PHYSICAL REVIEW D VOLUME 16, NUMBER 4 15 AUGUST 1977

Spherically symmetric monopoles*

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(Received 18 April 1977)

The task of finding all spherically symmetric three-dimensional point monopoles in a gauge theory with arbitrary compact semisimple group is completely formulated in an Abelian gauge, where the problem is purely group-theoretical. The form of the gauge transformation to the spherically symmetric gauge is explicitly given in the general case. For SU(N) groups this result is reduced to a simple diagrammatic method which yields all such point monopoles by inspection. Also for SU(N) groups, a technique is given for efficient construction of the radial differential equations satisfied by the corresponding finite-energy solutions.

I. INTRODUCTION

The striking feature of the 't Hooft[1]-Polyakov[2] monopole in SU(2) theory is that the solution is expressed in a gauge where it is manifestly spherically symmetric under combined space and isospin rotations. It is this property which makes it possible to construct a finite-energy ansatz for which the Yang-Mills[3] equations reduce to ordinary differential equations in the radial variable. Following Wu and Wu,[4] the concept of spherical symmetry may be generalized to the case of a general semisimple compact gauge group G by considering SU(2) subgroups. Expressing the vector field $\vec{A}$ and scalar field Φ as matrices in some faithful representation $D(G)$, we say a solution is spherically symmetric under $\vec{L}+\vec{T}$ if

$$[L_i+T_i,A_j]=i\epsilon_{ijk}A_k\,,\qquad [L_i+T_i,\Phi]=0\,, \tag{1.1}$$

where $\vec{L}=-i\vec{r}\times\nabla$, and T_i generate some SU(2) subgroup of G.

The first step in constructing such solutions is to find the spherically symmetric point monopoles, since they provide the asymptotic boundary conditions for the finite-energy solutions. The work of Weinberg and Guth[5] showed that the only spherically symmetric point monopole in SU(2) is that of 't Hooft and Polyakov, but in larger groups there may be many such monopoles. One way in which these solutions may be constructed is to write down the most general ansatz satisfying (1.1), and then to look for point-monopole solutions of the Yang-Mills equations of the theory.[6,7] Although simple in principle, this method is in practice very laborious, and to obtain all the spherically symmetric solutions it must be repeated for each of the possible embeddings of SU(2) in G.

By contrast, in an Abelian gauge with Φ constant and $\vec{A}$ expressed in terms of a singular Dirac[8] vector potential, it is a trivial matter to write down all the possible point-monopole solutions.[9] Our main result, given in Sec. II, is to give a simple necessary and sufficient condition for transforming such a solution to a nonsingular gauge in which it is spherically symmetric in the sense of (1.1). The form of the required gauge transformation is explicitly given. The method stresses the importance of the physical charge-pole angular momentum, and is a generalization of a previous result of the present authors.[10] Our interest in this problem was rekindled by a recent paper of Bais and Primack,[11] who speculated that our previous method gave the most general class of spherical point monopoles. However, already in the SU(4) solutions of Brihaye and Nuyts[7] there may be found a counterexample, which of course is included by our new procedure. As stressed by these latter authors, an important concept is that of the little group which transforms the spherically symmetric solutions among themselves; our second result of Sec. II is to show that the action of this group on the spherically symmetric point monopoles has a simple and natural counterpart in the Abelian gauge. Using these results we are able to formulate the problem of finding all the inequivalent spherically symmetric point monopoles in purely group-theoretical terms. All that is required is knowledge of the possible embeddings of SU(2) in G, a topic discussed in the literature.[11]

In Sec. III we specialize to SU(N) groups and reduce our previous result to a simple diagrammatic algorithm. Using this method, one may immediately rederive all the SU(3) solutions found by Corrigan, Olive, Fairlie, and Nuyts[6] and by Dereli and Swank,[12] and the SU(4) solutions given by Brihaye and Nuyts.[7] We give the details only in the case of the $4\to 3+1$ embedding in SU(4), for which we find one solution apparently missed by Brihaye and Nuyts.

In Sec. IV we consider finite-energy solutions in SU(N) groups. By performing the construction on the z axis, we obtain the most general spherically symmetric ansatz for a given $\vec{T}$ without explicit reference to the tensor structure of the various terms. The computation of the energy density, and hence the radial differential equations, is extremely simple in this representation. On the z axis the little group of the spherically symmetric solutions is just the little group of T_3. By analogy with a theorem of classical mechanics, we show that when this little group is an n-parameter group, it is possible to eliminate a total of either $2n-1$ or $2n$ parameters from the most general spherically symmetric ansatz, depending on whether or not T_3 has both integer and half-integer eigenvalues. Section V contains a brief discussion.

II. POINT SOLUTIONS FOR GENERAL GROUPS

We consider a spontaneously broken gauge theory for an arbitrary compact semisimple group G, with the scalar field (or fields) in the adjoint representation. Since the nonsingular monopoles of the theory are just those of the covering group G_{cov},[13] there is no loss of generality in considering G itself to be simply connected. We express the vector field A_μ and scalar field Φ as matrices in some faithful unitary representation $D(G)$ of G, so that the Lagrangian density may be written

$$\mathcal{L} = -\tfrac{1}{2}\mathrm{Tr}G_{\mu\nu}G^{\mu\nu} - \mathrm{Tr}D_\mu\Phi D^\mu\Phi - V(\Phi)\,, \qquad (2.1)$$

where

$$\begin{aligned} G_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu - ie[A_\mu, A_\nu]\,, \\ D_\mu\phi &= \partial_\mu\Phi - ie[A_\mu, \Phi]\,, \end{aligned} \qquad (2.2)$$

and $V(\Phi)$ is a G-invariant quartic polynomial. For an arbitrary position-dependent $\Lambda(x)$ in G, the local gauge invariance of the theory is expressed by

$$\begin{aligned} eA_\mu &\to \Lambda e A_\mu \Lambda^{-1} + i\Lambda\partial_\mu\Lambda^{-1}\,, \\ \Phi &\to \Lambda\Phi\Lambda^{-1}\,. \end{aligned} \qquad (2.3)$$

We are interested in time-independent solutions with $A_0 = 0$ so that the Yang-Mills tensor has only magnetic components $B_i = \frac{1}{2}\epsilon_{ijk}G_{jk}$, and the field equations may be written

$$\begin{aligned} \epsilon_{ijk}D_j B_k &= ie[\Phi, D_i\Phi]\,, \\ D_i D_i \Phi &= \frac{\partial V}{\partial \Phi}\,. \end{aligned} \qquad (2.4)$$

A point-monopole solution of these equations is one for which $\partial V/\partial\Phi$ and $\vec{D}\Phi$ vanish everywhere. A simple way to construct such solutions is first to write them in a singular Abelian gauge by means of a Dirac[8] vector potential $\vec{A}_D$ with a string along the negative z axis:

$$\vec{A}_D = \hat{\phi}(1-\cos\theta)/r\sin\theta\,, \qquad (2.5)$$

where (r,θ,ϕ) are spherical polar coordinates. Since $\vec{A}_D$ is a solution of the source-free Maxwell equations (except on the string) it is clear that a solution of the Yang-Mills equations (2.4) is

$$\begin{aligned} \Phi &= \Phi_0\,, \\ e\vec{A} &= Q\vec{A}_D\,, \end{aligned} \qquad (2.6)$$

where Φ_0 is a constant matrix which minimizes the scalar potential, and Q is a constant matrix which commutes with Φ_0. Provided the eigenvalues of Q are all integers or half integers [i.e., $\exp(4\pi i Q) = 1$] the string is unobservable and, since G is simply connected, may always be eliminated by means of a singular gauge transformation. An important concept is that of the physical charge-pole angular momentum $\vec{J}$ defined by

$$\vec{J} = \vec{r}\times(-i\nabla - e\vec{A}) - er^2\vec{B}\,. \qquad (2.7)$$

Just as for monopoles in ordinary U(1) electromagnetism one may verify that the point solution (2.6) has the properties

$$\begin{aligned} [J_i, J_j] &= i\epsilon_{ijk}J_k\,, \\ [J_i, \partial_j - ieA_j] &= i\epsilon_{ijk}(\partial_k - ieA_k)\,, \\ [J_i, \Phi] &= 0\,. \end{aligned} \qquad (2.8)$$

