One-component formulation of Reggeon field theory*

Benjamin C. Harms
Department of Physics and Astronomy, University of Alabama, University, Alabama 35486

Chung-I Tan
Department of Physics, Brown University, Providence, Rhode Island 02912

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We present a nonperturbative formulation of the anti-Hermitian cubic Reggeon field theory (RFT) in terms of a single field χ. We analyze the structure of RFT as α_0 is increased above 1 and clarify the relation between the perturbative vacuum and the classical stationary points. A canonical transformation is performed so that the new Hamiltonian depends on the sign of $\Delta_0 \equiv 1 - \alpha_0$ only through a potential of the Landau-Ginzburg type. Our one-component theory is normal-ordered with respect to the original Pomeron field without tadpoles, and it allows a path-integral formalism with undistorted contours. For $|\Delta_0|/g_0$ large and $\Delta_0 < 0$, we formulate two different and yet equivalent analog models. We unambiguously derive an analog model in terms of a single classical spin at each rapidity–impact-parameter site. Through the use of an asymmetrical transfer matrix, we obtain a kinklike ground-state configuration for the $D = 0$ model. Alternatively, by going on a lattice for the impact-parameter space only, we arrive at a quantum lattice-spin model. We explicitly demonstrate that the quantum spin model at $D = 0$ is equivalent to the classical lattice spin model.

I. INTRODUCTION

Much of the recent work on the asymptotic behavior of high-energy hadron interactions has been carried out within the framework of Gribov's Reggeon field theory (RFT),[1] in which the bare Pomeron is treated as a quasiparticle and is associated with fields $\psi(t, \vec{b}), \bar{\psi}(t, \vec{b})$ in one "time" and two transverse-space dimensions. Renormalization-group analyses[2] using the ϵ-expansion technique strongly suggest that the infrared behavior of RFT is analogous to the scaling phenomenon of a many-body system at a second-order phase transition, and, in particular, the critical point occurs at $\alpha_0 > 1$.[3] However, the specific nature of the Pomeron critical phenomenon is still not understood, partly because the ϵ-expansion results are quantitatively unreliable. A further complication is that the bare perturbative expansion is not well defined in the case $\alpha_0 > 1$. Therefore, much of the current effort has been directed toward finding nonperturbative approaches to the study of critical behavior of RFT.[4]

In this paper, we present a nonperturbative formulation of RFT which not only allows a smooth continuation from $\alpha_0 < 1$ to $\alpha_0 > 1$, but also leads to an expansion capable of treating both regions on an equal footing. Instead of using two independent Pomeron fields ψ and $\bar{\psi}$, we reformulate RFT with an anti-Hermitian cubic interaction[5] by working exclusively with a single field χ, $\chi \equiv (\psi - \bar{\psi})/2i$. We analyze the structure of RFT as α_0 is increased above 1, and clarify the relation between the perturbative vacuum and the classical stationary points. We concentrate on the "kinematic" aspects of our nonperturbative treatment, and do not in the present analysis attempt to discuss the actual critical behavior of RFT at $D = 2$.

The key to our alternative formulation of RFT is the identification of an appropriate gauge transformation so that the transformed Hamiltonian, $\tilde{H}$, has its kinetic and potential components separated. Our theory has no tadpoles and is normal-ordered with respect to ψ and $\bar{\psi}$. Furthermore, the kinetic energy component depends only on Δ_0^2, $(\Delta_0 \equiv 1 - \alpha_0)$, and the dependence on the sign of Δ_0 is isolated in a single potential function. We explicitly exhibit that the evolution from $\Delta_0 > 0$ to $\Delta_0 < 0$ is primarily controlled by a transition of the "Landau-Ginzburg" type in which the potential switches, as Δ_0 changes sign, from that of a single minimum at $\chi = 0$ to that of two symmetric minima at $\chi \simeq \pm |\Delta_0|/g_0$ (see Fig. 1). As a consequence, we are able to show that, in the path-integral formalism of RFT, once the theory is defined for $\alpha_0 < 1$ with integration paths along real axes, no contour distortion is necessary as α_0 is increased above 1.

Our formulation is nonperturbative with respect to the original bare expansion. Nevertheless, a new quantum expansion is derived which treats both $\alpha_0 > 1$ and $\alpha_0 < 1$ regions on an equal footing. At $\Delta_0 > 0$ and $g_0/\Delta_0 \to 0^+$, this new expansion approaches the original free-field result; but, at $\Delta_0 < 0$ and $g_0/\Delta_0 \to 0^-$, a completely different limit is reached. This latter limit is just as simple in structure as the original $\Delta_0 > 0$ limit, and it is related to the bare perturbative region through

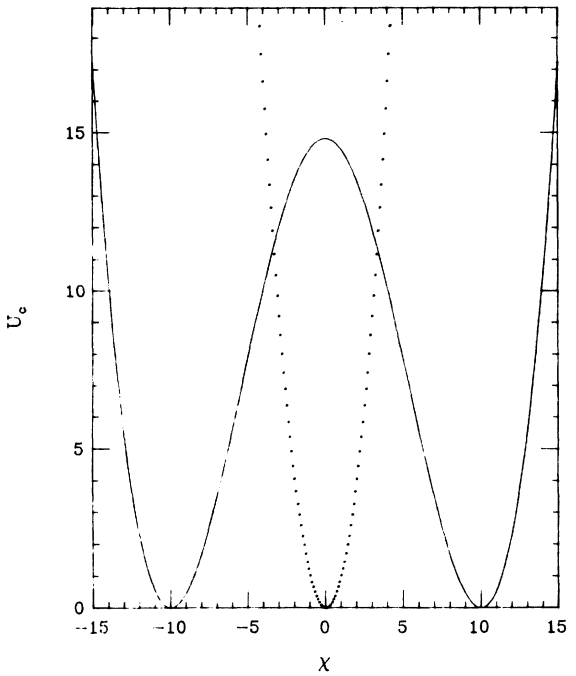

FIG. 1. Classical potential, $U_c(\chi)$, of RFT: $|\Delta_0| = 1$, $g_0 = 0.1$; dotted curve is for $\Delta_0 > 0$, solid curve is for $\Delta_0 < 0$.

analytic continuation in Δ_0 while holding $g_0 \neq 0$.

For $|\Delta_0|/g_0$ large and $\Delta_0 < 0$, our quantum expansion naturally suggests a "two-level truncation" approximation, as has been advocated earlier by many others.[6-10] Beginning at a common starting point, we formulate two different and yet equivalent analog models. By putting the rapidity-impact-parameter space on a lattice, we arrive unambiguously at a finite-temperature classical lattice-spin model with a single spin-$\frac{1}{2}$ variable at each site. We discuss the relation between the anti-Hermiticity of the triple-Pomeron coupling and the symmetries of classical RFT, and suggest that these properties can best be realized in a lattice formalism by always defining the system by an asymmetrical "transfer matrix." As an illustration, we explicitly demonstrate that, because of the anti-Hermitian interaction, the analog model at $D = 0$ has a kinklike ground-state configuration. However, the spin-spin correlation function is simply that of a (conventional) one-dimensional ferromagnetic Ising chain.

Alternatively, by going on a lattice for the impact-parameter space only, we arrive at the zero-temperature quantum lattice-spin model of Refs. 8–10. We explicitly demonstrate that the $D = 0$ quantum-spin model is equivalent to the classical lattice-spin model. Our "χ representation" for the $D = 0$ quantum RFT is particularly useful in revealing the relation between the classical fixed points of RFT and the evolution of left and right vacuums as Δ_0 varies. Our quantum expansion also allows us to explore the symmetry between left and right eigenvectors under $\chi \rightarrow -\chi$ so as to be able to construct trial functions for the excited states much more easily.

In order to motivate our procedure, we first perform in Sec. II an analogous gauge transformation for the classical RFT at $D = 0$, and show that this transformation can greatly simplify the description of classical motions in RFT. This classical analysis also provides us with an understanding of the type of "dispersion relation" our quantum field χ would obey at $\Delta_0 < 0$ if quantum tunneling effects could be turned off. We therefore keep the time variable $\tau = -iy$ real in the classical analysis, even though the quantum theory should be defined with the rapidity, y, real.

In Sec. III, the quantum gauge transformation is performed and the path-integral formalism is developed. To simplify the discussion we concentrate primarily on the $D = 0$ case. In Sec. IV, the path-integral formalism is used as a starting point for formulating a classical lattice-spin analog model of RFT. The construction of the quantum lattice-spin model is discussed in Sec. V. We review our work and comment on its relation to the works of others in Sec. VI.

In the remainder of this section, we comment briefly on the structure of RFT at the free-field level and on how RFT should be defined for $\Delta_0 < 0$. Denoting $\phi \equiv (\psi + \bar{\psi})/2$ and $\chi \equiv (\psi - \bar{\psi})/2i$, the two-component Lagrangian[1,5] becomes

$$L(\phi, \dot{\phi}; \chi, \dot{\chi}) = -\phi \partial_\tau \chi - \alpha' [(\vec{\nabla}_b \phi)^2 + (\vec{\nabla}_b \chi)^2] - \Delta_0 (\phi^2 + \chi^2) - i g_0 \phi (\phi^2 + \chi^2) . \quad (1.1)$$

With the anti-Hermitian cubic interaction turned off, $\bar{\psi}$ becomes the adjoint of ψ, so that ϕ and χ are respectively the real and imaginary parts of ψ. It is customary to assign ψ, which annihilates a bare Pomeron, to be a complex field with a *positive* energy component only (our convention[1] for L_0 forces upon us the definition $\tau \equiv -iy$, where y is the rapidity). The equations of motion for free fields are

$$\dot{\chi} = -(\Delta_0 - \alpha' \nabla_b^2)\phi , \quad \dot{\phi} = (\Delta_0 - \alpha' \nabla_b^2)\chi . \quad (1.2)$$

It follows that ϕ and χ separately satisfy field equations

$$[\partial_\tau^2 + (\Delta_0 - \alpha' \nabla_b^2)^2]\phi = 0 , \quad [\partial_\tau^2 + (\Delta_0 - \alpha' \nabla_b^2)^2]\chi = 0 , \quad (1.3)$$

and their Fourier components obey the dispersion relation $E^2 = (\Delta_0 + \alpha' \vec{k}^2)^2$, i.e., ϕ and χ both contain creation and annihilation components. However, the physical content of the theory is unaltered as

can be seen by the fact that for $y_2 > y_1$,

$$\langle 0|\psi(2)\bar{\psi}(1)|0\rangle = 4\langle 0|\phi(2)\phi(1)|0\rangle = 4\langle 0|\chi(2)\chi(1)|0\rangle .$$

So long as $\Delta_0 > 0$, it is easy to identify the positive-frequency component with the annihilation operators and the negative-frequency component with the creation operators, and no ambiguity can arise.

The situation is more involved when $\Delta_0 < 0$; for simplicity, let us consider the $\alpha' = 0$ limit so that $\dot{\chi} = -\Delta_0\phi$ and $\dot{\phi} = \Delta_0\chi$ for free fields. Reversing the sign of Δ_0 is seen to be *formally* equivalent to interchanging the positive- and negative-frequency components of ϕ and χ. This, in turn, leads to ψ being of negative frequency. However, the original construct of RFT requires ψ to be an annihilation operator, independent of the sign of Δ_0; care must be taken in applying the second-quantization procedure.

The above argument would be relevant if RFT were to be defined at $\Delta_0 < 0$ perturbatively. However, this view is not supported by phenomenological analyses based on the short-range correlation picture. Although the "two-component" analysis[11] justifies a perturbative treatment of the Pomeron interaction, the effective bare Pomeron intercept is less than 1 at low energies. Owing to the "multiperipheral threshold effects," the effective Δ_0 is energy-dependent; it is renormalized upward and probably has just passed $\alpha_0 = 1$ at ISR energies.[12] On the other hand, the triple-Pomeron coupling is known to be nonzero from inclusive experiments. Therefore, we adopt the view that, whereas RFT is defined perturbatively for $\Delta_0 > 0$, the $\Delta_0 < 0$ region must have a smooth continuation to the $\Delta_0 > 0$ region with the interaction left on. We shall show in Sec. II that, under this hypothesis, the switching of positive- and negative-frequency components does not take place, and the relation $\dot{\chi} = -\Delta_0\phi$ does not hold when $g_0 \to 0$ after the continuation in Δ_0.

II. ONE-COMPONENT CLASSICAL RFT

Our primary aim in this section is to provide an exposition on the stability of classical motion in RFT as one continues Δ_0 from a positive to a negative value. Since we are not at this moment interested in solitonlike solutions, the relevant features occur already in the limit $\alpha' = 0$; we therefore restrict ourselves to treating the $D = 0$ problem only. In our analysis, the time variable $\tau = -iy$ is kept real. Our strategy is first to find a Hamiltonian which leads to the correct equation of motion and then to perform a "gauge transformation" which transforms the "potential" into a "Landau-Ginzburg" form. This transformation then allows us to carry out the stability analysis for $\Delta_0 > 0$ and $\Delta_0 < 0$ on an equal footing.

A. $L(\phi,\dot{\phi})$ at $D = 0$

The choice of a χ field for our reformulation of RFT might seem arbitrary at first, since ϕ and χ are equivalent at the free-field level. For a cubic interaction, it is actually easier to start with the ϕ field since the equation of motion, for transverse dimensionality $D = 0$, has a relatively simple form

$$\frac{\ddot{\phi}}{\Delta_0 + ig_0\phi} - \frac{ig_0}{2}\frac{\dot{\phi}^2}{(\Delta_0 + ig_0\phi)^2} + \left(\Delta_0 + \frac{3ig_0}{2}\phi\right) = 0 , \tag{2.1}$$

and the χ field is given by

$$\chi = (\Delta_0 + ig_0\phi)^{-1}\dot{\phi} . \tag{2.2}$$

Equation (2.1) can be shown to follow from a Lagrangian,[13]

$$L_c(\phi,\dot{\phi}) = \frac{\dot{\phi}^2}{\Delta_0 + ig_0\phi} - \Delta_0\phi^2 - ig_0\phi^3 , \tag{2.3}$$

and the corresponding Hamiltonian is

$$H_c^{(\phi)}(\phi, P_\phi) = (\phi^2 + P_\phi^2/4)(\Delta_0 + ig_0\phi) , \tag{2.4}$$

where

$$P_\phi = \frac{\partial L}{\partial\dot{\phi}} = 2\chi . \tag{2.5}$$

For $\Delta_0 > 0$, the "ground state" corresponds to $\phi = \chi = 0$; an expansion in g_0 would lead to the classical perturbative result. However, as Δ_0 becomes negative, a degenerate pair of ground states occur at $\phi = i\Delta_0/g_0$, $\chi = \pm|\Delta_0|/g_0$. Since the coefficient of $\dot{\phi}^2$ in (2.3) diverges at these points, it becomes inconvenient to discuss the motion in terms of ϕ. As we shall explain in subsequent discussions, the most natural procedure for exhibiting the structure of RFT at $\Delta_0 < 0$ is by working exclusively with the χ field.[14]

B. One-component RFT at $D = 0$ and gauge transformation

Because of Eq. (2.5), the Hamiltonian for χ, $H_c^{(\chi)}(\chi, P_\chi)$, can be obtained from $H_c^{(\phi)}(\phi, P_\phi)$ by a canonical transformation via the generating function $F_1(\phi,\chi) = 2\phi\chi$. Using $P_\chi \equiv -\partial F_1/\partial\chi = -2\phi$, $P_\phi = \partial F_1/\partial\phi = 2\chi$, and Eq. (2.4), we obtain

$$H_c^{(\chi)}(\chi, P_\chi) = (\chi^2 + P_\chi^2/4)\left(\Delta_0 - \frac{ig_0}{2}P_\chi\right) . \tag{2.6}$$

The corresponding one-component Lagrangian can be obtained once one solves P_χ in terms of χ and $\dot{\chi}$. In what follows, the superscript χ will be dropped.

