Low-energy pion-nucleon scattering

M. K. Banerjee*
Department of Physics and Astronomy, University of Maryland, College Park, Maryland 20742

J. B. Cammarata†
Department of Physics, Virginia Polytechnic Institute and State University, Blacksburg, Virginia 24061
(Received 24 January 1977)

Using the divergence of the axial-vector current as the pion interpolating field and a soft-pion limit we derive a once-subtracted Low equation for the off-mass-shell πN amplitude. The σ commutator term and the once-subtracted z graphs are the main isoscalar and isovector driving terms. The calculated S-wave phase shifts are in good agreement with the experimental data when we use $|g_\pi(4M^2)| = 11.69$ and 25.5 MeV for the σ commutator term.

It has been more than twenty years since Low[1] and Chew and Low[2] (CL) developed a nonperturbative formalism which, with several approximations, describes the prominent dynamical features of the pion-nucleon P-wave interaction at low energy. But analyses of experiments carried out at the meson factories have repeatedly shown that a more complete description of the pion-nucleon interaction is needed for the proper interpretation of many pionic processes. To this end we have proposed[3] a theory of πN scattering which is a logical extension of the work of Chew and Low.

In this theory we use the Low equation obtained from the Lehmann-Symanzik-Zimmermann (LSZ) reduction formalism, but in contrast to CL we do not use the static approximation, and we retain the seagull terms and the antinucleon-intermediate-state contribution. We use the divergence of the axial-vector current as the interpolating pion field. A soft-pion limit is used to eliminate the isoscalar part of the seagull term and obtain a once-subtracted form of the Low equation. This equation allows the evaluation of both the physical πN amplitude as well as the off-mass-shell amplitude[4] once the remaining dynamical inputs are specified. These include the isovector part of the seagull term, the pion-nucleon form factor, and the σ commutator term which appears in the soft-pion limit. The theory describes all πN partial waves and formally is valid for all energies.

In this note we apply our theory to calculate the low-energy S-wave phase shifts.[5] Of the several earlier efforts[6] on this problem, two are close to ours in spirit. These are the works of Drell, Friedman, and Zachariasen[7] and Hackman,[8] both based on the Chew-Low formalism. Unlike the calculation of Drell *et al.*, our approach is not dependent on a phenomenological Lagrangian for the low-energy S-wave interactions. In contrast to the work of Hackman, we include nucleon recoil and antinucleon contributions in intermediate states and handle the seagull terms in a completely different manner. As a result we obtain better agreement in the S-wave phase shifts over a wider energy range than these earlier works.

Our approach is based on the off-mass-shell amplitude

$$F_{\beta\alpha}(P_i,P_f,k) = i\int d^4x\, e^{ik\cdot x}(\Box + m_\pi^2) \times \langle P_f|T(\phi_\beta(x)j_\alpha(0))|P_i\rangle, \quad (1)$$

with $|P_i\rangle$ and $|P_f\rangle$ physical nucleon states, and where the interpolating pion field is given in terms of the divergence of the axial-vector current by $\phi_\beta(x)=\sqrt{2}\,\partial^\mu A^\beta_\mu(x)/f_\pi$, $j_\alpha(x)=(\Box+m_\pi^2)\phi_\alpha(x)$, β and α are isospin indices, and f_π is the pion decay constant, taken to be $0.939\, m_\pi^3$. We can rewrite this as

$$F_{\beta\alpha}(k) = \int d^4x\, e^{ik\cdot x}\langle P_f|\,\delta(x_0)[\phi_\beta(x),j_\alpha(0)]k_0 + i\,\delta(x_0)[\dot{\phi}_\beta(x),j_\alpha(0)]\,|P_i\rangle + i\int d^4x\, e^{ik\cdot x}\langle P_f|T(j_\beta(x)j_\alpha(0))|P_i\rangle. \quad (2)$$

For brevity we exhibit only the pion four-momentum k as the argument of F. The first two terms on the right-hand side, the seagull terms, are taken to have the form

$$\langle P_f|i\epsilon_{\alpha\beta\lambda}(k+k')\cdot Y^\lambda(0) + \delta_{\alpha\beta}\Sigma(0)|P_i\rangle, \quad (3)$$

where Σ is a scalar-isoscalar operator, Y^λ is a vector-isovector operator and $k'=k+P_f-P_i$. In phenomenological Lagrangian models, Σ and Y are associated with the t-channel exchange of scalar-isoscalar and vector-isovector bosons, respectively.