Since both these properties and the definition (2.7) are gauge covariant, they must hold in any gauge. We now state and prove the following theorem:

Theorem 1. Let $\vec{T}$ be the generators of any SU(2) subgroup of G, not necessarily irreducible in $D(G)$. Then a necessary and sufficient condition for the existence of a gauge in which the Abelian gauge solution (2.6) becomes spherically symmetric under $\vec{L}+\vec{T}$ is that it is gauge equivalent to one for which the charge-product matrix Q has the form

$$Q = I_3 - T_3\,, \qquad (2.9)$$

where $\vec{I}$ is another embedding of SU(2) in G, also not necessarily irreducible in $D(G)$, with the properties

$$\begin{aligned} [\vec{I}, Q] &= 0\,, \\ [\vec{I}, \Phi_0] &= 0\,. \end{aligned} \qquad (2.10)$$

We first introduce some notation and prove a fundamental result which will be used throughout the paper. For an arbitrary SU(2) rotation described by Euler angles α, β, γ we write

$$\Omega(\alpha,\beta,\gamma) = e^{-i\alpha T_3}e^{-i\beta T_2}e^{-i\gamma T_3}\,. \qquad (2.11)$$

We also define a local gauge transformation $\Omega(\hat{r})$ with the property $\Omega(\hat{r})T_3\Omega^{-1}(\hat{r}) = \vec{T}\cdot\hat{r}$ by means of the rotation $(\phi, \theta, -\phi)$ which at each point (r,θ,ϕ) takes the $\hat{z}$ direction into the $\hat{r}$ direction:

$$\Omega(\hat{r})=\Omega(\phi,\theta,-\phi)\,. \tag{2.12}$$

Lemma. Let X_0 be any group element or generator which commutes with T_3. Then the quantity $X(\hat{r})$ defined by

$$X(\hat{r})=\Omega(\hat{r})X_0\Omega^{-1}(\hat{r}) \tag{2.13}$$

is a scalar with respect to $\vec{L}+\vec{T}$. Conversely, if $X(\hat{r})$ is a scalar under $\vec{L}+\vec{T}$, then $X_0\equiv X(\hat{z})$ commutes with T_3 and equals $\Omega^{-1}(\hat{r})X(\hat{r})\Omega(\hat{r})$.

Proof. Suppose X_0 commutes with T_3. Then $X(\hat{r})$ in (2.13) may be expressed as

$$\begin{aligned}X(\hat{r})&\equiv\Omega(\phi,\theta,-\phi)X_0\Omega^{-1}(\phi,\theta,-\phi)\\&=\Omega(\phi,\theta,\chi)X_0\Omega^{-1}(\phi,\theta,\chi)\,,\end{aligned} \tag{2.14}$$

where χ is an arbitrary angle. Thus we may conclude

$$X(\hat{r})=\Omega(R)X_0\Omega^{-1}(R)\,, \tag{2.15}$$

where R is *any* rotation which takes the $\hat{z}$ direction into the $\hat{r}$ direction, since the Euler-angle rotation (ϕ,θ,χ) is the most general such rotation. Consider now an arbitrary rotation $S\equiv(\alpha,\beta,\gamma)$. Then

$$\begin{aligned}\Omega(S)X(\hat{r})\Omega^{-1}(S)&=\Omega(SR)X_0\Omega^{-1}(SR)\\&=X(S\hat{r})\end{aligned} \tag{2.16}$$

since SR takes $\hat{z}$ into $S\hat{r}$. Thus $X(\hat{r})$ is a scalar under $\vec{L}+\vec{T}$ as required. Conversely if $X(\hat{r})$ is a scalar under $\vec{L}+\vec{T}$ then the fact that L_3 vanishes on the z axis implies that $X_0\equiv X(\hat{z})$ commutes with T_3. It follows that X_0 equals $\Omega^{-1}(\hat{r})X(\hat{r})\Omega(\hat{r})$. We may now prove theorem 1:

Proof of sufficiency. Suppose the conditions (2.9), (2.10) hold. For the embedding $\vec{I}$ we define quantities $\omega(\alpha,\beta,\gamma)$ and $\omega(\hat{r})$ analogous to (2.11), (2.12):

$$\begin{aligned}\omega(\alpha,\beta,\gamma)&=e^{-i\alpha I_3}e^{-i\beta I_2}e^{-i\gamma I_3}\,,\\\omega(\hat{r})&=\omega(\phi,\theta,-\phi)\,.\end{aligned} \tag{2.17}$$

Then following Ref. 10 we find that by means of the combined gauge transformation $\Lambda(\hat{r})\equiv\Omega(\hat{r})\omega^{-1}(\hat{r})$ the fields (2.6) may be brought to the form

$$\begin{aligned}e\vec{A}&=[\vec{I}(\hat{r})-\vec{T}]\times\hat{r}/r\,,\\\Phi&=\Phi(\hat{r})\,,\end{aligned} \tag{2.18}$$

where $\vec{I}(\hat{r})$ and $\Phi(\hat{r})$ are given by

$$\begin{aligned}\vec{I}(\hat{r})&=\Omega(\hat{r})\omega^{-1}(\hat{r})\vec{I}\omega(\hat{r})\Omega^{-1}(\hat{r})\,,\\\Phi(\hat{r})&=\Omega(\hat{r})\Phi_0\Omega^{-1}(\hat{r})\,.\end{aligned} \tag{2.19}$$

We assert that the forms (2.18) are spherically symmetric with respect to $\vec{L}+\vec{T}$. For the scalar field this follows immediately from the lemma, while for the vector field it is sufficient to show that $\vec{I}(\hat{r})$ is a vector under $\vec{L}+\vec{T}$. Recalling the relations $[T_3,I_3]=0$ and $[T_3-I_3,\vec{I}]=0$ we obtain

$$\begin{aligned}\vec{I}(\hat{r})&=\Omega(\phi,\theta,-\phi)\omega^{-1}(\phi,\theta,-\phi)\vec{I}\omega(\phi,\theta,-\phi)\\&\quad\times\Omega^{-1}(\phi,\theta,-\phi)\\&=\Omega(\phi,\theta,\chi)\omega^{-1}(\phi,\theta,\chi)\vec{I}\omega(\phi,\theta,\chi)\\&\quad\times\Omega^{-1}(\phi,\theta,\chi)\,,\end{aligned} \tag{2.20}$$

where χ is an arbitrary angle. Thus we may conclude

$$\vec{I}(\hat{r})=\Omega(R)\omega^{-1}(R)\vec{I}\omega(R)\Omega^{-1}(R)\,, \tag{2.21}$$

