PHYSICAL REVIEW D VOLUME 16, NUMBER 4 15 AUGUST 1977

On-shell prescription in three-particle scattering

T. H. Rihan*
Atomic Energy Establishment, Tripoli-Libya
(Received 23 March 1976)

The scattering of three identical particles initiated from a particle incident on a bound state is considered. A prescription is suggested for approximate evaluation of the physical amplitude in terms of on-shell and half-on-shell two-particle amplitudes adopting the first-order Alt-Grassberger-Sandhas perturbation treatment. The half-on-shell two-particle amplitudes can in turn be given in terms of their on-shell versions and the knowledge of two-particle interactions using the Kowalski-Noyes prescription. Sloan's first-order unitary model is also considered. Using our prescription, it is shown that with such a model, the elastic and rearrangement amplitudes are completely determined from on-shell two-particle amplitudes, while for the break-up process, half-on-shell two-particle amplitudes inevitably appear.

I. INTRODUCTION

The separable approximation for the off-shell two-particle amplitudes appearing in the three-particle Faddeev[1] equations has enjoyed wide application.[2] With such an approximation the whole complexity of the problem is reduced to the solution of one-dimensional coupled integral equations.[3] The usual practice is then to consider only bound-state-pole and resonance-pole contributions in the two-body amplitudes.[4] These forms are then taken to represent practically the off-shell two-body amplitudes for all values of their relevant parametric energies. However, the two-particle resonance-pole dominance (in the three-body problem) was fully investigated by Bolle *et al.*[5] and was found rather defective. Moreover, based on Weinberg's quasiparticle theory,[6] Alt, Grassberger, and Sandhas[7] (hereafter referred to as AGS) showed that the above concept of bound-state-pole and resonance-pole dominance represents only the zeroth-order term in a more general iteration scheme. Actually, the kernels of the Faddeev-type equations,[7] being of the Hilbert-Schmidt type, can be approximated by a sum of finite-rank operators separable in all variables.[4] But in general this sum should be infinite. In any case, keeping only a few terms in this sum (that correspond to bound-state and resonance poles in the two-body subsystems) will necessarily have only relative success. Indeed such models predict unreliable results for polarization. Typically, in three-body calculations, the separability of the two-body transition amplitudes results from assuming the separability of the two-body interaction potentials. However, the separable potentials confine their dynamics to the long-range (eternal-triangle) region, and therefore may give reasonable results for the scattering process, but unreliable three-body binding energies.[8] Further, the construction of Weinberg states, essential for practical three-body calculations, proves to be complicated especially above the three-particle threshold.[9] In passing, we also note that in any realistic calculations of low-energy nucleon-deuteron scattering the inclusion of P-wave components of the two-nucleon amplitudes (which do not represent any bound or resonant states) seems to be essential.[10]

In the present work we shall consider the scattering process of three identical particles initiated from a particle incident on a bound state. However, the identity of particles will not constitute a barrier for other special cases of interest to be studied. What will concern us mostly is that the two-particle bound states appearing in the problem can be described by the same nuclear interaction. The underlying philosophy of our approach is to try to work directly with observable on-shell two-particle amplitudes whenever possible. It is of course more advantageous to make our three-body amplitudes depend on measurable or at least knowable two-body amplitudes, rather than assuming some form for the interparticle forces. However, this policy will lead us to face the old and difficult problem of determining average two-body amplitudes.[11] We shall in the present work treat this problem in a more reliable way that will contain corrections to previous results[11] and is more suitable in the presence of two-particle resonances. This will constitute our main approximation. We shall generally work within the AGS effective two-particle formulation of the three-body problem,[7] considering only the first-order perturbation approximation. However, matters will be much simpler and more convenient for practical calculations within Sloan's first-order unitary approximation.[12] On some occasions, especially for the break-up process, we shall be forced to consider half-on-shell two-particle am-

plitudes. These in turn can be given in terms of their corresponding on-shell amplitudes and the knowledge of the phenomenological two-particle interaction using the techniques of Kowalski[13] and Noyes[14] (hereafter referred to as KN). Our aim will be to make a detailed study of the nucleon-deuteron scattering process (say) in the first-order unitary model and compare our results with the already existing calculations[10] based on rank-one separable potentials. Calculations to that end are in progress.

In Sec. II we briefly review the AGS approach specifying our choice for the separable part of the two-particle amplitudes. We then discuss our main approximation for the evaluation of the physical scattering amplitude in terms of on-shell two-particle amplitudes assuming particle identity. In Sec. III we evaluate the three-particle scattering amplitude in the first-order unitary approximation adopting our prescription for identical particles.

II. APPROXIMATE THREE-BODY EQUATIONS

Throughout this work we shall consider three-identical-particle scattering initiated from a particle incident on a bound state. As is well known, the AGS three-body formulation represents the most practical basis for a perturbation treatment of the problem.[15] We shall therefore adopt that method. Further, since the three-body problem has been formulated many times in the AGS approach we can quote only (whenever possible) the final equations. On the other hand, to make the present treatment self-contained, we have to outline some aspects of that approach. Moreover, to keep our discussion as simple as possible, we shall neglect spin complications.[16] We shall label in the usual way $\vec{p}_\alpha$ the relative momentum of the pair $\beta\gamma$ and the particle α in the total center-of-mass system. $\vec{q}_\alpha$ will define the relative momentum of the particles β and γ in their own center-of-mass system. Consequently, the kinetic energy of the three free particles is $E_0=(\hbar^2/2\nu_\alpha)p_\alpha{}^2+(\hbar^2/2\mu_\alpha)q_\alpha{}^2$, where ν_α and μ_α are the corresponding reduced masses. Let us now call channel α the asymptotic configuration in which particle α is free while the other pair $\beta\gamma$ is bound, and channel 0 the configuration with all particles free. Therefore, the channel states $|\phi_{\alpha n}(E_{\alpha n})\rangle$ $(\alpha\neq 0)$ can be given by

$$|\phi_{\alpha n}(E_{\alpha n})\rangle = |\psi_{\alpha n}\rangle|\vec{p}_\alpha\rangle, \tag{2.1a}$$

with

$$E_{\alpha n}=\frac{\hbar^2}{2\nu_\alpha}p_\alpha{}^2+\epsilon_{\alpha n},$$

and where $|\psi_{\alpha n}\rangle$ corresponds to the two-body bound state with eigenvalue $\epsilon_{\alpha n}$ (<0). For $\alpha=0$ we then have

$$|\phi_0(E_0)\rangle = |\vec{q}_\alpha\rangle|\vec{p}_\alpha\rangle. \tag{2.1b}$$

We now proceed to follow the AGS three-body formulation. These authors introduce transition operators $U_{\beta\alpha}$ for the three-body scattering problem which are slightly different from those of Faddeev[1] and/or Lovelace[3]; however, they yield the same scattering amplitude when their matrix elements are put on the energy shell. These operators satisfy the Faddeev-type equations[7]

$$\begin{aligned}U_{\beta\alpha}(z)&=\bar{\delta}_{\beta\alpha}G_0{}^{-1}(z)+\sum_{\gamma\neq\beta}T_\gamma(z)G_0(z)U_{\gamma\alpha}(z)\\&=\bar{\delta}_{\beta\alpha}G_0{}^{-1}(z)+\sum_{\gamma\neq\alpha}U_{\beta\gamma}(z)G_0(z)T_\gamma(z),\end{aligned} \tag{2.2}$$

where $\bar{\delta}_{\beta\alpha}=1-\delta_{\beta\alpha}$, $G_0(z)$ is the three-free-particle propagator, and z is the complex parametric energy. $T_\alpha(z)$ is the two-particle scattering operator (defined in the three-particle Hilbert space). The physical three-particle transition amplitude $T_{\beta n,\alpha m}$ is then given in terms of the scattering operator $U_{\beta\alpha}$ via

$$\begin{aligned}T_{\beta n,\alpha m}(\vec{p}'_\beta,\vec{p}_\alpha)&=\langle\vec{p}'_\beta|\,T_{\beta n,\alpha m}|\vec{p}_\alpha\rangle\\&=\langle\phi_{\beta n}|\,U_{\beta\alpha}(E_{\alpha n}+i0)|\,\phi_{\alpha m}\rangle.\end{aligned} \tag{2.3}$$