We eliminate Σ by subtracting from (2) the soft-pion limit[9]

$$F_{\beta\alpha}(0)=\lim_{k_0\to 0}[\lim_{\vec{k}\to 0}F_{\beta\alpha}(k)]\,,$$

i.e, the limit in which one pion is soft while the other has momentum $k'=P_f-P_i$. This differs from the more familiar soft-pion limit[10] where one lets $(k+k')\to 0$. Using the Ward identity we find

$$F_{\beta\alpha}(0)=\left(\frac{\sqrt{2}}{f_\pi}\right)^2 m_\pi{}^2(t-m_\pi{}^2)\langle P_f|\sigma(0)|P_i\rangle\delta_{\alpha\beta}-g_\pi(t)g_\pi(0)\bar{u}(P_f)\left(\gamma_0\frac{(M-\not{P}_f)}{4MP_{f0}}\tau_\beta\tau_\alpha+\frac{(M-\not{P}_i)}{4MP_{i0}}\gamma_0\tau_\alpha\tau_\beta\right)u(P_i), \tag{4}$$

where $t=(P_f-P_i)^2$ and

$$\langle P_f|\sigma(0)|P_i\rangle\delta_{\alpha\beta}=i\int d^4x\langle P_f|[A_0^\beta(x),\partial^\mu A_\mu^\alpha(0)]|P_i\rangle\delta(x_0).$$

The other term in (4) is the nucleon pole term in the soft-pion limit. There are no other contributions; this expression for the soft-pion amplitude is exact. The quantity $g_\pi(t)$ is defined by

$$\langle P_f|j_\alpha(0)|P_i\rangle=ig_\pi(t)\bar{u}(P_f)\gamma_5\tau_\alpha u(P_i)\,.$$

From the definitions of ϕ_α and j_α we have $g_\pi(0)=\sqrt{2}Mm_\pi{}^2g_A(0)/f_\pi$ and using $g_A(0)=1.25$ gives $g_\pi(0)=12.7$. Upon subtraction we have

$$F_{\beta\alpha}(k)=F_{\beta\alpha}(0)+\langle P_f|2i\epsilon_{\alpha\beta\lambda}k\cdot Y^\lambda(0)|P_i\rangle+i\int d^4x\,e^{ik\cdot x}\langle P_f|T(j_\beta(x)j_\alpha(0))|P_i\rangle-i\int d^4x\langle P_f|T(j_\beta(x)j_\alpha(0))|P_i\rangle\,. \tag{5}$$

For the integrals in (5) we insert a complete set of physical states between the current operators in both terms of the time-ordered product. We retain those states felt to be the most important. These include the states $|N\rangle$, $|\pi N\rangle$ and the disconnected parts (z graphs) arising from the $|\bar{N}NN\rangle$ terms, where N = nucleon, $\bar{N}$ = antinucleon. In the c.m. frame, the first integral in (5) becomes