where R is *any* rotation which takes the $\hat{z}$ direction into the $\hat{r}$ direction. Consider now an arbitrary rotation $S\equiv(\alpha,\beta,\gamma)$. Then

$$\begin{aligned}\Omega(S)I_i(\hat{r})\Omega^{-1}(S)&\\&=\Omega(SR)\omega^{-1}(SR)\omega(S)I_i\omega^{-1}(S)\omega(SR)\Omega^{-1}(SR)\\&=S_{ij}^{-1}\Omega(SR)\omega^{-1}(SR)I_j\omega(SR)\Omega^{-1}(SR)\\&=S_{ij}^{-1}I_j(S\hat{r})\,,\end{aligned} \tag{2.22}$$

since SR takes $\hat{z}$ into $S\hat{r}$. Thus $\vec{I}(\hat{r})$ is a vector under $\vec{L}+\vec{T}$ as required. Note that the magnetic field is given by

$$e\vec{B}=\hat{r}\cdot[\vec{I}(\hat{r})-\vec{T}]\hat{r}/r^2\,, \tag{2.23}$$

so that in the new gauge the physical angular momentum (2.7) takes the form

$$\vec{J}=\vec{L}+\vec{T}-\vec{I}(\hat{r})\,. \tag{2.24}$$

This equation will be used as the starting point for the proof of the converse part of the theorem.

Proof of necessity. We first show that any spherically symmetric vector potential $\vec{A}$ may be brought to a form orthogonal to $\hat{r}$ by a gauge transformation which commutes with $\vec{L}+\vec{T}$. Let us write the most general such potential as a sum of transverse and radial parts:

$$e\vec{A}=[\vec{M}(\vec{r})-\vec{T}]\times\hat{r}/r+N(\vec{r})\hat{r}\,. \tag{2.25}$$

The quantities $\vec{M}(\vec{r})$ and $N(\vec{r})$ are, respectively, a vector and scalar with respect to $\vec{L}+\vec{T}$, but otherwise unrestricted. Under the gauge transformation $\Omega^{-1}(\hat{r})$ this becomes

$$\begin{aligned}e\vec{A}&=T_3\vec{A}_D+\Omega^{-1}(\hat{r})\vec{M}(\vec{r})\Omega(\hat{r})\times\hat{r}/r\\&\quad+N_0(r)\hat{r}\,,\end{aligned} \tag{2.26}$$

where $N_0(r)\equiv N(r\hat{z})$ commutes with T_3. Let us define a gauge transformation $\Lambda_0(r)$ by

$$\Lambda_0(r)=R\exp\left[-i\int_0^r N_0(r')dr'\right]\,, \tag{2.27}$$

where the R indicates that the terms in the expansion of the exponential should be ordered so that the smallest argument r appears on the left. The transformation $\Lambda_0(r)$ commutes with T_3 [since

$N_0(r)$ does] and is the formal solution of the equation

$$\frac{d\Lambda_0}{dr} = -i\Lambda_0(r)N_0(r)\,. \tag{2.28}$$

Transforming the vector potential (2.26) with $\Lambda_0(r)$ just removes the radial component:

$$e\vec{A} = T_3\vec{A}_D + \Lambda_0(r)\Omega^{-1}(\hat{r})M(\vec{r})\Omega(\hat{r})\Lambda_0^{-1}(\hat{r})\,. \tag{2.29}$$

If we now apply the gauge transformation $\Omega(\hat{r})$, we find that by means of the combined transformation $\Lambda(\vec{r}) \equiv \Omega(\hat{r})\Lambda_0(r)\Omega^{-1}(\hat{r})$ the gauge potential (2.25) has been brought to the form

$$e\vec{A} = [\vec{\mathfrak{M}}(\vec{r}) - \vec{T}]\times\hat{r}/r\,, \tag{2.30}$$

where

$$\vec{\mathfrak{M}}(\hat{r}) = \Lambda(\hat{r})\vec{M}(\vec{r})\Lambda^{-1}(\vec{r})\,. \tag{2.31}$$

By the lemma the gauge transformation $\Lambda(\vec{r})$ commutes with $\vec{L}+\vec{T}$ and so the new $\vec{\mathfrak{M}}(\vec{r})$ is still a vector under $\vec{L}+\vec{T}$.

Suppose now that the point solution (2.6) can be brought to a spherically symmetric form. Without loss of generality we may assume that the vector potential is purely transverse. Since any r dependence of the gauge transformation from the Abelian gauge would induce a radial term in $\vec{A}$, we conclude that in the spherically symmetric gauge the quantities $r\vec{A}$, $r^2\vec{B}$, and Φ are functions of direction $\hat{r}$ only. In the latter gauge we may thus write the physical angular momentum $\vec{J}$ as

$$\begin{aligned}\vec{J} &= \vec{r}\times(-i\nabla - e\vec{A}) - er^2\vec{B}\\ &= \vec{L}+\vec{T}-\vec{I}(\hat{r})\,,\end{aligned} \tag{2.32}$$

where $\vec{I}(\hat{r})$ is now *defined* by this equation. By the spherical symmetry the quantity $\vec{I}(\hat{r})$ must be a vector under $\vec{L}+\vec{T}$. From the relations $[J_i, J_j] = i\epsilon_{ijk}J_k$ and $[\vec{J}, \vec{J}\cdot\hat{r}] = 0$ there follows

$$\begin{aligned}&[I_i(\hat{r}), I_j(\hat{r})] = i\epsilon_{ijk}I_k(\hat{r})\,,\\ &[\vec{I}(\hat{r}), \hat{r}\cdot[\vec{I}(\hat{r}) - \vec{T}]] = 0\,.\end{aligned} \tag{2.33}$$

Since the magnetic field is purely radial we obtain

$$\begin{aligned}e\vec{B} &= -\hat{r}\cdot\vec{J}\hat{r}/r^2\\ &= \hat{r}\cdot[\vec{I}(\hat{r}) - \vec{T}]\hat{r}/r^2\,.\end{aligned} \tag{2.34}$$

As the vector potential is transverse, its form is uniquely determined from (2.32), (2.34):

$$e\vec{A} = [\vec{I}(\hat{r}) - \vec{T}]\times\hat{r}/r\,. \tag{2.35}$$

One may easily verify that this potential does give the magnetic field $\vec{B}$ in (2.34). For the scalar field the relations $\vec{D}\Phi = 0$ and $[\vec{L}+\vec{T}, \Phi] = 0$ imply

$$\begin{aligned}&[\vec{I}(\hat{r}), \Phi(\hat{r})] = 0\,,\\ &[\hat{r}\cdot[\vec{I}(\hat{r}) - \vec{T}], \Phi(\hat{r})] = 0\,.\end{aligned} \tag{2.36}$$

Let us define $\vec{I} = \vec{I}(\hat{z})$, $\Phi_0 = \Phi(\hat{z})$, and $Q = I_3 - T_3$, so that $[\vec{I}, Q] = 0$, $[\vec{I}, \Phi_0] = 0$, and $[Q, \Phi_0] = 0$. Then by the reverse of the previous argument it is clear that the gauge transformation $\omega(\hat{r})\Omega^{-1}(\hat{r})$ brings the solution to the form

$$\begin{aligned}&e\vec{A} = Q\vec{A}_D\,,\\ &\Phi = \Phi_0\,.\end{aligned} \tag{2.37}$$

Thus the original Abelian gauge solution is gauge equivalent to one obeying the conditions (2.9), (2.10) as required. We remark that if two Abelian gauge solutions of the type (2.6) are related by a gauge transformation, then it must be a trivial constant transformation.