We begin our alternative formulation of RFT by performing a gauge transformation which shifts P_χ to $\bar{P}_\chi$ by

$$\bar{P}_\chi \equiv P_\chi + \left(\frac{2}{3ig_0}\right)[\Delta(\chi) - \Delta_0] \equiv P_\chi - \frac{\partial \Lambda}{\partial \chi}, \tag{2.7}$$

where

$$\Delta(\chi) \equiv (\Delta_0^2 + 3g_0^2\chi^2)^{1/2}. \tag{2.8}$$

the generating function for this transformation is $F_2(\chi, \bar{P}_\chi) = \chi\bar{P}_\chi + \Lambda(\chi)$, and the new Hamiltonian becomes

$$\bar{H}_c(\chi, \bar{P}_\chi) = \tfrac{1}{4}\Delta(\chi)\bar{P}_\chi^2 - \frac{ig_0}{8}\bar{P}_\chi^3 + U_c(\chi), \tag{2.9}$$

where the "classical potential" of RFT is

$$U_c(\chi) = \tfrac{2}{3}\Delta_0\chi^2 + \left(\frac{2}{27g_0^2}\right)[\Delta(\chi)^3 - \Delta_0^3]. \tag{2.10}$$

The potential possesses the following interesting features: (i) It is an even function of χ. (ii) It is non-negative and has zeros for its minima. (iii) For $\Delta_0 > 0$, it has a unique minimum at $\chi = 0$. (iv) For $\Delta_0 < 0$, it has symmetric minima at $\chi = \pm|\Delta_0|/g_0$. These features are schematically represented in Fig. 1. The transition from a single minimum to two symmetric minima as Δ_0 changes sign is the characteristic feature of a Landau-Ginzburg potential. Since $\Delta(\chi)$ is even in χ and $\Delta(\chi) \propto |\chi|$ as $\chi \to \pm\infty$, Eq. (2.10) is similar in structure to its counterpart in a ϕ^4 theory, $U = m^2\phi^2 + \lambda\phi^4$.

C. Boundary conditions for RFT

In order to appreciate our motivation for performing the gauge transformation, (2.7), and to gain a better understanding on the structure of RFT, let us return to the original Hamiltonian, Eq. (2.6), whose canonical equations are

$$\dot{\chi} = \frac{\partial H_c}{\partial P_\chi} = -\frac{3ig_0}{8}P_\chi^2 + \frac{\Delta_0}{2}P_\chi - \frac{ig_0}{2}\chi^2, \tag{2.11}$$

$$\dot{P}_\chi = -\frac{\partial H_c}{\partial \chi} = -\chi(2\Delta_0 - ig_0P_\chi). \tag{2.12}$$

Equation (2.11) allows us to solve for P_χ, which leads to two roots. This indicates that our RFT formally contains several branches; the branch on which the correct physics lies must be specified by additional boundary conditions. It is clear that the correct description must possess a smooth perturbative limit, $g_0 \to 0$, when $\Delta_0 > 0$. As we have already emphasized in the Introduction, the phenomenological success of the short-range-correlation picture at current machine energies suggests that the $\Delta_0 < 0$ RFT should be defined as a smooth continuation of $\Delta_0 > 0$ RFT with g_0 held *nonzero*. In the limit $g_0 \to 0$, $\Delta_0 > 0$, an inspection of Eq. (2.6) indicates that the minimum-energy configuration corresponds to $\chi = P_\chi = \dot{\chi} = \dot{P}_\chi = 0$. This is sufficient to fix the relation between χ, $\dot{\chi}$, and P_χ, and the desired branch for RFT is specified by

$$P_\chi = \left(\frac{2}{3ig_0}\right)[\Delta_0 - (\Delta_0^2 + 3g_0^2\chi^2 - 6ig_0\dot{\chi})^{1/2}]. \tag{2.13}$$

For $\dot{\chi}$ small, P_χ becomes

$$P_\chi \simeq \left(\frac{2}{3ig_0}\right)[\Delta_0 - \Delta(\chi)] + \frac{2}{\Delta(\chi)}\dot{\chi} + O(\dot{\chi}^2). \tag{2.14}$$

Near $\chi = 0$, and for $\Delta_0 > 0$, we find $P_\chi \simeq (2/\Delta_0)\dot{\chi}$, in agreement with the free-field result, Eq. (1.2). On the other hand, when $\Delta_0 < 0$ and $\chi \simeq \pm|\Delta_0|/g_0$, the relation becomes $P_\chi \simeq 2i|\Delta_0|/g_0 + \dot{\chi}/|\Delta_0|$, a result quite different from that naively expected based on (1.2). Note that *the coefficient of* $\dot{\chi}$ *in (2.14) always remains positive independent of the sign of* Δ_0. This guarantees that positive-energy states would remain positive as Δ_0 becomes negative and no Reggeons with zero-energy intercept greater than 1 would appear. The price one pays is that the relation between P_χ and $\dot{\chi}$ is not always of the canonical form for $\dot{\chi}$ small at $\Delta_0 < 0$.

We have chosen our gauge transformation, (2.7), so that the condition $\bar{P}_\chi \to 0$ as $\dot{\chi} \to 0$ is automatically satisfied for the shifted momentum, $\bar{P}_\chi$; it follows from (2.13) and (2.8) that

$$\bar{P}_\chi(\chi, \dot{\chi}) = (2/3ig_0)\{\Delta(\chi) - [\Delta(\chi)^2 - 6ig_0\dot{\chi}]^{1/2}\} \tag{2.15}$$

$$\simeq \frac{2}{\Delta(\chi)}\dot{\chi} + \frac{3ig_0}{\Delta(\chi)^3}\dot{\chi}^2 - \frac{9g_0^2}{\Delta(\chi)^5}\dot{\chi}^3 + \cdots. \tag{2.16}$$

This property is correlated with the absence of a term linear in $\bar{P}_\chi$ for our transformed Hamiltonian. We emphasize that either Eq. (2.13) or Eq. (2.15), with $\Delta(\chi)$ non-negative, uniquely specifies RFT at $\Delta_0 < 0$.

Since $\bar{P}_\chi \to 0$ as $\dot{\chi} \to 0$ and since $\bar{P}_\chi$ depends only on Δ_0^2 [through $\Delta(\chi)$], we have achieved in Eq. (2.9) a clean separation between the kinetic and potential components of our classical Hamiltonian, where the only dependence on the sign of Δ_0 is in the classical potential, $U_c(\chi)$. Equation (2.9) is particularly useful for studying small oscillations of classical RFT and their dependence on Δ_0.

D. Stationary points of RFT

We emphasize that, because of our choice for P_χ, Eq. (2.13), the stable motions of the system, for $\Delta_0 > 0$ or $\Delta_0 < 0$, are completely specified. In general, possible stable solutions occur about stationary points of the system, defined by $\dot{\chi} = \dot{P}_\chi = 0$; from (2.11), (2.12), and $P_\chi = -2\phi$, they are

$$\begin{aligned}&\text{(a)}\ \chi=0,\quad \phi=0,\\ &\text{(b)}\ \chi=0,\quad \phi=2i\Delta_0/3g_0,\\ &\text{(c)}\ \chi=|\Delta_0|/g_0,\quad \phi=i\Delta_0/g_0,\\ &\text{(d)}\ \chi=-|\Delta_0|/g_0,\quad \phi=i\Delta_0/g_0.\end{aligned}\tag{2.17}$$

In terms of χ and $\bar{P}_\chi$, by using (2.7), they become

$$\begin{aligned}&\text{(a)}\ \chi=0,\quad \bar{P}_\chi=(2/3ig_0)(|\Delta_0|-\Delta_0),\\ &\text{(b)}\ \chi=0,\quad \bar{P}_\chi=(2/3ig_0)(|\Delta_0|+\Delta_0),\\ &\text{(c)}\ \chi=|\Delta_0|/g_0,\quad \bar{P}_\chi=(4/3ig_0)(|\Delta_0|+\Delta_0),\\ &\text{(d)}\ \chi=-|\Delta_0|/g_0,\quad \bar{P}_\chi=(4/3ig_0)(|\Delta_0|+\Delta_0).\end{aligned}\tag{2.18}$$

These stationary points have "energies" $E_a=0$, $E_b=-\frac{4}{27}(\Delta_0^3/g^2)$, $E_c=0$, $E_d=0$, respectively. However, some do not satisfy our RFT boundary condition Eq. (2.13), or Eqs. (2.15) and (2.16). For $\Delta_0>0$, fixed points (b), (c), and (d) are incompatible with Eq. (2.16); the only candidate left is the "perturbative" stationary point, (a). Although $E_b<E_a$, for $\Delta_0>0$, point (b) does not lie on the correct branch because it does not admit a perturbative solution. As we continue to $\Delta_0<0$, point (a) is no longer compatible with Eq. (2.16); all three remaining stationary points are of the "nonperturbative" type. We shall show shortly that points (c) and (d), for $\Delta_0<0$, are "stable" in the same sense that the perturbative stationary point, (a), is stable for $\Delta_0>0$. On the other hand, point (b), at $\Delta_0<0$, with $E_b>0$, is seen to be unstable. Our result should be contrasted with that of Cardy and Sugar,[6] who first arrived at a similar conclusion for the quantum RFT by analyzing the path integrals defining the theory. We shall come back to this point in the next section. On the other hand, our analysis differs from that of Ref. 8 because their classical theory is already Wick-rotated.

The shift from (a) to (c) and (d) as Δ_0 changes sign corresponds to a shift in the ground state of the system; thus, formally, a change in the "vacuum" has taken place. However, this view is incorrect quantum mechanically, owing to the anti-Hermiticity of $H(\chi,P_\chi)$. We show, in Sec. V, that quantum states corresponding to solutions (c) and (d) for $\Delta_0<0$ are, in a certain sense, the same quantum states corresponding to (a) at $\Delta_0>0$, and, in the $\alpha'=0$ limit, the "perturbative vacuum" always remains a nondegenerate vacuum of RFT.

E. Small oscillations

The final form for our one-component Lagrangian can now be written as

$$\begin{aligned}\bar{L}_c(\chi,\dot{\chi})&=\dot{\chi}\bar{P}_\chi-\bar{H}_c(\chi,\bar{P}_\chi)=\tfrac{2}{3}\dot{\chi}\bar{P}_\chi+\tfrac{1}{12}\Delta(\chi)\bar{P}_\chi^{\,2}-U_c(\chi)\\ &=\left[\frac{\dot{\chi}^2}{\Delta(\chi)}+\frac{ig_0}{\Delta(\chi)^3}\dot{\chi}^3-\frac{9g_0^{\,2}}{4\Delta(\chi)^5}\dot{\chi}^4+\cdots\right]-U_c(\chi).\end{aligned}\tag{2.19}$$

Since $\Delta(\chi)$ is of the order $|\Delta_0|$, it suggests that we treat those $\dot{\chi}^3,\dot{\chi}^4,\ldots$ terms in (2.19) as a perturbation where the expansion parameter is effectively $g_0/|\Delta_0|$. Since $\Delta(\chi)$ is always positive, our formalism permits treating the limits $g_0/\Delta_0\to 0^+$ and $g_0/\Delta_0\to 0^-$ on a mathematically equal footing.

Equation (2.19) is particularly suitable for discussing motions about stationary points of RFT. For $|\dot{\chi}|$ sufficiently small, Eq. (2.19) can be approximated by

$$\bar{L}_c(\chi,\dot{\chi})\simeq\frac{\dot{\chi}^2}{\Delta(\chi)}-U_c(\chi).\tag{2.20}$$

Since the kinetic term is strictly positive, the stability criterion is completely decided by the structure of $U_c(\chi)$ at a stationary point. We have four cases to consider: (a) $\Delta_0>0$, $\chi\simeq 0$, (b) $\Delta_0<0$, $\chi\simeq 0$, (c) $\Delta_0<0$, $\chi\simeq|\Delta_0|/g_0$, (d) $\Delta_0<0$, $\chi\simeq-|\Delta_0|/g_0$. Consider the case (b) first. Since $U_c(\chi)$ has a local maximum at $\chi=0$, for $\Delta_0<0$, point (b) of (2.17) is therefore not a stable stationary point of RFT. The situation is just the opposite for (a), (c), and (d); small oscillations about these points are stable.

To have a better understanding of the nature of these small oscillations, let us keep $\bar{L}_c(\chi,\dot{\chi})$ to the $\dot{\chi}^3$ order and expand $\bar{L}_c$ about fixed points (a), (c), and (d):

$$\bar{L}_c^{\Delta_0>0}(\chi,\dot{\chi})\simeq\left(\frac{1}{\Delta_0}\dot{\chi}^2+\frac{ig_0}{\Delta_0^{\,3}}\dot{\chi}^3\right)-\Delta_0\chi^2,\quad \chi\simeq 0,\tag{2.21}$$

$$\begin{aligned}\bar{L}_c^{\Delta_0<0}(\chi,\dot{\chi})&\simeq\frac{1}{2}\left(\frac{1}{|\Delta_0|}\dot{\chi}^2+\frac{ig_0}{4|\Delta_0|^3}\dot{\chi}^3\right)\\ &\quad-\tfrac{1}{2}|\Delta_0|\left(\chi\mp\frac{|\Delta_0|}{g_0}\right)^2,\quad \chi\simeq\pm\frac{|\Delta_0|}{g_0}.\end{aligned}\tag{2.22}$$

We first note that, aside from numerical differences, Eq. (2.21) and Eq. (2.22) are essentially the same. It follows that the stability of motion about points (c) and (d) for $\Delta_0<0$ is precisely the same as that about point (a) for $\Delta_0>0$; we therefore only need to discuss the motion about the perturbative fixed point (a).