$$\begin{aligned}
i\int d^4x\,e^{ik\cdot x}\langle P_f|T(j_\beta(x)j_\alpha(0))|P_i\rangle=&\frac{g_\pi((P_f-P)^2)g_\pi((P_i-P)^2)}{2(k_0+P_{f0}-M)}\bar{u}(P_f)(1-\gamma_0)\tau_\beta\tau_\alpha u(P_i)\\
&+\frac{g_\pi((P_f-l)^2)g_\pi((P_i-l)^2)}{2l_0(k_0+l_0-P_{i0})}\bar{u}(P_f)[(l_0-P_{i0}-P_{f0})\gamma_0+M]\tau_\alpha\tau_\beta u(P_i)\\
&-\frac{\bar{g}_\pi((P_f+P)^2)\bar{g}_\pi((P_i+P)^2)}{2(k_0+P_{f0}+P_0)P_0}\bar{u}(P_f)(\not{P}+\not{P}_f)\tau_\beta\tau_\alpha u(P_i)\\
&-\frac{\bar{g}_\pi((P_f+\tilde{l})^2)\bar{g}_\pi((P_i+\tilde{l})^2)}{2\tilde{l}_0(P_{i0}+\tilde{l}_0-k_0)}\bar{u}(P_f)(\tilde{l}+\not{P}_i)\tau_\alpha\tau_\beta u(P_i)\\
&+\frac{\Sigma_\gamma\Sigma_s\int d^3q_\pi d^3q_N}{(2\pi)^3 2q_{\pi 0}}\frac{M}{q_{N0}}\frac{\langle P_f|j_\beta(0)|q_Ns,q_\pi\gamma\rangle\langle q_Ns,q_\pi\gamma|j_\alpha(0)|P_i\rangle}{q_{N0}+q_{\pi 0}-P_{f0}-k_0-i\eta}\delta^3(\vec{q}_\pi+\vec{q}_N+\vec{p}_f+\vec{k})\\
&+\Sigma_\gamma\Sigma_s\int\frac{d^3q_\pi d^3q_N}{(2\pi)^3 2q_{\pi 0}}\frac{M}{q_{N0}}\frac{\langle P_f|j_\alpha(0)|q_Ns,q_\pi\gamma\rangle\langle q_Ns,q_\pi\gamma|j_\beta(0)|P_i\rangle}{q_{N0}+q_{\pi 0}-P_{i0}+k_0}\delta^3(\vec{q}_\pi+\vec{q}_N+\vec{k}-\vec{p}_i)\,,
\end{aligned} \tag{6}$$

where l, $\tilde{l}$, and P are four-momenta of physical nucleons with $\vec{l}=-\vec{\tilde{l}}=-\vec{k}-\vec{k}'$ and $\vec{P}=0$. The last two terms are the direct and the crossed one-pion–one-nucleon contributions. The first two terms are the direct and crossed nucleon pole terms. The third and the fourth are the crossed and direct antinucleon terms (z graphs). In the original expression, these terms contained factors like

$$\tfrac{1}{2}[\langle 0|j_\alpha(0)|N(P_i),\bar{N}\rangle_{\text{in in}}\langle N(P_f),\bar{N}|j_\beta(0)|0\rangle+\langle 0|j_\alpha(0)|N(P_i),\bar{N}\rangle_{\text{out out}}\langle N(P_f),\bar{N}|j_\beta(0)|0\rangle]$$

which involve $\mathrm{Re}[g_\pi((P_i+\bar{N})^2)g_\pi^*((P_f+\bar{N})^2)]$. Since there is considerable uncertainty in our knowledge of g_π for $t>4M^2$,[11] in Eq. (6) we have made the simplifying assumption of replacing the real part of the product of the two complex form factors by the product of two real functions, described further below.

The corresponding expression for the final integral in Eq. (5) is obtained from Eq. (6) by dropping the nucleon-pole terms, setting $(k_0,\vec{k})=0$, and replacing the 4-vector P by $(P_{f0},-\vec{P}_f)$ and $\tilde{l}$ by $(P_{i0},-\vec{P}_i)$.

By omitting the S-wave inelastic states from the complete sum we have limited the range of applicability of our theory to the elastic region. Because the scattering amplitude is given by an equation with a once-subtracted form, the neglect of these high-mass states will have a weaker effect on the low-energy elastic amplitude than what occurs in an unsubtracted equation (e.g., the Chew-Low equation).

Using (6) and its analog for the $k \to 0$ limit (soft terms) in (5) and setting $k_0 = (\vec{k}^2 + m_\pi^2)^{1/2}$, we obtain a nonlinear integral equation for the off-mass-shell amplitude ${}_{\text{out}}\langle P_f, k\beta | j_\alpha(0) | P_i \rangle$. At present we have studied the solution of this equation for the S waves only. It should be noted that because we include nucleon recoil the integral equation for each partial-wave amplitude is coupled to all partial waves. This coupling arises from the crossed one-pion-one-nucleon term in (6) and, more importantly, from the soft terms, both direct and crossed.

The P-wave contribution to the S-wave equation is calculated by including inelasticity and parametrizing the absorptive part in a factorable form as $[4\pi W(q)/M]\sigma_\nu(q)\phi(k)\phi(k')/\phi^2(q)$ with the form factor

$$\phi(q) = q(1 + q^2/\mu^2)^{-5/2}, \tag{7}$$

where ν is the P-wave channel index, σ_ν is the total cross section in channel ν, and $W(q)$ is the c.m. energy. σ_ν is computed using the CERN theoretical fit.[12] As expected, the higher partial waves were found to contribute negligibly to the low-energy S-wave amplitude.

