Nonlinear-σ-model Padé calculation of $\pi\pi$ phase shifts. II

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The s- and p-wave $\pi\pi$ phase shifts from the elastic threshold up to the $K\bar{K}$ inelastic threshold are calculated by means of the unitary Padé approximant constructed from the perturbation series for the partial-wave amplitudes computed from a chiral [SU(2) $\times$ SU(2)] Lagrangian containing pions, kaons, and nucleons. Significant phenomenological improvement over previous calculations is obtained.

I. INTRODUCTION

We have been engaged for some time in a program[1] to compute phase shifts and resonance parameters for meson-meson and meson-baryon systems starting from a chiral Lagrangian (σ model). The essential ingredients of these calculations are as follows: (i) A renormalizable chiral-invariant [SU(2) $\times$ SU(2)] Lagrangian containing pseudoscalar mesons and spin-$\frac{1}{2}$ baryons, and the scalar mesons required for a linear realization of the chiral symmetry and renormalizability. (ii) Symmetry breaking which yields partial conservation of axial-vector current (PCAC) for the pion and the associated low-energy theorems for processes with external pions. (iii) A truncation procedure in which the scalar-meson masses are taken to be very large. Thus these unstable particles are removed from the theory. In particular, they are not identified with any scalar-meson resonances claimed in the Particle Data Group Tables. Any such physical resonances are to be results of the calculation, not input. But the (undetermined) input scalar-meson masses remain as effective cutoff parameters. [We will refer to calculations which incorporate (iii) as nonlinear σ-model (NLσM) calculations—as opposed to linear σ-model (LσM) calculations which attempt to identify the input scalar mesons with physical resonances.] (iv) Padé approximants constructed from the perturbation series for the partial-wave amplitudes. These enforce unitarity so that one has an approximation in which one can compute phase shifts and find resonances as results of the calculation.

The use of Padé approximants for low-energy meson-meson and meson-baryon scattering calculations predates[2] the advent of chiral dynamics, but very little phenomenological success was had before the successes of PCAC and SU(2) $\times$ SU(2) current algebra made clear the importance of (softly broken) chiral invariance in pion physics. Then Basdevant and Lee[3] did a LσM calculation of $\pi\pi$ scattering based on the simplest LσM Lagrangian, i.e., containing only π and σ. This calculation got the general properties right—positive δ_{00}, small negative δ_{20}, and δ_{11} which passes through 90°—but was not quantitatively successful; the calculated width of the p-wave resonance was much too small (35 MeV) and the s-wave phase shift reached a maximum at 700 or 800 MeV (c.m. energy). Then Jhung and Willey (JW, Ref. 1) did a NLσM calculation of $\pi\pi$ scattering starting from a LσM Lagrangian which included the nucleon as well as π and σ. The inclusion of the N led to dramatic improvement in the calculated δ_{11}. With essentially one adjustable parameter, both m_ρ and Γ_ρ were obtained in agreement with the experimental values. The calculated δ_{00} was slightly improved over that of Ref. 3, but it still flattened out at a value of about 90° at 700 or 800 MeV. In the meantime phase-shift analyses of high-statistics production experiments revealed the dramatic effect (S^*) associated with the opening of the $K\bar{K}$ channel at 990 MeV—δ_{00} shoots up rapidly through 180° and there is a sharp drop in the elasticity function η_{00}. This suggested that it would be important to include the K in calculation of the $I=0$ s-wave $\pi\pi$ phase shift. There are two sources of kaon contribution: (i) Feynman diagrams including kaon loops contribute to the perturbation series for the $\pi\pi\to\pi\pi$ amplitude from which the Padé approximants are computed. These terms contribute both above and below the $K\bar{K}$ threshold. (ii) Above the $K\bar{K}$ threshold, unitarity requires a coupled-channel approach. Coupled-channel unitarity can be incorporated in a matrix Padé calculation. We have carried out these calculations and find that δ_{00} below the $K\bar{K}$ threshold can be successfully calculated including just (i) above, and the dramatic behavior of δ_{00} above the $K\bar{K}$ threshold can be calculated by including (ii) also. In this paper we give the chiral Lagrangian including π, K, and N, which is the starting point of all of these calculations, outline the renormalization and calculation of all one-loop diagrams which contribute to the elastic channel, and give the results of the single-channel calculation up to the $K\bar{K}$ threshold. In a later paper we will report the renormalization and calculation of the one-loop diagrams which contribute to the inelastic channels and the matrix Padé calculation of δ_{00} above the $K\bar{K}$ threshold, as well as the πK s-wave phase shifts.

II. CHIRAL LAGRANGIAN

We ask for a renormalizable chiral SU(2) × SU(2) invariant Lagrangian including π, K, N. Renormalizability requires a linear realization of the chiral symmetry and hence the addition of scalar mesons: an isoscalar σ [chiral partner of the $\vec{\pi}$ in (1, 1)] and an isospinor κ [chiral partner of K in $(\frac{1}{2},0)+(0,\frac{1}{2})$]. The general Lagrangian satisfying these criteria, i.e., including all SU(2) × SU(2) invariant couplings of these particles of dimension less than or equal to four is

$$\mathcal{L}_{\rm inv}=\tfrac{1}{2}[(\partial\vec{\phi})^2+(\partial\chi)^2]-\tfrac{1}{2}\mu_0^{\,2}(\vec{\phi}^2+\chi^2)+\partial k^\dagger\partial k+\partial\xi^\dagger\partial\xi-\mu_1^{\,2}(k^\dagger k+\xi^\dagger\xi)+\bar{\psi}i\gamma\cdot\partial\psi-\tfrac{1}{4}\lambda_0(\vec{\phi}^2+\chi^2)^2-\lambda_1(k^\dagger k+\xi^\dagger\xi)^2$$
$$-\lambda_2(k^\dagger\xi+\xi^\dagger k)^2-\lambda_3(\vec{\phi}^2+\chi^2)(k^\dagger k+\xi^\dagger\xi)-h_0(\xi^\dagger\chi\xi-k^\dagger\xi k-i\xi^\dagger\vec{\tau}\cdot\vec{\phi}k+ik^\dagger\vec{\tau}\cdot\vec{\phi}\xi)-g_0\bar{\psi}(\chi-i\gamma_5\vec{\tau}\cdot\vec{\phi})\psi\,. \qquad (2.1)$$

Here ϕ is the canonical unrenormalized π field, and similarly $\chi(\sigma)$, $k(K)$, $\xi(\kappa)$, $\psi(N)$. In the context of this Lagrangian model, PCAC is simply achieved by adding the symmetry-breaking term

$$\mathcal{L}_{\rm SB}=c_0\chi\,. \qquad (2.2)$$

We assume that in the symmetry limit ($c_0=0$) the chiral symmetry is realized in the Nambu-Goldstone mode, i.e., $\mu_0^{\,2}<0$ and $\langle\Omega|\chi|\Omega\rangle=v_0$. The Lagrangian is rewritten in terms of the translated scalar field

$$\chi=\hat{\chi}+v_0\,,\quad \langle\Omega|\chi|\Omega\rangle=0\,. \qquad (2.3)$$

Then in the symmetry limit the pion is massless and the nucleon has acquired a mass. The pion mass is proportional to the symmetry-breaking parameter c_0.