Let us recall for the sake of completeness that the transition operator $T_\alpha(z)$ acting on three-particle space is related to the actual two-body transition matrix t_α (in the two-body momentum space) by[17]

$$\begin{aligned}&T_\alpha(\vec{p}'_\alpha\vec{q}'_\alpha,\vec{p}_\alpha\vec{q}_\alpha;z)\\&\quad=(2\pi)^3\delta(\vec{p}'_\alpha-\vec{p}_\alpha)t_\alpha\left(\vec{q}'_\alpha,\vec{q}_\alpha;z-\frac{\hbar^2}{2\nu_\alpha}p_\alpha{}^2\right).\end{aligned} \tag{2.4}$$

In the AGS perturbation approximation,[7] an essential aspect is the decomposition of the two-particle transition operator T_α into a separable part T^s_α and a weak nonseparable part T'_α via

$$T_\alpha(z)=T^s_\alpha(z)+T'_\alpha(z). \tag{2.5}$$

The separable part is then treated in an exact fashion, while the nonseparable part is treated perturbatively. This procedure leads to the following set of equations for the operators $U_{\beta\alpha}$:

$$U_{\beta\alpha}=U'_{\beta\alpha}+\sum_\gamma U'_{\beta\gamma}G_0T^s_\gamma G_0U_{\gamma\alpha}, \tag{2.6a}$$

while $U'_{\beta\alpha}$ will be given by

$$U'_{\beta\alpha}=\bar{\delta}_{\beta\alpha}G_0{}^{-1}+\sum_{\gamma\neq\beta}T'_\gamma G_0U'_{\gamma\alpha}. \tag{2.6b}$$

Namely, $U'_{\beta\alpha}$ satisfies the same Faddeev-type equations (2.2) as does $U_{\beta\alpha}$, but with the (assumed)

small nonseparable part T'_γ instead of the full operator T_γ. Therefore it is expected that a determination of $U'_{\beta\alpha}$ by iteration is justified. Choosing the separable part T^s_γ in correspondence with all bound and resonant states will ensure the convergence of the perturbation treatment for the nonperturbative part T'_γ.[7] However, as was mentioned in the introduction, the concept of two-particle resonance-pole dominance in three-particle scattering is rather vague.[5] Moreover, various attempts to introduce resonant states in the same way as bound states[18] in the full off-shell two-particle amplitudes (in the sense of the spectral representation), although very appealing in the description of the two-particle resonance behavior off the energy shell, will complicate the analytic structure of the nonseparable background term which also seems unknowable within these formulations. Further, since the definition of any two-particle transition operator (with parametric energy z) in the three-particle Hilbert space should involve an integration over the actual two-particle transition operator with respect to its parametric energy [see Eq. (2.4)] which will range (in our case) from $-\infty+i0$ to z, then the bound-state contribution should enter exactly in any decomposition of that transition operator. Therefore, apart from introducing Weinberg states[6] and/or using separable potentials in the decomposition of the two-body amplitudes, the only way that seems reliable is to stick to the usual well-defined spectral representation (Low equation) which reads (in our notations)

$$t_\alpha(z)=\sum_m \frac{v_\alpha|\psi_{\alpha m}\rangle\langle\psi_{\alpha m}|v_\alpha}{z+|\epsilon_{\alpha m}|}+\tilde{t}_\alpha(z), \tag{2.7}$$

where v_α is the interaction between the pair $\beta\gamma$, while $\langle\vec{q}|v_\alpha|\psi_{\alpha m}\rangle\equiv g_{\alpha m}(\vec{q})$ is the bound-state form factor. We may now write, taking into account the relation (2.4), the corresponding decomposition for the amplitude $T_\alpha(z)$ in the form

$$T_\alpha(z)=T^s_\alpha(z)+\tilde{T}_\alpha(z). \tag{2.8}$$

The decomposition (2.7) is exact, and the only singularity (in $|\vec{q}\rangle$ space) the nonpole term $\tilde{t}_\alpha(\vec{q}'_\alpha,\vec{q}_\alpha;z)$ will have will be the discontinuity across the scattering cut for positive energy values. Although $\tilde{t}_\alpha(z)$ will not be off-shell unitary, yet no unitarity violation will be introduced in the three-particle equations (2.6) when adopting Eq. (2.7).[19] In other words, the minimum constraint imposed by unitarity[20] will be satisfied [cf. Eqs. (3.14) and (4.2) of Ref. 20]. However, the question of higher-order corrections to the first-order perturbation approximation [in Eqs. (2.6)] may become somewhat complicated and is beyond the scope of the present work. For the simplicity of the subsequent presentation, we shall always assume that in each two-body subsystem there is at least one bound state. However, in general some two-body amplitudes may not have bound states, and the extension to that case is straightforward by just dropping the separable terms in (2.8) for those amplitudes. Having established the reliability of our two-body splitting (2.8), we now proceed along the usual patterns. Inserting the decomposition (2.8) into Eqs. (2.6), then in the first-order perturbation approximation (in $\tilde{T}_\gamma$) we find for the three-particle transition amplitude $T_{\beta n,\alpha m}$ (in the $|\vec{p}\rangle$ space representation) the multichannel Lippmann-Schwinger equation

$$T_{\beta n,\alpha m}(\vec{p}'_\beta,\vec{p}_\alpha)=V_{\beta n,\alpha M}(\vec{p}'_\beta,\vec{p}_\alpha)+\sum_{\gamma\tau}\int d^3p_\gamma\,\frac{V_{\beta n,\gamma\tau}(\vec{p}'_\beta,\vec{p}_\gamma)T_{\gamma\tau,\alpha m}(\vec{p}_\gamma,\vec{p}_\alpha)}{z+|\epsilon_{\gamma\tau}|-(\hbar^2/2\nu_\gamma)p_\gamma^{\,2}}, \tag{2.9}$$

where one can easily find (using the relation $G_0v_\alpha|\phi_{\alpha n}\rangle=|\phi_{\alpha n}\rangle$) that

$$V_{\beta n,\alpha m}(\vec{p}'_\beta,\vec{p}_\alpha)=\left\langle\phi_{\beta n}(\vec{p}'_\beta)\right|\left(\delta_{\beta\alpha}G_0^{-1}(z)+\sum_{\gamma\neq\alpha,\beta}\tilde{T}_\gamma(z)\right)\left|\phi_{\alpha m}(\vec{p}_\alpha)\right\rangle. \tag{2.10}$$

Equations (2.9) and (2.10) will be our starting point. We now proceed further to obtain explicit practical expressions for various terms in the effective potential $V_{\beta n,\alpha m}$.

