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Fermions and gauge vector mesons at finite temperature and density. II. The ground-state energy of a relativistic electron gas*

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We calculate the ground-state energy of a relativistic electron gas up to and including effects of order $\alpha^2\log\alpha$ and α^2. Cutting rules are developed which relate a vacuum-graph expansion for the thermodynamic potential to phase-space integrals over Feynman amplitudes. Overlapping infrared divergences, which cancel when all contributions of order α^2 are summed, are treated by performing a dimensional continuation to $4+\epsilon$ dimensions. Ultraviolet divergences associated with electron wave-function and charge renormalizations are rendered finite by use of a Landau gauge appropriate to $4+\epsilon$ dimensions.

INTRODUCTION

As an application of the formal techniques developed in our first paper, we will calculate here the ground-state energy of a relativistic electron gas up to and including effects of order α^2. A relativistic electron gas is of interest primarily since the techniques developed for and the results derived from this system are directly applicable to the physically interesting system of a quark gas. An electron gas is described by an Abelian gauge theory, and is not complicated by the non-Abelian structure of the theory which describes quark interactions. However, we will show in a later paper that the non-Abelian structure of quantum chromodynamics presents no new technical difficulties, and that many of the results derived from a consideration of a relativistic electron gas may be used for the description of a quark gas.

The ground-state energy of an electron gas may be found from the thermodynamic potential, $\Omega(\beta,\mu)$, in the zero-temperature limit ($1/\beta\to 0$). Since electrons in a relativistic gas are highly energetic, high-order Feynman graphs are important contributions to Ω. These graphs grow as powers of logarithms of the energy in the high-energy limit. Although our calculations are explicit to only order α^2, it is possible to include the dominant effects of these high-order graphs by using the renormalization group.[2-6] The use of the renormalization group allows the extension of our results far beyond order α^2, and is essentially nonperturbative. In a later paper, this extension will be performed.

The plasmon effect must also be handled nonperturbatively.[7,8] This effect arises in the spontaneous generation of a photon mass due to interactions. Since the photon becomes massive, the long-distance interaction of electrons becomes modified to a Yukawa interaction. The infrared structure of interactions is thus greatly modified.

The generation of a photon mass introduces infrared divergences in a direct, perturbative evaluation of the thermodynamic potential. These divergences stem from the fact that the plasmon mass is of order α, and an expansion in powers of the plasmon mass is singular. Nevertheless, the plasmon contribution may be found by summing an infinite class of Feynman graphs. These graphs occur naturally in an expression derived for the thermodynamic potential in Ref. 1, and yield an effect of order $\alpha^2\log\alpha$.

The organization of the body of this paper is as follows:

In the first section, the thermodynamic potential is discussed in the zero-temperature limit. We make essential use of a functional expression, derived in Ref. 1, which determines the thermodynamic potential as a functional of fully renormalized propagators and vertices. This functional expression is used to break the thermodynamic potential into several pieces which may be directly evaluated. One piece, the exchange energy, is of order α. The remaining pieces are the photon correlation energy, which contains the contributions of the plasmon effect and vacuum polarization, and contributions of order α^2 which are nonsingular in the zero-electron-mass limit. The nonsingular contributions include rescattering and vertex insertion corrections to the exchange energy, as well as three-body effects.

The second section consists of an evaluation of the exchange energy. The exchange energy provides a simple example of graphical techniques which are useful in determining higher-order contributions.

The photon correlation energy is considered in the third section. Here the effects of the plasma oscillation and vacuum polarization are determined in the small-electron-mass limit. A contour-integral representation for the photon polarization tensor discussed in Appendixes A and B is used to perform the calculation of the plasmon effect.

In the fourth section, the nonsingular contributions of order α^2 are obtained as phase-space integrals over Feynman amplitudes. The sum of these phase-space integrals is free of ultraviolet and infrared divergences. The sum is also gauge invariant and nonsingular in the massless-electron limit. However, the individual terms in the sum are not gauge invariant, are singular in the massless-electron limit, and contain infrared divergences.

In the fifth section, the results of the fourth section are determined in the small-electron-mass limit. To handle graphs with infrared divergences and singularities in the massless limit, we work in $4+\epsilon$ dimensions.[9-12] In $4+\epsilon$ dimensions, the singularities arising from infrared divergences and the zero-electron-mass limit are transformed into singularities of the $\epsilon\to 0$ limit. All such singularities cancel when the graphs are summed, giving a finite result. Ultraviolet divergences associated with the electron wave-function renormalization and the charge renormalization are made finite by a Landau gauge generalized to $4+\epsilon$ dimensions.[13]

In the sixth section, all contributions to the thermodynamic potential are grouped together. The implications for non-Abelian gauge theories of the results derived in the preceding sections are briefly discussed. In a later paper, the results of this last section will be extended to higher orders in perturbation theory by an application of the renormalization group.

