Bound-state effective potential formulation of dynamical symmetry breaking

Richard W. Haymaker
Department of Physics and Astronomy, Louisiana State University, Baton Rouge, Louisiana 70803
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We propose an approximation procedure for studying dynamical symmetry breaking that closely parallels scalar field models of spontaneous symmetry breaking. We focus our attention on the role a deep scalar bound state plays in effecting a phase transition. We show that a viable approximation to the effective potential must contain all one-particle-reducible bound-state pole structures. This dictates a Dyson equation for the self-energy even in the simplest approximation. For theories with 4-field interactions this can reduce to a closed-form Hartree approximation. We look at trilinear interactions where the Dyson integral equation is intractible because of its nonlinearity. Without linearizing the Dyson equation we extract a bound-state contribution to the effective potential. We end up with a generalized effective potential that is a function of classical fields representing the bound state. We show that this contribution displays the proper phase transition when the theory becomes unstable due to a composite tachyon.

I. INTRODUCTION

Spontaneous symmetry breaking in theories with no scalar fields can occur via the effect known as dynamical symmetry breaking.[1-3] The asymmetry of the vacuum manifests itself through the nonvanishing vacuum expectation value of a composite operator rather than an elementary field. In quark–vector-gluon models of hadrons it is presumably this effect that is responsible for the breaking of chiral symmetry and gauge symmetries. It is also an important option in weak-interaction model building.[4] The implementation of this idea in realisitic models poses difficult problems. The symmetry breaking arises from an instability in the normal vacuum caused by driving a scalar bound state carrying quantum numbers to negative squared mass, analogous to choosing a negative-squared-mass Lagrangian parameter in scalar field models. In this paper we focus our attention on the bound state and propose an approximation scheme that displays the desired phase transition in hopes of making studies of dynamical symmetry breaking more tractable.

A model of dynamical symmetry breaking must contain a deep bound state, and hence this is a strong-coupling situation with all its ensuing problems. Further, the redefinition of the vacuum is based on a presumed nonperturbative solution of a nonlinear Dyson equation for the matrix elements of the composite operator (e.g., for chiral symmetry one looks for solutions of the Dyson equation for the self-energy that give a nonzero quark mass even though an iterative solution would give zero). The prototype model of this effect is that by Nambu and Jona-Lasinio[1] (NJL), but it side-steps both problems. The kernel of their Bethe-Salpeter equation is just a four-point coupling, making it separable and hence soluble. Because of the separability, the nonlinear Dyson equation becomes an algebraic equation (the gap equation). Although the model displays these interesting effects in a simple way, the binding after all is due to a contact interaction. The binding of quarks to produce hadrons is presumably due to strong forces at large distances. If we model this even by single gluon exchange we are back to the nonlinear-integral-equation difficulties.

The complexity of this is in sharp contrast to studies of spontaneous symmetry breaking in scalar field models. There one can calculate the effective potential $V(\phi)$ to a desired order and survey it for local minima, choosing the lowest one to define the true vacuum. We feel it would be highly desirable to have a formulation of dynamical symmetry breaking that closely parallels scalar-field models where possible. We feel that by studying the role of the bound states we can find approximations that separate the ponderous integral equations from symmetry considerations to make it more amenable to phenomenological analysis.

Basic questions to ask to this end are the following: What approximation to a given field theory is complete enough to (i) generate a bound state (if the theory has one), and (ii) provide a stable vacuum (via a phase transition) even if the bound-state mass m_B^2 goes negative. Such an approximation is a candidate for dynamical symmetry breaking. Clearly for (i) perturbation theory is inadequate, but a ladder approximation or bubble sum may work. For (ii) neither of these is sufficient but a Hartree approximation may work. This is clear from recent work on the O(N) σ model.[5-7] We showed[6,7] on general grounds that as a bound-state mass m_B^2 approaches zero, the effective potential $V(\phi)$ develops a branch point in ϕ, that there is another minimum on the second branch, and that the minimum drops below the old minimum as

$m_B{}^2$ goes negative. We further showed that the large-N limit of the O(N) σ model (which is a Hartree approximation) is complete enough to display this effect. The bound state is an O(N) singlet, and hence no symmetry is broken. However, the approximation is identical to the NJL[1] model in which the bound state is a chiral doublet and chiral symmetry is broken. Both models have a four-point interaction term, a bound state from a bubble sum, and are solved in the Hartree approximation.

In this paper we generalize these results to the case in which the binding is due to particle exchange rather than a contact interaction. An example would be a $\bar{q}\gamma_\mu q v^\mu$ vector-gluon–quark model. However, we stick to a scalar version, $\phi^2\sigma$, of this coupling. The reasons are simplicity in formalism and renormalization, to keep close contact with the O(N) σ model, and because we feel the appropriate approximations can be found this way. We examine the simplest approximation that displays the desired effect. (By "simplest approximation" we mean the minimal subset of Feynman graphs.) To find the effective potential one must still solve a nonlinear Dyson equation which is intractable. However, we show how to isolate the bound-state contribution to the effective potential without neglecting the nonlinear effects. We further show that this contribution is responsible for the phase transition where $m_B{}^2$ goes negative. We end up with an approximate effective potential that is very similar in form to that of the O(N) σ model for large N.

We are using a model in this paper that is unstable, i.e., the $\phi^2\sigma$ interaction. $V(\phi,\sigma)$ almost certainly has no lower bound for large ϕ and σ. However, we are interested only in small ϕ and σ and how the choice of subsets of Feynman graphs dictate a phase transition there. The same Feynman-graph topology is applicable to trilinear coupling theories that probably are stable such as the $\bar{q}\gamma_\mu q v^\mu$ theories. The O(N) σ model in the large-N limit is also unstable in that the real part of $V(\phi)$ goes to $-\infty$ for large ϕ. However, it still has interesting small-ϕ behavior.