After doing some intermediate momentum integrations, we arrive at the following expression for the first term in Eq. (2.10) (to be denoted by $V^0_{\beta n,\alpha m}$):

$$V^0_{\beta n,\alpha m}(\vec{p}'_\beta,\vec{p}_\alpha)=\langle\phi_{\beta n}(\vec{p}'_\beta)|\,\delta_{\beta\alpha}G_0^{-1}(z)\,|\phi_{\alpha m}(\vec{p}_\alpha)\rangle$$
$$=-\delta_{\beta\alpha}\,\frac{g^*_{\beta n}(\vec{q}'_\beta)g_{\alpha m}(\vec{q}_\alpha)}{(\hbar^2/2\mu_\alpha)q_\alpha^{\,2}+|\epsilon_{\alpha m}|}, \tag{2.11}$$

where (note that from particle identity we can write $\mu_i=\mu=\frac{1}{2}M$ and $\nu_i=\nu=\frac{2}{3}M$ for $i=\alpha,\beta,\gamma$ with M the particle mass)

$$\vec{q}_\alpha=\tfrac{1}{2}\vec{p}_\alpha+\vec{p}'_\beta,\quad \vec{q}'_\beta=-\tfrac{1}{2}\vec{p}'_\beta-\vec{p}_\alpha.$$

The second term in Eq. (2.10) (to be denoted by $V'_{\beta n,\alpha m}$), which has the form

$$V'_{\beta n,\alpha m}(\vec{p}'_\beta,\vec{p}_\alpha)=\sum_{\gamma\neq\alpha,\beta}\langle\phi_{\beta n}(\vec{p}'_\beta)|\,\tilde{T}_\gamma\,|\phi_{\alpha m}(\vec{p}_\alpha)\rangle$$

can be similarly handled. But, for the sake of clarity we shall write down explicitly the expres-

sion for only one of the terms contained in $V'_{\beta n,\alpha m}$, say that with the amplitude $\tilde{T}_3$ [here we adopt the ordering convention that channel 1 denotes particle 1 plus the ordered pair (2,3), etc.] and let this term be denoted by $\mathcal{T}_{1n,1m}$. Consequently, we shall have

$$\mathcal{T}_{1n,1m}(\vec{p}',\vec{p}) \equiv \langle \phi_{1n}(\vec{p}') \,|\, \tilde{T}_3(z) \,|\, \phi_{1m}(\vec{p}) \rangle = \frac{1}{(2\pi)^3} \int d^3k\, \tilde{\phi}^*_{1n}(\vec{k}+\tfrac{1}{2}\vec{q})\tilde{t}_3(\vec{p}_f,\vec{p}_i;\mathcal{E}_3) \times \tilde{\phi}_{1m}(\vec{k}), \tag{2.12}$$

where

$$\tilde{\phi}_{\alpha m}(\vec{k}) = -\left(\frac{\hbar^2}{2\mu}k^2 + |\epsilon_{\alpha m}|\right) g_{\alpha m}(\vec{k}),$$

$$\vec{p}_i = \tfrac{3}{4}\vec{p} - \tfrac{1}{2}\vec{k},$$

$$\vec{p}_f = \tfrac{3}{4}\vec{p}' - \tfrac{1}{2}\vec{k} - \tfrac{1}{4}\vec{q},$$

$$\vec{q} = \vec{p} - \vec{p}',$$

$$\mathcal{E}_3 = z - \frac{\hbar^2}{2\nu}(\tfrac{1}{2}\vec{p}+\vec{k})^2,$$

and as usual for physical amplitudes,

$$z = E_{1m} + i0 = \frac{\hbar^2}{2\nu}p^2 - |\epsilon_{1m}| + i0 = E_{1n} + i0 = \frac{\hbar^2}{2\nu}p'^2 - |\epsilon_{1n}| + i0.$$

Further, in the case of break-up processes, the first term in Eq. (2.10) $V^0_{0,\alpha m}$ will vanish on the mass shell, and we shall be left with amplitudes of the form

$$V_{0,\alpha m} = V'_{0,\alpha m} = \sum_{\gamma\neq\alpha} \langle \phi_0(\vec{q}_\gamma,\vec{p}_\gamma) \,|\, \tilde{T}_\gamma(z) \,|\, \phi_{\alpha m}(\vec{p}_\alpha) = \sum_{\gamma\neq\alpha} \frac{\tilde{t}_\gamma(\vec{q}_\gamma,\vec{q}'_\gamma;(\hbar^2/2\mu)q_\gamma^2+i0)g_{\alpha m}(\vec{q}_\alpha)}{(\hbar^2/2\mu)q_\alpha^2 + |\epsilon_{\alpha m}|} \tag{2.13}$$

where

$$\vec{q}'_\gamma = \tfrac{1}{2}\vec{p}_\gamma + \vec{p}_\alpha, \quad \vec{q}_\alpha = -\tfrac{1}{2}\vec{p}_\alpha - \vec{p}_\gamma,$$

and it is obvious that $V'_{0,\alpha m}$ contains $\tilde{t}_\gamma$ half on the mass shell.

Now as our main objective is to work directly with on-shell two-body amplitudes (whenever possible), an outstanding problem is to find some plausible approximation for determining average values for the two-body amplitudes appearing in Eqs. (2.12) and (2.13). For this purpose we shall explicitly consider the evaluation of expression (2.12). Other terms (with different $\tilde{t}_\alpha$) corresponding to elastic and rearrangement processes can be treated in exactly the same way. First of all, we note that the expression (2.12) involves the nonpole two-body amplitude $\tilde{t}_3(\vec{p}_f,\vec{p}_i;\mathcal{E}_3)$ with the corresponding parametric energy $\mathcal{E}_3 = z - (\hbar^2/2\nu)(\frac{1}{2}\vec{p}+\vec{k})^2$. Thus the integration with respect to $\vec{k}$ will cause $\mathcal{E}_3$ to follow a contour parallel to and above the real axis extending from $-\infty + i0$ to $(\hbar^2/8\nu)p^2 - |\epsilon_{1m}| + i0$ [note the expression for z in (2.12)]. However, the amplitude $\tilde{t}_3$ as a function of its parametric energy is everywhere analytic in the upper-energy plane except for the right-hand (unitarity) cut, and is supposed bounded on the upper rim of that cut. Consequently, regarding $\tilde{t}_3$ as a function of its parametric energy $\mathcal{E}_3$, we can make use of its Taylor expansion around the value $\mathcal{E}_i = (\hbar^2/2\mu)\vec{p}_i^{\,2}$. One will then have [up to first-order terms in the energy shift $(\mathcal{E}_3 - \mathcal{E}_i)$]

$$\tilde{t}_3(\vec{p}_f,\vec{p}_i;\mathcal{E}_3) = \tilde{t}_3(\vec{p}_f,\vec{p}_i;\mathcal{E}_i) + \omega(\vec{k})\frac{\partial}{\partial\mathcal{E}_i}\tilde{t}(\vec{p}_f,\vec{p}_i;\mathcal{E}_i), \tag{2.14}$$

where

$$\omega(k) \equiv \mathcal{E}_3 - \mathcal{E}_i = \left(\frac{\hbar^2}{2M}k^2 + |\epsilon_{1m}|\right).$$

Equation (2.14) seems to be a plausible approximation for our nonpole two-body amplitude, since the presence of the bound-state form factor in (2.12) will restrict the range of important values of the momentum $\vec{k}$ in the integrand. Further, the quantity $\omega(k)$ may be regarded as the expectation value of the two-body interaction (which is of short-range character) and will be small in comparison with $\mathcal{E}_i$ except near the two-body binding energies. Even in the last circumstance, one may consider some higher-order terms in the Taylor expansion. With this approximation, we shall have the amplitude $\tilde{t}_3(\vec{p}_f,\vec{p}_i;\mathcal{E}_i)$ still not on the energy shell, and $\mathcal{T}_{1n,1m}$ is expressible as the following three-dimensional integral:

$$\mathcal{T}_{1n,1m} = \frac{1}{(2\pi)^3}\int d^3k\, \tilde{\phi}^*_{1n}(\vec{k}+\tfrac{1}{2}\vec{q})\tilde{\phi}_{1m}(\vec{k}) \times\left\{\left[1+\omega(\vec{k})\frac{\partial}{\partial\mathcal{E}_i}\right]\tilde{t}_3(\vec{p}_f,\vec{p}_i;\mathcal{E}_i)\right\}. \tag{2.15}$$