To carry out the calculations it is necessary to have the renormalized perturbation theory, consistent with the (broken) chiral symmetry. There are by now several different approaches to renormalization, all presumably leading to the same physical consequences. For the purpose of calculation with a theory possessed of a broken symmetry, perhaps the most convenient is the Bogoliubov-Parasiuk-Hepp-Zimmermann (BPHZ) scheme[4] in which the (finite) counterterms are determined by a combination of conventional renormalization conditions and the Ward identities formally implied by the chiral invariance of the Lagrangian.[5] The BPHZ effective Lagrangian which effects the renormalization of (2.1), (2.2), and (2.3) is

$$\mathcal{L}_{\rm eff}=\tfrac{1}{2}(1-\beta_\pi)(\partial\vec{\pi})^2-\tfrac{1}{2}(\mu^2+\alpha_\pi)\vec{\pi}^2+\tfrac{1}{2}(1-\beta_\sigma)(\partial\sigma)^2-\tfrac{1}{2}(M_\sigma^{\,2}+\alpha_\sigma)\sigma^2+(1-\beta_K)\partial K^\dagger\partial K-(\mu_K^{\,2}+\alpha_K)K^\dagger K$$
$$+(1-\beta_\kappa)\partial\kappa^\dagger\partial\kappa-(M_\kappa^{\,2}+\alpha_\kappa)\kappa^\dagger\kappa+(1-\beta_N)\bar{N}i\gamma\cdot\partial N-(m+\alpha_N)\bar{N}N-(G-\delta)\bar{N}(\sigma-i\gamma_5\vec{\tau}\cdot\vec{\pi})N$$
$$-(\lambda_\pi-\gamma_\pi)v\sigma(\vec{\pi}^2+\sigma^2)-\tfrac{1}{4}(\lambda_\pi-\gamma_\pi)(\vec{\pi}^2+\sigma^2)^2-(\lambda_{\pi K}-\gamma_{\pi K})(\vec{\pi}^2+\sigma^2)(K^\dagger K+\kappa^\dagger\kappa)-[2(\lambda_{\pi K}-\gamma_{\pi K})v-(h-\delta')]\sigma K^\dagger K$$
$$-[2(\lambda_{\pi K}-\gamma_{\pi K})v+(h-\delta')]\sigma\kappa^\dagger\kappa+i(h-\delta')(\kappa^\dagger\vec{\tau}\cdot\vec{\pi}K-K^\dagger\vec{\tau}\cdot\vec{\pi}\kappa)-(\lambda_K-\gamma_K)(K^\dagger K+\kappa^\dagger\kappa)^2-(\lambda_K'-\gamma_K')(K^\dagger\kappa+\kappa^\dagger K)^2\,.$$
$$(2.4)$$

The effective Lagrangian (2.4) contains 28 constants. They are accounted for as follows: β_π, α_π, $\beta_K, \alpha_K, \beta_N, \alpha_N$ are fixed by the conventional on-shell renormalization conditions for the pion, kaon, and nucleon propagators. The Ward identities which follow from the SU(2) × SU(2) commutation relations and PCAC provide the relations[6] (through one-loop order)

$$\beta_\sigma=\beta_\pi\,,\quad \alpha_\sigma=\alpha_\pi-2\gamma_\pi v^2\,,$$
$$\beta_\kappa=\beta_K\,,\quad \alpha_\kappa=\alpha_K-2\delta' v\,,$$
$$\lambda_\pi=\frac{1}{2v^2}(M_\sigma^{\,2}-\mu^2)\,,\quad h=\frac{1}{2v}(M_\kappa^{\,2}-\mu_K^{\,2})\,, \qquad (2.5)$$
$$G=\frac{m}{v}\,,\quad v\simeq f_\pi\,.$$

[The Ward identity $D_\pi(q^2=0) = -v/f_\pi \mu^2$ provides a nonlinear algebraic equation for v in terms of f_π, μ, m, etc. To an accuracy of 1% the solution through one-loop order is the last formula of (2.5)]. The counterterms γ'_k, δ do not enter through the one-loop order. The counterterms $\gamma_{\pi K}, \gamma_K, \delta'$ and the coupling constants λ_K, λ'_K do not enter the $\pi\pi \to \pi\pi$ amplitude through the one-loop order. (They will enter the $\pi\pi \to K\bar{K}$ and $K\bar{K} \to K\bar{K}$ amplitudes needed for the coupled-channel calculation.) The masses μ, μ_K, m are known. This leaves four undetermined parameters $M_\sigma, M_\kappa, \lambda_{\pi K}, \gamma_\pi$.