Let $\chi=\chi_0+g_0\chi_1+g_0^{\,2}\chi_2+\cdots$. A perturbative analysis for the Euler-Lagrange equation of (2.21) leads to

$$\begin{aligned}&\ddot{\chi}_0+\Delta_0^{\,2}\chi_0=0,\\ &\ddot{\chi}_1+\Delta_0^{\,2}\chi_1=+3i\chi_0\dot{\chi}_0.\end{aligned}$$

We immediately note that, owing to the imaginary cubic interaction, the small oscillation about the perturbative fixed point involves a complex amplitude; this has nothing to do with the sign of Δ_0. The frequency of small oscillation is $|\Delta_0|$, and this can be shown to hold also for the stationary points (c) and (d).

To summarize: We have shown that the only

allowed stable motions in RFT with a purely imaginary cubic interaction are small oscillations about point (a) for $\Delta_0>0$ and points (c) and (d) for $\Delta_0<0$. Motion about the stationary points of (2.17) under different circumstances are either unstable or not allowed. (We emphasize that the above picture depends crucially on the fact that we have not performed the Wick rotation.) Our classical stability analysis, of course, ignores quantum tunneling effects, which will be our main concern in the next three sections.

III. ONE-COMPONENT QUANTUM RFT

A. Quantum Hamiltonian

Although we are primarily interested in the situation when $|\Delta_0|/g_0$ is large and $\Delta_0<0$, our formulation must possess a smooth continuation to the $\Delta_0>0$ region. In quantizing the classical RFT for $\Delta_0>0$, the bare perturbative expansion dictates that the Hamiltonian be normal-ordered with respect to the bare Pomeron, and the theory is defined without tadpoles. Rewriting χ and P_χ in Eq. (2.6) in terms of $\psi=-P_\chi/2+i\chi$, $\bar\psi=-P_\chi/2-i\chi$, and using the quantization condition $[\chi,P_\chi]=i\hbar$, the normal-ordered version of our one-component quantum RFT Hamiltonian at $D=0$ becomes

$$H(\chi,P_\chi)=\Delta_0(P_\chi^2/4+\chi^2-\tfrac{1}{2}\hbar)-\tfrac{1}{2}ig_0[P_\chi^3/4+\chi^2P_\chi-i\hbar(\chi-iP_\chi)]\,. \tag{3.1}$$

Equation (3.1) serves as the starting point for our alternative formulation of quantum RFT. It can be viewed either as a quantized theory for $H_c(\chi,P_\chi)$ with a special "no tadpole" ordering or as a theory derived directly from the original $D=0$ RFT Hamiltonian, $\Delta_0\bar\psi\psi+(ig_0/2)\bar\psi(\bar\psi+\psi)\psi$, with $[\psi,\bar\psi]=\hbar$. Which particular view one should adopt is a matter of taste.

We begin our alternative formulation by performing a quantum gauge transformation defined by

$$e^{-i\Lambda_q(\chi)}P_\chi e^{i\Lambda_q(\chi)}=\frac{\hbar}{i}\frac{\partial}{\partial\chi}+\frac{\partial\Lambda_q(\chi)}{\partial\chi}\equiv\bar P_\chi+\Lambda_q'\,, \tag{3.2}$$

where

$$\Lambda_q'(\chi)\equiv\left(\frac{2}{3ig_0}\right)[\Delta_0-\Delta_q(\chi)] \tag{3.3}$$

and

$$\Delta_q(\chi)\equiv[(\Delta_0^2-3g_0^2\hbar)+3g_0^2\chi^2]^{1/2}\,. \tag{3.4}$$

The transformed Hamiltonian becomes

$$\tilde H(\chi,\bar P_\chi)\equiv e^{-i\Lambda_q}H(\chi,P_\chi)e^{i\Lambda_q} \tag{3.5a}$$

$$=H(\chi,\bar P_\chi+\Lambda_q') \tag{3.5b}$$

$$=\tfrac{1}{4}\Delta_q^{1/2}\bar P_\chi^{\,2}\Delta_q^{1/2}-\tfrac{1}{8}ig_0\bar P_\chi^{\,3}+U_q(\chi)\,, \tag{3.5c}$$

where

$$U_q(\chi)=\left\{\tfrac{2}{3}\Delta_0\chi^2+\left(\frac{2}{27g_0^{\,2}}\right)[\Delta_q^{\,3}(\chi)-\Delta_0^{\,3}]\right\}-\hbar\{\tfrac{1}{6}\Delta_0+\tfrac{1}{3}\Delta_q(\chi)+\tfrac{1}{4}g_0^{\,2}\hbar[9g_0^{\,2}\chi^2-5\Delta_q(0)^2]\Delta_q(\chi)^{-3}\}\,. \tag{3.6}$$

The basic structure of (3.5c) remains unchanged from its classical counterpart, Eq. (2.9). The quantum potential, $U_q(\chi)$, is dominated by the terms in the first curly brackets of (3.6) and it still possesses the following properties: (i) It is even in χ. (ii) For $\Delta_0>0$, it has a unique minimum at $\chi=0$. (iii) For $\Delta_0<0$, it has two symmetric minima near $\chi=\pm|\Delta_0|/g_0$. However, owing to normal-ordering, the minimum takes on a value $\sim-\hbar|\Delta_0|/2$.

If we substitute $(\hbar/i)\partial/\partial\chi$ for P_χ in Eq. (3.1), the Hamiltonian acts as a differential operator with its domain in L^2. The adjoint can then be found by partial integration:

$$H^\dagger(\chi,P_\chi)=\Delta_0(P_\chi^{\,2}/4+\chi^2-\tfrac{1}{2}\hbar)+\tfrac{1}{2}ig_0[P_\chi^{\,3}/4+\chi^2P_\chi-i\hbar(\chi-iP_\chi)]\,. \tag{3.7}$$

Since $H^\dagger\neq H$, care must be taken in interpreting the spectrum of H. In particular, one must distinguish between the space on which H operates and its dual. For instance, on the dual space, $\bar\psi$ acts as an annihilation operator, and yet $\bar\psi\neq\psi^\dagger$. In what follows, we denote the left vacuum by $|0\rangle$ and its dual by $\langle\bar 0|$.

The adjoint of our transformed Hamiltonian is

$$\tilde H^\dagger=\tfrac{1}{4}\Delta_q^{\,1/2}\bar P_\chi^{\,2}\Delta_q^{\,1/2}+\tfrac{1}{8}ig_0\bar P_\chi^{\,3}+U_q(\chi)\,. \tag{3.8}$$

In both cases of (3.7) and (3.8), we note that the corresponding adjoints can be obtained by simply substituting g_0 by $-g_0$. This is a consequence of the anti-Hermitian triple-Pomeron interaction where $H\to H^\dagger$ under $P_\chi\to-P_\chi$, $\chi\to-\chi$. For our transformed Hamiltonian, the anti-Hermiticity becomes $\tilde H\to\tilde H^\dagger$ under $\bar P_\chi\to-\bar P_\chi$, $\chi\to-\chi$. We emphasize that, in Eq. (3.5c), the only remaining anti-Hermitian interaction is contained in a single "interaction" term.

Lastly, we generalize the formalism to $D\neq0$. The normal-ordered Hamiltonian density is

$$H(\chi,P_\chi)=H^0(\chi,P_\chi)+\alpha'(-\tfrac{1}{2}\overrightarrow{\nabla}_b P_\chi-i\overrightarrow{\nabla}_b\chi)(-\tfrac{1}{2}\overrightarrow{\nabla}_b P_\chi+i\overrightarrow{\nabla}_b\chi)$$

$$=H^0(\chi,P_\chi)+\alpha'[\tfrac{1}{4}(\overrightarrow{\nabla}_b P_\chi)^2+(\overrightarrow{\nabla}_b\chi)^2-\tfrac{1}{2}D\hbar]\,, \tag{3.9}$$

where the superscript refers to the corresponding $D=0$ expression. Under the gauge transformation, it becomes

$$\tilde{H}(\chi,\overline{P}_\chi)=\tilde{H}^0(\chi,\overline{P}_\chi)+\alpha'[\tfrac{1}{4}(\vec{\nabla}_b\overline{P}_\chi+\vec{\nabla}_b\Lambda')^2+(\vec{\nabla}_b\chi)^2-\tfrac{1}{2}D\hbar]\,. \tag{3.10}$$

Equation (3.10) can further be simplified when we move onto a lattice.

B. Green's functions

The connection between RFT and physical observables is through the asymptotic expansion for the nonforward elastic amplitude (assuming point sources),

$$A_{ab}(Y,B)=-is\sum_{n,m=1}^{\infty}(i)^n(i)^m g_a^{(n)}g_b^{(m)}\times\langle\overline{0}|\psi(Y,\vec{B})^n\overline{\psi}(0,\vec{0})^m|0\rangle\,, \tag{3.11}$$

where $g_a^{(n)}$ is the coupling of n Pomerons to the external particle a. Since the positive-frequency component of χ is the same as that of $-i\psi/2$, and the negative component is the same as that of $i\overline{\psi}/2$, Eq. (3.11) can be written as

$$A_{ab}(Y,\vec{B})=-is\sum_{n,m=1}^{\infty}2^{n+m}(-1)^n g_a^{(n)}g_b^{(m)}\times\langle\overline{0}|\chi(Y,\vec{B})^n\chi(0,\vec{0})^m|0\rangle\,. \tag{3.12}$$

Note that no ordering ambiguity exists since all of the emissions (absorptions) are assumed to take place at the same rapidity point. Equation (3.12) indicates that our one-component formulation is equivalent to the original RFT, and there is no difficulty in relating the contents of the χ formulation to physical observables.

It is more convenient, for what follows, to work in a Schrödinger picture, where

$$A_{ab}(Y,\vec{B})=-is\sum_{n,m=1}^{\infty}2^{n+m}(-1)^n g_a^{(n)}g_b^{(m)}\times\langle\overline{0}|\chi(\vec{B})^n e^{-Y\tilde{H}}\chi(\vec{0})^m|0\rangle\,. \tag{3.13}$$

In terms of our gauge-transformed theory, Eq. (3.12) becomes

$$A_{ab}(Y,\vec{B})=-is\sum_{n,m=1}^{\infty}(2)^{n+m}(-1)^n g_a^{(n)}g_b^{(m)}\times\langle\overline{\psi}_0|\chi(\vec{B})^n e^{-Y\tilde{H}}\chi(\vec{0})^m|\psi_0\rangle\,, \tag{3.14}$$

where we have introduced the gauge-transformed left- and right-vacuum states

$$|\psi_0\rangle=e^{-i\Lambda_q}|0\rangle\,,\quad \langle\overline{\psi}_0|=\langle\overline{0}|e^{i\Lambda_q}\,. \tag{3.15}$$

The object of physical interest, therefore, is the quantum time-evolution operator $e^{-Y\tilde{H}}$, or $e^{-Y\tilde{H}}$ One procedure for calculating it is by diagonalizing or partially diagonalizing the Hamiltonian. This will be done in Sec. V, leading to a quantum-lattice-spin model. Another way of proceeding is to specify the matrix elements of $e^{-Y\tilde{H}}$ first for Y small, then generalizing to finite Y by a path integral.

C. Path-integral formalism

We start by first considering the $D=0$ case. We evaluate the matrix elements of $e^{-Y\tilde{H}}$ in our χ basis:

$$\langle\chi',y'|\chi,y\rangle\equiv\langle\chi'|e^{-(y'-y)\tilde{H}}|\chi\rangle\,. \tag{3.16}$$

For $(y'-y)$ sufficiently small, (3.16) can be defined directly in terms of the matrix element of $\tilde{H}$. A standard analysis then leads to a path-integral representation at finite values of $(y'-y)$,

$$\langle\chi',y'|\chi,y\rangle=\int\delta\chi\int\delta\overline{P}_\chi\exp\left\{\frac{1}{\hbar}\int_y^{y'}[i\overline{P}_\chi\partial_y\chi-\tilde{H}(\chi,\overline{P}_\chi)]dy\right\} \tag{3.17}$$

$$\equiv\int\delta\chi\exp\left[\frac{1}{\hbar}\int_y^{y'}\overline{L}_{\rm eff}(\chi,\partial_y\chi)dy\right]\,. \tag{3.18}$$

In writing (3.18), $\overline{L}_{\rm eff}$ is *derived* from $\tilde{H}$ by performing the $\overline{P}_\chi$-functional integration.

Because the kinetic terms in $\tilde{H}$ also depend on χ, the effective Lagrangian, $\overline{L}_{\rm eff}$, differs from the classical Lagrangian by a set of quantum corrections. These quantum corrections can be obtained from (3.18) directly if we substitute $\overline{L}_c(\chi,\partial_y\chi)$ for $\overline{L}_{\rm eff}$ [with $\Delta(\chi)$ and $U_c(\chi)$ replaced by $\Delta_q(\chi)$ and $U_q(\chi)$], and supply a normalization factor, $N(\chi,\partial_y\chi)$. This factor can be determined by the locality condition

$$\lim_{\epsilon\to 0}\langle\chi',y+\epsilon|\chi,y\rangle=\delta(\chi'-\chi)\,,$$

thus leading to[15]

$$\overline{L}_{\rm eff}=L_c+\hbar\delta(0)\ln N\,. \tag{3.19}$$

We have verified that this procedure leads to the same result as the Hamiltonian formalism. Since the Hamiltonian formalism also turns out to be more convenient for the $D\neq 0$ case, we adopt it here directly.

Substituting Eq. (3.5c) into (3.17), we have

$$\int\delta\chi\left(\exp\left[-\frac{1}{\hbar}\int U_q(\chi)dy\right]\right)\int\delta\overline{P}_\chi\left(\exp\left(\int\frac{ig_0}{8\hbar}\overline{P}_\chi{}^3dy\right)\right)\left(\exp\left\{\frac{1}{\hbar}\int[i\overline{P}_\chi\partial_y\chi-\tfrac{1}{4}\overline{P}_\chi\Delta_q(\chi)\overline{P}_\chi]dy\right\}\right)\,. \tag{3.20}$$

The convergence for the χ integration is controlled by $U_q(\chi)$ and for the $\overline{P}_\chi$ integration by the $\overline{P}_\chi \Delta_q(\chi) \overline{P}_\chi$ term. (We assume first that the result of the $\overline{P}_\chi$ integration does not alter the convergence in χ. Although this is clearly true, we nevertheless justify this assertion *a posteriori*). Since $\Delta_q(\chi)$ is always positive, for χ large, the χ integration can be kept along the real axis. Similarly, the $\overline{P}_\chi$ integration path is also along the real axis. (A small negative imaginary part should also be added in the case $|\Delta_0|^2 < 3g_0^2$.) Therefore, the positivity of $\Delta_q(\chi)$ assures us that no contour distortion is required as α_0 is increased above 1.