The theorem shows that the Abelian gauge form of all spherically symmetric point monopoles for a given $\vec{T}$ may be obtained by constructing all possible pairs $\vec{I}, \Phi_0$, where $\vec{I}$ is an SU(2) embedding and the conditions $[\vec{I}, Q] = 0$, $[\vec{I}, \Phi_0] = 0$, and $[Q, \Phi_0] = 0$ are all satisfied with $Q \equiv I_3 - T_3$. The corresponding spherically symmetric solutions are specified by the pairs $\vec{I}(\hat{r}), \Phi(\hat{r})$, the relation between the two gauges being given by (2.19). We say that two spherically symmetric solutions are equivalent if they are related by a gauge transformation which commutes with $\vec{L}+\vec{T}$; this definition ensures that spherically symmetric finite-energy solutions with the asymptotic boundary conditions of two equivalent point monopoles are themselves gauge equivalent. The r dependence of such gauge transformations was determined when we removed the radial component of the vector potential, but we are still free to make transformations commuting with $\vec{L}+\vec{T}$ which are functions of direction $\hat{r}$ only. The importance of this "little group" of the spherically symmetric solutions was emphasized by Brihaye and Nuyts. As noted by these authors, the quantity $\vec{\mathfrak{M}}(\vec{r})$ appearing in the vector potential $\vec{A}$ in (2.30) transforms covariantly under this group. In particular the quantities $\vec{I}(\hat{r}), \Phi(\hat{r})$ of our spherically symmetric point monopoles transform covariantly. The next theorem shows that this statement has a simple and natural analog in the Abelian gauge:

Theorem 2. Two spherically symmetric solutions $\vec{I}(\hat{r}), \Phi(\hat{r})$ and $\vec{I}'(\hat{r}), \Phi'(\hat{r})$ are related by a gauge transformation $V(\hat{r})$ which commutes with $\vec{L}+\vec{T}$ if and only if the corresponding Abelian gauge solutions $\vec{I}, \Phi_0$ and $\vec{I}', \Phi_0'$ are related by a constant gauge transformation V_0 which commutes with T_3. The correspondence between $V(\hat{r})$ and V_0 is given by

$$\begin{aligned}V_0 &= V(\hat{z})\\ &= \Omega^{-1}(\hat{r})V(\hat{r})\Omega(\hat{r})\,.\end{aligned} \tag{2.38}$$

Proof. Suppose $\vec{I}, \Phi_0$ and $\vec{I}', \Phi_0'$ are related by a constant transformation V_0 which commutes with T_3, and let R be any rotation which takes the $\hat{z}$ direction into the $\hat{r}$ direction. Writing $\Omega = \Omega(R)$ we find

$$\begin{aligned}\Phi'(\hat{r}) &\equiv \Omega\Phi_0'\Omega^{-1}\\ &= \Omega V_0 \Phi_0 V_0^{-1}\Omega^{-1}\\ &= \Omega V_0 \Omega^{-1}\Phi(\hat{r})\Omega V_0^{-1}\Omega^{-1}\,.\end{aligned} \tag{2.39}$$

Similarly,

$$\begin{aligned}I_i'(\hat{r}) &\equiv R_{ij}\Omega I_j'\Omega^{-1}\\ &= R_{ij}\Omega V_0 I_j V_0^{-1}\Omega^{-1}\\ &= \Omega V_0\Omega^{-1} I_i(\hat{r})\Omega V_0^{-1}\Omega^{-1}\,.\end{aligned} \tag{2.40}$$

Thus $\vec{I}'(\hat{r}), \Phi'(\hat{r})$ are related to $\vec{I}(\hat{r}), \Phi(\hat{r})$ by $V(\hat{r}) \equiv \Omega V_0 \Omega^{-1}$, which by the lemma commutes with $\vec{L} + \vec{T}$. The converse result is proved in a similar manner.

In order to give simple rules for determining all the spherically symmetric monopoles of the theory we introduce the terminology that two sets of generators Y_i, Y_i' are equivalent if they are related by a group transformation (inner automorphism). More generally, if X_j is another set of generators, then Y_i and Y_i' are equivalent (modulo X_j) if they are related by a gauge transformation which leaves each of the X_j invariant. The point monopoles spherically symmetric with respect to $\vec{L} + \vec{T}$ are now obtained by the following purely group-theoretical prescription:

(1) Write down all the inequivalent (modulo T_3) SU(2) embeddings $\vec{I}$ with the property that $Q \equiv I_3 - T_3$ commutes with $\vec{I}$.

(2) For each such $\vec{I}$ write down all the inequivalent (modulo T_3, $\vec{I}$) generators Φ_0 which commute with T_3, $\vec{I}$ (subject of course to the requirement that Φ_0 minimizes the scalar potential).

To find all the spherically symmetric monopoles of the theory the above construction must be repeated for each of the inequivalent SU(2) embeddings $\vec{T}$. The spherically symmetric form of the solution is in each case obtained by means of the gauge transformation $\Omega(\hat{r})\omega^{-1}(\hat{r})$.

III. POINT SOLUTIONS FOR SU(N) GROUPS

The problem of finding the spherically symmetric point monopoles for SU(N) groups is greatly simplified by the fact that, apart from an overall phase, every unitary transformation is a gauge transformation. This means in particular that any SU(2) embedding is equivalent to one which is explicitly decomposed into irreducible SU(2) representations, with diagonal T_3 and standard angular momentum phase conventions in each block (i.e., matrix elements of $T_\pm$ are real and positive). Let $\vec{T}$ be any such "standard" SU(2) embedding and consider a possible point monopole, i.e., a pair $\vec{I}, \Phi_0$ satisfying $[\vec{I}, Q] = 0, [\vec{I}, \Phi_0] = 0$ and $[Q, \Phi_0] = 0$, where $Q \equiv I_3 - T_3$. Then there exists a unitary transformation U which brings $(\vec{I}, Q, \Phi_0)$ to a form $(\vec{I}', Q', \Phi_0')$, where $\vec{I}'$ is a standard embedding and Q' and Φ_0' are each multiples of the unit matrix within each irreducible block of $\vec{I}'$. Let us write $T_3' = I_3' - Q' = UT_3U^{-1}$. By construction, T_3' is diagonal and so we may recover the original form of T_3 by the action of a permutation matrix P which merely rearranges the eigenvalues:

$$\begin{aligned}T_3 &= PT_3'P^{-1}\\ &= PUT_3(PU)^{-1}\,.\end{aligned} \tag{3.1}$$

Thus by means of the gauge transformation $V = PU$ we see that the original solution is equivalent to one for which $(\vec{I}, Q, \Phi_0)$ differ only by the action of a permutation matrix from a form $(\vec{I}', Q', \Phi_0')$ with $\vec{I}'$ standard and Q' and Φ_0' multiples of the unit matrix within each block. If T_3 has degenerate eigenvalues there will always be more than one such permutation and it is sufficient to pick any one of them.