I. THE THERMODYNAMIC POTENTIAL TO ORDER α^2

It was shown in Ref. 1 that the piece of the thermodynamic potential arising from interactions, Ω_I, could be written as a functional of the full propagators, S and D, the full vertex, Γ, and the contributions to the skeleton-graph expansion of the Bethe-Salpeter kernel, K_{2n}, involving $2n$ vertices. This functional is

$$\beta V\Omega_I(\beta,\mu)=\mathrm{Tr}\,\ln S_0^{-1}S-\tfrac{1}{2}\mathrm{Tr}\,\ln D_0^{-1}D-e^2\Gamma S\Gamma SD+\tfrac{1}{2}\sum_{n=1}^{\infty}e^{2n+2}\left(1+\frac{1}{n+1}\right)S\Gamma SK_{2n}S\Gamma SD\,, \tag{1.1}$$

and is shown diagrammatically in Fig. 1. In this expression, all Feynman integrals are in coordinate space, and the traces are over coordinates and spin indices.

The functional expression of Eq. (1.1) is simplified by performing a Fourier transformation to momentum space. In momentum space, the terms in Eq. (1.1) involving logarithms of propagators are diagonal in momenta and off-diagonal only in spin indices. The trace over coordinates can be performed immediately, leaving a trace over spin indices and yielding an overall factor of βV. The remaining terms in Eq. (1.1) also acquire an overall factor of βV upon Fourier-transforming to momentum space. Extracting this factor, Eq. (1.1) becomes in the zero-temperature limit

$$\begin{aligned}\Omega_I(\mu)=&\int\frac{d^4p}{(2\pi)^4}\,\mathrm{tr}\,\ln S_0^{-1}(p\,|\,\mu)S(p\,|\,\mu)\\&-\tfrac{1}{2}\int\frac{d^4q}{(2\pi)^4}\,\mathrm{tr}\,\ln D_0^{-1}(q)D(q\,|\,\mu)\\&-e^2\Gamma S\Gamma SD\\&+\tfrac{1}{2}\sum_{n=1}^{\infty}e^{2n+2}\left(1+\frac{1}{n+1}\right)S\Gamma SK_{2n}S\Gamma SD\,.\end{aligned} \tag{1.2}$$

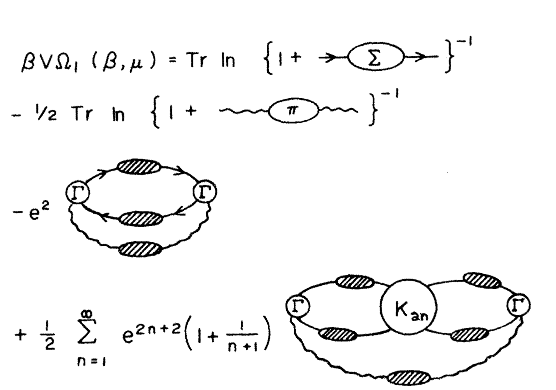

FIG. 1. The thermodynamic potential for quantum electrodynamics as a sum of vacuum graphs.

We will now turn to a perturbative evaluation of Eq. (1.2). In such an evaluation, the term involving the trace of the logarithm of the photon propagator must be handled carefully. If we were naively to use the Schwinger-Dyson equation,[14,15]

$$D^{-1}(q\,|\,\mu)=D_0^{-1}(q)+\Pi_R(q\,|\,\mu)\,, \tag{1.3}$$

and expand the logarithm in powers of $\Delta\Pi_R(q\,|\,\mu)$, we would encounter infrared divergences. These divergences are a consequence of the nonzero value of $\Delta\Pi_R(q\,|\,\mu)$ at $q^2=0$. This circumstance is due to the plasmon effect, by which the photon acquires a mass from interactions.

The diagrams which yield infrared divergences are associated with a singular expansion in powers of Π_R. These diagrams are shown in Fig. 2. An nth-order iteration diverges in the infrared as $(1/q^2)^n$ and is nonsingular when integrated over d^4q

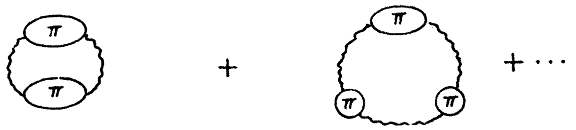

FIG. 2. Infrared-divergent diagrams arising from a singular expansion in powers of the plasmon mass.

for $n<2$. Only diagrams involving two or more iterations of the plasmon mass, therefore, are infrared divergent.

To treat the infrared difficulties associated with the plasmon effect properly, we must observe that although the expansion of the logarithm in Eq. (1.2) is singular, the integral over the logarithm is finite. This observation suggests that all terms which yield a singular expansion must be grouped together and handled nonperturbatively.[7,8] The expression

$$\Omega_I^{(c)} = \tfrac{1}{2} \int \frac{d^4q}{(2\pi)^4} \operatorname{tr}\{\ln[1+e^2 D_0(q)\Pi_R^{(2)}(q\,|\,\mu)] - e^2 D_0(q)\Pi_R^{(2)}(q\,|\,\mu)\} \tag{1.4}$$