The path-integral formalism of the original RFT can be written in terms of ϕ and χ; the expression analogous to (3.20) is

$$\int \delta\chi \int \delta\phi \exp\left(\frac{1}{\hbar}\int [-2i\phi\partial_y\chi - (\Delta_0 + ig_0\phi)(\phi^2+\chi^2)]dy\right). \tag{3.21}$$

It was emphasized in Ref. 6 that for $\Delta_0 > 0$ the integration paths in (3.21) are along real axes thus defining the bare perturbation theory. Since the functions inside the functional integrals are to be treated classically, one can consider that Eq. (3.20) is obtained from (3.21) by a change of variable (the difference due to normal-ordering is unimportant so far as the convergence question is concerned):

$$\overline{P}_\chi \equiv -2\left\{\phi + \frac{i}{3g_0}[\Delta_q(\chi) - \Delta_0]\right\}. \tag{3.22}$$

The ϕ integration runs along the real axis and corresponds to a $\overline{P}_\chi$ path parallel to the real axis but with a constant imaginary part $-(2i/3g_0)[\Delta_q(\chi) - \Delta_0]$. Conversely, by continuing the $\overline{P}_\chi$ integration path to the real axis, the ϕ path is forced to have, for fixed χ, a constant imaginary part $-(i/3g_0)[\Delta_q(\chi) - \Delta_0]$. It is a matter of straightforward algebra to show that, in Eq. (3.21), lowering the ϕ contour into the lower ϕ plane always helps the convergence. Since $-(1/3g_0)[\Delta_q(\chi) - \Delta_0]$ is always negative except for a small χ interval where it ranges from 0 to

$$\left(\frac{1}{3g_0}\right)[\Delta_0 - (\Delta_0^2 - 3g_0^2\hbar)^{1/2}],$$

one easily verifies that this brief venture into the upper ϕ plane also leads to convergence. Therefore, for $\Delta_0 > 0$, Eq. (3.21) can be distorted into Eq. (3.20). For $\Delta_0 < 0$, Eq. (3.21) requires a contour distortion, which can again be distorted into Eq. (3.20) with the χ and $\overline{P}_\chi$ integration paths along the real axes.

In Eq. (3.20), we have written the integrand as a product of three factors: The first factor involves the potential and is taken out of the $\overline{P}_\chi$ integral. The third factor is Gaussian in $\overline{P}_\chi$, and it can be integrated explicitly if the second factor is absent. This suggests that we expand $\exp[(ig_0/8\hbar)\overline{P}_\chi^3]$ in a power series thus developing a quantum perturbation expansion.

Introducing a source term,

$$\exp\left[\frac{1}{\hbar}\int k(y)\overline{P}_\chi dy\right],$$

into the third factor, we can replace $\overline{P}_\chi$ in the second factor by $(\hbar/i)\partial/\partial k(y)$, which can then be pulled out of the functional integral. The remaining integral yields

$$(\text{const}) \times \exp\left\{-\frac{1}{\hbar}\int [(\partial_y\chi + k)\Delta_q(\chi)^{-1}(\partial_y\chi + k) + \tfrac{1}{2}\hbar\delta(0)\ln\Delta_q(\chi)]dy\right\}. \tag{3.23}$$

After functionally differentiating with respect to $k(y)$ to the desired order and then setting $k = 0$, Eq. (3.20) is now expressed as a sum in increasing powers of $\partial_y\chi$. In order to get back to $\overline{L}_{\text{eff}}$, a "logarithm" has to be taken. It is clear that only those "connected terms," i.e., terms with a single y integration, can come from the first-order expansion of $\exp[(1/\hbar)\int \overline{L}_{\text{eff}} dy]$. Keeping terms to order $(\partial_y\chi)^4$, we obtain

$$\overline{L}_{\text{eff}} \simeq \left\{\left[\frac{-(\partial_y\chi)^2}{\Delta_q(\chi)} + \frac{g_0(\partial_y\chi)^3}{\Delta_q(\chi)^3} - \frac{4g_0^2(\partial_y\chi)^4}{9\Delta_q(\chi)^5} + \cdots\right] - U_q(\chi)\right\}$$
$$- \hbar\delta(0)\left\{\left[\frac{3g_0\partial_y\chi}{2\Delta_q(\chi)^2} - \frac{9g_0^2(\partial_y\chi)^2}{\Delta_q(\chi)^4} + \cdots\right] + \tfrac{1}{2}\ln\Delta_q(\chi)\right\}. \tag{3.24}$$

The expression in the first curly brackets is precisely the expansion (2.19) for the classical Lagrangian, with $\dot{\chi}$ replaced by $i\partial_y\chi$, $\Delta(\chi)$ by $\Delta_q(\chi)$, and $U_c(\chi)$ by $U_q(\chi)$. The second curly brackets represents a quantum correction; note that it is inversely proportional to the lattice spacing, a_t.[15]

We also note that the χ integration is always convergent so long as the expansion, Eq. (3.24), is defined.

The generalization to the $D \neq 0$ case is straightforward and will be done in Sec. IV after the discussion on the formulation of a $D = 0$ classical

spin-$\frac{1}{2}$ lattice analog of RFT.

The above quantum expansion also treats the $\Delta_0>0$ and $\Delta_0<0$ regions on an equal footing. Furthermore, since it is in correspondence with our classical expansion, (2.19), our expansion automatically picks out the branch of RFT defined by Eq. (2.13). This is not necessarily an obvious result since, in arriving at the classical Lagrangian, (2.19), Eq. (2.13) was used. However, in deriving Eq. (3.24), no reference to (2.13) was made. In order to understand this result, we refer to Eq. (2.18), where classical stationary points are listed. We observe that for $\Delta_0>0$ only the perturbative stationary point (a) lies in the region where our quantum RFT, Eq. (3.20), is defined. For $\Delta_0<0$, point (a) moves out of the range of definition for Eq. (3.20), and stationary points (b), (c), and (d) move onto the real axes. Therefore, in the region where our quantum expansion converges, the quantum theory is in exact accordance with the boundary condition, Eq. (2.13), for our classical RFT.

IV. FINITE-TEMPERATURE ONE-COMPONENT LATTICE-SPIN FORMULATION OF RFT

Efforts[6,16] in attempting to utilize the renormalization-group technique on a lattice for studying the infrared behavior of RFT began soon after the work of Ref. 2 appeared; so far, most of these efforts have not met with much success. The major obstacles involved are (a) the uncertainty in the number of ordered parameters required, (e.g., are there two or simply one spin at each site?), and (ii) the unusual symmetries of RFT. We concentrate in this section on formulating an analog model of RFT in terms of an interacting classical spin-$\frac{1}{2}$ system on a rapidity–impact-parameter lattice. Our detailed analysis leads to a model which is similar to, but different from that of Ref. 6. We pay special attention to the relation between the anti-Hermiticity of the triple-Pomeron coupling and the symmetries of classical RFT.

The standard procedure[16,17] for converting a quantum theory into a classical lattice theory involves the following steps: (1) Define a Wick-rotated quantum theory via a path-integral formalism. (2) After converting the space-time continuum into lattice points, reinterpret the path integral as defining a partition function. (3) Simplify, if possible, the resulting "Hamiltonian" by restricting the continuous spin at each site to discrete values. The first step has already been performed in Sec. III; exhaustive discussions on these steps can also be found in Refs. 16 and 18. We concentrate in what follows only on those aspects that are peculiar to RFT.

There are two problems which require our special attention. The first concerns how a spin-$\frac{1}{2}$ lattice model can emerge from our model, and, under such an approximation, what are the possible effects of those quantum corrections in Eq. (3.24) arising from our "nonlinear" Lagrangian? The second concerns the effects of an anti-Hermitian interaction and how this property can best be realized in a lattice formalism. Since both problems can best be discussed in the $D=0$ limit, we treat the $D=0$ model in detail before generalizing to the $D\neq 0$ case.

A. Two-level truncation and one-dimensional Ising system

Our starting point is the path-integral representation for the Green's function, Eq. (3.18), with the effective Lagrangian given by Eq. (3.24). We formulate a lattice-spin model by first going onto a rapidity lattice, with a lattice spacing a_t. We next replace $\partial_y\chi$ in $\bar{L}_{\rm eff}$ by $(\chi_i-\chi_{i-1})a_t^{-1}$ and $\delta(0)$ by a_t^{-1}.

Since $\bar{L}_{\rm eff}$ depends only on χ^2 and $\partial_y\chi$, the lattice interaction will be of two general types: (i) self-interactions at each site and (ii) interactions between nearest-neighbor sites. Regarding (3.18) as the partition function for a classical lattice-spin system, we find

$$Z=\int\prod_i d\mu(\chi_i)e^{-H_{\rm eff}(\chi)}, \tag{4.1}$$

where $H_{\rm eff}$ contains only the nearest-neighbor interactions, and the measure, $d\mu$, a statistical weight given by

$$d\mu=\left\{\exp\left[-\frac{a_t}{h}U_q(\chi)+2\ln\Delta_q(\chi)-\frac{2\chi^2}{ha_t\Delta_q(\chi)}+\frac{2g_0^{\,2}\chi^2}{a_t^{\,2}\Delta_q(\chi)^4}+\cdots\right]\right\}d\chi, \tag{4.2}$$

includes all self-interaction terms. Note that our classical spin takes on values from $-\infty$ to $+\infty$ and $d\mu(\chi)$ is an even function of χ.

In the limit $|\Delta_0|/g_0\to\infty$, $d\mu(\chi)$ is dominated by the potential, $U_q(\chi)$, *provided that* a_t *is not taken to zero first*. For $\Delta_0>0$, $d\mu(\chi)$ is peaked at $\chi=0$, and, for $\Delta_0<0$, it is peaked symmetrically about $\chi=\pm|\Delta_0|/g_0$. For the physically interesting case where $\Delta_0<0$, $d\mu(\chi)$ approaches in this limit

$$[\delta(\chi-|\Delta_0|/g_0)+\delta(\chi+|\Delta_0|/g_0)]d\chi.$$

It is then simpler to work with a classical spin,

$$\sigma_i\equiv\left(\frac{g_0}{|\Delta_0|}\right)\chi_i \tag{4.3}$$

for each lattice site where σ_i takes on values $+1$ and -1. The partition function can then be written

as

$$Z = \sum_{\{\sigma=\pm 1\}} e^{-H_{\rm eff}}, \tag{4.4}$$

where

$$H_{\rm eff} = -K \sum_i \sigma_{i+1}\sigma_i . \tag{4.5}$$

This represents a one-dimensional classical Ising chain.

The coupling constant, K, plays the role of an inverse temperature. It receives contributions from most terms in Eq. (3.24), including those quantum corrections, i.e., terms proportional to $\delta(0)$. Since, under the two-level truncation, terms with odd powers of $(\partial_y\chi)$ [i.e., $(\partial_y\chi)^{2n+1}$, $n = 0, 1, 2, \ldots,$] do not contribute to Z when summed over $\{\sigma_i\}$, the leading order contributions are

$$K = \frac{|\Delta_0|}{\hbar a_t g_0^{\,2}} - \frac{9}{8}\frac{1}{a_t^{\,2}|\Delta_0|^2} + \cdots . \tag{4.6}$$

The first term in (4.6) comes from $-(\partial_y\chi)^2/\Delta_q(\chi)$ in (3.24), and the second term comes from a quantum correction. *In the limit* $|\Delta_0|/g_0 \to \infty$ *and* $a_t \neq 0$, *we find that the quantum corrections are not important for our arriving at a classical spin-*$\frac{1}{2}$ *lattice analog model of RFT, and their contributions can be dropped.* In this limit, $K > 0$, so that the interaction is ferromagnetic. In what follows, we set $\hbar = 1$.

B. Anti-Hermiticity and transfer matrix

The leading order contribution to $\bar{L}_{\rm eff}$ from the anti-Hermitian interaction in $\tilde{H}$ is $g_0[\partial_y\chi/\Delta_q(\chi)]^3$. Under a two-level truncation approximation, $(\partial_y\chi)^3$ becomes $4|\Delta_0|^3(g_0 a_t)^{-3}(\sigma_i - \sigma_{i-1})$; its contribution to the partition function vanishes when summed over all spins. The same holds true for the anti-Hermitian interaction coming from the quantum correction term, $-\hbar\delta(0)(3g_0/2)[\partial_y\chi/\Delta_q(\chi)^2]$. It is interesting to note that terms such as $\partial_y\chi$, $(\partial_y\chi)^3$, etc., are not invariant under the transformation $S(\chi \to -\chi)$ and $T(y \to -y)$ separately. However, they are invariant under the combined transformation ST; this is precisely the symmetry of the classical RFT.

Under a quantum-mechanical time-reversal transformation, the Hamiltonian of a quantum system is mapped into its adjoint, $H^\dagger$. For a non-Hermitian system, $H \neq H^\dagger$, T is not an invariant. In the special case of an anti-Hermitian interaction ($H \to H^\dagger$ under S: $\chi \to -\chi$, $P_\chi \to -P_\chi$), the quantum symmetry of the system is the combined quantum transformation ST. Since the time-reversal invariance is broken in both classical and quantum analyses, it must also enter in our lattice formulation of RFT. In order to accommodate T noninvariance, it is desirable to adopt a lattice formalism in which a preferred rapidity direction is specified. This is indeed the case if we reinterpret Eq. (4.4) as defining a lattice system through a "transfer matrix"; an asymmetrical transfer matrix automatically leads to a T noninvariant theory.

Be retaining the anti-Hermitian term, the effective Hamiltonian of our analog model becomes

$$H_{\rm eff} = -K \sum_i \sigma_{i+1}\sigma_i - (B/2)\sum_i (\sigma_{i+1} - \sigma_i) + 2C \sum_i \sigma_i^2 . \tag{4.7}$$

In Eq. (4.7), we have also added a constant "background energy" per site, $2C$. Recall that, because of normal-ordering, the ground state of our quantum RFT is the perturbative vacuum, with $\epsilon_0 = 0$. As we shall see shortly, the partition function should be interpreted as the matrix elements of the time-evolution operator, $\exp(-y\tilde{H})$, as $y \to \infty$; the condition $\epsilon_0 = 0$ requires Z approaching a finite value in the limit. Therefore, for our present purpose, C is an adjustable free parameter. By examining $U_q(\chi)$, we expect C to be of the order $(a_t)(|\Delta_0|^3/g_0^{\,2})$; this addition should have accompanied our replacing $d\mu(\chi_i)$ by δ functions.