To proceed further, we shall adopt the so-called form-factor approximation.[21] This method relies on the mathematical observation that the function

$$F(\vec{k}) = \left\{\left[1+\omega(k)\frac{\partial}{\partial\mathcal{E}_i}\right]\tilde{t}_3(\vec{p}_f,\vec{p}_i;\mathcal{E}_i)\right\}$$

may be expanded in a Taylor series around some preferable value (say) $\vec{k}_0$, and if the product $\tilde{\phi}_{1n}(\vec{k}+\frac{1}{2}\vec{q})\tilde{\phi}_{1m}(\vec{k})$ in the integrand (2.15) is peaked near this value, then it is reasonable to take out the function $F(\vec{k})$ from under the integral sign at that value $\vec{k}_0$ (i.e., neglect all but the first term

in its Taylor expansion). Up to the second-order terms in the Taylor expansion, the optimal choice for the value $\vec{k}_0$ can be obtained by requiring that

$$\int d^3k(\vec{k}-\vec{k}_0)\tilde{\phi}^*_{\beta n}(\vec{k}+a\vec{q})\tilde{\phi}_{\alpha m}(\vec{k})=0 \tag{2.16}$$

(here we consider the general case for any term in the effective potential $V_{\beta n,\alpha m}$).

Since, because of particle identity, all the two-body bound-state wave functions satisfy the same Schrödinger equation except for the binding energies of different states, it can be easily shown (see Appendix A) that the optimal value $\vec{k}_0$ resulting from the requirement (2.16) can be given by

$$\vec{k}_0=-\tfrac{1}{2}a\vec{q}\left[1-\frac{2\mu}{a^2\hbar^2q^2}(|\epsilon_{\alpha m}|-|\epsilon_{\beta n}|)\right]. \tag{2.17}$$

For the sake of completeness, we have also investigated (see Appendix B) second-order terms in the expansion of $F(\vec{k})$ around the value $\vec{k}_0$ given above (higher-order terms can be treated on the same footing). It is shown that the above approximation procedure is reliable in the sense that, depending on the shapes of the bound-state wave functions and the two-body interactions, the second-order term will give negligible contribution except, perhaps, for very large angles.

We can now finally rewrite Eq. (2.15) in the form

$$\mathcal{T}_{1n,1m}=S_{1n,1m}(\tfrac{1}{2}\vec{q})\left\{\left[1+\omega(\vec{k}'_0)\frac{\partial}{\partial\mathcal{E}_i(\vec{k}'_0)}\right]\bar{t}_3(\vec{p}_f(\vec{k}'_0),\vec{p}_i(\vec{k}'_0);\mathcal{E}_i(\vec{k}'_0))\right\}, \tag{2.18}$$

with

$$S_{1n,1m}(\tfrac{1}{2}\vec{q})=\frac{1}{(2\pi)^3}\int d^3k\,\tilde{\phi}^*_{1n}(\vec{k}+\tfrac{1}{2}\vec{q})\tilde{\phi}_{1m}(\vec{k})$$

and

$$\vec{k}'_0=-\tfrac{1}{4}\vec{q}\left[1-\frac{\mu}{2\hbar^2q^2}(|\epsilon_{1m}|-|\epsilon_{1n}|)\right].$$

With this value $\vec{k}'_0$, it is easily verified that in Eq. (2.18)

$$\frac{\hbar^2}{2\mu}\vec{p}_i^{\,2}(\vec{k}'_0)=\frac{\hbar^2}{2\mu}\vec{p}_f^{\,2}(\vec{k}'_0)=\mathcal{E}_i(\vec{k}'_0),$$

which is the on-shell condition [note that from energy conservation we shall have $p^2=p'^2+(2\nu/\hbar^2)(|\epsilon_{1m}|-|\epsilon_{1n}|)$]. Similar on-shell expressions will be obtained for various terms in $V'_{\beta n,\alpha m}$ using the same techniques. The $\bar{t}_\alpha$ amplitudes can be then given in terms of the actual two-body amplitudes using the relation (2.7). Consequently, the effective potentials are completely determined (in our prescription) from on-shell two-body amplitudes except for the break-up process [see Eq. (2.13)]. The effective potentials appearing in the break-up amplitude are seen from Eq. (2.13) to be given in terms of half-on-shell two-body amplitudes, and our averaging prescription clearly does not work in this case. Moreover, the effective potentials $V_{\beta n,\alpha m}$ also appear in the kernels of our three-body coupled integral equations (2.9), and with our averaging procedure it is clear that we shall be also left with half-on-shell two-body amplitudes under the integral sign.

However, after partial-wave decomposition, the half-on-shell partial-wave two-body amplitudes $t_l(p,k;s+i0)\,[k=(2\mu s/\hbar^2)^{1/2}]$ can be given in terms of their on-shell counterparts using the KN prescription,[13,14] viz.,

$$t_l(p,k;s+i0)=f_l(p,s)t_l(k), \tag{2.19}$$

where $t_l(k)=t_l(k,k;(\hbar^2/2\mu)k^2+i0)$ is the fully on-shell two-body amplitude, and the half-on-shell function $f_l(p,s)$ [with the normalization $f_l(k,s)=1$] satisfies an integral equation with a Fredholm kernel, namely,

$$f_l(p,s)=\frac{v_l(p,k)}{v_l(k,k)}+\int_0^\infty\Lambda(p,q;s+i0)f_l(q,s)q^2dq, \tag{2.20}$$

where the kernel $\Lambda(p,q;s+i0)$ is given by

$$\Lambda(p,q;s+i_0)=\frac{2}{\pi}\frac{1}{s-(\hbar^2/2\mu)q^2+i0}\times\left[v_l(p,q)-\frac{v_l(p,k)v_l(k,q)}{v_l(k,k)}\right]$$

and $v_l(p,q)$ is the corresponding lth partial-wave projection of the two-body potential. Since $f_l(p,s)$ satisfies the integral equation (2.20) with a nonsingular kernel, then the Born series may converge rapidly and one may adopt the approximation

$$f_l(p,s)\cong\frac{v_l(p,k)}{v_l(k,k)} \tag{2.21}$$

as a first step towards evaluating the complicated three-body scattering amplitude. One more word should be added here in connection with the application of Eq. (2.9) together with our prescription in the case of nucleon-deuteron scattering. It is well known that, following Lovelace,[4] Eq. (2.9) can be reduced to a single equation for identical particles. Such a situation is always assumed in nucleon-deuteron scattering provided one adopts an isospin convention. The spin, isospin, and

kinematical structure of the matrix elements of the type (2.12) with respect to deuteron-plus-free-nucleon states has been studied extensively.[10] However, most calculations in that case are based on separable potentials.[22] In our approach, we have on-shell two-particle amplitudes, so that we can directly make use of the phase-shift parametrization for our amplitudes (after necessary extrapolation). A final comment is called for concerning our approximation (2.14) for the two-particle amplitude. This approximation may be regarded as a farther step beyond the usual fixed-scatterer (impulse) approximation. Further, in the two-body problem, the magnitude of the time delay is proportional to the energy derivative of the on-shell amplitude. Such a time delay will become important in the presence of resonances in the two-body amplitude. Therefore, our approximate two-body amplitudes in Eq. (2.18) will be more accurate near two-body resonances than the usual (kind of impulse) form-factor approximation.