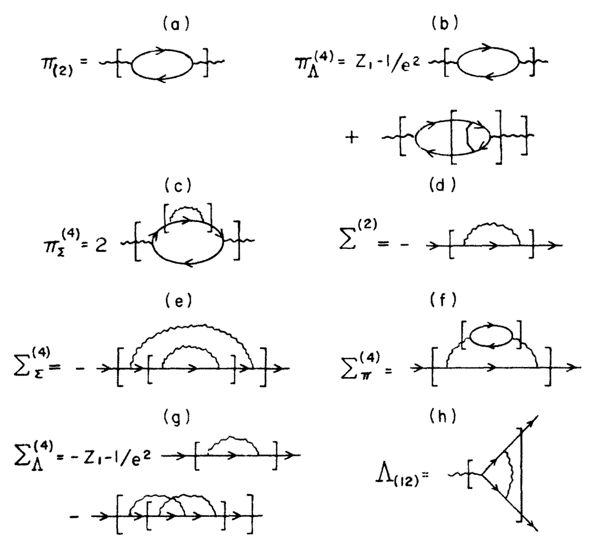

FIG. 3. The parts of the kernels Π_R, Σ_R, and Λ_R needed for an order-e^4 evaluation of the thermodynamic potential. The brackets, [ ], indicate the renormalization of the enclosed ultraviolet-divergent subintegration by the procedure discussed in Ref. 1. The order-e^2 charge renormalization constant is denoted by Z_1. (a) The order-e^2 photon polarization tensor. (b) The piece of the order-e^4 photon polarization tensor due to vertex insertions. (c) The piece of the e^4 photon polarization tensor due to a fermion self-mass insertion. (d) The order-e^2 fermion self-mass kernel. (e) The piece of the e^4 fermion self-mass kernel due to a fermion self-mass insertion. (f) The piece of the order-e^4 self-mass kernel due to a photon polarization tensor insertion. (g) The piece of the order-e^4 self-mass kernel due to a vertex insertion. (h) The order-e^2 vertex insertion.

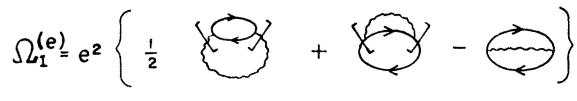

FIG. 4. The exchange contribution to the thermodynamic potential.

groups together all such singular terms up to and including effects of order α^2. In this expression, the renormalized photon polarization tensor to order e^2 is denoted by $\Pi_R^{(2)}$.

The remaining terms, which are infrared finite, may be written in terms of kernels of the Schwinger-Dyson equations. The inverse propagator, S^{-1}, and the vertex, Γ, are given by these kernels as

$$S^{-1}(p\,|\,\mu) = S_0^{-1}(p\,|\,\mu) + \Sigma_R(p\,|\,\mu) \tag{1.5}$$

and

$$\Gamma(p, p+q, q\,|\,\mu) = \gamma + \Lambda_R(p, p+q, q\,|\,\mu)\,. \tag{1.6}$$

The inverse photon propagator, D^{-1}, is given by Eq. (1.3).

The kernels Π_R, Σ_R, and Λ_R possess finite perturbation expansions in powers of e^2. To calculate the thermodynamic potential to order e^4, we need Σ_R and Π_R to order e^4, and Λ_R to order e^2. These kernels may thus be written as

$$\Pi_R = e^2\Pi^{(2)} + e^4(\Pi_\Sigma^{(4)} + \Pi_\Lambda^{(4)})\,, \tag{1.7}$$

$$\Sigma_R = e^2\Sigma^{(2)} + e^4(\Sigma_\Pi^{(4)} + \Sigma_\Sigma^{(4)} + \Sigma_\Lambda^{(4)}), \tag{1.8}$$

$$\Lambda_R = e^2\Lambda_{(2)}\,. \tag{1.9}$$

These equations are represented in Figs. 3(a)–3(h)

Using the definitions of Eqs. (1.7)–(1.9) and the Schwinger-Dyson equations, Eqs. (1.3), (1.5), and (1.6), the thermodynamic potential may be written as

$$\Omega_I = \Omega_I^{(e)} + \Omega_I^{(c)} + \Omega_I^{\Pi} + \Omega_I^{\Sigma} + \Omega_I^{\Lambda}. \tag{1.10}$$

The exchange contribution, $\Omega_I^{(e)}$, contains all effects of order e^2, and is represented by

$$\Omega_I^{(e)} = e^2(\tfrac{1}{2}D_0\Pi^{(2)} - S_0\Sigma^{(2)} - \gamma S_0 \gamma S_0 D_0)\,. \tag{1.11}$$

This equation is shown diagrammatically in Fig. 4.

The plasmon effect and a piece of vacuum polarization corrections to the exchange energy are contained in the correlation contribution, $\Omega_I^{(c)}$. This contribution is given by Eq. (1.4) and is represented in Fig. 5. As discussed above, $\Omega_I^{(c)}$ includes

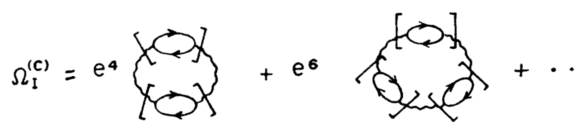

FIG. 5. The photon correlation contribution to the thermodynamic potential.