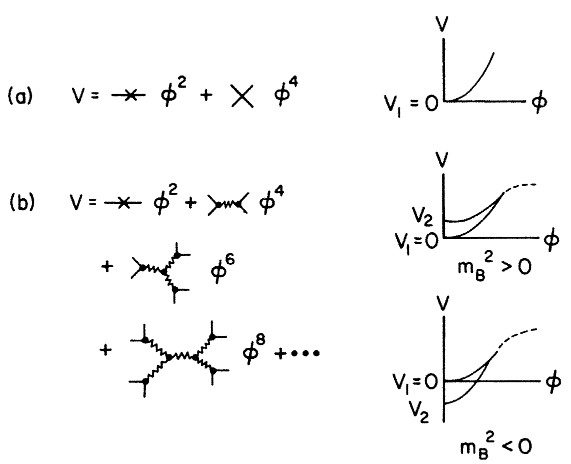


FIG. 1. Schematic behavior of the effective potential (a) for weak coupling, (b) for the case in which deep bound state (mass m_B) exists. For case (b) the effective potential has two branches as shown. As $m_B{}^2$ goes negative the minimum on the second branch, V_2, drops below the original minimum, V_1.

In Sec. II we make a heuristic connection between dynamical approximations and a phase transition. It is partly a review of earlier work[6,7] and is intended to motivate the formalism that follows. In Sec. III we give an implicit expression for the effective potential, V. This is taken from a paper by Cornwall, Jackiw, and Tomboulis[3] and adapted to our problem. In Sec. IV we extract the bound-state contribution to V. In Sec. V we show that the resulting V has a phase transition, and we compare it to results in the O(N) σ model.

II. DEEP BOUND STATES AND PHASE TRANSITIONS

We would like to review briefly how a deep bound manifests itself in the effective potential and how a phase transition can occur when the bound-state mass, $m_B{}^2$, goes negative.[6,7] Our intention is to show in a simple heuristic way how these considerations lead to the standard models of dynamical symmetry breaking. If a theory has a deep bound state, it is possible to isolate the bound-state contribution to the effective potential. It is then easy to see what subsets of Feynman graphs will produce this contribution and hence what sets allow for a phase transition where $m_B{}^2$ goes negative.

Let us consider a massive ϕ^4 theory. For weak coupling, the effective potential reduces to the tree approximation:

$$V_{\rm tree}(\phi) = \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4 . \tag{2.1}$$

If λ and m^2 are positive this potential has a minimum at $\phi = 0$ as indicated in Fig. 1(a). Let us assume that the coupling is increased and that a deep bound state of two ϕ's is formed.[8] Even if we cannot calculate $V(\phi)$ for strong coupling, one can argue that the bound-state contribution dominates for $m_B{}^2$ sufficiently small. To see this we note the following: $V(\phi)$ is the generating function for one-ϕ-particle-irreducible n-point functions at zero momentum $\Gamma^{(n)}(0,\ldots,0)$. The $\Gamma^{(n)}(\{P_i\})$ do not have ϕ poles but do have bound-state poles. If we isolate the pole structure in $\Gamma^{(n)}(\{P_i\})$ with the maximum number of poles, then at zero momentum it will have the maximum power of $1/m_B{}^2$. Since we are interested in small $m_B{}^2$, this will give the leading bound-state contribution to $V(\phi)$.[9] In Ref. 6 we isolated this contribution as shown in Fig. 1(b) and it leads to the following expression for V:[10]

$$V(\phi)=\frac{m^2}{2}\phi^2-\frac{m_B^6}{\gamma^2}\xi^2\sum_{i=0}^{\infty}\frac{(2i)!\,\xi^i}{2^{2i+2}i!\,(i+2)!}, \quad (2.2)$$

where

$$\xi=\frac{\beta\gamma}{m_B^2}\phi^2, \quad (2.3)$$

and β is the bound-state–2ϕ coupling, γ the 3-bound-state coupling. Summing the series gives

$$V(\phi)=\frac{m^2}{2}\phi^2-\frac{m_B^6}{6\gamma^2}[3\xi-2+2(1-\xi)^{3/2}]. \quad (2.4)$$

This function has a branch point at $\phi^2=m_B^2/\beta\gamma$ and is complex for ϕ^2 larger than this value. The important features of this result[6,7] are that (i) there is a second real branch to $V(\phi)$ [see Fig. 1(b)], (ii) the second branch has a minimum at $\phi=0$ [V_2 in Fig. 1(b)], and (iii) the minimum V_2 drops below the old minimum V_1 as m_B^2 goes negative. Hence, as the theory becomes unstable due to a composite tachyon, a new vacuum state at lower energy appears. This bound-state contribution to $V(\phi)$ is sufficient to show the effect. One can further show that in the new vacuum the composite particle has positive mass.

Let us contrast this with the situation in which the small-mass particle is elementary with a corresponding field χ. Then the effective potential would be a function of the two fields ϕ and χ of the form

$$V(\phi,\chi)=\frac{m^2}{2}\phi^2+\frac{m_B^2}{2}\chi^2+\frac{\beta}{2}\phi^2\chi+\frac{\gamma}{3!}\chi^3+\cdots. \quad (2.5)$$

For $m_B^2>0$ the minimum of V is at $\phi=\chi=0$. For $m_B^2<0$ the minimum shifts in the usual way to $\phi=0$, $\chi\neq 0$, defining a new vacuum with positive mass excitations.

Any difference between the bound-state and elementary-field cases in this discussion is illusory. The two expressions for V—Eqs. (2.4) and (2.5)—are simply related by the constraint

$$V(\phi)=V(\phi,\chi(\phi)), \quad (2.6)$$

where $\chi(\phi)$ is given by the equation

$$\frac{\partial V(\phi,\chi)}{\partial\chi}=0. \quad (2.7)$$

This constraint procedure to eliminate χ simply sums up the one-χ-particle-*reducible* pole terms that must be present in $V(\phi)$ if χ is not an elementary field. Clearly all the stationary points of $V(\phi,\chi)$ must lie on the constraint Eq. (2.7). Hence, all the stationary points of $V(\phi,\chi)$ are given by the solution to the equation $dV(\phi)/d\phi=0$. For either case the gross features of the new vacuum are determined by the couplings of ϕ and χ.