III. ONE-LOOP PERTURBATION CALCULATION OF THE $\pi\pi$ AMPLITUDE

The momentum and isospin variables are illustrated in Fig. 1 along with the decomposition of the $\pi\pi \to \pi\pi$ amplitude into its σ-pole terms and its one-particle irreducible part. The isospin decomposition is

$$A_{abcd}(q_1,q_2,q_3,q_4) = \delta_{ab}\delta_{cd} A(q_1,q_2,q_3,q_4) + \delta_{ac}\delta_{bd} A(-q_4,q_2,q_3,-q_1) + \delta_{ad}\delta_{bc} A(-q_3,q_2,-q_1,q_4)\,, \qquad (3.1)$$

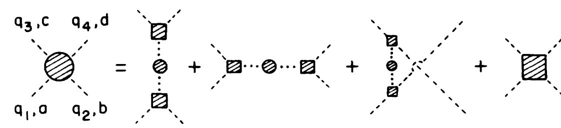

FIG. 1. Kinematics of the $\pi\pi \to \pi\pi$ invariant matrix element and its decomposition into σ-pole terms and one-particle irreducible terms.

$$s = (q_1+q_2)^2\,, \quad t = (q_1-q_3)^2\,, \quad u = (q_1-q_4)^2\,. \qquad (3.2)$$

The chiral perturbation expansion is

$$A = A^{(0)} + A_p^{(1)} + A_{\rm IR}^{(1)} + (\text{2-loop}) \pm \cdots\,. \qquad (3.3)$$

The Born term (tree diagram) is

$$A^{(0)}(q_1,q_2,q_3,q_4) = -2\lambda_\pi\left(1 - \frac{2\lambda_\pi v^2}{M_\sigma^2 - s}\right)\,. \qquad (3.4)$$

The one-loop contributions computed from the Lagrangian (2.4) with the renormalization conditions and Ward-identity constraints given in the previous section are[6]

$$A_p^{(1)}(q_1,q_2,q_3,q_4) = -(2\lambda_\pi v)^2 \frac{1}{(s-M_\sigma^2)^2}\Sigma_\sigma^{(1)}(s) + 2\lambda_\pi v \frac{1}{s-M_\sigma^2}[V_{\pi\pi\sigma}^{(1)}(q_1,q_2) + V_{\pi\pi\sigma}^{(1)}(q_3,q_4)]\,, \qquad (3.5)$$

$$\Sigma_\sigma^{(1)}(s) = 2\gamma_\pi v^2 + \alpha_\pi + \beta_\pi s + 6\lambda_\pi{}^2 v^2 I_{\pi\pi}^{(0)}(s) + 18\lambda_\pi{}^2 v^2 I_{\sigma\sigma}^{(0)}(s) + 2f_{K\sigma}{}^2 I_{KK}^{(0)}(s) + 2f_{\kappa\sigma}{}^2 I_{\kappa\kappa}^{(0)}(s) + 4G^2(s-4m^2) I_{NN}^{(0)}(s)\,, \qquad (3.6)$$

$$\alpha_\pi = -4\lambda_\pi{}^2 v^2[I_{\sigma\pi}^{(0)}(\mu^2) - \mu^2 I'_{\sigma\pi}(\mu^2)] - 4h^2[I_{K\kappa}^{(0)}(\mu^2) - \mu^2 I'_{K\kappa}(\mu^2)] + 4G^2\mu^4 I'_{NN}(\mu^2)\,, \qquad (3.7a)$$

$$\beta_\pi = -4\lambda_\pi{}^2 v^2 I'_{\sigma\pi}(\mu^2) - 4h^2 I'_{K\kappa}(\mu^2) - 4G^2[I_{NN}^{(0)}(\mu^2) + \mu^2 I'_{NN}(\mu^2)]\,. \qquad (3.7b)$$

We have defined the combinations

$$f_{K\sigma} = 2\lambda_{\pi K} v - h\,, \quad f_{\kappa\sigma} = 2\lambda_{\pi K} v + h\,, \qquad (3.8)$$

$$\begin{aligned} V_{\pi\pi\sigma}^{(1)}(q_1,q_2) = {} & 2\gamma_\pi v - 8\lambda_\pi{}^3 v^3[K_{\sigma\pi\pi}^{(0)}(q_1,q_2) + 3K_{\pi\pi\sigma}^{(0)}(q_1,q_2)] - 10\lambda_\pi{}^2 v I_{\pi\pi}^{(0)}(s) - 6\lambda_\pi{}^2 v I_{\sigma\sigma}^{(0)}(s) \\ & - 4\lambda_\pi{}^2 v[I_{\sigma\pi}^{(0)}(q_1{}^2) + I_{\sigma\pi}^{(0)}(q_2{}^2)] - 4f_{K\sigma} h^2 K_{\kappa KK}^{(0)}(q_1,q_2) - 4f_{\kappa\sigma} h^2 K_{K\kappa\kappa}(q_1,q_2) - 4f_{K\sigma}\lambda_{\pi K} I_{KK}^{(0)}(s) \\ & - 4f_{\kappa\sigma}\lambda_{\pi K} I_{\kappa\kappa}^{(0)}(s) + 8G^3 m[2I_{NN}^{(0)}(s) + (s - q_1{}^2 - q_2{}^2)K_{NNN}(q_1,q_2)]\,. \end{aligned} \qquad (3.9)$$

$$\begin{aligned} A_{\rm IR}^{(1)}(q_1,q_2,q_3,q_4) = {} & 2\gamma_\pi - (2\lambda_\pi)^2[\tfrac{7}{2} I_{\pi\pi}^{(0)}(s) + I_{\pi\pi}^{(0)}(t) + I_{\pi\pi}^{(0)}(u) + \tfrac{1}{2} I_{\sigma\sigma}^{(0)}(s)] - (2\lambda_{\pi k})^2[2I_{KK}^{(0)}(s) + 2I_{\kappa\kappa}^{(0)}(s)] \\ & - (2\lambda_\pi)^3 v^2[K_{\pi\sigma\sigma}^{(0)}(q_1,q_2) + K_{\pi\sigma\sigma}^{(0)}(q_3,q_4) + K_{\sigma\pi\pi}^{(0)}(q_1,q_2) + K_{\sigma\pi\pi}^{(0)}(q_3,q_4) + K_{\sigma\pi\pi}^{(0)}(q_1,-q_4) \\ & + K_{\sigma\pi\pi}^{(0)}(q_2,-q_3) + K_{\sigma\pi\pi}^{(0)}(q_1,-q_3) + K_{\sigma\pi\pi}^{(0)}(q_2,-q_4)] \\ & - 8\lambda_{\pi K} h^2[K_{\kappa KK}^{(0)}(q_1,q_2) + K_{\kappa KK}^{(0)}(q_3,q_4) + K_{K\kappa\kappa}^{(0)}(q_1,q_2) + K_{K\kappa\kappa}^{(0)}(q_3,q_4)] \\ & - (2\lambda_\pi v)^4[H_{\pi\sigma\pi\sigma}^{(0)}(q_1,q_2,q_3,q_4) + H_{\pi\sigma\pi\sigma}^{(0)}(q_1,q_2,q_4,q_3)] \\ & - 4h^4[H_{\kappa K\kappa K}^{(0)}(q_1,q_2,q_3,q_4) + H_{\kappa K\kappa K}^{(0)}(q_1,q_2,q_4,q_3) - H_{\kappa K\kappa K}^{(0)}(q_1,-q_4,q_3,-q_2) \\ & + H_{K\kappa K\kappa}^{(0)}(q_1,q_2,q_3,q_4) + H_{K\kappa K\kappa}^{(0)}(q_1,q_2,q_4,q_3) - H_{K\kappa K\kappa}^{(0)}(q_1,-q_4,q_3,-q_2)] \\ & + 4G^4[4I_{NN}^{(0)}(s) + 2(s - q_1{}^2 - q_2{}^2)K_{NNN}(q_1,q_2) + 2(s - q_3{}^2 - q_4{}^2)K_{NNN}(q_3,q_4) \\ & + (q_1{}^2 q_4{}^2 + q_2{}^2 q_3{}^2 - st)H_{NNNN}(q_1,q_2,q_3,q_4) + (q_1{}^2 q_3{}^2 + q_2{}^2 q_4{}^2 - su)H_{NNNN}(q_1,q_2,q_4,q_3) \\ & - (q_1{}^2 q_2{}^2 + q_3{}^2 q_4{}^2 - ut)H_{NNNN}(q_1,-q_4,q_3,-q_2)]\,. \end{aligned} \qquad (3.10)$$