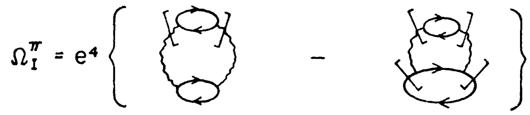

FIG. 6. The contribution to the thermodynamic potential, Ω_I^Π.

contributions from all orders in perturbation theory. In Sec. III, $\Omega_I^{(c)}$ will be evaluated in combination with Ω_I^Π. The contributions of the vacuum polarization corrections to the exchange energy not included in $\Omega_I^{(c)}$ are included in Ω_I^Π. These contributions possess the integral representation

$$\Omega_I^\Pi = e^4(\gamma S_0 \gamma S_0 D_0 \Pi_{(2)} D_0 - S_0 \Sigma_\Pi^{(4)}) , \tag{1.12}$$

which is shown in Fig. 6. Although Ω_I^Π is topologically similar to $\Omega_I^{(c)}$, Ω_I^Π has no infrared divergences since the minus sign in Eq. (1.12) cancels these potential difficulties.

The terms Ω_I^Σ and Ω_I^Λ are associated with fermion self-mass kernel insertions and vertex insertions. These contributions to the thermodynamic potential possess the integral representations

$$\Omega_I^\Sigma = e^4(\tfrac{1}{2} S_0 \Sigma_{(2)} S_0 \Sigma_{(2)} - S_0 \Sigma_{(4)}^\Sigma + \tfrac{1}{2} D_0 \Pi_{(4)}^\Sigma + 2\gamma S_0 \Sigma_{(2)} \gamma S_0 D_0) \tag{1.13}$$

and

$$\Omega_I^\Lambda = e^4(\tfrac{1}{2} D_0 \Pi_\Lambda^{(4)} - S_0 \Sigma_\Lambda^{(4)} - 2\Lambda^{(2)} S_0 \gamma S_0 D_0 + \tfrac{3}{4} S_0 \gamma S_0 D_0 S_0 \gamma S_0 D_0) . \tag{1.14}$$

These equations are represented graphically in Figs. 7 and 8. In Sec. IV, Ω_I^Σ and Ω_I^Λ will be related to phase-space integrals over Feynman amplitudes. In Sec. V, these phase-space integrals will be evaluated in a massless-electron limit. It is important to consider Ω_I^Σ and Ω_I^Λ together since, as we shall see, only their sum is gauge invariant.

II. THE EXCHANGE ENERGY

The exchange energy, shown in Fig. 4, provides a simple example of graphical techniques which will later be used to calculate order-α^2 corrections to the thermodynamic potential. Since these techniques are important in the later analysis, we shall consider the exchange energy in detail here.

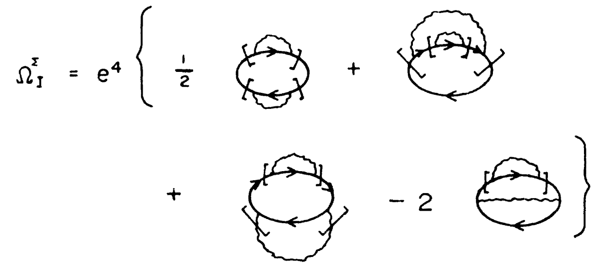

FIG. 7. The contributions to the thermodynamic potential, Ω_I^Σ.

The general procedure we shall use involves the analyticity properties in the energy of Feynman graphs. We will assume that in the vacuum Feynman-graph representation for the thermodynamic potential, it is always possible to evaluate energy integrals before integrals over three-momenta. Using contour-integral techniques, we will analytically continue energy integrals so as to obtain finite contributions and divergent terms associated with the energy density of the vacuum. Since the energy density of the vacuum is to be subtracted from the renormalized thermodynamic potential, the remaining finite contributions give the entire result.[1]

In Fig. 9(a), the exchange energy is represented with the dependences on the energies of the propagators made explicit. The only difference between this diagram and the exchange energy of the vacuum is that at finite density, the energy, p^0, in fermion propagators is replaced by $p^0 + i\mu$, where μ is the chemical potential.

The finite-density expression may be converted to finite terms and a vacuum term by analytically deforming the fermion energy integral down by $i\mu$ in the complex p^0 plane. The remaining contour integral at $p^0 - i\mu$ vanishes upon subtraction of the energy density of the vacuum. However, in performing this contour deformation, poles of the fermion propagators are encountered. In the complex energy plane, these poles are at

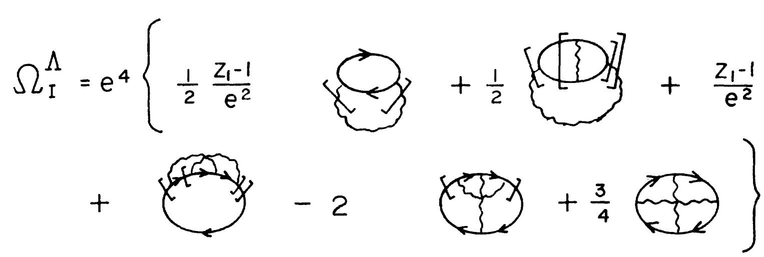

FIG. 8. The contributions to the thermodynamic potential, Ω_I^Λ.

