. 43. For earlier work, see L. M. Chounet, J. M. Gaillard, and M. K. Gaillard, Phys. Rep. 4C, 199 (1972).

[46]G. Goldhaber *et al.*, Phys. Rev. Lett. 37, 255 (1976).

[47]L. M. Seghal, Nuovo Cimento 45, 785 (1966); B. R. Martin, E. de Rafael, and J. Smith, Phys. Rev. D 2, 179 (1970); G. Farrar and S. B. Treiman, *ibid.* 4, 257 (1971). For a review of these calculations and further references, see M. K. Gaillard and H. Stern, Ann. Phys. (N.Y.) 76, 580 (1973).

[48]See the discussion in Particle Data Group, Rev. Mod. Phys. 48, S1 (1976). The number is obtained from a combination of the results of the Columbia and Princeton experiments: W. C. Carrithers *et al.*, Phys. Rev. Lett. 30, 1336 (1973); 31, 1025 (1973); Y. Fukushima *et al.*, *ibid.* 36, 348 (1976).

[49]B. C. Barish *et al.*, Phys. Rev. Lett. 38, 577 (1977); 38, 1037(E) (1977).

[50]A. Benvenuti *et al.*, Phys. Rev. Lett. 38, 1110 (1977); 38, 1183 (1977).

[51]The literature on the production and decay of heavy leptons is quite considerable. A partial list of references is Y.-S. Tsai, Phys. Rev. D 4, 2821 (1971); J. D. Bjorken and C. H. Llewellyn-Smith, *ibid.* 7, 887 (1973); M. L. Perl, Lectures at the Institute of Particle Physics Summer School, McGill University, 1975 (unpublished); M. Perl *et al.*, Ref. 23; C. H. Albright, Phys. Rev. Lett. 28, 1150 (1972); Phys. Rev. D 7, 63 (1973); 13, 2508 (1977); C. H. Albright and C. Jarlskog, Nucl. Phys. B84, 467 (1975); C. H. Albright, C. Jarlskog, and E. L. Wolfenstein, *ibid.* B84, 493 (1975); L. N. Chang, E. Derman, and J. Ng, Phys. Rev. Lett. 35, 6 (1975); Phys. Rev. D 12, 3539 (1975); A. Pais and S. B. Treiman, Phys. Rev. Lett. 35, 1206 (1975), Phys. Rev. D 14, 293 (1976); K. Fujikawa and N. Kawamoto, Phys. Rev. Lett. 35, 1560; Phys. Rev. D 13, 2534 (1976); 14, 59 (1976); A. Soni, *ibid.* 11, 624 (1975); F. Wilczek and A. Zee, Nucl. Phys. B106, 461 (1976); S. Y. Pi and A. I. Sanda, Phys. Rev. Lett. 36, 1 (1976);

[52]M. K. Gaillard, B. W. Lee, and J. L. Rosner, Ref. 7; I. Karliner, Phys. Rev. Lett. 36, 759 (1976).

[53]S. Weinberg and G. Feinberg, Phys. Rev. Lett. 3, 111 (1959); 3, 244(E) (1959).

[54]W. J. Marciano and A. I. Sanda, Phys. Rev. Lett. 38, 1512 (1977).

[55]For early discussions, see G. Feinberg and S. Weinberg, Phys. Rev. 123, 1439 (1961); B. Pontecorvo, Zh. Eksp. Teor. Fiz. 33, 549 (1957) [Sov. Phys.—JETP 6, 429 (1958)].