In this discussion so far nothing has been said about symmetry. The generalization is clear: The effective potential Eq. (2.5) would become an invariant function of ϕ_i and χ_a, and the constraints would read

$$\frac{\partial V(\phi_i,\chi_a)}{\partial\chi_b}=0. \quad (2.8)$$

The vacuum is defined by the stationary points of $V(\phi_i,\chi_a)$ and the Goldstone phenomenon follows exactly as in the χ_a field case.

In looking for models that display this type of phase transition it is clear that the effective potential must contain the pole structure shown in Fig. 1(b). An example is the Hartree approximation.[5] In ϕ^4 theory this approximation consists in summing graphs of the type shown in Fig. 2(a). (These are vacuum graphs containing couplings to external classical field ϕ.) The characteristic feature of this set of graphs is that the bubbles are hooked together to give the topology of trees. (There are no "loops of bubbles.") One can sum the chains of bubbles and if there is a bound state in the chain it will lead to a pole structure shown in Fig. 2(b). This is just what we need[11] according to the above discussion.

An example of this model with an internal symmetry is the O(N) σ model. However, the bound state is an O(N) singlet and hence no symmetry is

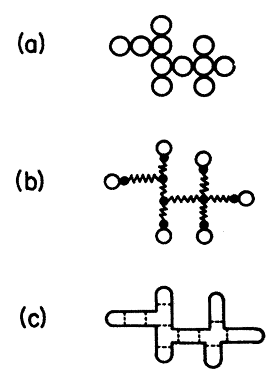


FIG. 2. (a) A typical vacuum graph that occurs in the Hartree approximation for theories with four-field interactions. These are characterized by bubbles hooked together to form trees. (b) Summing the chains of bubbles can form bound states denoted by wavy lines. (c) Corresponding graph in the generalization of the Hartree approximation to theories with trilinear interactions. This also has the bound-state structure (b) arising from a ladder sum.

broken by the phase transition. The Nambu–Jona-Lasinio[1] model has a four-fermion interaction and sums the same set of graphs. Here the bound state is a chiral doublet and chiral symmetry is spontaneously broken.

This phase transition occurs in ϕ^4 theory in a particulary simple approximation, and one can get a closed form for $V(\phi)$. The reason is that the Bethe-Salpeter kernel that generates the chain of bubbles is just a contact interaction which is separable. The momentum integrals for each bubble factor. The simplest generalization to a $\phi^2\sigma$ theory can be obtained by replacing the contact interaction by 1σ exchange as shown in Fig. 2(c).[12] If we sum the ladders and there is a bound state we will again generate the desired pole structure. The problem is that now we must solve a nonlinear integral equation to obtain $V(\phi)$. The class of graphs are those with one ϕ loop and planar σ corrections across the interior of the loop. This set of graphs is the same as the standard quark–vector-gluon models of dynamical symmetry breaking with $\phi \to q$, $\sigma \to V^{\mu 2}$ as we will show later.

A closed-form solution even in this approximation appears to be out of the question. However, the discussion in this section suggests a further approximation that will at least preserve the phase transition and that is to do a spectral decomposition of the ladder sum and keep only the lowest-lying bound state. This approximation to the ladder sum is separable simply because the pole factorizes. The remainder of this paper develops this idea. We can get a closed form for the effective potential that displays the phase transition in terms of (unknown) wave function which is very similar to the ϕ^4 Hartree approximation.

III. VARIATIONAL PRINCIPLE FOR V

The qualitative arguments of the previous section suggest a class of graphs in $\phi^2\sigma$ theory that will lend to a phase transition if the exchange of a σ between two ϕ's produces a deep bound state. The effective potential graphs consist of the sum of all vacuum graphs with one ϕ loop and planar σ lines connecting across the interior of the loop. We can get an expression for $V(\phi,\sigma)$ in this approximation from a paper by Cornwall, Jackiw, and Tomboulis[3] on the generating functional for composite operators. Their generalized effective potential is a stationary functional of the one-particle Green's function $G(P)$. The stationary condition gives a Dyson equation for G. In this section we apply their results to our problem.

We will give this generalized effective potential for the theory defined by the Lagrangian

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \tfrac{1}{2}m_0^2\phi^2 + \tfrac{1}{2}(\partial_\mu\sigma)(\partial^\mu\sigma) - \tfrac{1}{2}\mu^2\sigma^2 + \tfrac{1}{2}g\,\phi^2\sigma\,. \tag{3.1}$$

Let us introduce a matrix index $\Phi_1=\phi$, $\Phi_2=\sigma$. We then replace Φ by $\Phi+\Phi_c$, where Φ_c is a translation-invariant classical field. Treating it as a parameter we can defined a modified mass term: $-\frac{1}{2}\Phi_i M^2{}_{ij}\Phi_j$, where

$$M^2{}_{ij} = \begin{pmatrix} m_0^2 - g\sigma_c & g\phi_c \\ g\phi_c & \mu^2 \end{pmatrix}. \tag{3.2}$$

The modified propagator is

$$i\mathfrak{D}^{-1} = \begin{pmatrix} P^2 - m_0^2 - g\sigma_c & -g\phi_c \\ -g\phi_c & P^2-\mu^2 \end{pmatrix}. \tag{3.3}$$

The ordinary propagator is D:

$$iD^{-1} = \begin{pmatrix} P^2 - m_0^2 & 0 \\ 0 & P^2-\mu^2 \end{pmatrix}. \tag{3.4}$$

With these definitions, the generalized effective potential is[3]

$$V(\Phi_c,G) = U(\Phi_c) - \frac{i}{2}\int\frac{d^4P}{(2\pi)^4}\ln\det[D(P)G^{-1}(P)] - \frac{i}{2}\int\frac{d^4P}{(2\pi)^4}\mathrm{Tr}[\mathfrak{D}^{-1}(P)G(P)-1] + V_2(\Phi_c,G)\,, \tag{3.5}$$

where

$$U(\Phi_c) = \frac{m_0^2}{2}\phi_c^2 + \frac{\mu^2}{2}\sigma_c^2 - \frac{g}{2}\phi_c^2\sigma_c\,, \tag{3.6}$$

and V_2 is the sum of all two particle-irreducible vacuum graphs in which the propagator is G (not the free propagator). V is a function of Φ_c and a functional of G. G is arbitrary. It is fixed, however, by the stationary condition on V:

$$\frac{\delta V}{\delta G(P)} = 0 \Rightarrow G = G\,|_{\text{stat}}\,. \tag{3.7}$$