The Feynman integrals which occur in these formulas are

$$I_{ab}(s)=i\int (dk)\,\frac{1}{[(k+q_1)^2-M_a{}^2][(k-q_2)^2-M_b{}^2]}\ , \tag{3.11a}$$

$$K_{abc}(q_1,q_2)=i\int (dk)\,\frac{1}{[k^2-M_a{}^2][(k+q_1)^2-M_b{}^2][(k-q_2)^2-M_c{}^2]}\ , \tag{3.11b}$$

$$H_{abcd}(q_1,q_2,q_3,q_4)=i\int (dk)\,\frac{1}{[k\ -M_a{}^2][(k+q_1)^2-M_b{}^2][(k+q_1-q_3)^2-M_c{}^2][(k-q_2)^2-M_d{}^2]}\ , \tag{3.11c}$$

where

$$\int (dk)=\int \frac{d^4k}{(2\pi)^4}\ .$$

The superscript zero in parentheses indicates one subtraction at zero external four-momentum, e.g.,

$$K_{abc}^{(0)}(q_1,q_2)=K_{abc}(q_1,q_2)-K_{abc}(0,0)\ ,\ \text{etc.} \tag{3.12}$$

A detailed discussion of the properties of these integrals is given in Appendix C of JW.

Equations (3.4) and (3.5) include pole terms corresponding to a real stable scalar meson. Since there is no real stable σ particle, we want to eliminate it. We do this by letting M_σ become very large. In the no-loop terms we can take the limit $M_\sigma\to\infty$. The fifth Ward-identity constraint of (2.5) requires that λ_π also goes to infinity in this limit. Then

$$A^{(0)}(q_1,q_2,q_3,q_4)=\frac{1}{v^2}\,(s-\mu^2) \tag{3.13}$$

which is just the Weinberg[7] low-energy $\pi\pi$ amplitude or the Born term of the nonlinear σ model. In the one-loop terms, one cannot simply take the limit of M_σ (and M_κ) going to infinity, because that limit does not exist—a reflection of the conventional nonrenormalizability of the nonlinear σ model.

The procedure to eliminate the σ and the κ from the one-loop terms is the following. First, all the Feynman momentum integrals are evaluated (the $M_\sigma\to\infty$ limit cannot be taken inside the infinite-momentum integrations). Then asymptotic expansions are made for the Feynman-parameter integrals. This leads to a double power series in $M_\sigma{}^2$ and $\ln M_\sigma$ (or $M_\kappa{}^2$ and $\ln M_\kappa$). All negative powers are legislated to zero. This leaves terms independent of M and also terms with positive powers of M^2 and $\ln M$. When all the terms from (3.5) to (3.10) are added together, all but the $\ln M$ terms (and the terms independent of M) cancel out. These are kept as two ($\ln M_\sigma$, $\ln M_\kappa$) arbitrary parameters of the theory. This procedure was originally proposed by Bessis and Zinn-Justin[8] as a regularization procedure for the nonlinear σ model. A detailed description of the procedure, asymptotic expansions of the integrals, etc., as well as some improvement with regard to the treatment of the chiral-invariant ($\mu\to 0$) limit is contained in JW. To carry this procedure through to eliminate the κ-particle contributions, as well as the σ-particle contributions, is tedious but offers no new features. (This is not true of the $\pi\pi\to K\bar{K}$ and $K\bar{K}\to K\bar{K}$ amplitudes, but this discussion is deferred to the subsequent paper.) The result is (for all $q_i{}^2=\mu^2$)

$$\begin{aligned}
A(s,t,u)={}&\frac{1}{v^2}\,(s-\mu^2)\\
&+\frac{1}{v^4}\Big\{-\tfrac{1}{2}(s^2-\mu^4)I_{\pi\pi}^{(0)}(s)+[\tfrac{1}{6}st+\tfrac{1}{3}tu+\tfrac{1}{3}\mu^2(s-u)-\mu^4]I_{\pi\pi}^{(0)}(t)\\
&\quad+[\tfrac{1}{6}su+\tfrac{1}{3}ut+\tfrac{1}{3}\mu^2(s-t)-\mu^4]I_{\pi\pi}^{(0)}(u)\\
&\quad+[-\tfrac{1}{8}(s-2\mu^2)^2+\beta(s-\mu^2)(s-2\mu^2)-2\beta^2(s-\mu^2)^2]I_{K\bar{K}}^{(0)}(s)\\
&\quad+[-\tfrac{1}{24}t(s-u)+\tfrac{1}{6}\mu_K{}^2(s-u)]I_{K\bar{K}}^{(0)}(t)+[-\tfrac{1}{24}u(s-t)+\tfrac{1}{6}\mu_K{}^2(s-t)]I_{K\bar{K}}^{(0)}(u)-4m^2sI_{NN}^{(\pi)}(s)\\
&\quad+4(2\mu^4-st)m^4H_N(s,t,u)+4(2\mu^4-su)m^4H_N(s,u,t)-4(2\mu^4-ut)m^4H_N(u,t,s)+4m^2(s-\mu^2)\mu^2I'_{NN}(\mu^2)\\
&\quad+\frac{1}{16\pi^2}\Big[(-\tfrac{5}{6}L+\tfrac{31}{18}+2\Gamma_\pi)s^2+(-\tfrac{2}{3}L+\tfrac{16}{9})tu+(\tfrac{7}{3}L-\tfrac{41}{9}-4\Gamma_\pi)\mu^2s+(-\tfrac{7}{6}+2\Gamma_\pi)\mu^4\\
&\qquad+L'(\tfrac{1}{12}s^2-\tfrac{1}{12}tu-\tfrac{1}{3}\mu^2s+\tfrac{1}{2}\mu^4)+\beta L'(s-\mu^2)(2\mu^2-s)\\
&\qquad+(2\beta-\tfrac{1}{2}\alpha)(s-\mu^2)(s-2\mu^2)-\tfrac{1}{6}s^2+\tfrac{1}{4}tu+\mu^2s-\tfrac{5}{3}\mu^4\Big]\Big\}\ .
\end{aligned} \tag{3.14}$$