We label the lattice sites from left to right as $N', N'-1, N'-2, \ldots, -N+1, -N$ (i.e., $+\hat{y}$ direction points to the left), and consider the limit $N', N \to \infty$. Treating the lattice as an open chain, the partition function $Z_{N'+N+1}$ can be written as

$$Z_{N'+N+1} = \sum_{\{\sigma=\pm 1\}} M_{\sigma_{N'}\sigma_{N'-1}} M_{\sigma_{N'-1}\sigma_{N'-2}} \cdots M_{\sigma_{-N+1}\sigma_{-N}} \equiv \sum_{\sigma_{N'}} \sum_{\sigma_{-N}} Z(N', -N), \tag{4.8}$$

where

$$M_{\sigma'\sigma} \equiv \exp[-C + K\sigma'\sigma + (B/2)(\sigma' - \sigma)]$$

is the transfer matrix

$$M = \begin{pmatrix} M_{++} & M_{+-} \\ M_{-+} & M_{--} \end{pmatrix} = \begin{pmatrix} e^{-C+K} & e^{-C-K+B} \\ e^{-C-K-B} & e^{-C+K} \end{pmatrix}, \tag{4.9}$$

and $Z(N', -N) \equiv M^{N'+N}$ is a 2×2 matrix defined by matrix multiplications.

The system described by (4.8) has the following features: (i) the ferromagnetic Ising interaction tries to align neighboring spins, and it raises the energy of the system by $2K$ whenever a "spin-flip" occurs between neighboring sites. (ii) Moving on the lattice in the $+\hat{y}$ direction, the anti-Hermitian interaction raises the energy of the system by B if a spin flip from $\sigma_i = +1$ to $\sigma_{i+1} = -1$ occurs, and it

lowers the energy by the same amount if $\sigma_i = -1$ and $\sigma_{i+1} = +1$. (iii) Moving in the $-\hat{y}$ direction, the effect of the B interaction is reversed. It is easy to show that, for B large, σ_∞ tends to align statistically at $+1$, as if there were a magnetic field present at $N' = +\infty$. Conversely, $\sigma_{-\infty}$ tends to align at -1. Therefore, the ground-state configuration, as a result of the anti-Hermitian interaction, is kinklike, starting near -1 at $y = -\infty$ and changing to $+1$ at $+\infty$.

C. Structure of ground state

Because of translational invariance, $\langle\sigma\rangle = 0$ holds. On the other hand, the spin-spin correlation function, $\langle\sigma_k\sigma_l\rangle$, $k > l$, is nonzero, and its dependence on $|k-l|$ provides a measure on the scale over which the "kink transition" takes place, i.e., the correlation length. Since the spin-spin correlation function is the analog of a two-point in RFT, we show next how it is obtained in a lattice formalism; this also serves to clarify the structure of our kink state.

It follows from (4.8) that $Z(N', -N)$ satisfies a matrix recursion relation

$$Z(N', -N) = MZ(N'-1, -N) = Z(N', -N+1)M, \tag{4.10}$$

and it can be found by diagonalizing the transfer matrix M. Since $M \neq M^\dagger$, one must distinguish between the left and right eigenvectors. Denoting the eigenvalues by λ_i, $\lambda_0 > \lambda_1$, and their left and right eigenvectors by $|\bar{\psi}_i\rangle$ and $|\psi_i\rangle$ respectively, we find

$$Z(N', -N) = \lambda_0{}^{N'+N}|\psi_0\rangle\langle\bar{\psi}_0| + \lambda_1{}^{N'+N}|\psi_1\rangle\langle\bar{\psi}_1|, \tag{4.11}$$

or, its matrix elements, when expressed in terms of components of $\psi_i, \bar{\psi}_i$,

$$Z(\sigma_{N'}, \sigma_{-N}) = \sum_{i=0}^{1} \lambda_i{}^{N'+N}\psi_i(\sigma_{N'})\bar{\psi}_i(\sigma_{-N}), \tag{4.12}$$

where we have adopted the normalization condition

$$\langle\bar{\psi}_i|\psi_j\rangle = \delta_{i,j}. \tag{4.13}$$

It follows from (4.9) that

$$\lambda_0 = 2e^{-C}\cosh K, \quad \lambda_1 = 2e^{-C}\sinh K \tag{4.14}$$

and

$$\begin{aligned} \bar{\psi}_0(\pm) &= +\frac{1}{\sqrt{2}}e^{\mp B/2}, \quad \psi_0(\pm) = +\frac{1}{\sqrt{2}}e^{\pm B/2}, \\ \bar{\psi}_1(\pm) &= \mp\frac{1}{\sqrt{2}}e^{\mp B/2}, \quad \psi_1(\pm)\frac{1}{\sqrt{2}}e^{\pm B/2}. \end{aligned} \tag{4.15}$$

Because $\lambda_0 > \lambda_1$, in the limit $N', N \to \infty$, the partition function is controlled by the λ_0 term. The spin state, as $N' \to \infty$, approaches $|\psi_0\rangle$ (with a $\sigma_\infty = +1$ to $\sigma_\infty = -1$ statistical ratio e^B). Conversely, σ_{-N}, $N \to \infty$, approaches the state $|\bar{\psi}_0\rangle$ (the statistical weight for $\sigma_{-\infty} = -1$ to $\sigma_{-\infty} = +1$ is also e^B), thus confirming our intuitive analysis given earlier. We also note that the requirement that the ground-state energy be zero corresponds to $\lambda_0 = 1$. In the limit $K \to \infty$, Eqs. (4.6) and (4.14), together with $C \propto a_t|\Delta_0|^3/g_0{}^2$, lead to the constraint

$$a_t = O(|\Delta_0|^{-1}). \tag{4.16}$$

The average magnetization at the kth site can be written in terms of the $Z(\sigma_i, \sigma_j)$ matrix as

$$\langle\sigma_k\rangle = (Z_{N'+N+1})^{-1}\sum_{\sigma_{N'}}\sum_{\sigma_k}\sum_{\sigma_{-N}} Z(\sigma_{N'}, \sigma_k)\sigma_k Z(\sigma_k, \sigma_{-N}). \tag{4.17}$$

In the limit $N', N \to \infty$, the λ_1 contribution [see (4.11)] diminishes so that

$$\begin{aligned} \langle\sigma_k\rangle &= \sum_{\sigma_k}\bar{\psi}_0(\sigma_k)\sigma_k\psi_0(\sigma_k) \\ &= \langle\bar{\psi}_0|\tilde{\sigma}|\psi_0\rangle, \end{aligned} \tag{4.18}$$

where $\tilde{\sigma}$ is the diagonal matrix σ_z. From (4.15), we see that $\langle\sigma_k\rangle = 0$, as expected.

The spin-spin correlation function can also be written in terms of $Z(M, N)$ as

$$\begin{aligned} \langle\sigma_k\sigma_l\rangle &= \lim_{N', N\to\infty}(Z_{N'+N+1})^{-1}\left[\sum_{\sigma_{N'}}\sum_{\sigma_k}\sum_{\sigma_l}\sum_{\sigma_{-N}} Z(\sigma_{N'}, \sigma_k)\sigma_k Z(\sigma_k, \sigma_l)\sigma_l Z(\sigma_l, \sigma_{-N})\right] \\ &= \lambda_0{}^{-(k-l)}\langle\bar{\psi}_0|\tilde{\sigma}M^{k-l}\tilde{\sigma}|\psi_0\rangle \\ &= \lambda_0{}^{-(k-l)}\langle\bar{\psi}_0|\tilde{\sigma}e^{-(k-l)H_{\text{eff}}}\tilde{\sigma}|\psi_0\rangle. \end{aligned} \tag{4.19}$$

This representation clearly demonstrates that $\langle\sigma_k\sigma_l\rangle$ is precisely the two-point function of RFT. In (4.19) and in what follows, λ_0 is understood to be 1.

In the basis where M is diagonal, the spin matrix $\tilde{\sigma}$ is no longer diagonal. Under our convention (4.15), $\tilde{\sigma}$ is $\sum_{ij}\sigma_{ij}|\psi_i\rangle\langle\bar{\psi}_j|$, where

$$\{\sigma_{ij}\} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}. \tag{4.20}$$

It then follows that

$$\begin{aligned} \langle\sigma_k\sigma_l\rangle &= \lambda_0{}^{-(k-l)}\langle\bar{\psi}_1|e^{-(k-l)H_{\text{eff}}}|\psi_1\rangle \\ &= (\tanh K)^{k-l}, \end{aligned} \tag{4.21}$$

so that the inverse correlation length δ, measured in terms of the lattice spacing is $-a_t^{-1}\ln(\tanh K)$. It is interesting to point out that the two-point function depends only on the eigenvalues of M, and it is the same as what we would have obtained by treating the analog model as a conventional Ising model. The anti-Hermiticity of RFT only enters in the description of the left and right eigenvectors of M (and hence the peculiar ground-state configuration.)

The inverse lattice spacing, a_t^{-1}, provides a cutoff in E. At $D=0$, the energy scale of RFT is $|\Delta_0|$; a lattice analog model can be meaningful only if $a_t^{-1} \gtrsim |\Delta_0|$. Since a_t cannot be set equal to zero first in order to arrive at a spin-$\frac{1}{2}$ system, we can therefore replace a_t in (4.6) by $\gamma|\Delta_0|^{-1}$, where γ is of order $O(1)$. This is in agreement with Eq. (4.16). For a one-dimensional Ising chain, it is well known that a phase transition does not take place and the correlation length, δ^{-1}, is finite. Since δ is also the energy gap between the first excited state and the ground state, we find, for K large,

$$\delta \propto \exp(-2|\Delta_0|^2/\gamma g_0^2),$$

in qualitative agreement with the result of Ref. 7. Apparently, the lattice spacing enters into the analog model in a nontrivial fashion, and one cannot appeal to the universality principle for help at $D=0$.

Our spin space is two-dimensional; the natural basis is the one in which $\tilde{\sigma}$ is diagonal ($\sigma=\pm1$). We could equally well have used the left and right ground state $\bar{\psi}_0$ and ψ_0 as bases, in terms of which, for B large,

$$|+\rangle \simeq \sqrt{2}e^{-B/2}|\psi_0\rangle - \sqrt{2}e^{-3B/2}|\bar{\psi}_0\rangle,$$

$$|-\rangle \simeq \sqrt{2}e^{-B/2}|\bar{\psi}_0\rangle - \sqrt{2}e^{-3B/2}|\psi_0\rangle. \tag{4.22}$$

This representation will become useful in Sec. V.

D. Classical lattice-spin formulation

The quantum Hamiltonian density at $D\neq 0$ is

$$H(\chi, P_\chi) = H^0(\chi, P_\chi) + \alpha'(\vec{\nabla}_b\chi)^2 + \tfrac{1}{4}\alpha'(\vec{\nabla}_b P_\chi)^2 \tag{4.23}$$

and, after the gauge transformation, (3.2), it becomes

$$\tilde{H}(\chi, \bar{P}_\chi) = \tilde{H}^0(\chi, \bar{P}_\chi) + \alpha'(\vec{\nabla}_b\chi)^2 + \tfrac{1}{4}\alpha'(\vec{\nabla}_b\Lambda_q')^2 + \tfrac{1}{2}\alpha'\vec{\nabla}_b\Lambda_q'\cdot\bar{P}_\chi + \tfrac{1}{4}\alpha'(\vec{\nabla}_b\bar{P}_\chi)^2. \tag{4.24}$$

In (4.23) and (4.24), the superscript "0" refers to the corresponding $D=0$ expression, and we have also dropped a constant term.

We next perform the functional integration, with Eq. (4.24) substituted for the Hamiltonian density. We anticipate a similar result that, for $\Delta_0<0$ and large, χ is restricted to $\pm|\Delta_0|/g_0$ at each lattice site. Since $\Lambda_q'(\chi)$ is a function of χ^2 only, $\vec{\nabla}_b\Lambda_q'$ vanishes when it is replaced by a finite difference between values a neighboring sites; we therefore drop terms in (4.24) containing the factor $\vec{\nabla}_b\Lambda_q'$ in what follows. Writing the integrand as a product of three factors and introducing a classical source $(i/\hbar)\int k(t,k)\bar{P}_\chi\, dy\, d^D\vec{b}$ as was done earlier, the resulting Gaussian integral analogous to Eq. (3.22) now yields

$$\text{const}\times\exp\left(-\frac{1}{\hbar}\int dy\, d^D\vec{b}\{(\partial_y\chi+k)[\Delta_q(\chi)-\alpha'\nabla_b^2]^{-1}(\partial_y\chi+k)\}\right). \tag{4.25}$$

Next we perform functional differentiations and then exponentiate those "connected" terms; the effective Lagrangian becomes

$$L_{\text{eff}} \simeq (-\partial_y\chi[\Delta_q(\chi)-\alpha'\nabla_b^2]^{-1}\partial_y\chi - \alpha'(\vec{\nabla}_b\chi)^2 + g_0\{[\Delta_q(\chi)-\alpha'\nabla_b^2]^{-1}\partial_y\chi\}^3+\cdots) - U_q(\chi). \tag{4.26}$$

In (4.25) and (4.26), the quantum corrections are not exhibited. Although the quantum corrections contain nontrivial dependence on the lattice spacing, they are unimportant for arriving at the dominant interaction for the $D=0$ lattice model, Eq. (4.6). So long as lattice spacings are kept nonzero, the quantum corrections need not be kept explicitly. Expanding Eq. (4.26) in α' and anticipating that, for $\Delta_0<0$, $\chi^2\simeq|\Delta_0|^2/g_0^2$, we obtain, up to first order in α',[18]

$$L_{\text{eff}} \simeq \left[\frac{-1}{2|\Delta_0|}(\partial_y\chi)^2 - \alpha'(\vec{\nabla}_b\chi)^2 + \frac{\alpha'}{4|\Delta_0|^2}(\vec{\nabla}_b\partial_y\chi)^2 + \frac{g_0}{8|\Delta_0|^3}(\partial_y\chi)^3 + \frac{3g_0\alpha'}{16|\Delta_0|^4}(\partial_y\chi)^2\nabla_b^2(\partial_y\chi)+\cdots\right] - U_q(\chi). \tag{4.27}$$

Our classical lattice-spin model will be formulated starting from the above expansion.

By introducing a lattice spacing, a_s, in the impact-space direction, we can move onto the rapidity–impact-parameter lattice (we have already performed the Wick rotation.) We note in particular the presence of $U_q(\chi)$ in (4.27); an analysis similar to that performed for $D=0$ would justify replacing the integration measure $d\mu(\chi)$ by

$$[\delta(\chi-|\Delta_0|/g_0)+\delta(\chi+|\Delta_0|/g_0)]d\chi$$

in the limit $|\Delta_0|/g_0\to\infty$ and $\Delta_0<0$. Under such an

approximation, it is again convenient to introduce an Ising spin S for each site, $S \equiv (g_0/|\Delta_0|)\chi$, so that S takes on values ± 1. The resulting interacting Hamiltonian can be obtained from Eq. (4.27).