The ordinary effective potential $V(\phi)$ is then

$$V(\Phi_c) = V(\Phi_c, G\,|_{\text{stat}})\,. \tag{3.8}$$

$G\,|_{\text{stat}}$ is a function of Φ_c through Eq. (3.7). To obtain the Green's function of the theory we further demand that $V(\Phi_c)$ is stationary in Φ_c:

$$\frac{dV(\Phi_c)}{d\Phi_c} = 0 \Rightarrow \Phi_c = \Phi_c\,|_{\text{stat}}\,. \tag{3.9}$$

Using the stationary value of Φ_c, G_{stat} is the full one-particle Green's function of the theory.

This formulation, Eq. (3.5) to Eq. (3.8) is an alternative definition of the effective potential. The advantage of this form is that one can truncate $V_2(\Phi,G)$ and obtain interesting approximations. G

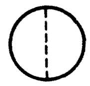
V_2 = 

FIG. 3. Lowest-order two-particle-irreducible vacuum graph used as the driving term in the variational principle for V.

is no longer the full one-particle Green's function, but contains prescribed corrections. For any choice of the driving term V_2, this procedure sums up the one-particle-irreducible graphs. Cornwall, Jackiw, and Tomboulis[3] show that the lowest-order driving term gives the Hartree approximation in the O(N) σ model. They also show that one can get dynamical symmetry breaking of chiral symmetry in a quark–vector gluon model. Our interest is to use this to extract a bound-state contribution to V which contains the desired phase transition.

We will choose the lowest-order graph for the driving term V_2 which is shown in Fig. 3. This graph is two-particle-irreducible. We will not look at the Green's function with σ external lines or possible vacuums for nonzero σ_c, and hence we set $\sigma_c = 0$. Equation (3.5) then becomes

$$V(\Phi,G)=\frac{m_0^2}{2}\Phi^2-\frac{i}{2}\int\frac{d^4P}{(2\pi)^4}\ln\det[D(P)G^{-1}(P)]-\frac{i}{2}\int\frac{d^4P}{(2\pi)^4}\mathrm{Tr}[\mathfrak{D}^{-1}(P)G(P)-1]-\frac{i}{4}g^2\int\frac{d^4P}{(2\pi)^4}\int\frac{d^4q}{(2\pi)^4}G_{11}(P)\times D_{22}(P-q)G_{11}(q)\,. \tag{3.10}$$

(We have dropped the c subscript on Φ and will do so below.) Taking the variational derivative and setting it equal to zero gives the following:

$$iG^{-1}(P)_{11}=i\mathfrak{D}^{-1}(P)_{11}+ig^2\int\frac{d^4q}{(2\pi)^4}D_{22}(P-q)G_{11}(q)\,, \tag{3.11a}$$

$$iG^{-1}(P)_{12}=i\mathfrak{D}^{-1}(P)_{12}\,, \tag{3.11b}$$

$$iG^{-1}(P)_{22}=i\mathfrak{D}^{-1}(P)_{22}\,. \tag{3.11c}$$

One stationary point of V has $\Phi=0$. There $\mathfrak{D}$ becomes the ordinary propagator D and there is no mixing, and Eq. (3.11a) is a truncated Dyson equation for the propagator. For $\Phi\neq 0$, the equations couple but can be reduced to a single equation for G_{11} say: Define

$$\Sigma(P)=-ig^2\int\frac{d^4q}{(2\pi)^4}D_{22}(P-q)G_{11}(q)\,, \tag{3.12}$$

then

$$iG^{-1}=\begin{pmatrix}P^2-m_0^2-\Sigma & g\phi\\ g\phi & P^2-\mu^2\end{pmatrix} \tag{3.13}$$

and

$$G_{11}=i\left(P^2-m_0^2-\Sigma-\frac{g^2\phi^2}{P^2-\mu^2}\right)^{-1}. \tag{3.14}$$

Hence the desired integral equation for G_{11} is

$$\frac{i}{G_{11}}=P^2-m_0^2-\frac{g^2\phi^2}{P^2-\mu^2}+ig^2\int\frac{d^4q}{(2\pi)^4}D_{22}(P-q)G_{11}(q)\,. \tag{3.15}$$

For convenience we define χ as follows:

$$P^2-\chi=P^2-m_0^2-\Sigma-\frac{g^2\phi^2}{P^2-\mu^2}=\frac{i}{G_{11}}\,. \tag{3.16}$$

Then

$$V(\phi,\chi)=\frac{m_0^2}{2}\phi^2-\frac{i}{2}\int\frac{d^4P}{(2\pi)^4}\ln\left(\frac{P^2-\chi(P)}{P^2-m_0^2}\right)-\frac{i}{2}\int\frac{d^4P}{(2\pi)^4}\frac{P^2-m^2-g^2\phi^2/(P^2-\mu^2)}{P^2-\chi(P)}+\frac{1}{4}g^2i\int\frac{d^4P}{(2\pi)^4}i\int\frac{d^4q}{(2\pi)^4}\frac{1}{P^2-\chi(P)}\frac{1}{(P-q)^2-\mu^2}\frac{1}{q^2-\chi(q)}\,. \tag{3.17}$$

$\delta V/\delta\chi=0$ gives

$$\chi(P)=m_0^2+\frac{g^2\phi^2}{P^2-\mu^2}+g^2i\int\frac{d^4q}{(2\pi)^4}\frac{1}{(P-q)^2-\mu^2}\frac{1}{q^2-\chi(q)}\,. \tag{3.18}$$

This theory is superrenormalizable and only requires a mass counterterm:

$$m_0^2=m^2-ig^2\int\frac{d^4q}{(2\pi)^4}\frac{1}{(q^2-M^2)^2}\,. \tag{3.19}$$

M is the renormalization mass and can be chosen as one pleases. Our approximation does not generate self-energy for the σ and hence μ^2 requires no counterterm. $V(\phi,\chi)$ is not yet finite but the divergence is contained in $V(0,\chi_{\phi=0})$, where $\chi_{\phi=0}$ is the solution to Eqs. (3.18), (3.19) with $\phi=0$, as will be discussed below. The Green's function generated by $V(\phi,\chi)$ are all finite.