In order to construct systematically all the possible solutions for a given $\vec{T}$ it is convenient to introduce the following diagrammatic notation, the full generality of which is illustrated in Fig. 1 for the case where $\vec{T}$ is the $8 \to 3+1+2+2$ embedding in SU(8). The irreducible blocks of $\vec{T}$ are written in columns side by side in such a way that equal eigenvalues of T_3 lie on the same horizontal row and the columns with integer T_3 eigenvalues are separated from those with half-integer eigenval-

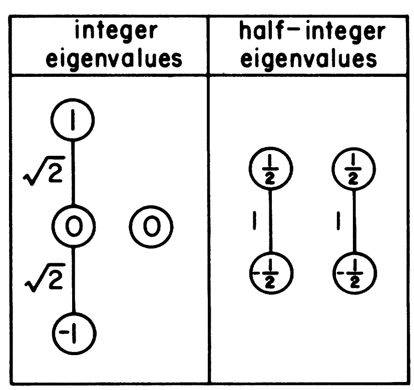

FIG. 1. Diagram for the $8 \to 3+1+2+2$ embedding of SU(2) in SU(8). The irreducible blocks of $\vec{T}$ are written in columns side by side with equal eigenvalues of T_3 on the same horizontal row. The elements of the diagram are labeled by the eigenvalues of T_3, and the connecting lines by the matrix elements of $T_\pm$.

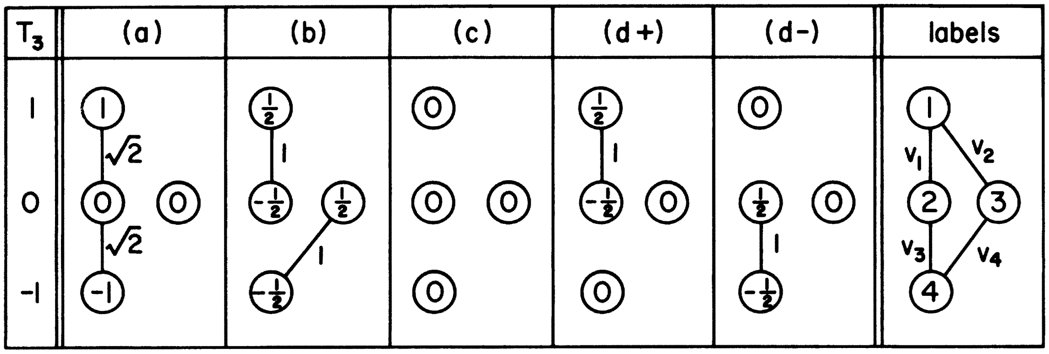

FIG. 2. Diagrams showing the five inequivalent SU(2) embeddings $\vec{I}$ satisfying (2.9), (2.10) when $\vec{T}$ is the $4\to 3+1$ embedding in SU(4). The elements are labeled by the eigenvalues of I_3, and the connecting lines by the matrix elements of $I_\pm$. Diagram (a) corresponds to $\vec{I}=\vec{T}$. The charge-product matrix Q in (2.6) is in each case given by $Q=I_3-T_3$, while the scalar field Φ_0 can be any traceless diagonal matrix which is constant along each connecting string of the diagram. The last diagram shows both the order of the diagonal elements and the labeling of the matrix elements of $I_\pm$ used in constructing Table I. The solutions for the other possible embeddings $\vec{T}$ in SU(4) are obtained in a similar manner.

ues. The vertical lines show the action of the raising and lowering operators $T_\pm$. All the spherically symmetric point monopoles may now be obtained by the following rules, illustrated in Fig. 2 for the case of the $4\to 3+1$ embedding in SU(4):

(1) First draw the diagram for $\vec{T}$, but without the connecting lines. Since the possible embeddings $\vec{I}$ are such that $Q\equiv I_3-T_3$ is constant in each block of $\vec{I}$, every such block may be represented by a "string" passing downward through the diagram (not necessarily vertically) beginning and ending at an arbitrary point. A string of length l represents a block of spin $l/2$ and dimension $l+1$. An embedding $\vec{I}$ consists of a set of such strings with at most one string intersecting each element of the diagram; elements with no strings passing through them are singlets under $\vec{I}$. Diagrams so constructed which differ only by permutation of the elements in a given row (or rows) correspond to equivalent solutions and only one of them should be counted.

(2) For a given $\vec{I}$, the possible scalar fields Φ_0 are just those traceless diagonal matrices which are constant within each block of $\vec{I}$, i.e., constant along each connecting string of the diagram for $\vec{I}$. Two choices Φ_0 and Φ_0' which are related by interchange of identical strings of $\vec{I}$ (ones which connect the same rows) are equivalent, and only one of them should be counted; this never happens for the example in Fig. 2.

When $\vec{T}$ is the $3+1$ embedding in SU(4) we see from Fig. 2 that there are five inequivalent choices for $\vec{I}$. The solution (a) corresponds to $\vec{I}=\vec{T}$ so that there is no magnetic field, while the solution (c) is $\vec{I}=0$, for which the angular momentum $\vec{J}$ in the spherically symmetric gauge is just $\vec{L}+\vec{T}$. The solutions (d±) are related by reflection about the horizontal line $T_3=0$ and are antimonopoles of each other; the other solutions are self-conjugate in this sense.

The same information, together with the various possibilities for the scalar field, is shown in Table I. We have taken the diagonal elements from the diagram for $\vec{T}$ not in column order, which would correspond to $\vec{T}$ being explicitly reduced, but rather in row order so that the equal eigenvalues of T_3 appear together. The quantities v_1, v_2, v_3, and v_4 refer to the matrix elements of $I_\pm$, where

TABLE I. Details of the spherically symmetric point monopoles when $\vec{T}$ is the $4\to 3+1$ embedding in SU(4). The solutions correspond to the five embeddings $\vec{I}$ shown in Fig. 2. The meaning of the diagonal matrices Q and Φ_0 is given in (2.6), while the quantities v_α are the matrix elements of $I_\pm$; the labeling is given in the last diagram of Fig. 2. The quantities a, b, c for the scalar field are arbitrary, but will be restricted in practice by the requirement that Φ_0 is a minimum of the scalar potential.

	(a)	(b)	(c)	(d+)	(d−)
Q	0	$-\frac{1}{2}$	-1	$-\frac{1}{2}$	-1
	0	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2}$
	0	$\frac{1}{2}$	0	0	0
	0	$\frac{1}{2}$	1	1	$\frac{1}{2}$
Φ_0	a	a	a	a	$-(2a+b)$
	a	a	b	a	a
	$-3a$	$-a$	c	b	b
	a	$-a$	$-(a+b+c)$	$-(2a+b)$	a
v_1	$\sqrt{2}$	1	0	1	0
v_2	0	0	0	0	0
v_3	$\sqrt{2}$	0	0	0	1
v_4	0	1	0	0	0

the labeling is given by the last diagram of Fig. 2. These results are in agreement with the analysis of Brihaye and Nuyts, except that these authors apparently missed the solutions (d±).

If the lines of a diagram can be drawn so that they are all vertical, then each block of $\vec{I}$ acts entirely within a single block of $\vec{T}$, and following the terminology of Bais and Primack we say that $\vec{I}$ is a subembedding of $\vec{T}$. In such cases it is possible to express the rotated quantities $\vec{I}(\hat{r})$ and $\Phi(\hat{r})$ in (2.19) as explicit vector and scalar functions of $\hat{r}$ and $\vec{T}_\alpha$, where $\vec{T}_\alpha$ denotes the αth irreducible block of $\vec{T}$. Some examples of this construction are given in our previous paper,[10] and by Wilkinson,[14] and by Bais and Primack.[11] Such subembedding situations were the only ones considered in these papers but, as we see from solution (b) of Fig. 2, this is not the most general possibility. When $\vec{I}$ is not a subembedding, an explicit demonstration that $\vec{I}(\hat{r})$ is a vector under $\vec{L}+\vec{T}$ requires the use of tensor quantities which cannot be expressed as functions of the irreducible components $\vec{T}_\alpha$ of $\vec{T}$. However, we shall see in the next section that even for the purpose of constructing the radial differential equations for the corresponding finite-energy solutions, it is never necessary to display this tensor structure explicitly.