The first term in (4.27) leads to a nearest-neighbor (nn) interaction in the space direction. When the third term is expanded, in addition to modifying the nn interactions, it adds a next-nearest-neighbor (nnn) interaction. All three terms of the two-spin-interaction type; they are invariant under transformations $T(y \to -y)$, $S(\chi \to -\chi)$, and $P(\vec{b} \to -\vec{b})$ separately. The next two terms in (4.27) involve three-spin couplings, which are invariant only under P and the combined transformation TS. As was emphasized in Ref. 6, invariance under TS but not T, S separately is the symmetry of "classical" RFT; expansion (4.27) is thus sufficient as far as determining the critical behavior of RFT is concerned. In what follows, we ignore the added structure due to the B and C terms in (4.7); the fourth term in (4.27) can therefore be dropped. The partition function can then be represented symbolically as

$$Z = \sum_{\{S_i\}} e^{-H_{\text{eff}}},$$

where

$$H_{\text{eff}} = -K_1 \sum_{\text{nn}(y)} S_i S_j - K_2 \sum_{\text{nn}(s)} S_i S_j - K_3 \sum_{\text{nnn}} S_i S_j \pm L \sum_{\langle ijk \rangle} S_i S_j S_k . \tag{4.28}$$

These four types of interactions are illustrated in Fig. 2; note, in particular, that the three-spin coupling, "$\pm L$", properly reflects the ST invariance of RFT.

In terms of the original parameters of RFT and lattice spacings, the coupling constants in (4.28), with $\hbar = 1$, are

$$K_1 = \frac{|\Delta_0|}{g_0^2}\frac{a_s^D}{a_t} - \frac{\alpha' a_s^{D-2}}{g_0^2 a_t},$$

$$K_2 = \frac{2\alpha' |\Delta_0|^2 a_t a_s^{D-2}}{g_0^2} - \frac{\alpha' a_s^{D-2}}{g_0^2 a_t^2}, \tag{4.29}$$

$$K_3 = \frac{\alpha'}{2g_0^2}\frac{a_s^{D-2}}{a_t}, \quad L = \frac{3\alpha'}{4|\Delta_0| g_0^2}\frac{a_s^{D-2}}{a_t^2}.$$

E Origin of lattice interactions

In the limit $|\Delta_0|/g_0$ large, $\Delta_0 < 0$, and α', a_t, a_s nonzero, coupling constants K_1, K_2, and K_3 in (4.29) are positive. The first three terms in H_{eff} represent asymmetrical ferromagnetic Ising interactions, and the last term characterizes the unique symmetry of the classical RFT.

Since the quantum correction is not needed in arriving at (4.26), the origin of these interactions can be directly traced by examining the classical Lagrangian [we will not bother to substitute $\Delta_q(\chi)$ for $\Delta(\chi)$ and U_q for U_c]

$$L(\chi, \dot{\chi}) = \dot{\chi}\bar{P}_\chi - \bar{H}_c(\chi, \bar{P}_\chi) \simeq \dot{\chi}\bar{P}_\chi - \tfrac{1}{4}\Delta(\chi)\bar{P}_\chi^2 - \alpha'(\vec{\nabla}_b\chi)^2 - \tfrac{1}{4}\alpha'(\vec{\nabla}_b\bar{P}_\chi)^2 + \tfrac{1}{8}ig_0\bar{P}_\chi^3 - U_c(\chi). \tag{4.30}$$

In writing (4.30), we have dropped terms proportional to $\vec{\nabla}_b\Lambda'$. Recall that $\phi = -\frac{1}{2}P_\chi$; we define a "shifted" field by

$$\bar{\phi}(\chi, \dot{\chi}) \equiv \phi - \frac{i}{3g_0}(\Delta_0 - \Delta(\chi)) = -\tfrac{1}{2}\bar{P}_\chi(\chi, \dot{\chi}), \tag{4.31}$$

so that Eq. (4.30) can be written as

$$\bar{L}(\chi, \dot{\chi}) \simeq -2\bar{\phi}\dot{\chi} - \Delta(\chi)\bar{\phi}^2 - \alpha'[(\vec{\nabla}_b\chi)^2 + (\vec{\nabla}_b\bar{\phi})^2] - ig_0\bar{\phi}^3 - U_c(\chi). \tag{4.32}$$

In comparing (4.32) with the original "two-component" Lagrangian, Eq. (1.1), we see that, in addition to replacing Δ_0 by $\Delta(\chi)$, the contribution due to the shift (4.31) has been grouped into the classical potential, $U_c(\chi)$.

In Eq. (4.31), $\bar{\phi}$ is a function of χ and $\dot{\chi}$; it can be obtained by solving the canonical equation $\dot{\chi} = \delta\bar{H}(\chi, \bar{P}_\chi)/\delta\bar{P}_\chi$ perturbatively. To match (4.26), we expand to first order in g_0 while keeping $\Delta(\chi)$ fixed

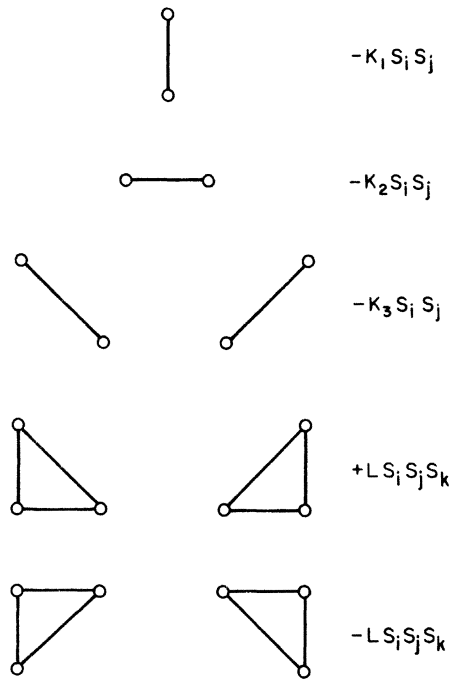

FIG. 2. Schematic representation of lattice Hamiltonian couplings for $D = 1$. Space runs across and rapidity runs up the page.

$$\bar{\phi}(\chi,\dot{\chi}) \simeq -(\Delta(\chi)-\alpha'\nabla_b^2)^{-1}\dot{\chi} - \tfrac{1}{2}3ig_0[\Delta(\chi)-\alpha'\nabla_b^2]^{-1}\{[\Delta(\chi)-\alpha'\nabla_b^2]\chi\}^2 . \tag{4.33}$$

By substituting (4.33) into (4.32), we explicitly verify that (i) The $\bar{\phi}\dot{\chi}$ and $\Delta(\chi)\bar{\phi}^2$ terms lead to the K_1 interaction. (ii) The $(\vec{\nabla}_b\chi)^2$ term leads to the K_2 interaction. (iii) The $(\vec{\nabla}_b\bar{\phi})^2$ term contributes to lower-order terms in K_1 and K_2. It, in addition, leads to the K_3 interaction. (iv) The three-spin interaction, $\pm L$, in (4.28), comes exclusively from the $\bar{\phi}^3$ term. Contributions to the $\pm L$ interaction from $\bar{\phi}\dot{\chi}$, $\bar{\phi}^2$, and $(\vec{\nabla}_b\bar{\phi})^2$ cancel identically to the order (4.33).

Our result, (4.29), is similar to, but differs in several important aspects from that of Cardy and Sugar.[6] We note that their result does not contain the K_1 and K_3 terms. Whereas we agree on the K_2 interaction, the three-spin coupling differs in that their coupling strength L is $(\alpha'|\Delta_0|/g_0^2)a_s^{D-2}$. It is unclear to us as to the significance of this difference in view of the condition (4.16) and the usual universality argument. However, we would like to point out that, in reaching their conclusion, Cardy and Sugar have made use of the specific form of the L coupling as well as arguments based on letting $a_t \to 0$. Our analysis indicates that the validity of those steps is in doubt.

Our confidence in the result, Eq. (4.29), is further enhanced by the observation that, whereas our lattice-spin model posseses a smooth $D=0$ limit, the model of Cardy and Sugar does not. Because of the absence of the K_1 interaction, the possibility that the critical indices of RFT are simply that of a conventional asymmetrical Ising system was ruled out in Ref. 6; this is *a priori* unjustified.

V. ONE-COMPONENT QUANTUM LATTICE-SPIN FORMULATION OF RFT

In this section we construct, in terms of the χ field, a quantum spin analog of RFT on a two-dimensional impact-parameter lattice. In this approach, the rapidity variable is kept continuous, and the quantum mechanics for $D=0$ is treated first.

Our treatment, not surprisingly, leads to the nearly identical lattice-spin models of Refs. 8–10. In those analyses, the spectrum of the single-site Hamiltonian in truncated by retaining only the two lowest-energy states, thus introducing a quantum Ising spin for each lattice site. The justification for this procedure lies in the realization that, for $\Delta_0<0$ and $|\Delta_0|/g_0 \to \infty$. the quantum splitting between two degenerate, classical ground states is much smaller than the level spacing above each ground state.[7] However, in Refs. 7–10, ψ and $\bar{\psi}$ fields are utilized. We believe that our choice of using the χ field as the basis of discussion is more natural and can provide a more "visual" description of the physics involved.

A. Ground state of quantum RFT at $D=0$

Recall that the quantum Hamiltonian in our χ representation is given by Eq. (3.1). Since $H(\chi,P_\chi)$ is non-Hermitian, one must distinguish between its right and left eigenvectors. Denote, in ascending order, the energy eigenvalues by $\{\epsilon_n\}$, $n=0, 1, 2, \ldots$, and the corresponding right and left eigenvectors by $\{\phi_n\}$ and $\{\bar{\phi}_n\}$, respectively, i.e., $H\phi_n=\epsilon_n\phi_n$ and $H^\dagger\bar{\phi}_n=\epsilon_n\bar{\phi}_n$. It is convenient to normalize the eigenvectors so that

$$\langle\bar{\phi}_n|\phi_m\rangle=\delta_{n,m}, \tag{5.1}$$

and the completeness relation becomes

$$\sum_n |\phi_n\rangle\langle\bar{\phi}_n| = I . \tag{5.2}$$

It has been proved in Ref. 7 that $\{\epsilon_n\}$ are nondegenerate and non-negative for both $\Delta_0>0$ and $\Delta_0<0$; a heuristic argument for the validity of (4.6) is also provided in Sec. V B.

It has been emphasized[7-9] that, owing to normal-ordering, the bare vacuum always remains the ground state of RFT with $\epsilon_0=0$. Since $H^\dagger$ also annihilates the bare vacuum, the "right vacuum" is the same as the "left vacuum", i.e., $|0\rangle=|\bar{\phi}_0\rangle=|\phi_0\rangle$, whose coordinate representation is given by the ground-state harmonic oscillator wave function

$$\phi_0(\chi)=\bar{\phi}_0(\chi)=(2/\pi)^{1/4}e^{-\chi^2} . \tag{5.3}$$

We stress that Eq. (5.3) holds for both $\Delta_0>0$ and $\Delta_0<0$. The removal of "zero-point energy," i.e., $\epsilon_0=0$, is accomplished by the quantum correction in $U_q(\chi)$, a fact alluded to earlier.

Let us consider the shifted Hamiltonian $\tilde{H}(\chi,P_\chi)$ next. Denoting its eigenvectors by $\{\psi_n\}$, it follows from (3.2) that

$$\psi_n(\chi)=e^{F(\chi)}\phi_n(\chi) \tag{5.4}$$

and

$$\tilde{H}\psi_n=\epsilon_n\psi_n , \tag{5.5}$$

where we have introduced a *real* function, $F(\chi)\equiv -i\Lambda_q(\chi)$, so that[19]

$$F'(\chi)=-i\Lambda_q'=\left(\frac{2}{3g_0}\right)(-\Delta_0+\Lambda_q(\chi)) . \tag{5.6}$$

Similarly, the eigenvectors of $\tilde{H}^\dagger\equiv e^{-F}H^\dagger e^F$ are

$$\bar{\psi}_n=e^{-F}\bar{\phi}_n , \tag{5.7}$$

where $\tilde{H}^\dagger\bar{\psi}_n=\epsilon_n\bar{\psi}_n$. In terms of these vectors, the

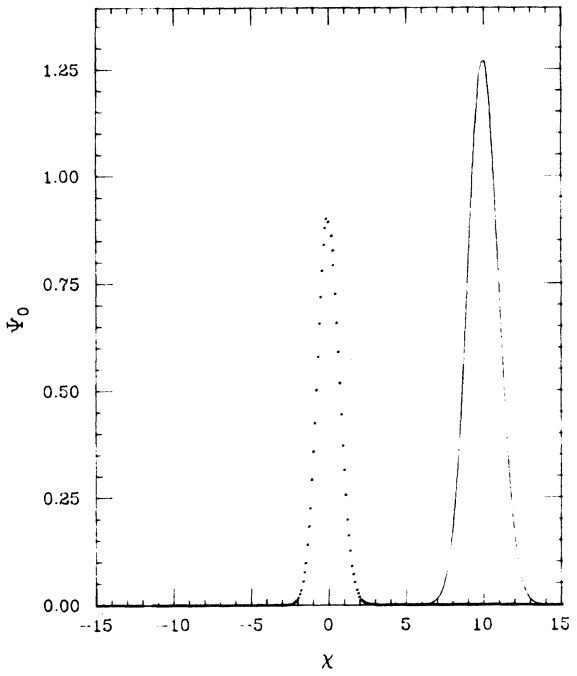

FIG. 3. Ground-state wave function, $\psi_0(\chi)$: $|\Delta_0|=1$, $g_0=0.1$; dotted curve is for $\Delta_0>0$, solid curve is for $\Delta_0<0$. The solid curve has been scaled down by a factor of 10^{25}.

orthogonality and completeness conditions become

$$\langle\bar{\psi}_n|\psi_m\rangle=\delta_{n,m},\quad \sum_n|\psi_n\rangle\langle\bar{\psi}_n|=I. \tag{5.8}$$

For any operator, A, the transformed operator is defined by $\tilde{A}=e^F A e^{-F}$. It follows from (5.4) and (5.7) that

$$\langle\bar{\psi}_n|\tilde{A}|\psi_m\rangle=\langle\bar{\phi}_n|A|\phi_m\rangle. \tag{5.9}$$

Therefore, matrix elements of an operator are invariant under our gauge transformation. The right and left vacuums are now represented respectively by wave functions

$$\psi_0(\chi)=(2/\pi)^{1/4}e^{F(\chi)-\chi^2},\quad \bar{\psi}_0(\chi)=(2/\pi)^{1/4}e^{-F(\chi)-\chi^2}. \tag{5.10}$$

Because of (5.6), one can show that $\psi_0(\chi)$ and $\bar{\psi}_0(\chi)$ are functions with a single "peak"; this is schematically represented in Fig. 3. For $\Delta_0>0$, both ψ_0 and $\bar{\psi}_0$ are peaked at $\chi\simeq 0$, and $\psi_0\simeq\bar{\psi}_0\simeq(2/\pi)^{1/4}e^{-\chi^2}$ in the limit $|\Delta_0|/g_0$ is large. However, for $\Delta_0<0$, ψ_0 is peaked at $\chi\simeq|\Delta_0|/g_0$, and $\bar{\psi}_0$ at $\chi\simeq-|\Delta_0|/g_0$. Furthermore, the half-width for ψ_0 and $\bar{\psi}_0$ has been increased by a factor of $\sqrt{2}$, i.e., near their respectively maxima,

$$\psi_0(\chi)\propto\exp[-\tfrac{1}{2}(\chi-|\Delta_0|/g_0)^2],\qquad \bar{\psi}_0(\chi)\propto\exp[-\tfrac{1}{2}(\chi+|\Delta_0|/g_0)^2]. \tag{5.11}$$

B. Non-Hermiticity and nature of spectrum

Our gauge transformation, Eq. (3.2), has been particularly useful in "unmasking" the non-Hermitian nature of RFT. Although ψ_0 and $\bar{\psi}_0$ are both peaked at $\chi=0$ when $\Delta_0>0$, they are unmistakably distinct after Δ_0 is continued to a negative value. This, in turn, signals that a non-Hermitian Hamiltonian is at work.