IV. BOUND-STATE APPROXIMATION

In this section we show how to isolate a bound-state contribution to $V(\phi,\chi)$. The organization is as follows: We first look at the four-point function defined from V and see that it satisfies a Bethe-Salpeter equation:

$$T = g^2 K + g^2 KGT\,, \tag{4.1}$$

where K is a σ exchange kernel and G is the product of two ϕ propagators (containing self-energy corrections). We assume the homogeneous equation has a solution

$$f_1 = g_1^2 KGf_1 \tag{4.2}$$

corresponding to a bound state at zero energy. We then project the kernel onto the state f_1. This gives a separable kernel expressed in terms of the bound-state wave function f_1. Inserting the separable kernel back into the variational principle for V we obtain the desired bound-state piece.

Let us then examine the four-point function implied by V to find the Bethe-Salpeter equation which it satisfies. We know it is a ladder approximation but we must identify the corrections to the propagators. The one-particle-irreducible four-point function at zero momentum is

$$\Gamma^4(\{P_i\})\,|_{P_i=0} = -V''''(\phi)\,|_{\phi=0}\,, \tag{4.3}$$

where a prime denotes $d/d\phi$. In Eq. (3.17), ϕ enters explicitly and implicitly in χ via the constraint Eq. (3.18). Note

$$V'(\phi) = \frac{\partial V}{\sigma\phi} + \int d^4P \frac{\delta V}{\delta\chi}\chi'\,, \tag{4.4}$$

and since the variational derviative is zero

$$V' = \frac{\partial V}{\partial\phi} = \phi\left(\chi(0) + \frac{g^2\phi^2}{\mu^2}\right). \tag{4.5}$$

Therefore,

$$V'''' = 3\left(\chi''(0) + \frac{2g^2}{\mu^2}\right) + \phi\chi'''(0)\,. \tag{4.6}$$

Specializing to $\phi = 0$ gives

$$V'''' = 3\left(\chi''(0) + \frac{2g^2}{\mu^2}\right) \tag{4.7}$$

and

$$\chi_0''(P) = \frac{2g^2}{P^2-\mu^2} + ig^2\int \frac{d^4q}{(2\pi)^4}\frac{1}{(P-q)^2-\mu^2}\frac{\chi_0''(q)}{[q^2-\chi_0(q)]^2}\,. \tag{4.8}$$

The subscript zero indicates ϕ is zero. We see that $\chi_0''(P)$ satisfies a Bethe-Salpeter equation in which the ϕ propagators are $[q^2-\chi_0(q)]^{-1}$. χ_0 is a solution of the equation

$$\chi_0(P) = m_0^2 + ig^2\int\frac{d^4q}{(2\pi)^4}\frac{1}{(P-q)^2-\mu^2}\frac{1}{q^2-\chi_0(q)}\,. \tag{4.9}$$

The inhomogeneous term in Eq. (4.8) contains a factor of 2 corresponding to the two cross channel exchanges. $V''''|_{\phi=0}$ is one-particle irreducible, and the one-particle exchanges are subtracted out of the ladder sum. The factor of 3 comes from the ladder sum in the three channels. If g is such that there is a bound state at zero energy, then the eigenvalue equation has a solution

$$f_1(P) = ig_1^2\int\frac{d^4q}{(2\pi)^4}\frac{1}{(P-q)^2-\mu^2}\frac{f_1(q)}{[q^2-\chi_0(q)]^2}\,. \tag{4.10}$$

(The subscript 1 refers to the fact that it is the first or lowest bound state.)

Returning to the variational principle for $V(\phi,\chi)$, Eq. (3.17), let us separate off the $\phi=0$ part. Define

$$\hat\chi(P) \equiv \chi(P) - \chi_0(P)\,,\qquad \hat V(\phi,\hat\chi) \equiv V(\phi,\chi) - V(0,\chi_0)\,. \tag{4.11}$$

Substituting this in Eq. (3.17) gives

$$\hat V(\phi,\hat\chi) = \frac{1}{2}\phi^2\left(\chi_0(0) + ig^2\int\frac{d^4P}{(2\pi)^4}\frac{1}{P^2-\mu^2}A(P)\right) - \frac{i}{2}\int\frac{d^4P}{(2\pi)^4}\left[\ln\left(1-\frac{\hat\chi(P)}{P^2-\chi_0(P)}\right) + \frac{\hat\chi(P)}{P^2-\chi_0(P)} + \hat\chi(P)A(P)\right]$$
$$+\frac{g^2}{4}i\int\frac{d^4P}{(2\pi)^4}\,i\int\frac{d^4q}{(2\pi)^4}A(P)\frac{1}{(P-q)^2-\mu^2}A(q)\,, \tag{4.12}$$

where

$$A(P) \equiv \frac{\hat\chi(P)}{[P^2-\chi_0(P)-\hat\chi(P)][P^2-\chi_0(P)]}\,. \tag{4.13}$$

$\hat\chi(P)$ satisfies the equation

$$\hat\chi(P) = \frac{g^2\phi^2}{P^2-\mu^2} + g^2 i\int\frac{d^4q}{(2\pi)^4}\frac{A(q)}{[(P-q)^2-\mu^2]}\,. \tag{4.14}$$

We still have a nonlinear integral equation to solve— Eq. (3.18)—which is intractable. In the discussion that follows we will not need a solution of this, but will need $\chi_0(P)$ which satisfies an equally bad equation—Eq. (4.9) which is obtained by setting $\phi=0$. However, we are interested in the ϕ dependence of V, and since χ_0 has none its detailed form does not play a role in demonstrating the phase transition. Hence, we will treat the propagator $[q^2 - \chi_0(q)]^{-1}$ as a given function of q. χ_0 also depends on g which we will be varying in the neighborhood of g_1. For definiteness let us imagine fixing g to be g_1 in this propagator.