IV. FINITE-ENERGY SOLUTIONS IN SU(N)

The ultimate goal of classical monopole theory is to find fields $\vec{A}, \Phi$ such that the energy E given by

$$E = \int d^3\vec{x}\,[\mathrm{Tr}\vec{B}^2 + \mathrm{Tr}(\vec{D}\Phi)^2 + V(\Phi)] \tag{4.1}$$

has the minimum value for a given topological configuration of the scalar fields at infinity. Since any $\vec{L}+\vec{T}$ is a symmetry of the full Yang-Mills theory, a solution of the field equations (although not necessarily even a local minimum of the energy) is always obtained by writing the most general spherically symmetric ansatz for $\vec{A}$ and Φ, substituting into the energy expression (4.1), and then varying with respect to the radial functions appearing in the ansatz. This reduces the problem to a set of ordinary differential equations. As shown in Sec. II, the gauge freedom of the theory may be used to remove the radial terms in $\vec{A}$, so that we may write

$$\begin{aligned} e\vec{A} &= [\vec{\mathfrak{M}}(r,\hat{r}) - \vec{T}] \times \hat{r}/r\,, \\ \Phi &= \Phi(r,\hat{r})\,, \end{aligned} \tag{4.2}$$

in which the $\hat{r}$ dependence of $\vec{\mathfrak{M}}$ and Φ is carried by the various basis elements of the ansatz, and the r dependence by the radial functions. From the spherical symmetry there follows

$$\begin{aligned} e\vec{B} &= -i\hat{r}\cdot(\vec{\mathfrak{M}}\times\vec{\mathfrak{M}} - \vec{T}\times\vec{T})\hat{r}/r^2 \\ &\quad + \hat{r}\times(\vec{\mathfrak{M}}'\times\hat{r})/r\,, \\ \vec{D}\phi &= -i[\vec{\mathfrak{M}},\Phi]\times\hat{r}/r + \hat{r}\Phi'\,, \end{aligned} \tag{4.3}$$

where primes denote partial differentiation with respect to r with $\hat{r}$ held fixed. For the purposes of computing the energy density in (4.1) it is sufficient to consider these quantities on the positive z axis, where we obtain

$$\begin{aligned} eB_3 &= (\tfrac{1}{2}[M_+,M_-] - T_3)/r^2\,, \\ eB_\pm &= M'_\pm/r\,, \\ D_3\Phi &= \Phi'_0\,, \\ D_\pm\Phi &= \mp[M_\pm,\Phi_0]/r\,, \end{aligned} \tag{4.4}$$

where we have written

$$\begin{aligned} \vec{M}(r) &= \vec{\mathfrak{M}}(r,\hat{z})\,, \\ \Phi_0(r) &= \Phi(r,\hat{z})\,. \end{aligned} \tag{4.5}$$

All that remains is to construct an ansatz for the quantities $\vec{M}$ and Φ_0. Since L_3 vanishes on the z axis and $\vec{\mathfrak{M}}(\vec{r})$ and $\Phi(\vec{r})$ are, respectively, a vector and scalar under $\vec{L}+\vec{T}$, there follow

$$\begin{aligned} [T_3,\Phi_0(r)] &= 0\,, \\ [T_3,M_\pm(r)] &= \pm M_\pm(r)\,. \end{aligned} \tag{4.6}$$

We shall argue that given the most general $\Phi_0(r)$ and $M_\pm(r)$ with these properties (and $M_\pm^\dagger = M_\mp$) it is always possible to reconstruct scalar and vector quantities $\Phi(\vec{r})$ and $\vec{\mathfrak{M}}(\vec{r})$ which reduce to these $\Phi_0(r)$ and $M_\pm(r)$ on the z axis. For Φ_0 this is clear, since if we write

$$\Phi(\vec{r}) = \Omega(\hat{r})\Phi_0(r)\Omega^{-1}(\hat{r})\,, \tag{4.7}$$

it follows from the lemma of Sec. II that $\Phi(\vec{r})$ is a scalar under $\vec{L}+\vec{T}$ and satisfies $\Phi(r\hat{z}) = \Phi_0(r)$. From (4.6) the most general $M_\pm(r)$ may be written

$$\begin{aligned} M_+(r) &= \sum_\alpha v_\alpha(r) I_{\alpha+}\,, \\ M_-(r) &= \sum_\alpha v_\alpha^*(r) I_{\alpha-}\,, \end{aligned} \tag{4.8}$$

where the v_α are arbitrary complex radial functions, and the quantities $\vec{I}_\alpha$ are a particular subset of the SU(2) embeddings $\vec{I}$ which were used to construct the point solutions, namely those which consist of a single two-dimensional representation. In terms of our diagrams this means that they have a single line connecting two adjacent rows. The four possibilities for the $4 \to 3+1$ embedding in SU(4) are given by the four lines of the last diagram of Fig. 2. Note that for the purpose of constructing an ansatz it is necessary to consider all such two-dimensional $\vec{I}$'s, not just those which are inequivalent modulo T_3. Equation (4.8) may be rewritten in the form

$$M_\pm(r)=\sum_\alpha\,[a_\alpha(r)I_{\alpha\pm}+b_\alpha(r)(\hat z\times\vec{\mathrm{I}}_\alpha)_\pm]\,, \tag{4.9}$$

where $v_\alpha=a_\alpha+ib_\alpha$. It follows that $\vec{\mathfrak{M}}(\vec{\mathrm{r}})$ is given by

$$\vec{\mathfrak{M}}(\vec{\mathrm{r}})=\sum_\alpha\,[a_\alpha(r)\vec{\mathrm{I}}_\alpha(\hat r)+b_\alpha(r)\hat r\times\vec{\mathrm{I}}_\alpha(\hat r)]\,, \tag{4.10}$$

where $\vec{\mathrm{I}}_\alpha(\hat r)$ is the rotated version of $\vec{\mathrm{I}}_\alpha$:

$$\vec{\mathrm{I}}_\alpha(\hat r)=\Omega(\hat r)\omega_\alpha^{-1}(\hat r)\vec{\mathrm{I}}_\alpha\omega_\alpha(\hat r)\Omega^{-1}(\hat r)\,. \tag{4.11}$$

By the arguments of Sec. II, $\vec{\mathrm{I}}_\alpha(\hat r)$ and hence $\vec{\mathfrak{M}}(\vec{\mathrm{r}})$ in (4.10) are vector quantities under $\vec{\mathrm{L}}+\vec{\mathrm{T}}$.