Under the simultaneous interchange of $P_\chi\rightarrow-P_\chi$ and $\chi\rightarrow-\chi$, $H(\chi,P_\chi)$ is transformed into $H^\dagger(\chi,P_\chi)$. This symmetry is a consequence of the non-Hermitian cubic interaction, and this transformation can formally be accomplished by simply reversing the sign of g_0 in (3.1). Similarly, $\tilde{H}\rightarrow\tilde{H}^\dagger$ under $g_0\rightarrow-g_0$. However, since eigenvalues are common to $\tilde{H}$ and $\tilde{H}^\dagger$, we can immediately conclude that $\{\epsilon_n\}$ must be functions of ${g_0}^2$. We can also conclude that it is possible to choose the phase so that $\psi_n(\chi,g_0)=\bar{\psi}_n(\chi,-g_0)$, in agreement with (5.10).[20] The fact that $\psi(\chi)$ centers around $\chi=|\Delta_0|/g_0$ for $\Delta_0<0$ is purely due to the convention that $g_0>0$, and no physical significance should be attached to this apparent symmetry.

To gain a qualitative understanding of the structure of the spectrum in the limit $g_0/|\Delta_0|\rightarrow 0$, let us first consider $\Delta_0>0$ and expand $\tilde{H}(\chi,P_\chi)$ about $\chi=0$:

$$\begin{aligned}\tilde{H}(\chi,P_\chi)&\simeq\left[-\frac{\Delta_0}{4}\frac{\partial^2}{\partial\chi^2}+(g_0)\left(\frac{1}{8}\frac{\partial^3}{\partial\chi^3}+\frac{3\chi g_0}{4\Delta_0}\frac{\partial}{\partial\chi}\right)+\cdots\right]\\&\quad+[\Delta_0(\chi^2-\tfrac{1}{2})+\cdots]\\&\simeq\Delta_0\left(\left(-\frac{1}{4}\frac{\partial^2}{\partial\chi^2}+\chi^2-\frac{1}{2}\right)+O(g_0/\Delta_0)\right).\end{aligned} \tag{5.12}$$

One immediately recognizes that the spectrum of $\tilde{H}$ is that of a harmonic oscillator with an energy-spacing Δ_0. This is the perturbative result, appropriate for $\Delta_0>0$. The ground-state energy is $\epsilon_0=0$ and the wave function is approximately $\psi_0\propto e^{-\chi^2}$, in agreement with the exact result, Eq. (5.10)

Next we consider the limit $g_0/|\Delta_0|\rightarrow 0$ with $\Delta_0<0$, and expand $\tilde{H}$ about $\chi=\pm|\Delta_0|/g_0$:

$$\begin{aligned}\tilde{H}_\pm(\chi,P_\chi)&\simeq\left[-\frac{|\Delta_0|}{2}\frac{\partial^2}{\partial\chi^2}+(g_0)\left(\frac{1}{8}\frac{\partial^3}{\partial\chi^3}\mp\frac{3}{8}\frac{\partial}{\partial\chi}\right)+\cdots\right]+\left\{\frac{|\Delta_0|}{2}\left[\left(\chi\mp\frac{|\Delta_0|}{g_0}\right)^2-1\right]+\cdots\right\}\\&\simeq|\Delta_0|\left\{\left[-\frac{1}{2}\frac{\partial^2}{\partial\chi^2}+\tfrac{1}{2}(\chi\mp|\Delta_0|/g_0)^2-\tfrac{1}{2}\right]+O(g_0/|\Delta_0|)\right\}.\end{aligned} \tag{5.13}$$

Since this expansion leads to an identical approximation as that of (5.12), we conclude that the spectrum of $\tilde{H}$ would consist of two sets of degenerate oscillator states where the energy-spacing is again $|\Delta_0|$. This degeneracy is lifted by a tunneling effect. However, because the potential barrier increases with $|\Delta_0|/g_0$, the splitting between the two nearly degenerate vacuums is exponentially small.

A similar analysis can also be carried out for $\tilde{H}^\dagger$, leading to an identical conclusion. This qualitative picture thus provides a heuristic justification for the hypothesis under which the discussion of Sec. V A was carried out. In particular, we emphasize that, for $D=0$, the vacuum is always nondegenerate so long as $|\Delta_0|/g_0$ remains finite.

C. First excited state

For $\Delta_0<0$, it follows from Eq. (5.13) that the ground-state wave function behaves near $\chi=\pm|\Delta_0|/g_0$ as

$$\psi_R \propto \exp[-\tfrac{1}{2}(\chi-|\Delta_0|/g_0)^2], \quad \chi\simeq|\Delta_0|/g_0 \tag{5.14a}$$

$$\psi_L \propto \exp[-\tfrac{1}{2}(\chi+|\Delta_0|/g_0)^2], \quad \chi\simeq-|\Delta_0|/g_0 . \tag{5.14b}$$

Since the first excited state differs from the true vacuum only by a tunneling correction, it is reasonable to choose a trial wave function by a linear combination of functions satisfying conditions (5.14a) and (5.14b). Denoting the wave functions for $|\psi_1\rangle$ and $|\bar{\psi}_1\rangle$ by $\psi_1(\chi)$ and $\bar{\psi}_1(\chi)$, respectively, the trial functions must also obey orthonormal condition, i.e.,

$$\begin{aligned}&\langle\bar{\psi}_1|\psi_1\rangle=\langle\bar{\psi}_0|\psi_0\rangle=1,\\ &\langle\bar{\psi}_1|\psi_0\rangle=\langle\bar{\psi}_0|\psi_1\rangle=0.\end{aligned} \tag{5.15}$$

Equation (5.11) indicates that $\psi_0(\chi)$ and $\bar{\psi}_0(\chi)$, satisfy the constraints (5.14a) and (5.14b); it is natural to try using them as the basis for constructing $\psi_1(\chi)$ and $\bar{\psi}_1(\chi)$. Let us write

$$\begin{aligned}&\psi_1(\chi)\equiv a[\bar{\psi}_0(\chi)-b\psi_0(\chi)]=a(e^{-F}\phi_0-be^{F}\phi_0),\\ &\bar{\psi}_1(\chi)\equiv\bar{a}[\psi_0(\chi)-\bar{b}\bar{\psi}_0(\chi)]=\bar{a}(e^{F}\phi_0-\bar{b}e^{-F}\phi_0).\end{aligned} \tag{5.16}$$

The normalization constants $a, \bar{a}, b, \bar{b}$ can be fixed by enforcing (5.15); we obtain

$$b=\bar{b}=\lambda^{-1}, \quad a=-\bar{a}=\lambda(1-\lambda^2)^{-1/2}, \tag{5.17}$$

where

$$\begin{aligned}\lambda^{-1}&\equiv\langle\psi_0|\psi_0\rangle=\langle\bar{\psi}_0|\bar{\psi}_0\rangle\\ &=\int_{-\infty}^{\infty}d\chi\, e^{2F(\chi)}\phi_0^{\,2}(\chi).\end{aligned} \tag{5.18}$$

In the limit $|\Delta_0|/g_0\to\infty$, λ vanishes as $(1/\sqrt{2})e^{-1.17|\Delta_0|^2/g_0^2}$.

To first order in λ, our trial wave functions for the first excited state become

$$\begin{aligned}\psi_1(\chi)&=(1-\lambda^2)^{1/2}[-\psi_0(\chi)+\lambda\bar{\psi}_0(\chi)]\\ &\simeq-\psi_0(\chi)+\lambda\bar{\psi}_0(\chi),\\ \bar{\psi}_1(\chi)&=(1-\lambda^2)^{1/2}[\bar{\psi}_0(\chi)-\lambda\psi_0(\chi)]\\ &\simeq\bar{\psi}_0(\chi)-\lambda\psi_0(\chi).\end{aligned} \tag{5.19}$$

D. Two-level truncation

In the limit $|\Delta_0|/g_0\to\infty$ and $\Delta_0<0$, we truncate the spectrum of $H(\chi,P_\chi)$ by retaining only the ground state and the first excited state. Given an arbitrary operator A, its restriction to this truncated space can be represented by a 2×2 matrix, $A_{ij}\equiv\langle\bar{\phi}_i|A|\phi_j\rangle$, $i,j=0,1$, so that

$$A_{\text{truncated}}=\sum_{i,j=0,1}A_{ij}|\phi_i\rangle\langle\bar{\phi}_j|.$$

Alternatively, the restriction of the transformed operator, $A=e^{F}Ae^{-F}$, can be written as

$$\sum_{i,j=0,1}A_{ij}|\psi_i\rangle\langle\bar{\psi}_j|,$$

where A_{ij} is also given by $\langle\bar{\psi}_i|\tilde{A}|\psi_j\rangle$, owing to (5.9). However, since most operators have nonnegligible off-diagonal elements connecting $|\phi_0\rangle$, $|\phi_1\rangle$ to higher excited states, one cannot replace an operator product by a multiplication of 2×2 matrices.

In order to find the truncated spectrum, $\langle\bar{\phi}_i|H(\chi,P_\chi)|\phi_j\rangle=\epsilon_i\delta_{ij}$, $i,j=0,1$, we need to calculate the matrix elements for operators $P_\chi^{\,3}$, $P_\chi^{\,2}$, χ^2P_χ, χ^2, P_χ, and χ as they appear in Eq. (3.1). Because of (5.9), it is easier to work with the transformed operators $\tilde{P}_\chi^{\,3}, \tilde{P}_\chi^{\,2}, \ldots$, and we directly calculate the matrix elements in terms of our trial wave functions (5.19). For example, by using (5.3), (5.10), and (5.19), we find that in the limit $|\Delta_0|/g_0\to\infty$,

$$\chi\simeq-\frac{|\Delta_0|}{g_0}\begin{pmatrix}0&1\\1&0\end{pmatrix}$$

and (5.20)

$$\tilde{P}_\chi\simeq\frac{2i|\Delta_0|}{g_0}\begin{pmatrix}0&1\\-1&2\end{pmatrix}.$$

Although χ and $\tilde{P}_\chi$ are not diagonal in our $|\psi_i\rangle\langle\bar{\psi}_j|$ representation, Eq. (5.20) correctly produces the desired eigenvalues as anticipated by our classical stability analysis, thus indicating that our trial functions, (5.19), are acceptable. This perhaps should not come as a surprise since only the behavior of $\psi_1(\chi)$ and $\bar{\psi}_1(\chi)$ near $\chi=\pm|\Delta_0|/g_0$ enters

in the determination of the dominant matrix elements of χ and P_χ; this has been explicitly verified by choosing other trial functions. So long as they satisfy (5.14a), (5.14b), and (5.15), equivalent representations for χ and $\tilde{P}_\chi$ are obtained.

Substituting analogous matrix representations for P_χ^3, P_χ^2, ..., into Eq. (3.1), we find that $H(\chi, P_\chi) \simeq 0$ so that $\epsilon_1 \simeq \epsilon_0 = 0$. To obtain ϵ_1 more accurately, we need to keep next-order terms for all relevant matrix elements. Using our trial functions ψ_1 and $\bar{\psi}_1$, (5.19), we find that

$$\epsilon_1 \propto \lambda^2 = e^{-2.35|\Delta_0|^2/g_0^2},$$

a result differing slightly from the exact result

$$\epsilon_1 \propto e^{-2\Delta_0^2/g_0^2}$$

obtained in Ref. 7. Clearly, the calculation for ϵ_1 is sensitive to the structure of trial wave functions in the "tunneling" region; the exact result for ϵ_1 is not expected to emerge for "simple" choices as (5.19). In the subsection where we discuss the $D \neq 0$ case, we simply write the $D = 0$ Hamiltonian at each site as

$$H^{D=0} = (\tfrac{1}{2}\delta)(1 - \sigma_z), \tag{5.21}$$

where $\delta = \epsilon_1$.

E. Derivation of the classical lattice analog of RFT

The classical lattice analog of RFT is derived in Sec. IV by applying a two-level truncation approximation after we first convert the representation for the matrix element of the evolution operator, $e^{-y\tilde{H}}$, into a path integral. Our quantum lattice model, on the other hand, is obtained via a different two-level truncation procedure, where we first diagonalize the operator $\tilde{H}$. We now demonstrate their equivalence, thus establishing that the critical behavior of these analog models, when they occur, belong to the same universality class[21]. This also justifies the duplication of notations between Sec. IV and Sec. V.