We note here that $\hat{V}(\phi, \hat{\chi})$ is finite. $\chi_0(P)$ is finite because of the mass renormalization, Eq. (3.19). $\hat{\chi}(P) \sim 1/P^2$ for large P, as follows from Eq. (4.14). Simple power counting shows all the integrals are finite.

Next we project the kernel of the Bethe-Salpeter equation onto the lowest bound state. We can do this by replacing $[(P-q)^2 - \mu^2]^{-1}$ in Eq. (4.10) by a separable form in P and q that will give the same eigenvalue g_1^2 and eigenfunction f_1:

$$\frac{1}{(P-q)^2 - \mu^2} \to f_1(P) c f_1(q) , \tag{4.15}$$

where c is a constant determined by the above requirement:

$$f_1(P) = g_1^2 f_1(P) c i \int \frac{d^4q}{(2\pi)^4} \frac{f_1(q)^2}{[q^2 - \chi_0(q)]^2} . \tag{4.16}$$

Therefore, we should make the replacement

$$\frac{1}{(P-q)^2 - \mu^2} \to f_1(P) \frac{-1}{g_1^2 (f_1 G f_1)} f_1(q) , \tag{4.17}$$

where

$$(f_1 G f_1) \equiv -i \int \frac{d^4q}{(2\pi)^4} \frac{f_1(q)^2}{[q^2 - \chi_0(q)]^2} . \tag{4.18}$$

Substituting Eq. (4.17) in Eq. (4.14) gives

$$\hat{\chi}(P) = \frac{-f_1(P) g^2}{(f_1 G f_1) g_1^2} \times \left(\phi^2 f_1(0) + i \int \frac{d^4q}{(2\pi)^4} f_1(q) A(q) \right) . \tag{4.19}$$

Hence, we see that $\hat{\chi}(P)$ is proportional to the bound-state wave function $f_1(P)$. So let us write

$$\hat{\chi}(P) = \tilde{\chi} f_1(P) . \tag{4.20}$$

Inserting this in Eq. (4.19) gives an equation relating ϕ on $\tilde{\chi}$.

$$\tilde{\chi} g_1^2 (f_1 G f_1) = g^2 [-\phi^2 f_1(0) + \tilde{\chi} (f_1 G_{\tilde{\chi}} f_1)] , \tag{4.21}$$

where

$$(f_1 G_{\tilde{\chi}} f_1) \equiv -i \int \frac{d^4q}{(2\pi)^4} \frac{f_1(q)^2}{[q^2 - \chi_0(q) - \tilde{\chi} f_1(q)][q^2 - \chi_0(q)]} . \tag{4.22}$$

Finally let us insert the separable kernel in the equation for $\hat{V}$, Eq. (4.12), giving

$$\hat{V}(\phi, \tilde{\chi} f_1) = \frac{\phi^2}{2} \left(\chi_0(0) + \tilde{\chi} \frac{g^2}{g_1^2} \frac{(f_1 G_{\tilde{\chi}} f_1)}{(f_1 G f_1)} f_1(0) \right) - \frac{i}{2} \int \frac{d^4P}{(2\pi)^4} \left[\ln\left(1 - \frac{\tilde{\chi} f_1(P)}{P^2 - \chi_0(P)}\right) + \frac{\tilde{\chi} f_1(P)}{P^2 - \chi_0(P)} \right]$$
$$- \frac{g^2}{4} \tilde{\chi}^2 (f_1 G_{\tilde{\chi}} f_1) \frac{g^2}{g_1^2 (f_1 G_{\tilde{\chi}} f_1)} (f_1 G_{\tilde{\chi}} f_1) . \tag{4.23}$$

Using Eq. (4.21) we can eliminate $(f_1 G_{\tilde{\chi}} f_1)$ in favor of $(f_1 G f_1)$. $(f_1 G f_1)$ is the normalization integral for the eigenvalue equation in g^2, Eq. (4.10). Let us denote it by N:

$$N \equiv -i \int \frac{d^4P}{(2\pi)^4} \left(\frac{f_1(P)}{P^2 - \chi_0(P)} \right)^2 = (f_1 G f_1) . \tag{4.24}$$

Our final form for $\hat{V}$ is then

$$\hat{V}(\phi, \tilde{\chi} f_1(P)) = \tfrac{1}{2} \phi^2 (\chi_0(0) + \tilde{\chi} f_1(0)) - \frac{i}{2} \int \frac{d^4P}{(2\pi)^4} \left[\ln\left(1 - \frac{\tilde{\chi} f_1(P)}{P^2 - \chi_0(P)}\right) + \frac{\tilde{\chi} f_1(P)}{P^2 - \chi_0(P)} + \frac{1}{2} \left(\frac{\tilde{\chi} f_1(P)}{P^2 - \chi_0(P)} \right)^2 \right]$$
$$+ \frac{\phi^4 g^2}{4 g_1^2} \frac{f_1^2(0)}{N} + \frac{(g_1^2 - g^2) \tilde{\chi}^2 N}{4 g^2} . \tag{4.25}$$

This approximation has determined the P dependence of $\tilde{\chi}(P)$ to be $f_1(P)$. The variation of $\hat{V}$ with respect to $\hat{\chi}(P)$ is now reduced to the ordinary derivative of $\hat{V}$ with respect to $\tilde{\chi}$. $\partial \hat{V} / \partial \tilde{\chi} = 0$ gives

$$0 = \frac{\partial \hat{V}}{\partial \tilde{\chi}} = \frac{1}{2} \phi^2 f_1(0) + \frac{i}{2} \int \frac{d^4P}{(2\pi)^4} \frac{\tilde{\chi} f_1(P)^2}{(P^2 - \chi_0)[P^2 - \chi_0 - \tilde{\chi} f_1(P)]} + \frac{g_1^2 \tilde{\chi}}{2 g^2} N . \tag{4.26}$$

This is the gap equation which we already obtained above, Eq. (4.21), directly from the variational constraint.