We assume that the isovector seagull term $Y^\lambda = 0$ (no ρ-meson terms).[13] The σ commutator is parametrized as

$$\langle P_f | \sigma(0) | P_i \rangle = \frac{\bar{u}(P_f) u(P_i) g_\sigma}{\left(1 - \frac{t}{\mu_1^2}\right)\left(1 - \frac{t}{\mu_2^2}\right)\left(1 - \frac{t}{\mu_3^2}\right)}. \tag{8}$$

For $\bar{g}_\pi(t)$, with which we approximate the role of $g_\pi(t)$ for $t > 4M^2$, we use

$$\bar{g}_\pi(t) = \bar{g}_\pi[1 + (t - 4M^2)/4m_0^2]^{-1}, \tag{9}$$

where $\bar{g}_\pi = |g_\pi(4M^2)|$. For $g_\pi(t)$, $t < 0$, we use

$$g_\pi(t) = g_\pi(0)[1 + t(t - 4M^2)/4M^2 m_0^2]^{-1}. \tag{10}$$

Because of the small contribution of the nucleon-pole term to the low-energy S-wave problem we use the same form-factor mass in (10) and (9). Using Padé approximants we succeeded in constructing solutions of the nonlinear integral equation for the S-wave amplitudes $f_{2I,2J}(k,k')$, k and k' being the magnitudes of the on- and off-shell momenta. When substituted into the integral equation the solutions reproduced themselves within 5% for $k, k' < 2.2 m_\pi$. We believe that our phase shifts are reliable to 1%. The numerical method will be described in a longer paper which is in preparation.

The S-wave phase shifts are very sensitive to the parameters g_σ and $\bar{g}_\pi$. δ_{11} is particularly sensitive to these quantities because of the cancellation between the repulsive σ commutator term and the attractive, once-subtracted z graphs. The phase shifts are less sensitive to the form-factor masses. We settled on the values $\mu = 8m_\pi$ (P wave), $\mu_1 = \mu_2 = 8.24 m_\pi$, $\mu_3 = 7.5 m_\pi$ (σ commutator), and $m_0 = 8.6 m_\pi$ [$g_\pi(t)$, and $\bar{g}_\pi(t)$]. As discussed below our value of g_σ is close to the value of 26 MeV obtained by Huang *et al.*[14] and is in the range given by Reya.[15] The form-factor masses μ_1, μ_2, μ_3 are compatible with the mass of the broad, scalar-isoscalar $\pi\pi$ resonance[16] $\epsilon(1200)$.

We demanded a very good fit to δ_{31} for pion lab energy $T_\pi < 100$ MeV. In the figure we present our calculated phase shifts (dashed curves) for three sets of g_σ and $\bar{g}_\pi$. These are (a) 25 MeV and 11.85, (b) 25.5 MeV and 11.69, and (c) 26 MeV and 11.54. The three sets give identical results for δ_{31}. The solid curves are the energy-dependent fits of Zidell, Roper, and Arndt.[17] The flagged circles are the energy-independent fits of Carter, Bugg, and Carter.[18] Our agreement with the experimental δ_{31} is excellent up to $T_\pi = 100$ MeV. For δ_{11} we note first the marked sensitivity of our calculated phase shifts to the parameters g_σ and $\bar{g}_\pi$. The ex-