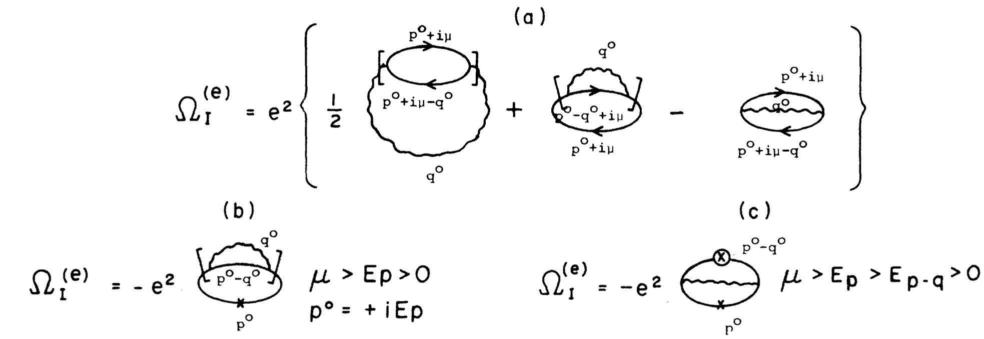

FIG. 9. (a) The exchange energy with dependence on the energy of the fermion propagators made explicit. (b) The result of deforming the energy integration over p^0 and subtracting off the vacuum energy density. (c) The nonvanishing contribution to the exchange energy.

$$p^0 = \pm iE_p = \pm i(\vec{p}\,^2 + m^2)^{1/2} \tag{2.1}$$

and

$$p^0 = q^0 \pm iE_{p-q} = q^0 \pm i[(\vec{p} - \vec{q})^2 + m^2]^{1/2}, \tag{2.2}$$

and will be encountered if $\mu > E_p > 0$ or $\mu > E_{p-q} > 0$. The remaining finite contributions are, therefore, given by the residues of these poles. Since the downward deformation in the complex energy plane encircles the poles of the fermion propagators in a clockwise direction, these residues appear with an overall minus sign.

The result of the contour deformation is shown in Fig. 9(b). In arriving at this result, we have used the fact that in order α a discontinuity of the fermion self-mass kernel, or of the photon polarization tensor, does not require renormalization. This fact is a consequence of the circumstance that the renormalization of $\Pi_{(2)}$ and $\Sigma_{(2)}$ introduces no new singularities in the complex energy plane, and only shifts the position of the pole in the fermion propagator from the value of the bare mass to the value of the physical electron mass. This shift is taken into account in Eqs. (2.1) and (2.2). The change of variables, $\vec{q} \to \vec{p} - \vec{q}$, has also been performed in graphs where the pole at $p^0 = q^0 \pm iE_{p-q}$ is encircled.

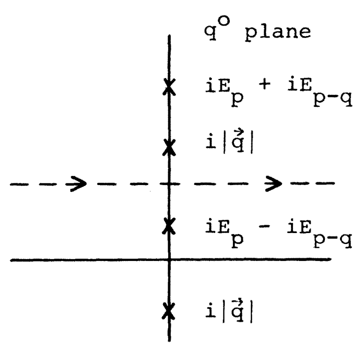

FIG. 10. The q^0 contour-integral representation for $\Sigma_R(p)$ is shown with a dashed line. The contour integral for Fig. 9(b) is along the real Eucilidean axis.

In Fig. 9(b), only one term remains which is associated with a fermion pole at $p^0 = iE_p$ for $\mu > E_p > 0$. Now, since p^0 is at iE_p, $p^2 = -m^2$. Since the fermion self-mass kernel vanishes at $p^2 = -m^2$, we might conclude—wrongly—that this term would vanish under these conditions. However, in the analytic continuation of the integral representation for the self-mass kernel, $\Sigma_{(2)}(p^0, \vec{p})$, from real, Euclidean energy, p^0, to complex energy, the contour integral over the photon energy, q^0, must be deformed continuously so as always to split the positive- and negative-frequency poles of the integrand in the complex q^0 plane. This analytic continuation vanishes at $p^0 = iE_p$. In Fig. 9(b), the q^0 integration is along the real Euclidean axis, and does not split the positive- and negative-frequency poles of the integrand. The poles of the integrand are at

$$\begin{aligned} &q^0 = \pm i|\vec{q}|, \\ &q^0 = iE_p \pm iE_{p-q}, \end{aligned} \tag{2.3}$$

and a negative-frequency pole at $q^0 = i(E_p - E_{p-q})$ is in the upper half q^0 plane for $E_p > E_{p-q}$. This situation is shown in Fig. 10.

To obtain a finite term and a term which vanishes at $p^0 = iE_p$, the q^0 integration contour may be deformed from real q^0 to a contour which splits positive- and negative-frequency poles. This remaining contour integral vanishes for $p^0 = iE_p$. The finite contribution is the residue of the pole at $q^0 = i(E_p - E_{p-q})$, which gives a contribution for $E_p > E_{p-q}$. This contribution is shown in Fig. 9(c). To obtain this result, we have again used the fact that a discontinuity of $\Sigma_{(2)}$ is not in need of renormalization.