III. FIRST-ORDER UNITARY APPROXIMATION

Armed with the approximations presented in the preceding section, we can easily proceed to evaluate the three-body amplitude in the first-order unitary approximation. This approximation was first proposed by Sloan[12] and then recovered by Kowalski[23] within the context of a more general K-matrix formulation for the three-body problem. The essential feature and simplicity of this approximation lies in the clean division of the two-particle transition operator $T_\alpha(z)$ via

$$T_\alpha(z) = \overline{T}_\alpha(z) + \Delta T_\alpha(z), \tag{3.1}$$

where $\Delta T_\alpha(z)$ is that part of the two-particle transition operator $T_\alpha(z)$ (in the three-particle Hilbert space) which gives rise to the Dirac δ-function contribution of the bound-state pole, while $\overline{T}_\alpha(z)$ is its complementary part. Using the AGS perturbation techniques one finds, for the on-shell three-body transition amplitude, the following expression in the first-order approximation:

$$T_{\beta n,\alpha m}(\vec{p}',\vec{p}) = I_{\beta n,\alpha m}(\vec{p}',\vec{p}) - i\pi \sum_{\gamma\neq 0,\delta} \int d^3p_\gamma I_{\beta n,\gamma\delta}(\vec{p}',\vec{p}_\gamma) \times \delta(E - E_\gamma) T_{\gamma\delta,\alpha m}(\vec{p}_\gamma,\vec{p}). \tag{3.2}$$

Here $z = E + i0$ and E is the energy available in the initial and final three-body states, while E_γ is the energy associated with the three-body state $|\phi_{\gamma\delta}(\vec{p}_\gamma)\rangle$, and

$$I_{\beta n,\alpha m}(\vec{p}',\vec{p}) = \left\langle \phi_{\beta n}(\vec{p}') \middle| \left(\delta_{\beta\alpha} G_0^{-1} + \sum_{\gamma\neq\alpha,\beta} \overline{T}_\gamma(z) \right) \middle| \phi_{\alpha m}(\vec{p}) \right\rangle. \tag{3.3}$$

We should also note here (as in the preceding section) that if some two-body potential does not support a bound state, then the corresponding term (under the integral sign) on the right-hand side of (3.2) will vanish.

To evaluate the various terms in Eq. (3.2) is a rather simple problem, owing to the presence of the conservation δ function in the integrand. Matters will be much clearer after partial-wave decomposition. However, first of all we shall apply the approximation scheme developed in the preceding section to evaluate the nonlinear term $I_{\beta n,\alpha m}(\vec{p}',\vec{p})$. It is clear from Eq. (3.3) that the source term $\langle \phi_{\beta n}(\vec{p}') | \overline{\delta}_{\beta\alpha} G_0^{-1}(z) | \phi_{\alpha m}(\vec{p}) \rangle$ will be given by the same relation (2.11) and, for the break-up process, will vanish on the mass shell. Further, the evaluation of the second term $(\sum_{\gamma\neq 0,\alpha,\beta} \langle \phi_{\beta n}(\vec{p}') | \overline{T}_\gamma(z) | \phi_{\alpha m}(\vec{p}) \rangle)$ will follow the same pattern as in the preceding section [Eqs. (2.14) to (2.18)]. However, after such a procedure we shall obtain various on-shell matrix elements for $\overline{t}_\gamma$ (and its energy derivative) with positive parametric energies. Thus it will be unnecessary to distinguish between the actual two-body t_γ and $\overline{t}_\gamma$ (note that $\overline{t}_\gamma$ differs from t_γ only in that $\overline{t}_\gamma$ comes with the principal value of the bound-state-pole contribution). Similar arguments will hold for the break-up term $I_{0,\alpha m}$, since it is clear from Eq. (2.13) that the two-body $\overline{t}_\gamma$ amplitude appearing in this term will come also with positive parametric energy. Therefore, in our prescription, $I_{\beta n,\alpha m}(\vec{p}',\vec{p})$ will be evaluated in terms of full on-shell actual two-body amplitudes $t_\gamma(\gamma \neq 0,\alpha,\beta)$. Once the amplitudes $I_{\beta n,\alpha m}$ are known, Eq. (3.3) can readily be solved with the aid of partial-wave expansion. On taking $\alpha = 1$ in Eq. (3.2) (corresponding to particle 1 incident on a bound state of 2 and 3), one sees that $T_{1n,1m}$, $T_{2n,1m}$, and $T_{3n,1m}$ satisfy a set of three coupled equations, while $T_{0,1m}$ is expressible in terms of them. Finally, for the purpose of future calculations with our prescription in the context of this first-order unitary approximation, we shall give below explicit formulas for the evaluation of the amplitude $T_{\beta n,\alpha m}$ in the case of a bound-state scattering for a system of identical particles (spin complications will be neglected). In this case (as was mentioned in the preceding section) the symmetry requirements for either bosons and/or fermions, will lead to great simplifications in the coupled equations (3.2) without any further approximation. Following Lovelace,[4] the amplitudes $T_{\beta n,\alpha m}$ and $I_{\beta n,\alpha m}$ will then not depend on either α or β. The coupling

between channels can be removed in favor of diagonal amplitudes $T_{\alpha n,\alpha m}=T^D_{nm}$ and nondiagonal ones $T_{\beta n,\alpha m}=T^N_{nm}$ $(\beta\neq\alpha)$, and obviously the result holds for the amplitudes $I_{\beta n,\alpha m}$. Consequently, the symmetrized or anitsymmetrized on-shell amplitude will be given by

$$T_{nm}(\vec{p}',\vec{p})=I_{nm}(\vec{p}',\vec{p}) - i\pi\sum_{n''}\int d^3p''\,I_{nn''}(\vec{p}',\vec{p}) \times\delta(E-E'')T_{n''m}(\vec{p}'',\vec{p}), \tag{3.4}$$

where

$$E''=\frac{\hbar^2}{2\nu}p''^2-|\epsilon_{n''}|,$$

$$T_{nm}(\vec{p}',\vec{p})=T^D_{nm}(\vec{p}',\vec{p})+2T^N_{nm}(\vec{p}',\vec{p}).$$

Using the ordering convention, that channel 1 denotes particle 1 plus the ordered pair (2,3), etc., then

$$I_{nm}(\vec{p}',\vec{p})=2[\langle\phi_{1n}(\vec{p}')|\overline{T}_{12}(E+i0)|\phi_{1m}(\vec{p})\rangle +\langle\phi_{2n}(\vec{p}')|\overline{T}_{12}(E+i0)|\phi_{1m}(\vec{p})\rangle +\langle\phi_{2n}(\vec{p}')|v_{23}|\phi_{1m}(\vec{p})\rangle]. \tag{3.5}$$

The first and second terms are expressible as the three-dimensional integrals

$$L_1=\langle\phi_{1n}(\vec{p}')|\overline{T}_{12}(E+i0)|\phi_{1m}(\vec{p})\rangle =\frac{1}{(2\pi)^3}\int d^3k\,\tilde{\phi}^*_{1n}(\vec{k}+\tfrac{1}{2}\vec{q})\tilde{\phi}_{1m}(\vec{k})\overline{t}_{12}(\vec{P}_f,\vec{P}_i;\mathcal{E}_1), \tag{3.6a}$$

$$L_2=\langle\phi_{2n}(\vec{p}')|\overline{T}_{12}(E+i0)|\phi_{1m}(\vec{p})\rangle =\frac{1}{(2\pi)^3}\int d^3Q\,\tilde{\phi}^*_{2n}(\tfrac{1}{2}\vec{p}'-\tfrac{1}{2}\vec{p}-\vec{Q}) \times\tilde{\phi}_{1m}(\vec{Q})\overline{t}_{12}(\vec{P}'_f,\vec{P}'_i;\mathcal{E}_2), \tag{3.6b}$$

while the third term is given by

$$L_3=\langle\phi_{2n}(\vec{p}')|v_{23}|\phi_{1m}(\vec{p})\rangle =-\frac{g^*_{2n}(-\frac{1}{2}\vec{p}'-\vec{p})g_{1m}(\frac{1}{2}\vec{p}+\vec{p}')}{(\hbar^2/2\mu)(\frac{1}{2}\vec{p}'+\vec{p})^2+|\epsilon_{2n}|}. \tag{3.6c}$$