To see what this means explicitly, consider the familiar example of the $4\to3+1$ embedding in SU(4). As in Sec. III, let us use a matrix representation not where $\vec{\mathrm{T}}$ is explicitly reduced but, rather, one where $T_3=\mathrm{diag}(1,0,0,-1)$, i.e., the equal eigenvalues of T_3 appear together. The most general matrices $\Phi_0(r)$ and $M_\pm(r)$ satisfying (4.6) are then given by

$$\Phi_0(r)=\begin{pmatrix}\phi_1&&&\\&\phi_2&\psi&\\&\psi^*&\phi_3&\\&&&\phi_4\end{pmatrix},$$

$$M_+(r)=\begin{pmatrix}&v_1&v_2&\\&&&v_3\\&&&v_4\\&&&\end{pmatrix}, \tag{4.12}$$

$$M_-(r)=\begin{pmatrix}&&&\\v_1^*&&&\\v_2^*&&&\\&v_3^*&v_4^*&\end{pmatrix},$$

where v_α,ψ are complex radial functions, and ϕ_i are real radial functions. The constraint $\mathrm{Tr}\Phi_0=0$ may be put in either by hand or by adding a Lagrange-multiplier term $\lambda\sum_{i=1}^4\phi_i$ to the energy expression (4.1). The generalization of the forms (4.12) to a general embedding $\vec{\mathrm{T}}$ in a general SU(N) group is clear: With respect to the block structure defined by the equal eigenvalues of T_3, the matrix $\Phi_0(r)$ can have arbitrary matrix elements within the blocks, while $M_\pm$ can have matrix elements only between adjacent blocks differing by one unit of T_3. It is then a trivial matter to compute $\vec{\mathrm{B}}$ and $\vec{\mathrm{D}}\phi$ from (4.4) and insert them into the energy density in (4.1). The field equations are obtained by varying with respect to the functions v_α, ϕ_i, and ψ. The asymptotic boundary conditions are just those of the possible point monopoles in Sec. III. If the point monopole in the Abelian gauge is given by the pair $\vec{\mathrm{I}},\Phi_0$ then we find

$$\begin{aligned}M_\pm(\infty)&=I_\pm\,,\\ \Phi_0(\infty)&=\Phi_0\,.\end{aligned} \tag{4.13}$$

In terms of the matrices (4.12) this means that ψ vanishes, and ϕ_i, v_α are as given in Table I.

For the chosen example the energy density derived in this way is a functional of thirteen real variables. However, there is some residual gauge invariance in the problem, namely the little group of r-independent transformations which commute with $\vec{\mathrm{L}}+\vec{\mathrm{T}}$. On the z axis this is just the little group of T_3, and the fields $\Phi_0(r), M_\pm(r)$ afford a representation of this group under which the energy density (4.1), (4.4) is invariant. In the present example this little group is $\mathrm{SU}(2)\times\mathrm{U}(1)\times\mathrm{U}(1)$ and may be used to eliminate five of the thirteen variables from the formalism. This result is really just a theorem of classical mechanics, and we shall express it as such, though the language will be noticeably quantum mechanical. Consider a classical system with generalized coordinates x and Lagrangian $L(x,\dot x,t)$. Suppose that the x's afford a linear faithful representation of some n-parameter continuous group of constant transformations U which leave $L(x,\dot x,t)$ invariant, i.e.,

$$L(Ux,U\dot x,t)=L(x,\dot x,t)\,. \tag{4.14}$$

We remark that since this is a purely algebraic relation, it holds even if U is a function of the time. Now the symmetry under $x\to Ux$ leads to n conserved quantities K_i given by

$$K_i=-i\frac{\partial L}{\partial\dot x}\;Y_ix\,, \tag{4.15}$$

where the Y_i are the generators of the group. Suppose that we are interested in the particular type of solution for which the coordinates x are specified at some initial time t_1 and the action $\int_{t_1}^{t_2}L\,dt$ is stationary with respect to arbitrary variations. At t_2 we then have the natural boundary conditions $\partial L/\partial\dot x=0$, so the motion is such that the conserved quantities K_i vanish. Let us now exchange the coordinates x for a new set q,θ defined by

$$x=U(\theta)x_0(q)\,, \tag{4.16}$$

where the "invariant variables" q label the equivalence classes of configurations (orbits) under action of the group, $x_0(q)$ are specified functions giving a representative x_0 of each class q, and the n "group variables" θ represent the parameters of the group. Substituting the expression (4.16) into the Lagrangian and using (4.14) we obtain a new Lagrangian L' which is a function of the new coordinates:

$$L'(q,\dot{q},\theta,\dot{\theta},t)=L(x_0,\dot{x}_0-i\omega x_0,t)\,, \tag{4.17}$$

where $\dot{x}_0$ and $\omega\equiv\omega_i Y_i$ are defined by

$$\begin{aligned}\dot{x}_0&=\frac{\partial x_0}{\partial q_j}\dot{q}_j\,,\\ \omega&=iU^{-1}\dot{U}\\ &=iU^{-1}(\theta)\frac{\partial U}{\partial\theta_i}\dot{\theta}_i\,.\end{aligned} \tag{4.18}$$

Thus the dependence of the new Lagrangian L' on the group variables θ is only via the combination ω, i.e., $L'=L'(q,\dot{q},\omega,t)$. In order that the action constructed from the original Lagrangian L be stationary with respect to arbitrary variations of x, it is certainly sufficient that the action constructed from L' be stationary with respect to arbitrary variations of q,ω. The Euler-Lagrange equations for the ω_i are then just constraint equations $\partial L'/\partial\omega_i=0$, and may be used to eliminate ω from the Lagrangian, leaving an expression $L''(q,\dot{q},t)$ containing only the invariant variables q. Note that $\partial L'/\partial\omega_i$ may be expressed as

$$\begin{aligned}\frac{\partial L'}{\partial\omega_i}&=-i\frac{\partial L}{\partial\dot{x}}U(\theta)Y_iU^{-1}(\theta)x\\ &=-iR^{-1}{}_{ij}(\theta)\frac{\partial L}{\partial\dot{x}}Y_jx\\ &=R^{-1}{}_{ij}(\theta)K_j\,,\end{aligned} \tag{4.19}$$

where $R(\theta)$ is the adjoint representation of the group. Thus the solutions with $\partial L'/\partial\omega_j=0$ are just those which correspond to $K_i=0$. In the case where the group is Abelian these results may be phrased in more familiar language. If we write $U(\theta)=\exp(-i\theta_iY_i)$ then Eq. (4.18) gives simply $\omega_i=\dot{\theta}_i$, so that the above analysis reduces to the statement that the θ_i are ignorable coordinates.[15] The standard procedure in such cases is the Routhian method of performing a partial Legendre transformation which eliminates the variables $\dot{\theta}_i$ in favor of the canonical momenta $p_i\equiv\partial L'/\partial\dot{\theta}_i$, and then setting the p_i equal to constants. Using (4.19) we see that the p_i are just the conserved quantities K_i, since the adjoint representation of an Abelian group is trivial. When the p_i vanish the Routhian method is equivalent to using the constraint equation $\partial L/\partial\dot{\theta}_i=0$ to remove the variables $\dot{\theta}_i$ from the Lagrangian. In the non-Abelian case it is impossible to find coordinates such that the conserved quantities K_i are all canonical momenta, and there is no simple analog of the general Routhian procedure. However, as shown above, the particular motions for which the K_i vanish can be handled in exactly the same way as in the Abelian case. Note that for the purpose of calculating the action $\int_{t_1}^{t_2}L\,dt$, the reduced problem with the Lagrangian $L''(q,\dot{q},t)$ is sufficient; in order to determine the full motion of the system one must compute $\omega_i(t)$ from $\partial L'/\partial\omega_i=0$, and obtain $U(t)$ by solution of $U^{-1}\dot{U}=-i\omega$:

$$U(t)=U(t_1)T\exp\left[-i\int_{t_1}^{t}\omega(t')dt'\right]\,, \tag{4.20}$$

where T denotes a time ordering.