The graph of Fig. 9(c) can now be written as a phase-space integral over a Feynman amplitude. The result of some straightforward algebra is

$$\Omega_I^{(e)} = e^2 \int \frac{d^3p}{(2\pi)^3 2E_p} \frac{d^3p'}{(2\pi)^3 2E_{p'}} \theta(\mu - E_p)\theta(E_p - E_p') \operatorname{tr}(m - \not{p})\gamma^\mu(m - \not{p}')\gamma_\mu \frac{1}{(p-p')^2}\Bigg|_{\substack{p_0 = iE_p \\ p_0' = iE_{p'}}} . \tag{2.4}$$

The trace over (γ) matrices in Eq. (2.4) is easily performed. Moreover, the transformation $iE_p, iE_p' \to E_p, E_p'$ only converts the Euclidean metric, $\delta^{\mu\nu}$, to the Minkowski metric, $g^{\mu\nu}$. Thus, in the Minkowski metric, Eq. (2.4) is

$$\Omega_I^{(e)} = 32\pi\alpha \int \frac{d^4p}{(2\pi)^4} \frac{d^4p'}{(2\pi)^4} \theta(\mu - p_0)\theta(p_0 - p_0')\theta(p_0')2\pi\delta(p^2 + m^2)2\pi\delta(p'^2 + m^2)\left[\tfrac{1}{2} - \frac{m^2}{(p-p')^2}\right] . \tag{2.5}$$

This equation has the diagrammatic interpretation shown in Fig. 11.

The integrations over p and p' in Eq. (2.5) can be performed directly. The result is[8]

$$\Omega_I^{(e)} = \frac{1}{3\pi^2}\left(\frac{3}{2}\frac{\alpha}{\pi}\right)\frac{1}{4}\left\{3\left[\mu(\mu^2 - m^2)^{1/2} - m^2 \ln\frac{\mu + (\mu^2 - m^2)^{1/2}}{m}\right]^2 - 2(\mu^2 - m^2)^2\right\}, \tag{2.6}$$

which for $\mu \gg m$ becomes

$$\lim_{m\to 0}\Omega_I^{(e)} = \frac{1}{3\pi^2}\frac{3}{2}\frac{\alpha}{\pi}\tfrac{1}{4}\mu^4 . \tag{2.7}$$

III. PLASMONS AND VACUUM POLARIZATION

In this section, the plasmon contribution to the thermodynamic potential, and the vacuum polarization modifications of the order-e^2 exchange energy, will be evaluated. These contributions are $\Omega_I^{(c)}$ and Ω_I^{Π}, and possess the integral representations of Eqs. (1.4) and (1.12).

The part of $\Omega_I^{(c)}$ which is potentially infrared divergent may be isolated by writing

$$\Pi_{(2)}(k \mid \mu) = \Delta\Pi_{(2)}(k \mid \mu) + \Pi_{(2)}(k) . \tag{3.1}$$

Here $\Pi_{(2)}(k)$ is $\Pi_{(2)}(k \mid \mu)$ in the limit of zero chemical potential. The difference of these terms, $\Delta\Pi_{(2)}(k \mid \mu)$, is finite and not in need of renormalization. Using Eq. (3.1) in Eq. (1.4) for $\Omega_I^{(c)}$, we find

$$\Omega_I^{(c)} = \tfrac{1}{2}\int \frac{d^4k}{(2\pi)^4}\left\{\operatorname{tr}\ln\left[1 + e^2\frac{1}{1 + e^2D_0(k)\Pi_2(k)}D_0(k)\Delta\Pi_{(2)}(k \mid \mu)\right] + \operatorname{tr}\ln[1 + e^2D_0(k)\Pi_{(2)}(k)] - e^2D_0(k)\Pi_{(2)}(k \mid \mu)\right\} . \tag{3.2}$$

The second term in this equation is a contribution to the energy density of the vacuum and may be ignored.

To order e^4, all potentially infrared-divergent contributions to Eq. (3.2) are given by expanding in powers of $\Delta\Pi_{(2)}(k \mid \mu)$ with no $\Pi_{(2)}(k)$ insertions. To this order, ignoring contributions to the energy density of the vacuum, $\Omega_I^{(c)}$ is

$$\Omega_I^{(c)} = \tfrac{1}{2}\int \frac{d^4k}{(2\pi)^4}\{\operatorname{tr}\ln[1 + e^2D_0(k)\Delta\Pi_{(2)}(k \mid \mu)] - e^2D_0(k)\Delta\Pi_{(2)}(k \mid \mu) - e^4D_0(k)\Pi_{(2)}(k)D_0(k)\Pi_{(2)}(k \mid \mu)\} . \tag{3.3}$$

The first two terms in Eq. (3.3) give the plasmon contribution to the thermodynamic potential

$$\Omega_I^{\mathrm{pl}} = \tfrac{1}{2}\int \frac{d^4k}{(2\pi)^4}\{\operatorname{tr}\ln[1 + e^2D_0(k)\Delta\Pi_{(2)}(k \mid \mu)] - e^2D_0(k)\Delta\Pi_{(2)}(k \mid \mu)\} . \tag{3.4}$$

The remaining term in Eq. (3.3) combines with Ω_I^{Π} to yield vacuum polarization modifications of the exchange energy

$$\Omega_I^{\mathrm{pol}} = e^4\int \frac{d^4k}{(2\pi)^4}\left[(\gamma S_0\gamma S_0)(k \mid \mu)D_0(k)\Pi_{(2)}(k \mid \mu)D_0(k) - \tfrac{1}{2}D_0(k)\Pi_{(2)}(k)D_0(k)\Pi_{(2)}(k \mid \mu) - S_0(k \mid \mu)\Sigma_{(4)}^{\Pi}(k \mid \mu)\right] . \tag{3.5}$$