To recapitulate, we started with a variational principle for V giving a nonlinear Dyson equation for $\chi(P)$. We then imagine solving the problem for $\phi = 0$ giving $\chi_0(P)$ and $V(0, \chi_0)$. Of course we cannot solve this problem but the detailed form of $\chi_0(P)$ is not needed to exhibit the phase transition and $V(0, \chi_0)$ is an irrelevant constant. In Sec. II we claimed that if there is a deep bound state, it controls the small-ϕ behavior of V. We pick out the bound-state contribution by projecting the Bethe-Salpeter kernel on the lowest bound state. As a consequence we learn that $\chi(P)=\chi_0(P)+\tilde{\chi}f_1(P)$. Our expression $\tilde{V}$, Eq. (4.25), is now an ordinary function of $\phi, \tilde{\chi}$.

The effective potential is a quantity that has a meaning prior to choosing a vacuum. Our approximation scheme is based on the properties of a particular vacuum. This does not preclude us from then surveying the effective potential for other stationary points to find other vacuums. The existence of a deep bound state in one implies a nearby stationary points with a composite tachyon. As g^2 passes g_1^2, the vacuum must shift from one point to the other.

V. INTERPRETATION OF $\hat{V}$

Our expression for $\hat{V}(\phi,\tilde{\chi})$, Eq. (4.25), has a simple diagrammatic interpretation. It is the one-ϕ-loop generating function for one-particle-irreducible vertices of ϕ's and $\tilde{\chi}$'s at zero momentum, where $\tilde{\chi}$ represents the bound state. It is almost identical in form to the effective-potential expression derived by Coleman, Jackiw, and Politzer for the O(N) σ model, Ref. 5, Eq. (2.5). There the bound-state wave function is a constant in momentum space because the Bethe-Salpeter kernel was a contact interaction, whereas the wave function shows up explicitly in our problem. If we elminate $\tilde{\chi}$ via the constraint $\partial\hat{V}(\phi,\tilde{\chi})/\partial\tilde{\chi}=0$, we get $\hat{V}(\phi)\equiv\hat{V}(\phi,\tilde{\chi}(\phi))$ which sums up the one-bound-state-reducible graphs giving the generating function for one-ϕ-irreducible graphs.

There is one hitch in this interpretation of $\hat{V}(\phi,\tilde{\chi})$. We have not yet set the scale of $\tilde{\chi}$. Changes in scale of $\tilde{\chi}$ will drop out of $\hat{V}(\phi)$. We still have the normalization constant N at our disposal. The $\tilde{\chi}^2$ term in $\hat{V}(\phi,\tilde{\chi})$ should be $\frac{1}{2}m_B^2\tilde{\chi}^2$ for our interpretation to be correct. Hence, let us choose N such that

$$\frac{(g_1^2-g^2)N}{2}=m_B^2\,. \tag{5.1}$$

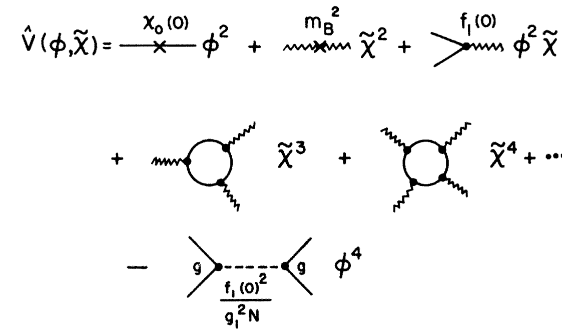


FIG. 4. Diagrammatic interpretation of our final expression for $\hat{V}$, Eq. (5.4). Wavy lines represent the bound state.

The mass m_B has not yet entered in our discussion because our eigenvalue problem was in g^2 rather than s. We did this in order to work at zero momentum. However, the eigenvalue problem at finite energy determines $m_B^2(g^2)$ or alternatively $g^2(m_B^2)$, where m_B is the mass of the lowest bound state. For small m_B^2,

$$g^2(m_B^2)\approx g_1^2+m_B^2\frac{dg^2}{dm_B^2}\,. \tag{5.2}$$

Therefore, Eq. (5.1) is approximately

$$\left(-\frac{dg^2}{dm_B^2}\right)\frac{N}{2g_1^2}=1\,, \tag{5.3}$$

where

$$N=-i\int\frac{d^4P}{(2\pi)^4}\left(\frac{f_1(P)}{P^2-\chi_0(P)}\right)^2.$$

With this condition, $\hat{V}$ becomes

$$\hat{V}(\phi,\tilde{\chi})=\tfrac{1}{2}\chi_0(0)\phi^2+\tfrac{1}{2}m_B^2\tilde{\chi}^2+\tfrac{1}{2}f_1(0)\phi^2\tilde{\chi}$$
$$-\frac{i}{2}\int\frac{d^4P}{(2\pi)^4}\left[\ln\left(1-\frac{\tilde{\chi}f_1(P)}{P^2-\chi_0(P)}\right)+\frac{\tilde{\chi}f_1(P)}{P^2-\chi_0(P)}+\frac{1}{2}\left(\frac{\tilde{\chi}f_1(P)}{P^2-\chi_0(P)}\right)^2\right]+\frac{g^2f_1^2(0)}{4g_1^2N}\phi^4\,. \tag{5.4}$$

Figure 4 gives the first few diagrams resulting from expanding $\hat{V}$ in ϕ and $\tilde{\chi}$. The propagators in the loops are $[q^2-\chi_0(q)]^{-1}$. The last graph needs some explanation. It comes from the ϕ^4 term in Eq. (5.4). To see that it is a one σ pole graph remember that we projected it onto the bound-state wave function Eq. (4.17). This graph must not appear in $\hat{V}(\phi)$ because it is one-σ-reducible. This graph shows up in $\hat{V}(\phi,\tilde{\chi})$ with the wrong sign, and it cancels the unwanted graph that is generated when $\tilde{\chi}$ is eliminated to give $\hat{V}(\phi)$.