Quantities not previously defined which appear in (3.14) are

$$\beta = \lim_{M_\sigma, M_K \to \infty} (\lambda_{\pi K}/\lambda_\pi - \tfrac{1}{2}\alpha)\,, \tag{3.15a}$$

$$\alpha = M_K{}^2/M_\sigma{}^2\,, \tag{3.15b}$$

$$L = \ln(M_\sigma{}^2/\mu^2)\,, \tag{3.15c}$$

$$L' = \ln(M_K{}^2/\mu_K{}^2) = L + \ln(\alpha\mu^2/\mu_K{}^2) \quad \text{(not an independent parameter)}, \tag{3.15d}$$

$$\gamma_\pi = (1/16\pi^2 v^4)\Gamma_\pi M_\sigma{}^4 + O(1/v^6) \quad (\text{definition of } \Gamma_\pi)\,, \tag{3.15e}$$

and

$$I_{NN}^{(\pi)}(s) = I_{NN}(s) - I_{NN}(\mu^2)\,,\quad H_N(s,t,u) = H_{NNNN}(q_1,q_2,q_3,q_4)_{q_i{}^2=\mu^2}\,. \tag{3.16}$$

A very important check on (3.14) follows from the observation that it is determined, except for the polynomial terms, by unitarity, crossing, analyticity, and the chiral Born terms for $\pi\pi\to\pi\pi$, $\pi\pi\to K\bar K$, and $\pi\pi\to N\bar N$. The chiral Born terms are obtained by computing the tree diagrams from the Lagrangian (2.4) and taking the limit $M_\sigma{}^2, M_K{}^2\to\infty$ (with the finite ratio α):

$$\pi\pi\to\pi\pi:\ A^{(0)}(s,t,u) = \frac{1}{v^2}(s-\mu^2)\,, \tag{3.17a}$$

$$\pi\pi\to K\bar K:\ M_{ab}(s,t,u) = \frac{1}{4v^2}\,\xi^x\{\delta_{ab}[4\beta(s-\mu^2)+t+u-2\mu_K{}^2] + \tfrac{1}{2}[\tau_a,\tau_b](u-t)\}\eta\,, \tag{3.17b}$$

$$\pi\pi\to N\bar N:\ M_{ab}(s,t,u) = G^2\bar u(p')\left\{\frac{1}{m}\,\delta_{ab} + \tfrac{1}{2}\gamma(q_1-q_2)\left[\delta_{ab}\left(\frac{-1}{(p-q_1)^2-m^2} + \frac{1}{(p-q_2)^2-m^2}\right)\right.\right.$$
$$\left.\left. + \tfrac{1}{2}[\tau_b,\tau_a]\left(\frac{-1}{(p-q_1)^2-m^2} - \frac{1}{(p-q_2)^2-m^2}\right)\right]\right\}v(p)\,. \tag{3.17c}$$

The general, on-shell polynomial (undetermined by unitarity, crossing, and analyticity) is

$$P(s,t,u) = \frac{1}{16\pi^2 v^4}(a + bs + cs^2 + \text{dtu})\,. \tag{3.18}$$

The chiral low-energy theorems (Adler, Adler-Weisberger-Weinberg) imply $a, b\to 0$ for $\mu\to 0$,

$$A(s,t,u) = \frac{1}{v^2}\,s + O(s^2,t^2,u^2) \quad \text{for } \mu = 0\,. \tag{3.19}$$

The truncation procedure which leads from the LσM amplitude (3.5) to (3.10) to the NLσM amplitude (3.14) provides the values of the coefficients in (3.18) in terms of the coupling constant and (finite) counterterms of the LσM:

$$\begin{aligned}
a &= [\tfrac{3}{2}L + (\tfrac{1}{2} - 2\beta)L' + 2\bar\Gamma_\pi - \tfrac{13}{6} + 4\beta - \alpha]\,\mu^4\,,\\
b &= [-\tfrac{2}{3}L + (-\tfrac{1}{3} + 3\beta)L' - 4\bar\Gamma_\pi - \tfrac{38}{9} - 6\beta + \tfrac{3}{2}\alpha]\,\mu^2\,,\\
c &= \tfrac{2}{3}L + (\tfrac{1}{12} - \beta)L' + 2\bar\Gamma_\pi + \tfrac{14}{9} + 2\beta - \tfrac{1}{2}\alpha\,,\\
d &= -\tfrac{2}{3}L - \tfrac{1}{12}L' + \tfrac{73}{36}\,.
\end{aligned} \tag{3.20}$$

Here

$$\Gamma_\pi = \tfrac{3}{4}L + \bar\Gamma_\pi\,.$$

$\bar\Gamma_\pi$ is finite in the limit $\mu\to 0$, otherwise arbitrary. This dependence of the counterterm on $\ln(M_\sigma{}^2/\mu^2)$ is determined by consideration of the chiral-symmetry limit ($\mu\to 0$; see JW). This limit provides no other constraints on the parameters in (3.14). Thus, at this point, the four undetermined parameters $M_\sigma, M_K, \lambda_{\pi K}, \gamma_\pi$ appear in the one-loop $\pi\pi\to\pi\pi$ amplitude in the combinations $L, \alpha, \beta, \bar\Gamma_\pi$ defined in (3.15) and (3.20). Chiral SU(3) × SU(3) invariance would fix the values of α and β. We do not impose this symmetry. However, our consideration of the $K\bar K$ channel does lead to a constraint relating[9] α and β.