The vacuum polarization modification, Ω_I^{pol}, may be related to a phase-space integral over a Feynman amplitude by the diagrammatic techniques of Sec. II. The analysis begins by using the diagrammatic representation of Eq. (3.5) shown in Fig. 12(a). As in Sec. II, the energy integrations must

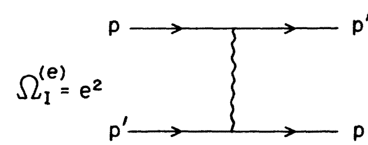

FIG. 11. The exchange energy.

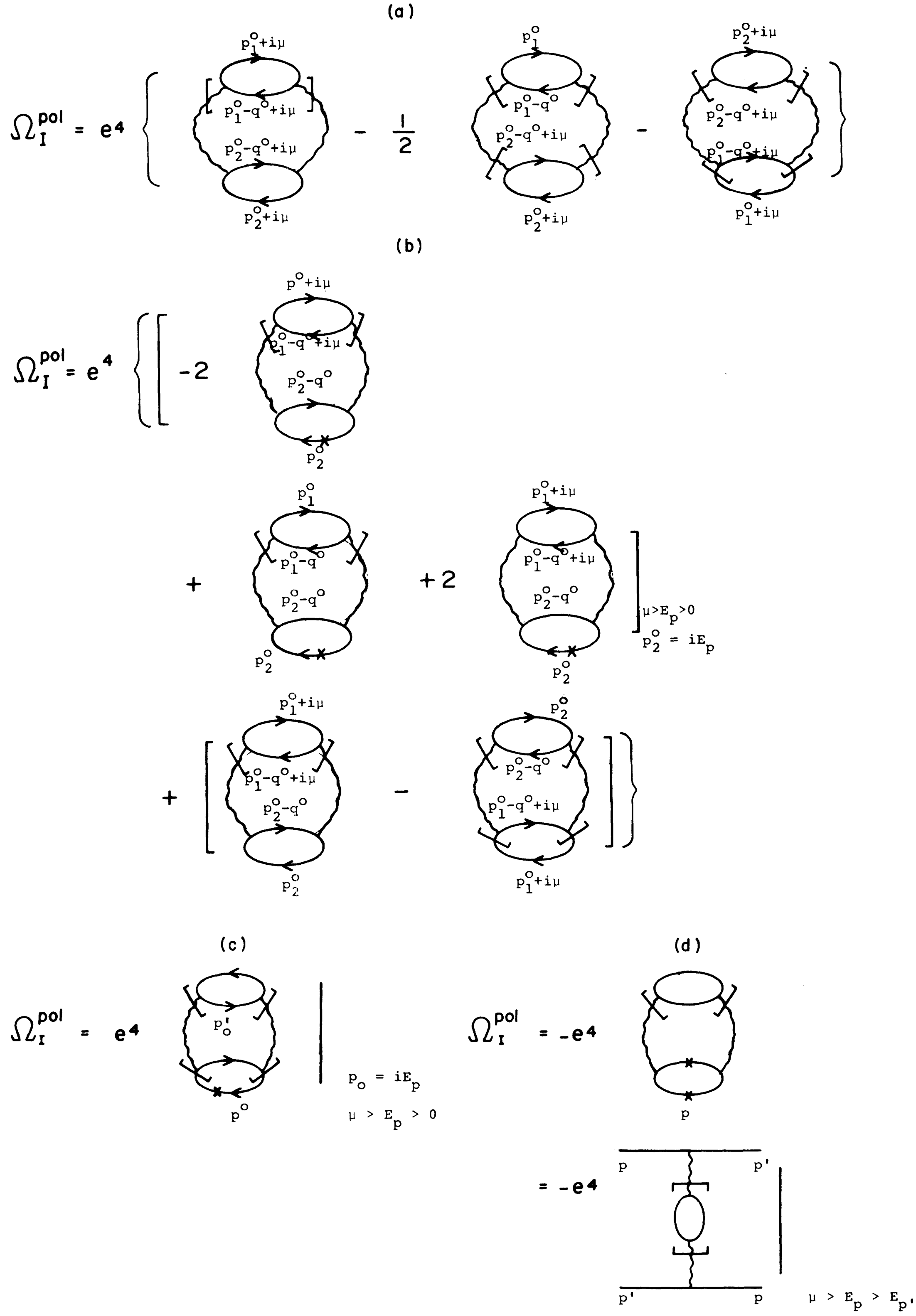

FIG. 12. (a) A diagrammatic interpretation of Eq. (3.5). (b) The result of deforming the p_2^0 integration by $-i\mu$ in the complex p_2^0 plane. (c) The result of deforming the p_2^0 contour. (d) The nonvanishing contribution to Ω_I^{pol}.

be analytically continued to obtain finite pieces and pieces associated with the vacuum energy density; the latter group may be ignored.