Here

$$\vec{q}=\vec{p}-\vec{p}',$$

$$\vec{P}_i=\tfrac{3}{4}\vec{p}-\tfrac{1}{2}\vec{k},$$

$$\vec{P}_f=\tfrac{3}{4}\vec{p}'-\tfrac{1}{2}\vec{k}-\tfrac{1}{4}\vec{q},$$

$$\vec{P}'_i=\tfrac{3}{4}\vec{p}-\tfrac{1}{2}\vec{Q},$$

$$\vec{P}'_f=-\tfrac{3}{4}\vec{p}+\tfrac{1}{2}\vec{Q}+\vec{q},$$

$$\mathcal{E}_1=E-\frac{\hbar^2}{2\nu}(\tfrac{1}{2}\vec{p}+\vec{k})^2+i0,\quad \mathcal{E}_2=E-\frac{\hbar^2}{2\nu}(\tfrac{1}{2}\vec{p}+\vec{Q})^2+i0,$$

while

$$E=\frac{\hbar^2}{2\nu}\vec{p}^2-|\epsilon_{1m}|.$$

Adopting our prescription developed in the preceding section to obtain average values for the two-body amplitudes in Eqs. (3.6), we shall have

$$L_1=\left\{\left[1+\omega(\vec{k}_0)\frac{\partial}{\partial\mathcal{E}_i(\vec{k}_0)}\right] \times t_{12}(\vec{P}_f(\vec{k}_0),\vec{P}_i(\vec{k}_0);\mathcal{E}_i(\vec{k}_0))\right\}S_{nm}(\tfrac{1}{2}\vec{q}), \tag{3.7a}$$

$$L_2=\left\{\left[1+\omega(\vec{k}_0)\frac{\partial}{\partial\mathcal{E}'_i(\vec{k}_0)}\right] \times t_{12}(\vec{P}'_f(\vec{k}_0),\vec{P}'_i(\vec{k}_0);\mathcal{E}'_i(\vec{k}_0))\right\}\tilde{S}_{nm}(-\tfrac{1}{2}\vec{q}), \tag{3.7b}$$

where

$$\omega(x)=\frac{\hbar^2}{M}x^2+|\epsilon_{1m}|,$$

$$\mathcal{E}_i(\vec{k}_0)=\frac{\hbar^2}{2\mu}\vec{P}_i^{\,2}(\vec{k}_0)+i0,$$

$$\mathcal{E}'_i(k_0)=\frac{\hbar^2}{2\mu}\vec{P}'^{\,2}_i(\vec{k}_0)+i0,$$

$$\vec{k}_0=-\tfrac{1}{4}\vec{q}\left[1-\frac{8\mu}{\hbar^2q^2}(|\epsilon_m|-|\epsilon_n|)\right],\quad |\epsilon_{\alpha m}|\equiv|\epsilon_m|,$$

$(\alpha=1,2,3)$ while

$$S_{nm}(\tfrac{1}{2}\vec{q})=\frac{1}{(2\pi)^3}\int d^3k\,\tilde{\phi}^*_{1n}(\vec{k}+\tfrac{1}{2}\vec{q})\tilde{\phi}_{1m}(\vec{k}),$$

$$\tilde{S}_{nm}(-\tfrac{1}{2}\vec{q})=\frac{1}{(2\pi)^3}\int d^3Q\,\tilde{\phi}^*_{2n}(-\tfrac{1}{2}\vec{q}-\vec{Q})\tilde{\phi}_{1m}(\vec{Q}).$$

We can thus calculate $I_{nm}=L_1+L_2+L_3$ as a function of the scattering angles in terms of fully on-shell free two-body amplitudes (3.7) and the knowledge of the two-body form factor $g_{\alpha m}(\vec{k})$. The next step is to consider the partial-wave decomposition of the amplitude $T_{nm}(\vec{p}',\vec{p})$, which from Eq. (3.4) can be readily obtained as

$$T^l_{nm}(p',p)=I^l_{nm}(p',p) -\frac{4i\pi^2\nu}{\hbar^2}P''\sum_{n''}I^l_{nn''}(p',P'')T^l_{n''m}(P'',p), \tag{3.8}$$

where

$$P''=\left[p^2+\frac{2\mu}{\hbar^2}(|\epsilon_{n''}|-|\epsilon_n|)\right]^{1/2}, \tag{3.9}$$

$$I^l_{nm}(p',p)=\tfrac{1}{2}\int_{-1}^{+1}d\cos\theta\,P_l(\cos\theta)I_{nm}(\vec{p}',\vec{p}),$$

with θ the angle between $\vec{p}$ and $\vec{p}'$ (scattering angle). We can easily infer from Eq. (3.8) that L_1 and L_2 are functions of the angles in their physical range, and in general $I^l_{nm}(p',p)$ can be easily handled. Moreover, the evaluation of I^l_{nm} depends also on knowing the two-body phase shifts for real physical energies (except below the two-particle bound-state energies). Finally, we wish to comment on the evaluation of L_1 and L_2 in case our two-body t matrices do not possess any resonance in the energy range of interest. In this case a further simplification for the terms L_1 and L_2 in Eq. (3.7) can be obtained noting that (on the mass shell)

$$\frac{\partial}{\partial \mathcal{E}_i} t_{12}(\vec{P}_f,\vec{P}_i;\mathcal{E}_i) \cong \frac{-2\mu}{\hbar^2 \vec{P}_i^{\,2}} t_{12}(\vec{P}_f,\vec{P}_i;\mathcal{E}_i); \qquad (3.10)$$

consequently, we shall have in this case

$$L_1 \cong \left[1-\frac{2\mu\omega(k_0)}{\hbar^2\vec{P}_i^{\,2}}\right] t_{12}(\vec{P}_f,\vec{P}_i;\mathcal{E}_i) S_{nm}(+\tfrac{1}{2}q),$$
$$L_2 \cong \left[1-\frac{2\mu\omega(k_0)}{\hbar^2\vec{P}_i'^{\,2}}\right] t_{12}(\vec{P}_f',\vec{P}_i';\mathcal{E}_i') \bar{S}_{nm}(-\tfrac{1}{2}\vec{q}). \qquad (3.11)$$

The terms in the brackets in Eq. (3.11) will give in general appreciable correction to the usual fixed-scatterer approximation,[11] especially in the backward direction in the scattering amplitude. However, it is reasonable to expect that the importance of such corrections will diminish at higher energies. Also, at higher energies, the importance of the neglected higher-order terms in our procedure for evaluating the three-body scattering amplitude may become less sound.

Although we have provided a phenomenological treatment which seems to be very reliable in evaluating bound-to-bound three-body scattering amplitude, the evaluation of the break-up amplitudes seems to still need more effort. A specific method for parametrizing the functions $f_l(p,s)$ of Eqs. (2.20) which preserves the fit to the two-body on-shell data has been described by Noyes.[14] Thus, in general the break-up amplitude may also be handled in a satisfactory manner.