IV. ISOSPIN AND PARTIAL-WAVE AMPLITUDES, PADÉ APPROXIMANTS, AND PHASE SHIFTS

The isospin and partial-wave amplitudes are

$$\begin{aligned}
M_0(s,t,u) &= 3A(s,t,u) + A(t,s,u) + A(u,t,s)\,,\\
M_1(s,t,u) &= A(t,s,u) - A(u,t,s)\,,\\
M_2(s,t,u) &= A(t,s,u) + A(u,t,s)\,,\\
&[A(s,t,u) = A(s,u,t)]
\end{aligned} \tag{4.1}$$

$$A_J(s) = \tfrac{1}{2}\int_{-1}^{1} dx\, P_J(x) M(s,t)\,, \tag{4.2}$$

$$\begin{aligned}
s &= W^2 = 4(q^2+\mu^2)\,,\quad t = -2q^2(1-x)\,,\\
u &= -2q^2(1+x)\,,\quad x = \cos\theta\,.
\end{aligned} \tag{4.3}$$

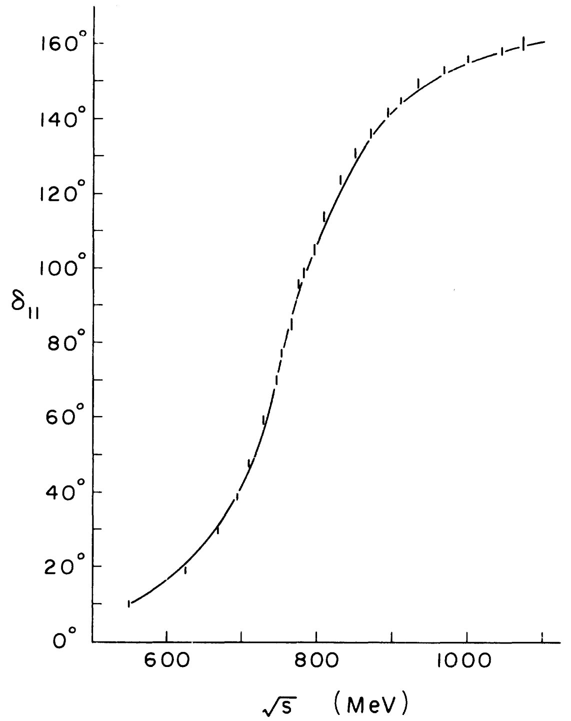

FIG. 2. The $I=1$, p-wave phase shift. The curve is the result of the calculation described in the text. The vertical dashes represent experimental values from Ref. 10.

Formulas for the integrals encountered in making the partial-wave projections (4.2) from the invariant matrix elements (4.1), determined from (3.14), are given in detail in JW.

The [1, 1] Padé approximant constructed from the first two terms in a perturbation series is

$$A_{IJ}^{[1,1]}(s)=\frac{[A_{IJ}^{(0)}(s)]^2}{A_{IJ}^{(0)}(s)-A_{IJ}^{(1)}(s)}\equiv\frac{B^2}{B-\mathcal{R}-i\mathcal{I}}, \quad (4.4)$$

where $B=A_{IJ}^{(0)}(s)$ is the (real) Born term and $\mathcal{R}$, $\mathcal{I}$ are the real and imaginary parts of the one-loop

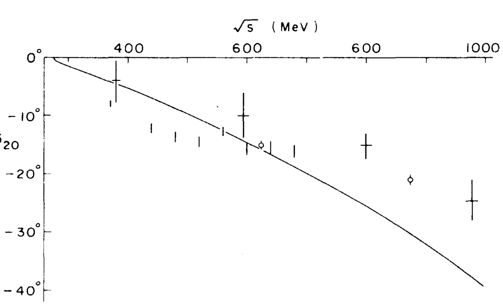

FIG. 3. The $I=2$, s-wave phase shift. The experimental values are form Ref. 12 (+), Ref. 13 (○), and Ref. 14 (|).

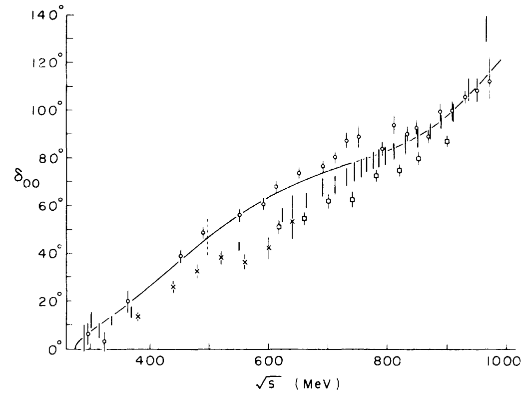

FIG. 4. The $I=0$, s-wave phase shift. The vertical bars above 400 MeV are values from Ref. 10. The vertical bars below 400 MeV are from Ref. 19. The circles above 400 MeV are from Ref. 15, and below 400 MeV are from Ref. 18. The boxes are from Ref. 16. The crosses are from Ref. 17. The dashed line at 496 MeV represents our estimate of δ_{00} from the $K\to 2\pi$ data.

perturbation amplitude. By virtue of perturbative unitarity

$$\mathrm{Im}A_{IJ}^{(1)}(s)=\frac{1}{16\pi}\,\frac{q}{w}\,|A_{IJ}^{(0)}(s)|^2$$

(for two identical particles) (4.5)

the [1, 1] Padé approximants satisfy elastic unitarity exactly

$$\mathrm{Im}A_{IJ}^{[1,1]}(s)=\frac{1}{16\pi}\,\frac{q}{w}\,|A_{IJ}^{[1,1]}(s)|^2. \quad (4.6)$$

Thus below inelastic thresholds (experimentally the $\pi\pi\to 4\pi$ inelasticity is very small below the $K\bar{K}$ threshold) the phase shift can be computed from

$$\tan\delta_{IJ}^{[1,1]}=\frac{\mathrm{Im}A_{IJ}^{[1,1]}}{\mathrm{Re}A_{IJ}^{[1,1]}}=\frac{\mathcal{I}}{B-\mathcal{R}}\ . \quad (4.7)$$

The resulting $I=0$ and 2 s-wave and $I=1$ p-wave phase shifts are given in Figs. 2–4.