Since $\{|\psi_i\rangle\}$ and $\{|\bar{\psi}_j\rangle\}$ are both spanned by the basis vectors, $|\psi_0\rangle$ and $|\bar{\psi}_0\rangle$, it is possible to transform them to a set of orthonormal bases on which the χ matrix becomes diagonal: Let

$$|\hat{e}_i\rangle = \sum_{j=0}^{1} S_{ji}|\psi_j\rangle = \sum_{k=0}^{1} (S^{-1})_{ik}|\bar{\psi}_k\rangle, \tag{5.22}$$

where

$$\langle\hat{e}_i|\hat{e}_j\rangle = \delta_{ij}, \quad i,j = 0, 1 \tag{5.23}$$

and

$$\chi' \equiv S^{-1}\chi S = \frac{|\Delta_0|}{g_0}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5.24}$$

It follows from (5.16), (5.17), (5.23), and (5.24) that the desired transformation is

$$S = \begin{pmatrix} c & d \\ -c & d \end{pmatrix}, \tag{5.25}$$

where, with λ defined by (5.18) and $\lambda \ll 1$,

$$c = \lambda^{1/2}(1-\lambda^2)^{1/2}[1 + (1-\lambda^2)^{1/2}]^{-1} \simeq \tfrac{1}{2}\lambda^{1/2}, \tag{5.26}$$

$$d = \lambda^{-1/2}(1-\lambda^2) \simeq \lambda^{-1/2}. \tag{5.27}$$

Because of (5.24), this new basis can be identified with the "spin basis" of Sec. IV, i.e.,

$$\begin{aligned} |+\rangle &\equiv |\hat{e}_0\rangle \simeq \lambda^{1/2}|\psi_0\rangle - \tfrac{1}{2}\lambda^{3/2}|\bar{\psi}_0\rangle, \\ |-\rangle &\equiv |\hat{e}_1\rangle \simeq \lambda^{1/2}|\bar{\psi}_0\rangle - \tfrac{1}{2}\lambda^{3/2}|\bar{\psi}_0\rangle, \end{aligned} \tag{5.28}$$

and $\chi'|\pm\rangle = \pm|\Delta_0|/g_0|\pm\rangle$. In order for (5.28) to agree with (4.22), we need

$$\lambda^{1/2} = \sqrt{2}e^{-B/2}. \tag{5.29}$$

Recall that, in our quantum spin formulation, $\lambda^{-1/2}$ is the length of the ground-state vectors, $|\psi_0\rangle$ and $|\bar{\psi}_0\rangle$. Similarly, $(1/\sqrt{2})\exp(B/2)$ is also the length of ψ_0 and $\bar{\psi}_0$ in the classical lattice model. In ooth cases, $|\psi_0| = |\bar{\psi}_0| \gg 1$ follows from the anti-Hermiticity of the triple-Pomeron interaction.

To complete the identification, let us rederive the transfer matrix of Sec. IV starting from Eq. (5.21), i.e., we calculate the matrix elements of the time-evolution operator, $e^{-y\tilde{H}}$, in the spin basis for $y = a_t$. The transformation $S^{-1}\exp(-a_t\tilde{H})S$ leads to a transfer matrix

$$M = \begin{pmatrix} 1 + e^{-a_t\epsilon_1} & (d/c)(1 - e^{-a_t\epsilon_1}) \\ (c/d)(1 - e^{-a_t\epsilon_1}) & 1 + e^{-a_t\epsilon_1} \end{pmatrix}, \tag{5.30}$$

where $\epsilon_1 = \delta > 0$. Equation (5.30) is identical to Eq. (4.9) provided that

$$e^{-C\pm K} = 1 \pm e^{-a_t\epsilon_1}, \tag{5.31}$$

$$e^B = d/c \simeq 2\lambda^{-1}. \tag{5.32}$$

Equation (5.32) follows from (5.29); Eq. (5.31) leads to $\delta = -a_t^{-1}\ln\tanh K$, in exact agreement with (4.21).

F. Quantum lattice-spin analog of RFT

At $D \neq 0$, the normal-ordered Hamiltonian density is

$$H(\chi, P_\chi) = H^{D=0}(\chi, P_\chi) + \alpha'(-\tfrac{1}{2}\overleftrightarrow{\nabla}_b P_\chi - i\overleftrightarrow{\nabla}_b\chi)(-\tfrac{1}{2}\overleftrightarrow{\nabla}_b P_\chi + i\overleftrightarrow{\nabla}_b\chi). \tag{5.33}$$

When replacing the impact-parameter space by discrete lattice sites, $\vec{n}$, one converts fields $\chi(t,\vec{b})$ and $P_\chi(t,\vec{b})$ to $\{\chi_{\vec{n}}(t)\}$ and $\{P_{\vec{n}}(t)\}$. The lattice spacing, a_b, is related to the "effective" cutoff in transverse momentum which is a parameter already present in the original RFT. After replacing the gradient, $\vec{\nabla}_b$, by a finite difference involving neighboring sites, the second term on the right of Eq. (5.33) becomes

$$\frac{\alpha'}{a_s^2}\sum_{\vec{e}_b}\{[\tfrac{1}{4}P_{\vec{n}+\vec{e}_b}{}^2(t)+\chi_{\vec{n}+\vec{e}_b}{}^2(t)-\tfrac{1}{2}\hbar]+[\tfrac{1}{4}P_{\vec{n}}{}^2+\chi_{\vec{n}}{}^2(t)-\tfrac{1}{2}\hbar]\}-\frac{\alpha'}{2a_s^2}\sum_{\vec{e}_b}(\tfrac{1}{4}P_{\vec{n}}P_{\vec{n}+\vec{e}_b}+\chi_{\vec{n}}\chi_{\vec{n}+\vec{e}_b}), \tag{5.34}$$

where $\{\vec{e}_b\}$ represents unit vectors for the lattice. The first sum in (5.34) does not involve intersite couplings; it can therefore be grouped with the single-site Hamiltonian density $H^{D=0}(\chi,P_\chi)$. In particular, it can simply be absorbed by the Pomeron mass term, $\Delta_0(\chi^2+P_\chi{}^2/4-\hbar/2)$, in (3.1), so that Δ_0 is replaced by an effective mass,

$$\Delta_{eff}\equiv\Delta_0+\frac{2D\alpha'}{a_s^2}. \tag{5.35}$$

In Eq. (5.35), the transverse dimensionality D is 2 and a_s^{-1} is of the order of 1 GeV/c. After integrating over the $\vec{b}$ space, the total Hamiltonian becomes

$$H(\chi,P_\chi)=\sum_{\vec{n}}a_s^D H^{D=0}(\chi_{\vec{n}},P_{\vec{n}},\Delta_{eff})-2\alpha' a_s^{D-2}\sum_{\vec{n}}\sum_{\vec{e}_b}(\chi_{\vec{n}}\chi_{\vec{n}+\vec{e}_b}+\tfrac{1}{4}P_{\vec{n}}P_{\vec{n}+\vec{e}_b}), \tag{5.36}$$

where we have explicitly exhibited that the single-site Hamiltonian is calculated by using Δ_{eff} instead of Δ_0.

We can next represent H on a basis constructed from the direct product of single-site states. For $|\Delta_{eff}|/g_0\gg 1$ and $\Delta_{eff}<0$, we can truncate the basis by keeping only $|\phi_0\rangle$ and $|\phi_1\rangle$ at each site. Introducing 2×2 spin matrices at each site by

$$S_\chi^{\vec{n}}=\frac{g_0}{|\Delta_{eff}|}\chi_{\vec{n}}\quad\text{and}\quad S_p^{\vec{n}}\equiv\frac{g_0}{|\Delta_{eff}|}P_{\vec{n}}, \tag{5.37}$$

and recalling (5.20), we can rewrite (5.36) as

$$H=\left[\left(\frac{\delta_{eff}a_s^D}{2}\right)\sum_{\vec{n}}(1-\sigma_z^{\vec{n}})\right]-\left[\left(2\alpha'\frac{|\Delta_{eff}|^2}{g_0^2}a_s^{D-2}\right)\sum_{\vec{n}}\sum_{\vec{e}_b}(S_\chi^{\vec{n}}S_\chi^{\vec{n}+\vec{e}_b}+\tfrac{1}{4}S_p^{\vec{n}}S_p^{\vec{n}+\vec{e}_b})\right]. \tag{5.38}$$

In Eq. (5.38), δ_{eff} is the new mass gap between $|\phi_1\rangle$ and $|\phi_0\rangle$ when Δ_{eff} is used in (3.1). Because of Eq. (5.20), $S_\chi^{\vec{n}}$ and $S_p^{\vec{n}}$ can be represented by Pauli matrices

$$S_\chi^{\vec{n}}=-\sigma_x^{\vec{n}},\quad S_p^{\vec{n}}=-2[\sigma_y^{\vec{n}}+i(1-\sigma_z^{\vec{n}})]. \tag{5.39}$$

After (5.39) is substituted into (5.38) and then rotated $\pi/2$ about the z axis so that $(\sigma_x,\sigma_y)\to(\sigma_y,-\sigma_x)$, the Hamiltonian for our quantum spin analog of RFT, with the exception of having Δ_{eff} in place of Δ_0, becomes identical to that of Refs. 8–10. This completes our analysis.

VI. DISCUSSION

The current central problem in high-energy hadron interaction has been the observation that all hadronic cross sections rise with increasing energies. It has been argued that the increase in cross sections at current machine energies can be attributed to the multiperipheral threshold effects within the short-range correlation picture of hadron production.[12] However, if this trend persists to higher energies, it would indicate that hadron dynamics at high energies is primarily controlled by unitarity constraints (both direct and cross channel).

Recent theoretical development favors a two-step approach to the study of high-energy hadron interaction. One first searches for a realistic production model capable of accounting quantitatively for the bulk of hadronic cross sections at current machine energies; this is the short-range correlation component of production amplitudes and the leading singularity so generated is the bare Pomeron. Assuming that the zero-energy intercept of the bare Pomeron is greater than 1, $\Delta_0\equiv 1-\alpha_0<0$, the asymptotic limit of hadronic cross sections is then determined by "renormalization" effects, i.e., the interaction between the bare Pomeron and its associated Regge cuts. The second step involves the implementation of t-channel and/or s-channel unitarity constraints. Much of the recent work of this program has been carried out within the framework of Reggeon field theory so that the t-channel constraints are enforced first.

To provide further insights into the structure of

RFT at $\alpha_0 > 1$, we have reformulated the anti-Hermitian cubic RFT in terms of a single field χ. Our treatment is nonperturbative with respect to the original bare expansion; a different expansion which treats both $\alpha_0 > 1$ and $\alpha_0 < 1$ regions on an equal footing is derived. Our one-component theory is anti-Hermitian, and it is normal-ordered with respect to ψ and $\bar{\psi}$ without tadpoles. Furthermore, our theory allows a path-integral formalism in which the integration paths are always along real axes and no contour distortion is required as α_0 is increased above 1.

The crucial step in our alternative formulation of RFT is the identification of a gauge transformation so that the transformed Hamiltonian has its kinetic and potential energy components separated, and the dependence on the sign of Δ_0 isolated. The structure of our gauge-transformed theory depends crucially on the fact that the gauge function, $\Lambda_q(\chi)$, is purely imaginary, which is a direct reflection of the anti-Hermiticity of the triple-Pomeron interaction.

We have elaborated our procedure by first considering an analogous gauge transformation for the classical RFT. The quantum gauge transformation is carried out in Sec. III; we have shown that many of the features of the classical RFT remain at the quantum level. In particular, our quantum expansion is seen to be in exact accordance with the boundary condition for the classical RFT; the evolution of classical fixed points as α_0 is increased above 1 is correlated with the quantum description through the analysis of the integration domain in a path-integral formalism. In our approach, the integration paths are always along real axes and no contour distortion is required as Δ_0 changes sign. Our formalism, therefore, represents a technical improvement over that of Cardy and Sugar in Ref. 6.

The quantum RFT also contains many usual feature not present in a classical analysis, e.g., the structure of the left and right vacuums at $\Delta_0 < 0$. We have concentrated on discussing the limit where $|\Delta_0|/g_0$ is large and $\Delta_0 < 0$ so that a two-level truncation approximation can be used. Particular attention has been paid to the consequences of anti-Hermiticity. We first formulate in Sec. IV a finite-temperature classical lattice-spin model with a single spin-$\frac{1}{2}$ variable at each lattice site.[22] The anti-Hermitician interaction is seen to lead to a kinklike ground-state configuration. In Sec. V, we have constructed a quantum lattice-spin model, thus making contact with the work of Amati *et al.* We have also explicitly demonstrated that, at $D = 0$, the quantum lattice-spin approach is isomorphic to our classical lattice-spin formalism.

We have not attempted in this paper to carry out an analysis on the critical behavior of RFT at $D = 2$. Since we have arrived at the quantum lattice-spin model of Refs. 8–10, this analysis would be identical to that of Amati *et al.* Because our treatment allows us to explore more easily the symmetry between left and right eigenvectors under $\chi \rightarrow -\chi$, we believe that our approach is more useful as a starting point for going beyond the two-level truncation approximation. Our analysis also suggests that a renormalization-group transformation approach to a classical lattice-spin analog of RFT can still provide insights into the nature of the Pomeron phase transition provided that the symmetries of classical RFT are taken into account. Preliminary analysis indicates that practical calculations can indeed be carried out. We have also been applying the generalization of Eq. (3.18) to a treatment of the "cut Reggeon field theory"[23] and to simplify the analyses of inclusive processes.[24]

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[14]It is interesting to note that if we perform a transformation $q\equiv(2\sqrt{2}/g_0)(\Delta_0+ig_0\phi)^{1/2}$, the Lagrangian for q is $L(q,\dot{q})=-\frac{1}{2}\dot{q}^2+\frac{1}{8}\Delta_0^2q^2[1-(g_0^2/8\Delta_0)q^2]^2$, and $\chi=(2i/g_0)\dot{q}/q$.

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[19]Note that the gauge function is purely imaginary. Since $\Lambda_q'(\chi)$ is even in χ, we fix the additivity constant in Λ_q by requiring $\Lambda_q(\chi)=-\Lambda_q(-\chi)$, that is,

$$\Lambda_q(\chi)=\frac{2}{3g_0}\int_0^\chi d\chi[-\Delta_0+\Delta_q(\chi)]$$
$$=\frac{\chi}{3g_0}[-2\Delta_0+\Delta_q(\chi)]+\frac{\Delta_q^2(0)}{3\sqrt{3}g_0^2}\ln\left[\frac{\Delta_q(\chi)+(3g_0^2)^{1/2}\chi}{\Delta_q(0)}\right].$$

[20]Because of convention $\Lambda_q(\chi)=-\Lambda_q(-\chi)$, it follows that $F(\chi,g_0)=-F(\chi,-g_0)=-F(-\chi,g_0)=F(-\chi,-g_0)$. However, we shall adopt this phase condition only for the ground-state wave functions, $\psi_0(\chi)$ and $\overline{\psi}_0(\chi)$.

[21]For $D=0$, a phase transition takes place as $|\Delta_0|/g_0\to\infty$. For $D\neq 0$, this assertion remains a conjecture. For a differeing view, see Ref. 9.

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