APPENDIX A

Let us consider the condition (2.16) for obtaining the optimal value of the average momentum $\vec{k}_0$. This condition can be rewritten as

$$\int d^3k\, \vec{k}\tilde{\phi}^*_{\beta n}(\vec{k}+a\vec{q})\tilde{\phi}_{\alpha m}(\vec{k}) = \vec{k}_0 \int d^3k\, \tilde{\phi}^*_{\beta n}(\vec{k}+a\vec{q})\tilde{\phi}_{\alpha m}(\vec{k}). \qquad (A1)$$

From the left-hand side of this equation (note that the only unintegrated vector is $\vec{q}$) we infer that $\vec{k}_0$ in general is proportional to (i.e., in the direction of) the vector $\vec{q}$. Thus, multiplying both sides of Eq. (A1) scalarly by $2a\vec{q}$, and rearranging terms, we arrive at

$$\int d^3k[(\vec{k}+a\vec{q})^2-k^2]\tilde{\phi}^*_{\beta n}(\vec{k}+a\vec{q})\tilde{\phi}_{\alpha m}(\vec{k}) = (a^2q^2+2a\vec{q}\cdot\vec{k}_0)\int d^3k\, \tilde{\phi}^*_{\beta n}(\vec{k}+a\vec{q})\phi_{\alpha m}(\vec{k}). \qquad (A2)$$

Now, because of the assumed particle identity, the Fourier transforms $\tilde{\phi}^*_{\beta n}$ and $\tilde{\phi}_{\alpha m}$ of the corresponding two-body bound-state wave functions, will satisfy (same two-body potentials)

$$k^2\tilde{\phi}_{\alpha m}(\vec{k}) = \frac{-2\mu}{\hbar^2}|\epsilon_{\alpha m}|\tilde{\phi}_{\alpha m}(\vec{k}) - \frac{1}{(2\pi)^3}\int d^3Q\, \Theta(\vec{k}-\vec{Q})\tilde{\phi}_{\alpha m}(\vec{Q}), \qquad (A3)$$

$$(\vec{k}+a\vec{q})^2\tilde{\phi}^*_{\beta n}(\vec{k}+a\vec{q}) = \frac{-2\mu}{\hbar^2}|\epsilon_{\beta n}|\tilde{\phi}^*_{\beta n}(\vec{k}+a\vec{q}) - \frac{1}{(2\pi)^3}\int d^3\kappa\, \tilde{\phi}^*_{\beta n}(\vec{\kappa})\Theta(\vec{k}+a\vec{q}-\vec{\kappa}), \qquad (A4)$$

where $\Theta(\vec{x}) = (2\mu/\hbar^2)v(\vec{x})$, and $v(\vec{x})$ is the Fourier transform of the two-body potentials.

Multiplying Eq. (A3) by $\tilde{\phi}^*_{\beta n}(\vec{k}+a\vec{q})$ and Eq. (A4) by $\tilde{\phi}_{\alpha m}(\vec{k})$ and then subtracting both results, we shall obtain, after integration with respect to vector $\vec{k}$, the following result:

$$\int d^3k[(\vec{k}+a\vec{q})^2-k^2]\tilde{\phi}^*_{\beta n}(\vec{k}+a\vec{q})\tilde{\phi}_{\alpha m}(\vec{k}) = \frac{2\mu}{\hbar^2}(|\epsilon_{\alpha m}|-|\epsilon_{\beta n}|)\int d^3k\, \tilde{\phi}^*_{\beta n}(\vec{k}+a\vec{q})\tilde{\phi}_{\alpha m}(\vec{k}) + \frac{1}{(2\pi)^3}\left\{\int d^3k d^3Q[\Theta(\vec{k}-\vec{Q})\tilde{\phi}^*_{\beta n}(\vec{k}+a\vec{q})\tilde{\phi}_{\alpha m}(\vec{Q}) - \Theta(\vec{k}-\vec{Q})\tilde{\phi}^*_{\beta n}(\vec{Q}+a\vec{q})\tilde{\phi}_{\alpha m}(\vec{k})]\right\} \qquad (A5)$$

The last term in large curly brackets in this equation will vanish identically, as the two-body potential is supposed to be spherically symmetric, i.e., $\Theta(\vec{x}-\vec{y}) \equiv \Theta(|\vec{x}-\vec{y}|)$ (make in the last integral the the replacement $\vec{k} \rightleftharpoons \vec{Q}$). So that, from our condition (A2), we can determine $\vec{k}_0$ from the relation

$$a^2q^2+2a\vec{q}\cdot\vec{k}_0 = \frac{2\mu}{\hbar^2}(|\epsilon_{\alpha m}|-|\epsilon_{\beta n}|), \qquad (A6)$$

and as the vector $\vec{k}_0$ is supposed to be in the direction of the vector $\vec{q}$, one finally obtains

$$\vec{k}_0 = -\tfrac{1}{2}a\vec{q}\left[1 - \frac{2\mu(|\epsilon_{\alpha m}| - |\epsilon_{\beta n}|)}{a^2\hbar^2 q^2}\right]. \tag{A7}$$

APPENDIX B

It turns out now to be very interesting to see under what conditions our form-factor approximation [see Eqs. (2.15) and (2.16)] will be satisfactory. For this purpose, one has to investigate higher-order terms in the expansion of $F(\vec{k})$ [see Eq. (2.15)] around the optimal value $\vec{k}_0$ given in Appendix A by Eq. (A7).

For the sake of simplicity, we shall investigate here the second-order term only. Other terms can be treated on the same footing. It can be easily shown (using Fourier transformation), that the second-order term in the expansion of $F(\vec{k})$ around $\vec{k}_0$ is given by

$$I = \frac{1}{2!}\frac{[(\vec{k}-\vec{k}_0)\cdot\vec{k}_0)]^2}{\vec{k}_0^{\,2}}\frac{\partial^2}{\partial\vec{k}_0^{\,2}}F(\vec{k}_0). \tag{B1}$$

Consequently, the contribution of this term to Eq. (2.15) will be given by (in the general case)

$$\mathcal{T}^{(2)}_{\beta\alpha} = \frac{1}{2!}\frac{\partial^2 F(\vec{k}_0)}{\partial\vec{k}_0^{\,2}}\int d^3k\,\frac{[(\vec{k}-\vec{k}_0)\cdot\vec{k}_0]^2}{\vec{k}_0^{\,2}} \times \tilde{\phi}^*_{\beta n}(\vec{k}+a\vec{q})\phi_{\alpha m}(\vec{k}) \tag{B2}$$

which, after making use of Eq. (A1), can be rewritten in the form

$$\mathcal{T}^{(2)}_{\beta\alpha} = \frac{1}{2!}\frac{\partial^2 F(\vec{k}_0)}{\partial k_0^{\,2}}\int d^3k\left[\frac{(\vec{k}\cdot\vec{k}_0)^2}{k_0^{\,2}} - k_0^{\,2}\right] \times \tilde{\phi}^*_{\beta n}(\vec{k}+a\vec{q})\phi_{\alpha m}(\vec{k}). \tag{B3}$$

Further, noting that from Eq. (A7) one has $\vec{k}_0 = \sigma a\vec{q}$, where

$$\sigma = -\tfrac{1}{2}\left[1 - \frac{2\mu(|\epsilon_{\alpha m}| - |\epsilon_{\beta n}|)}{a^2\hbar^2 q^2}\right],$$

then $\mathcal{T}^{(2)}_{\beta\alpha}$ will take the form

$$\mathcal{T}^{(2)}_{\beta\alpha} = \frac{1}{2!}\frac{\partial^2 F(\vec{k}_0)}{\partial k_0^{\,2}} \times \int d^3k\left\{\frac{[(\vec{k}+a\vec{q})^2 - k^2 - a^2q^2]^2}{4a^2q^2} - k_0^{\,2}\right\} \times \tilde{\phi}^*_{\beta n}(\vec{k}+a\vec{q})\tilde{\phi}_{\alpha m}(\vec{k}). \tag{B4}$$