V. RESULTS AND DISCUSSION

The calculated [1, 1] Padé phase shifts depend on the three parameters $L, \beta, \overline{\Gamma}_\pi$ which correspond to the cutoff which regulates the NLσM, the πK coupling constant, and the $\pi\pi\to\pi\pi$ finite counterterm (i.e., the value of the $\pi\pi\to\pi\pi$ amplitude at some conventionally chosen renormalization point). All three phase shifts depend on all three parameters; but roughly, β is chosen to fit the overall shape of δ_{00} (e.g., turning up below the $K\bar{K}$ threshold) and $\overline{\Gamma}_\pi$ and L are chosen to make δ_{11} pass through 90° at 773 MeV and δ_{00} pass through 90° somewhere between 800 and 900 MeV. The values of the parameters, and known physical constants, for the calculated phase shifts displayed in Figs. 2–4 are

$$\beta=1.0,\quad L=5.03,\quad \bar{\Gamma}_\pi=3.5075\quad (\alpha=16.5)\,, \tag{5.1a}$$

$$f_\pi=0.68\mu,\quad \mu=138\text{ MeV},\quad \mu_k=3.592\mu,\quad m=6.802\mu. \tag{5.1b}$$

The calculated p-wave shift plotted in Fig. 2 is seen to agree in detail with the rather well determined experimental values from 550 to 1200 MeV. We have plotted the experimental points from the phase-shift analysis of the LBL experiment[10] of Protopopescu *et al.* ($\pi^+p\to\Delta^{++}\pi^+\pi^-$). The phase-shift analyses of the high-statistics CERN-Munich experiment, Hyams *et al.*[11] ($\pi^-p\to\pi^+\pi^-n$) give very similar results for the p-wave phase shifts. We make a couple of comments on the difference between this calculation and the earlier JW claculation. In both calculations the p wave is relatively insensitive to the value of the parameter L. (In the chiral-invariant limit, $\mu\to 0$, it is independent of L. Se Appendix E of JW.) In the JW calculation a special s,t,u symmetry of the divergent part of the single-pion-loop unrenormalized integrals was used to determine $\bar{\Gamma}_\pi$ (discussion at the end of Appendix B of JW). Thus JW had essentially no parameters to fit δ_{11} and in fact had to adjust the effective πN coupling constant (with a hand-waving appeal to higher-order calculations) to get a fit to δ_{11} and the ρ parameters. In the present calculation, with the inclusion of kaon loops, loops, the special symmetry of the single-pion loops is lost, and $\bar{\Gamma}_\pi$ is a free parameter, which primarily determines the mass of the ρ. $G=m/f_\pi$ is fixed at its proper value to be the chiral pertrubation-expansion parameter, and the width of the ρ (≈ 160 MeV) comes out essentially with no free parameter (as mentioned above it is relatively insensitive to L) and β is chosen primarily for the shape of δ_{00}.

The calculated $I=2$ s-wave phase shift is plotted in Fig. 3 along with some of the more recently published experimental data.[12-14] The calculation and the experimental analyses agree that the phase shift is small and negative with negative slope. Above 700 to 800 MeV the calculated phase shift has become somewhat more negative than is indicated by the experimental analyses.

The calculated $I=0$ s-wave phase shift is plotted in Fig. 4 along with the results of two different phase-shift analyses[15,16] of the CERN-Munich experiment, the results of the phase-shift analysis of the LBL[10] experiment, and the results of the Argonne–Notre Dame[17] experiment. (The two analyses of the CERN-Munich experiment reproduced here represent the extremes of five different analyses presented in Ref. 16.) Also plotted is the value of δ_{00} at $\sqrt{s}=m_K$ determined from the analysis of the final-state interaction in $K\to 2\pi$ decays, and the low-energy values of δ_{00} from the K_{e4} decay experiments of Beier *et al.*[18] and Rosselet *et al.*[19] The existence of these experimental values determined from the final-state interactions of the two pions from K decays is important, not only because they cover a generally different (lower) energy range, but also because all of the high-statistics phase-shift analyses above 400 MeV are indirect, i.e., they are extracted from production processes with three hadrons in the final state and require extrapolation to the virtual pion pole after subtraction of background not due to single-pion exchange. Although the more recent experimental analyses have high statistical accuracy, the question of possible systematic error (in the analyses) is very difficult to assess[20] (c.f. our Fig. 4 or the original Fig. 31 of Ref. 16 in which phase shifts obtained from different analyses of the same experiment differ by considerably more than the statistical errors).

Comparison of the decay rates of the different charge states of $K\to 2\pi$ gives the value of $\delta_{00}-\delta_{20}$ at $\sqrt{s}=m_K$ if one assumes that the $\Delta I=\frac{5}{2}$ amplitude is negligible compared to the weak $\Delta I=\frac{1}{2}$ and $\frac{3}{2}$ amplitudes. In the conventional weak-interaction theory there is no lowest-order $\Delta I=\frac{5}{2}$ term in the weak nonloptonic Hamiltonian; it must be of order $G\alpha$. This would suffice except that, empirically, nonleptonic $\Delta I=\frac{3}{2}$ decays are suppressed (relative to $\Delta I=\frac{1}{2}$); so the question becomes arguable. Nelgecting the $\Delta I=\frac{5}{2}$ amplitude and using $K\to 2\pi$ data from the 1976 Particle Data Group compilation,[21] we find $\delta_{00}-\delta_{20}\simeq 58°$, consistent, within a fairly large uncertainty, with the values, for that energy, obtained from the production experiments. *If* we accept that δ_{20} is in the range $-8°$ to $-14°$ at $\sqrt{s}=m_K$, as indicated by the more recent production experiments,[12,14] then this value of δ_{00} favors the (larger) phase shift of the Estabrooks and Martin analysis[15] over the other analyses of the CERN-Munich experiment in the region of the kaon mass. It is also a little larger than the Argonne–Notre Dame result. Our calculated value is $47°+9°=56°$. The analysis to extract the low-energy $\pi\pi$ phase shifts from the final-state interaction of the two pions from K_{e4} decays is generally straightforward, the problem in this case being to get sufficiently good statistics from these rare decay events. It is seen from the figure that the calculated phase shift is consistent with the experimental phase shifts (from these three different sources) over the entire energy range from the elastic threshold (276 MeV) up to the inelastic threshold (991 MeV).