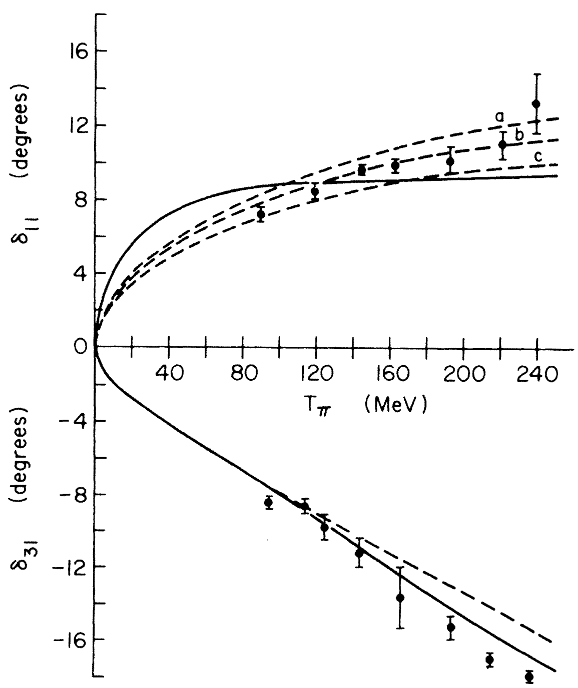

FIG. 1. Phase shifts δ_{11} and δ_{31}. The dashed lines are our results. The solid lines give the results of Ref. 17, while solid circles with error bars are from Ref. 18. Dashed lines (a), (b) and (c) represent different choices for g_σ and $\bar{g}_\pi$, as discussed in the text.

perimental δ_{11} obtained by the energy-dependent fit[17] shows a sharp change of slope at $T_\pi \sim 60$ MeV. The energy-independent fit,[18] using a smaller set of data, appears to have a gentler curvature, a feature which we easily obtain. We could not reproduce the qualitative features of δ_{11} of the energy-dependent fit without spoiling the agreement for δ_{31}, even with substantial changes of all parameters. We have chosen the set (b) as our "best" set because of the quality of agreement with the energy-independent fit. With this choice our scattering lengths and effective ranges are $-0.143 m_\pi^{-1}$ and $0.98 m_\pi^{-1}$ for S_{11} and $0.095 m_\pi^{-1}$ and $5.35 m_\pi^{-1}$ for S_{31}.

In our theory we have not used the current algebra but have only used the weaker condition that $\int \delta(x_0 - y_0)[A_0^\alpha(x), A_\mu^\beta(y)] d^4x$ is divergenceless. This fact establishes that the σ commutator term is isoscalar. So current algebra can still be used as a check of our dynamical theory. This is done as follows: Setting $\vec{P}_i = \vec{P}_f$ we evaluate the soft-pion limit of the quantity $[F_{\beta\alpha}(k) - F_{\alpha\beta}(k)]/k_0$ in two ways, by using the current algebra and by using the Low expansion [Eq. (2) and (6)]. Equating the two expressions for $\vec{P}_i = \vec{P}_f = 0$ one obtains the following sum rule for the off-mass-shell amplitudes $f_{2I,2J}(q, 0)$:

$$\left(\frac{\sqrt{2}}{f_\pi}\right)^2 \frac{m_\pi^4}{2} = \frac{1}{2M^2} \left|g_\pi(4M^2)\right|^2 + \frac{M}{6\pi^2} \int \frac{q^2 dq}{\omega(q)E(q)} \times \frac{|f_{11}(q,0)|^2 - |f_{31}(q,0)|^2}{[W(q) - M]^2} .$$

Our off-shell amplitudes satisfy the sum rule to better than 3%, and, naturally, our charge-exchange scattering length $a^{(-)} = 0.0793 m_\pi^{-1}$ agrees very well with the current-algebra result,[10] $0.0786 m_\pi^{-1}$.

In summary, we have presented a theory of πN scattering with σ commutator term and the once-subtracted z graphs as the main isoscalar and isovector driving terms in the S wave. The advantages of a once-subtracted Low equation over an unsubtracted one are also essential for a practical theory.

The current practice in the study of π nucleus scattering is to use a simple parametrized form for the fully off-mass-shell πN amplitude. The limitations of such approaches and the critical need for a better knowledge of the fully off-mass-shell πN amplitude have been pointed out.[19,20] We believe that the present theory provides the vital first step towards that goal.

The authors wish to thank V. Zidell, D. Roper, and R. Arndt for advance communication of their results. We are grateful also to R. Peccei for a useful discussion and to R. Zia for a critical reading of the manuscript. We thank the Computer Science Center of the University of Maryland for providing part of the computer time used for this work. Part of this work was completed while one of us (J.B.C.) participated in the Visitors Program of the University of Maryland's Nuclear Theory Group.

---

*Research supported in part by the U. S. Energy Research and Development Administration.

†Work supported in part by National Science Foundation.

[1] F. E. Low, Phys. Rev. 97, 1392 (1955).

[2] G. F. Chew and F. E. Low, Phys. Rev. 101, 1570 (1956).

[3] M. K. Banerjee and J. B. Cammarata, Bull. Am. Phys. Soc. 20, 1193 (1975).