The application of this result to the monopole problem is clear. The situation is entirely analogous in that we are interested in minimizing the energy with specified boundary conditions at infinity but free boundary conditions at $r=0$. (Actually the singularity of the energy density as $r\to0$ forces some of the variables to take fixed values there; however, this imposes no constraint on the group variables.) In the case of the $4\to3+1$ embedding in SU(4) the method allows one to reduce the problem from thirteen to eight coupled nonlinear equations; the details are straightforward but complicated, and we do not give them here.[16] In simpler cases where T_3 has no repeated eigenvalues and its little group is Abelian, there are no variables analogous to ψ in (4.12); the group variables θ are just the phases of the quantities v_α, while the invariant variables q are the ϕ_i and the magnitudes of the v_α. The equations $\partial L/\partial\theta_i'$ then always give $\theta_i'=0$, and the overall effect is that without loss of generality the v_α may be considered as real. Let us consider in detail the $3\to2+1$ embedding in SU(3), an example which includes the 't Hooft-Polyakov SU(2) monopole as a special case. In terms of the usual Gell-Mann[17] λ matrices we have $T_i=\frac{1}{2}\lambda_i$, $i=1,2,3$, and the ansatz analogous to (4.12) may be written

$$\begin{aligned}\Phi_0(r)&=\phi_1(r)\frac{\lambda_3}{2}+\phi_2(r)\frac{\lambda_8}{2}\,,\\ M_+(r)&=v(r)(\lambda_1+i\lambda_2)/2\,,\\ M_-(r)&=v^*(r)(\lambda_1-i\lambda_2)/2\,.\end{aligned} \tag{4.21}$$

Writing $v=ue^{i\theta}$, the energy expression becomes

$$E=4\pi\int_0^\infty dr[(u')^2+u^2(\theta')^2+\tfrac{1}{2}(1-u^2)^2/r^2+\tfrac{1}{2}r^2(\phi_1')^2+\tfrac{1}{2}r^2(\phi_2')^2+u^2\phi_1{}^2+r^2V(\phi_1,\phi_2)]\,, \tag{4.22}$$

where the gauge coupling e has been set equal to unity. As claimed, the variable θ is an ignorable coordinate and the constraint equation $\partial L/\partial\theta'=0$ gives simply $\theta'=0$; it is also clear that for a given choice of u, ϕ_1, and ϕ_2 the energy is minimized when θ' vanishes. Thus θ is constant and without loss of generality may be set equal to zero. Although our argument is somewhat different, this is the same conclusion as reached in the analysis of SU(3) monopoles by Corrigan, Olive, Fairlie, and Nuyts. The analogous result for the SU(2) 't Hooft-Polyakov monopole is obtained by omitting the field ϕ_2 from Eqs. (4.21), (4.22). The ansatz for this simplest case therefore requires only two real functions, one for the vector field and one for the scalar field.

Observe that in the above SU(3) example T_3 is $\text{diag}(\frac{1}{2},-\frac{1}{2},0)$ so that its little group is U(1) × U(1) generated by λ_3 and λ_8, but there is only one ignorable coordinate θ. This is an example of something which always happens when T_3 has both integer and half-integer eigenvalues: There are no quantities v_α connecting the two sectors, and the fields $M_\pm(r)$, $\Phi_0(r)$ do not afford a faithful representation of one of the U(1) factors of the little group of T_3. Thus only $n-1$ of the n parameters of this group may be used to reduce the number of variables. The overall conclusion is that, starting with the most general spherically symmetric ansatz (including a radial component of $\vec{A}$), the little group of the spherically symmetric solutions allows one to eliminate a total of either $2n-1$ or $2n$ parameters, depending on whether or not T_3 has both integer and half-integer eigenvalues.

V. DISCUSSION

The theorems of Sec. II and the specialization to SU(N) groups in Sec. III show that the spherically symmetric point monopoles are essentially kinematic objects, and that it is not necessary to construct the full dynamical equations in order to compute them. In particular the diagrammatic method for SU(N) groups allows one to obtain all the solutions by inspection.

Our technique for the finite-energy ansatz in Sec. IV has the great advantage that all SU(N) groups and all embeddings $\vec{T}$ are treated in the same way, whereas the usual method involves the explicit tensor structure of the various terms, and makes it appear that different cases are unrelated to each other. While our formalism is undoubtedly rather heavy for the simplest SU(2) 't Hooft-Polyakov monopole, the discussion at the end of Sec. IV does show why it is necessary to consider only the term $\vec{T}\times\hat{r}$ in the ansatz for the vector field, and not the terms $\hat{r}\times(\vec{T}\times\hat{r})$ and $(\hat{r}\cdot\vec{T})\hat{r}$. The usual parity argument had left open the possibility that these terms could lower the energy, but this is now excluded.

Of course the most important property of magnetic monopoles in spontaneously broken gauge theories is their topological stability.[9,18] This property has been completely ignored in our discussion, since it has little role in the actual construction of solutions to the Yang-Mills equations. The topological argument ensures that for each topological configuration of the scalar field at infinity there is a state of lowest energy, but it does not guarantee the existence of a finite-energy solution with the asymptotic boundary conditions of a given spherically symmetric point monopole. Indeed, in a recent calculation using a spherically symmetric ansatz for the $4\rightarrow 4$ embedding in SU(4), Wilkinson[14] found no such solution for a particular choice of asymptotic boundary condition. The requirement of topological stability does indirectly impose some constraints on the point monopoles, since if the scalar field configurations are to be identified with the elements of the second homotopy group $\pi_2(G/H)$, where H is the unbroken subgroup, then the potential $V(\Phi)$ must have the property that its minimum is uniquely determined up to a gauge transformation. For SU(N) groups, this is true if we consider the most general quartic potential $V(\Phi)$. As shown by Li,[19] minimization of such a potential always leads to a vacuum expectation value Φ_0 with just two distinct eigenvalues, so that the symmetry is broken to $SU(n_1)\times SU(n_2)\times U(1)$, with $n_1+n_2=N$. The topological conservation law is then equivalent to conservation of the U(1) component of the magentic charge.[18,20,14] To see what this means for the point solutions, consider the $3+1$ embedding in SU(4). By examining the form of the scalar fields in Table I, we see that for $n_1=n_2=2$ solution (a) is absent, while for $n_1=3$, $n_2=1$ solution (b) is absent. Similar restrictions obtain in other cases.

In this paper, then, we have given a universal prescription to find the spherically symmetric point monopoles for arbitrary groups, and a technique for efficient construction of minimum-energy spherically symmetric solutions for SU(N) groups. Although it seems very plausible that the lowest-energy configuration for a given topology is spherically symmetric, it remains to be proven whether such solutions are even local minima of the energy when variations outside the spherical symmetry are considered.[21] Perhaps our techniques could be helpful in attempting to answer such questions.

*Work supported in part by the National Science Foundation under Grant No. PHY-76-15328.

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[21]The only progress in this direction has been the demonstration by S. Coleman, S. Parke, A. Neveu, and C. Sommerfield [Phys. Rev. D 15, 544 (1977)] that the exact Prasad-Sommerfield [M. K. Prasad and C. Sommerfield, Phys. Rev. Lett. 35, 760 (1975)] solution in SU(2) saturates the topological lower bound on the energy in the limit of vanishing scalar potential. A similar result has been obtained by Wilkinson (Ref. 14) for two of the topological configurations in SU(4) broken to SU(2)×SU(2)×U(1). These results imply at least neutral stability of the spherically symmetric solutions in the absence of the scalar potential.