The success of the δ_{00} and δ_{11} calculations renders the calculated scattering lengths of some interest.

$$A_I^{[1,1]} = \lim_{q\to 0} \frac{1}{q^{2J+1}} \delta_{IJ}^{[1,1]}$$

$$\underset{J=0}{=} A_I^{(0)} \left[1 - \frac{A_{I0}^{(1)}(4\mu^2)}{A_{I0}^{(0)}(4\mu^2)}\right]^{-1} \quad (5.2a)$$

$$\underset{J=1}{=} A_1^{(0)} \left[1 - \frac{(d/dq^2)A_{11}^{(1)}(4\mu^2)}{(d/dq^2)A_{11}^{(0)}(4\mu^2)}\right]^{-1}, \quad (5.2b)$$

where the tree-diagram scattering lengths $A_I^{(0)}$ are the Weinberg[7] scattering lengths

$$A_0^{(0)} = \frac{1}{32\pi}\left(\frac{\mu^2}{f_\pi^2}\right) 7\mu^{-1} \simeq 0.15\mu^{-1},$$

$$A_1^{(0)} = \frac{1}{32\pi}\left(\frac{\mu^2}{f_\pi^2}\right) \tfrac{4}{3}\mu^{-3} \simeq 0.03\mu^{-3}, \quad (5.3)$$

$$A_2^{(0)} = \frac{1}{32\pi}\left(\frac{\mu^2}{f_\pi^2}\right)(-2\mu^{-1}) \simeq -0.04\mu^{-1}.$$

In the present calculation we find

$$A_0 = 1.70 A_0^{(0)} \simeq 0.26\mu^{-1},$$

$$A_1 = 1.40 A_1^{(0)} \simeq 0.04\mu^{-3}, \quad (5.4)$$

$$A_2 = 1.09 A_2^{(0)} \simeq -0.05\mu^{-1}.$$

(The large fractional change in the numerical value of A_2 from $A_2^{(0)}$ is largely due to rounding off.)

As already mentioned in the Introduction, in a subsequent paper we will report the results[9] of a coupled-channel ($\pi\pi, K\bar{K}$) matrix Padé calculation of δ_{00} above the $K\bar{K}$ threshold, and the πK s-wave phase shifts. Another direction in which this calculation can be expanded is to include the other "stable" pseudoscalar mesons η, η' (the η' does have a strong decay channel $\eta\pi\pi$, but it is very close to threshold and very narrow), and the other stable spin-$\frac{1}{2}$ baryons Λ, Σ, Ξ. This leads to a consideration of chiral SU(2) × SU(2) versus chiral SU(3) × SU(3). We believe the empirical case for chiral SU(2) × SU(2) as a good approximate (Nambu-Goldstone) symmetry of hadron physics is persuasive,[22] and we regard the success of the calculations reported here as further evidence in its favor (we are by now well beyond soft-pion theorems, particularly with the ρ and the coupled-channel calculation of the S^* phenomena). We regard the status of chiral SU(3) × SU(3) as unclear. There exists a very detailed LσM SU(3) × SU(3) Padé calculation by Chan and Haymaker[23] (CH) which is relevant. Chiral SU(3) × SU(3) symmetry, with only linear symmetry breaking, is very restrictive; thus CH have very few parameters. They are able to fit the observed pattern of pseudoscalar masses and a number of meson decay constants, but their calculated phase shifts are quite different from the experimental ones (δ_{00} reaches a maximum and turns over, as in the original LσM calculation of Ref. 3; there is no S^* effect, even in a coupled-channel calculation; the calculated ρ is very narrow—again, as in Ref. 3). We take these results of CH as an indication that chiral SU(3) × SU(3) is strongly broken, perhaps too strongly to be useful for this kind of dynamical calculation. There are, of course, other possibilities. The approximation of the partial-wave amplitudes by the [1,1] Padé approximants may be failing. A priori the convergence properties of the Padé approximants in a real field theory are unknown. But a posteriori there is the success of the phase-shift calculations of the present paper, achieved with just one more free parameter. As another example, in our calculation we find that the nucleon loop plays a non-negligible role in the generation of the ρ resonance (although it is no longer the dominant factor as in the previous JW calculation); the CH calculation includes only mesons. There is also the question of what would happen in an SU(3) × SU(3) NLσM calculation[24] [in the sense of (iii) of the Introduction of this paper]. In considering this possibility one immediately runs into the problem that chiral SU(3) × SU(3) with only linear symmetry breaking does not admit the mass pattern $\mu_\pi < \mu_K \ll M_\sigma, M_\kappa$ required for a NLσM calculation regulated by the scalar masses. Thus if a chiral SU(3) × SU(3) NLσM calculation is to be carried out it will require bilinear symmetry breaking as well as linear. Whether this can be consistently carried out is not yet known. The alternative is to include the additional particles, but only classify them in chiral SU(2) × SU(2) multiplets. Of course this leaves more free parameters (coupling constants and scalar masses) and no low-energy theorems for external pseudoscalars other than the pion. Finally, we note that our calculation is consistent with the indication from CH that chiral SU(3) × SU(3) is badly broken, in the sense that the values of our parameters which give successful fits are very far from the SU(3) × SU(3)-symmetry-limit values ($\beta_0 = \frac{1}{2}, \alpha_0 = 1$). Of course, in the SU(3) × SU(3) symmetry limit the calculation would include contributions from additional diagrams involving η, η', etc., so the best-fit parameters would presumably change some; how much cannot be determined before the calculation is done.

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$$0=-\tfrac{1}{6}\beta^2+\tfrac{1}{3}\beta^3-\tfrac{1}{6}\beta^4-\tfrac{1}{24}\beta\alpha-\tfrac{1}{4}\beta^2\alpha$$
$$+\tfrac{7}{32}\alpha+\tfrac{1}{48}\alpha^2+\rho_K'\left(-\tfrac{1}{8}-\tfrac{1}{3}\beta\right)-\frac{\rho_K'}{6\alpha},$$
$$\rho_K'=\frac{2\alpha(\beta-1)^2}{\alpha-1}\ \ln\alpha-2\alpha\beta-\tfrac{5}{4}\alpha.$$

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