[4] Here and subsequently the phase "off-mass-shell amplitude" refers to the four-point function in which one pion's four-momentum is off the mass shell.

[5] A study of the P-wave amplitude is in progress.

[6] For examples see B. Dutta-Roy, I. R. Lapidus, and M. J. Tausner, Phys. Rev. 181, 2091 (1969), and P. Nath and S. S. Kere, in Lectures from the LAMPF Summer School on Pion-Nucleus Scattering, 1973, Los Alamos Report No. LA-5443-C (unpublished), p. 226.

[7] S. D. Drell, M. H. Friedman, and F. Zachariasen, Phys. Rev. 104, 236 (1956).

[8] R. H. Hackman, Phys. Rev. D 8, 3920 (1973).

[9] The order in which the limits are carried out is important.

[10] S. Weinberg, Phys. Rev. Lett. 17, 616 (1966).

[11] The recent analyses of the πN form factors by W. Nutt and B. Loiseau, Nucl. Phys. B104, 98 (1976), and J. B. Cammarata and M. K. Banerjee, Bull. Am. Phys. Soc. 20, 1192 (1975), which give qualitatively similar results, are valid for $t < 4M^2$ only.

[12] D. H. Herndon, A. Barbaro-Galtieri, and A. H. Rosenfeld, LRL Report No. UCRL-20030 πN, 1970 (unpublished).

[13] The absence of a canonical ρ field in a Lagrangian is not incompatible with dynamical generation. See, for example, J. L. Basdevant and B. W. Lee, Phys. Rev. D 2, 1680 (1970), who generated ρ and f_0 resonances using the nonlinear-σ-model Lagrangian of M. Gell-Mann and M. Levy, Nuovo Cimento 16, 705 (1960).

[14] W. T. Huang, C. A. Levinson, and M. K. Banerjee, Phys. Rev. C 5, 651 (1972). This and other evaluations of the σ commutator term using the Fubini-Furlan method all give values ~26 MeV (see Ref. 15 for details). Various authors using a Cheng-Dashen (CD) type approach have obtained values ranging from 45 to 80 MeV. In this approach one equates g_σ to $(f_\pi/\sqrt{2})^2 m_\pi^4 f^{(+)}(\nu = 0, t = 2m_\pi^2, k^2 = k'^2 = m_\pi^2)$, where $\nu = (P_f + P_i)(k + k')/4M$, believing that terms $\sim k^4$ are negligible (Ref. 15). Assuming the validity of our form factors for timelike momentum transfers we get $g_\sigma = 29$ MeV from the CD amplitude. Two recent papers [W. Langbein, Nucl. Phys. B94, 519 (1975); D. C. Moir, R. J. Jacob, and G. E. Hite, *ibid.* B103, 477

(1976)] using the CD approach evaluate amplitudes related to $f^{(+)}$ for $t \leqslant 0$ and then extrapolate to $t = 2m_\pi^2$. We assert that the high values of g_σ obtained by the authors is purely the result of the *ad hoc* assumption that *all errors are statistical*. Langbein plots $\tilde{C}^{(+)}(\nu = 0, t \leqslant 0)$ in his Fig. 9. We can calculate this quantity from our theory. The amplitude is sensitive to the parameter μ [see Eq. (7)]. Using Langbein's values for $t > -0.05$ GeV^2 we deduce that g_σ lies between 28 and 33 MeV for $\mu = \infty$. $\mu = 14m_\pi$ yields g_σ between 24 and 27 MeV. We thank Dr. R. J. Jacob for providing us with the values of $D(0, t)$ and for useful comments.

[15]E. Reya, Rev. Mod. Phys. 46, 545 (1974).
[16]Particle Data Group, Rev. Mod. Phys. 48, S1 (1976).
[17]V. S. Zidell, L. D. Roper, and R. A. Arndt, unpublished work.
[18]J. R. Carter, D. V. Bugg, and A. A. Carter, Nucl. Phys. B58, 378 (1973).
[19]M. K. Banerjee, in *Meson-Nuclear Physics—1976*, proceedings of the International Topical Conference, Pittsburgh, edited by P. D. Barnes, R. A. Eisenstein, and L. S. Kisslinger (AIP, New York, 1976), p. 119.
[20]J. B. Cammarata and M. K. Banerjee, Phys. Rev. C 13, 299 (1976).