We have reduced the problem of exhibiting the phase transition to finding the stationary points of $\hat{V}$. We restrict our attention to stationary points on the line $\phi=0$. [This is a solution to $\partial\hat{V}(\phi,\tilde{\chi})/\partial\phi=0$.] Then

$$\hat{V}(0,\tilde{\chi})\approx\frac{m_B^2}{2}\tilde{\chi}^2+\frac{\gamma}{6}\tilde{\chi}^3+\cdots\,, \tag{5.5}$$

where

$$\gamma \equiv -i \int \frac{d^4P}{(2\pi)^4} \left(\frac{f_1(P)}{P^2 - \chi_0(P)} \right)^3 .$$

For m_B^2 positive, there is a local minimum at zero with curvature m_B^2. For m_B^2 negative, the local minimum shifts to $\bar{\chi} \approx -2m_B^2/\gamma$, with curvature $\approx -m_B^2$. This is the basic effect. For more detail on the small-m_B^2 limit we refer the reader to the earlier paper, Ref. 7.

We can write $\hat{V}$ in terms of ϕ and $(\hat{\chi} - \bar{\chi})$ and thereby exhibit the couplings in the new vacuum. We first get an expression for $\bar{\chi}$:

$$0 = m_B^2 + \frac{\bar{\chi} N}{4} + \frac{i}{2} \int \frac{d^4P}{(2\pi)^4} \frac{\bar{\chi} f_1(P)^2}{(P^2 - \chi_0)[P^2 - \chi_0 - \bar{\chi} f_1(P)]} . \tag{5.6}$$

We cannot solve this for $\bar{\chi}$ but can use $\bar{\chi}$ as a variable in lieu of m_B^2. The expansion about the new vacuum gives

$$\begin{aligned} \hat{V}(\phi, \hat{\chi}) = {} & \hat{V}(0, \bar{\chi}) + \tfrac{1}{2}[\chi_0(0) + \bar{\chi} f_1(0)]\phi^2 + \tfrac{1}{2} f_1(0)\phi^2(\hat{\chi} - \bar{\chi}) \\ & + \tfrac{1}{2}(\hat{\chi} - \bar{\chi})^2 \left\{ \frac{i}{2} \int \frac{d^4P}{(2\pi)^4} \frac{\bar{\chi} f_1^{\,3}(P)}{[P^2 - \chi_0(P) - \bar{\chi} f_1(P)]^2 [P^2 - \chi_0(P)]} \right\} \\ & - \frac{i}{2} \int \frac{d^4P}{(2\pi)^4} \left\{ \ln\left(1 - \frac{(\chi - \bar{\chi}) f_1(P)}{P^2 - \chi_0(P) - \bar{\chi} f_1(P)}\right) + \frac{(\chi - \bar{\chi}) f_1(P)}{P^2 - \chi_0(P) - \bar{\chi} f_1(P)} + \frac{1}{2}\left[\frac{(\chi - \bar{\chi}) f_1(P)}{P^2 - \chi_0(P) - \bar{\chi} f_1(P)}\right]^2 \right\} . \end{aligned}$$

For $m_B^2 < 0$, $\bar{\chi} > 0$ and $\chi = \bar{\chi}$ is the correct minimum. The coefficient of $\frac{1}{2}(\hat{\chi} - \bar{\chi})^2$ is $-m_B^2$ to first order in $\bar{\chi}$. In this vacuum, the ϕ propagator is $[P^2 - \chi_0(P) - \bar{\chi} f_1(P)]^{-1}$.

In a theory with internal symmetry $\hat{V}$ would be an invariant function of ϕ_i and $\hat{\chi}_a$. In the translated vacuum $\bar{\chi}_a$ would occur as a parameter in the couplings giving spontaneous symmetry breaking just as if $\hat{\chi}_a$ were an elementary field.

VI. SUMMARY AND CONCLUSION

In this paper we proposed an approximation procedure for studying dynamical symmetry breaking. We focus our attention on a presumed deep bound state in a theory and look for the simplest approximation that will give a phase transition if the mass square of the bound state, m_B^2, goes negative. One is led to a nonlinear Dyson integral equation for the self-energy. Even the simplest approximation is intractable for all theories except those with a four-field interaction such as the O(N) σ model and the Nambu–Jona-Lasinio model. We eschew linearizing the Dyson equation. We use the formulation of the effective potential V given by Cornwall, Jackiw, and Tomboulis[3] which is a functional of the self-energy. This is a variational principle for V with the self-energy as a trial function. We show how to isolate the bound-state contribution to V. This contribution displays the desired phase transition as m_B^2 goes negative. The nonlinearity of the Dyson equation is preserved by our approximation and plays an essential role in defining the new vacuum. The problem of dynamical symmetry breaking is reduced to surveying the effective potential for stationary points in the classical field variable representing the bound state just as one would do in a σ model.

In working with scalar fields we have brushed aside many of the important problems in the current view of hadrons based on quantum chromodynamics (QCD). However, we feel this has applications to QCD where $\phi \to$ quarks, $\sigma \to$ colored gluons, $\chi \to$ hadrons. This formulation provides a vehicle for relating gauge theories to the σ model. The SU_3 σ model gives a very good account of low-energy meson dynamics even at the tree-graph level. It is not a fundamental theory, but we feel it is not generally recognized that it could emerge from an approximation to a more fundamental theory.

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[8]We wish to side-step the question of the existence of a bound state in this theory with no internal symmetry. We use this model to illustrate the pole assumption.

[9]This implicitly assumes the couplings go to constants as m_B^2 goes to zero, This is not necessarily the case in massless theories, and these arguments must be modified.

[10]m^2 is shorthand for $-\Delta_R^{-1}(0)$, the renormalized propagator at $q^2=0$. ϕ is the renormalized field.

[11]This approximation also generates couplings of more than three bound states. These are nonleading for small m_B^2. See Ref. 9.

[12]This approximation is a leading term in the $1/N$ expansion in an $O(N)$-symmetric theory if we assign ϕ_i to the N-dimensional representation and σ_{ij} to the $[N(N+1)/2-1]$-dimensional representation σ_{ij} is traceless and symmetric. Then each closed loop shown in Fig. 2(c) picks up a factor of N.