Now, making use of Eqs. (A3) and (A4), then after some manipulations one finally arrives at

$$\mathcal{T}^{(2)}_{\beta\alpha} = a\frac{\partial^2 F(\vec{k}_0)}{\partial^2 k_0^{\,2}}\frac{1}{(2\pi)^3}\int d^3k d^3Q\,(\vec{q}\cdot\vec{Q})\Theta(\vec{Q}) \times \tilde{\phi}^*_{\beta n}(\vec{k}+a\vec{q}+\vec{Q})\tilde{\phi}_{\alpha m}(\vec{k}). \tag{B5}$$

Since the bound-state wave function ordinarily falls off much more rapidly than the potential in momentum space, we can in general neglect the dependence of $\tilde{\phi}^*_{\beta n}(\vec{k}+a\vec{q}+\vec{Q})$ on $\vec{Q}$ in the integrand (B5). As a consequence of that, we shall have $\mathcal{T}^{(2)}_{\beta\alpha} = 0$, since the potential is supposed spherically symmetric. Further, this argument will become more appropriate if one also assumes the potential to fall off rapidly for large momenta (which in turn implies that the potential is very smooth in coordinate space). However, it would be more interesting to have at least a qualitative idea of the order of magnitude of $\mathcal{T}^{(2)}_{\beta\alpha}$ with respect to $\mathcal{T}_{\beta\alpha} = F(\vec{k}_0)S_{\beta\alpha}(a\vec{q})$ given by Eqs. (2.18). For this purpose, we shall make the following simplifying assumptions (without any loss of generality):

(a) We shall assume for the moment that our two-body amplitudes do not possess resonances, so that we can make use of Eq. (3.10).

(b) We shall always neglect $\omega(k_0)$ with respect to the incident energy $\mathcal{E}_i$ [see Eq. (2.14)]. This assumption is made only to render the final equations look not complicated, and it has nothing to do with the results to be obtained.

In the light of these simplifications, one will obtain

$$\frac{\partial^2 F(k_0)}{\partial k_0^{\,2}} \cong F(\vec{k}_0)\left(\frac{3}{2p_i^{\,2}} + \frac{q^2}{32p_i^{\,4}}\right), \tag{B6}$$

where $p_i^{\,2} = (\frac{3}{4}\vec{p} - \frac{1}{2}\vec{k}_0)^2$ while $\vec{p}$ is the initial momentum and $\vec{q}$ is the momentum transfer. Further, the integral in (B5) can be (from mathematical observations) given by [since the only unintegrated vector in integrating with respect to $\vec{Q}$ will be $(\vec{k}+a\vec{q})$, and then with respect to $\vec{k}$ will be $\vec{q}$]

$$\frac{1}{(2\pi)^3}\int d^3k\,\tilde{\phi}_{\alpha m}(\vec{k})\int (a\vec{q}\cdot\vec{Q})\Theta(Q)\tilde{\phi}^*_{\beta n}(\vec{k}+a\vec{q}+\vec{Q})d^3Q \cong q^2\Psi_{\beta n,\alpha m}(\vec{q}), \tag{B7}$$

where Ψ is some function of $\vec{q}$.

Consequently, $\mathcal{T}^{(2)}_{\beta\alpha}$ can be given by (we take here also $|\sigma| = \frac{1}{2}$)

$$\mathcal{T}^{(2)}_{\beta\alpha} \cong F(\vec{k}_0)\left(\frac{3q^2}{2p_i^{\,2}} + \frac{q^4}{32p_i^{\,4}}\right)\Psi_{\beta n,\alpha m}(\vec{q}), \tag{B8}$$

and comparison with $\mathcal{T}_{\beta\alpha}$ yields

$$\frac{\mathcal{T}^{(2)}_{\beta\alpha}}{\mathcal{T}_{\beta\alpha}} \cong \left\{\left(\frac{3}{2}\frac{q^2}{p_i^{\,2}} + \frac{q^4}{32p_i^{\,4}}\right)\Big/ S_{\beta\alpha}(a\vec{q})\right\}\Psi_{\beta n,\alpha m}(q). \tag{B9}$$

Now, in general $S_{\beta\alpha}(a\tilde{q})$ will decrease as the angle increases, and then the term in large curly brackets will attain its maximum value at large angles. So that, if $\Psi_{\beta n,\alpha m}$ is of any appreciable size, then one expects $\mathcal{T}^{(2)}_{\beta\alpha}$ to have some contrition at large angles. Summing up, we can say that our approximation (2.18) will be very reliable except, perhaps, for very large angles.

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*Work supported in part by the von Humboldt-Stifting Foundation. Present address: Institut für Theoretische Physik, Frankfurt-am-Main, West Germany.

[1]L. D. Faddeev, Zh. Eksp. Teor. Fiz. 39, 1459 (1960) [Sov. Phys.-JETP 12, 1014 (1961)].

[2]For a comprehensive review of three-body aspects, see V. Vanzani in proceedings of the Extended Seminar on Nuclear Physics, Trieste, 1973 (unpublished).

[3]C. Lovelace, in *Strong Interactions and High-Energy Physics*, edited by R. C. Moorehouse (Oliver and Boyd, London, 1964).

[4]C. Lovelace, Phys. Rev. 135, B1225 (1964).

[5]D. Bolle and K. L. Kowalski, Nuovo Cimento 62, 523 (1970).

[6]S. Weinberg, Phys. Rev. 133, B232 (1964).

[7]E. O. Alt, P. Grassberger, and W. Sandhas, Nucl. Phys. B2, 167 (1967).

[8]H. P. Noyes, Phys. Rev. D 5, 1574 (1972).

[9]E. O. Alt, P. Grassberger, and W. Sandhas, Phys. Rev. D 1, 2581 (1970).

[10]S. C. Pieper and K. L. Kowalski, Phys. Rev. C 5, 306 (1972).

[11]K. L. Kowalski and D. Feldman, Phys. Rev. 130, 276 (1963).

[12]I. H. Sloan, Phys. Rev. 165, 1587 (1968).

[13]K. L. Kowalski, Phys. Rev. Lett. 15, 798 (1965).

[14]H. P. Noyes, Phys. Rev. Lett. 15, 538 (1965); H. P. Noyes, in *Nuclear Forces and the Three-Nucleon Problem* (Centro de Investigacion y de Estodios Avanzados del Instituto Politecnico Nacional, Mexico, D.F. 14, 1972) (this reference was not available to the author).

[15]K. L. Kowalski and S. C. Pieper, Phys. Rev. C 2, 324 (1972); G. Bencze and P. Doleshall, Phys. Lett. 44B, 235 (1973).

[16]Inclusion of spin is a straightforward extension which multiplies the number of channels and indices without introducing qualitatively different dynamical phenomena.

[17]It is more convenient (and mathematically more correct) to use the notations of integral kernels instead of matrix elements via

$$\langle p'_\alpha\, q'_\alpha \,|\, T_\alpha(z) \,|\, p_\alpha\, q_\alpha\rangle = T_\alpha(p'_\alpha\, q'_\alpha, p_\alpha\, q_\alpha\,;\, z).$$

[18]T. Berggren, Nucl. Phys. A109, 265 (1968); W. Romo, *ibid.* A116, 618 (1968).

[19]R. W. Finkel and L. Rosenberg, Phys. Rev. 168, 1841 (1968).

[20]K. L. Kowalski, Phys. Rev. 188, 2235 (1969).

[21]H. N. Pendleton, Phys. Rev. 131, 1833 (1963).

[22]A. G. Sitenkoand and V. F. Kharchenko, Nucl. Phys. 49, 15 (1963); R. Aaron and R. D. Amado, Phys. Rev. 150, 857 (1966); G. L. Schrenk and A. N. Mitra, Phys. Rev. Lett. 19, 530 (1967).

[23]K. L. Kowalski, Phys. Rev. D 5, 395 (1972).