The explicit μ dependence in the p_2^0 integrals of Fig. 12(a) may be removed by deforming downward by $i\mu$ in the complex p_2^0 plane. The result of this deformation is shown in Fig. 12(b). To obtain this result, contributions to the vacuum energy have been discarded. We have also made use of the fact that discontinuities associated with fermion propagators of $\Sigma_{(4)}^{\Pi}$ and $\Pi_{(2)}$ are not in need of renormalization.

In all remaining diagrams with μ dependence, we deform the p_1^0 integrations by $-i\mu$. The result of this deformation is shown in Fig. 12(c). A large number of cancellations can be performed until only one diagram remains. This remaining term is similar in structure to Fig. 9(b), the graph which gave the exchange energy.

At this point it is convenient to perform a Wick rotation on the integration from real Euclidean energy to imaginary energy. There are no singularities in the path of the contour integral for real, Euclidean q^0.

The contribution represented in Fig. 12(c) would vanish if the q^0 integral were to split the positive- and negative-frequency poles of the integrand. The graph is nonvanishing, however, because of the presence of a negative-frequency pole at

$$q^0 = i(E_p - E_{p-q}), \quad E_p > E_{p-q}. \tag{3.6}$$

Upon deforming the q^0 contour, we obtain the residue of this pole and a vanishing contribution.

The result of this deformation is shown in Fig. 12(d). This result is clearly a vacuum polarization correction to the exchange energy. It is finite and renormalized, and possesses the integral representation

$$\Omega_I^{\text{pol}} = -128\pi^2\alpha^2 \int \frac{d^4p}{(2\pi)^4}\frac{d^4p'}{(2\pi)^4}\theta(\mu - p_0)\theta(p_0 - p_0')\theta(p_0')$$
$$\times 2\pi\delta(p^2+m^2)2\pi\delta(p'^2+m^2)\left[\frac{1}{2} - \frac{m^2}{(p-p')^2}\right]\Pi_{(2)}((p-p')^2), \tag{3.7}$$

where $\Pi_{(2)}$ is given in terms of $\Pi_{(2)}^{\mu\nu}$ by

$$\Pi_{(2)}^{\mu\nu}(q) = (q^2 g^{\mu\nu} - q^\mu q^\nu)\Pi_{(2)}(q^2). \tag{3.8}$$

Here we have converted back from the Euclidean to the Minkowski metric.

The vacuum polarization modification of the exchange energy may be evaluated for the $\mu/m \gg 1$. In this limit, $\Pi_{(2)}$ may be approximated as[12]

$$\Pi_{(2)}(q^2) = -\frac{1}{4\pi^2}\left(\tfrac{1}{3}\ln\frac{q^2}{m^2} - \tfrac{5}{9}\right). \tag{3.9}$$

We then insert this expression into Eq. (3.7) and evaluate the phase-space integrals in the massless-electron limit, finding

$$\Omega_I^{\text{pol}} = \frac{1}{3\pi^2}\tfrac{1}{4}\mu^4\left(\frac{\alpha}{\pi}\right)^2\left(\tfrac{1}{2}\ln\frac{\mu^2}{m^2} + \ln 2 - \frac{11}{6}\right). \tag{3.10}$$

In this expression, a logarithm appears. This logarithm arises from the short-distance increase in the effective charge due to vacuum fluctuations. In a later paper, logarithms of μ/m will be summed nonperturbatively for high-order graphs by the renormalization group.

We turn now to an evaluation of Ω_I^{pl}. In Eq. (3.4), there is a trace to be performed over spin indices. If rotational invariance and current conservation are used to write

$$\Delta\Pi_{\mu\nu}^{(2)}(q\,|\,\mu) = (q^2\delta_{\mu\nu} - q_\mu q_\nu)\frac{1}{\vec{q}^{\,2}}\Delta\Pi_{00}^{(2)}(q\,|\,\mu) + \delta_{\mu k}(\vec{q}^{\,2}\delta_{kl} - q_k q_l)\delta_{l\nu}\frac{1}{2\vec{q}^{\,2}}\left[\Delta\Pi_{\mu\mu}^{(2)}(q\,|\,\mu) - \frac{3q^2}{\vec{q}^{\,2}}\Delta\Pi_{00}^{(2)}(q\,|\,\mu)\right], \tag{3.11}$$

the evaluation of the trace is straightforward. The result of this evaluation is

$$\Omega_I^{\text{pl}} = \tfrac{1}{2}\int\frac{d^4q}{(2\pi)^4}\left(\ln\left[1 + e^2\frac{\Delta\Pi_{00}(q\,|\,\mu)}{\vec{q}^{\,2}}\right] + 2\ln\left\{1 + e^2\frac{1}{2}\left[\frac{\Delta\Pi_{\mu\mu}(q\,|\,\mu)}{q^2} - \frac{\Delta\Pi_{00}(q\,|\,\mu)}{\vec{q}^{\,2}}\right]\right\} - e^2\frac{\Delta\Pi_{\mu\mu}(q\,|\,\mu)}{q^2}\right). \tag{3.12}$$

To evaluate Eq. (3.12), it is useful to introduce the Euclidean spherical variables $(q, \phi, \theta, \varphi)$. Here the variables θ and φ are ordinary three-dimensional angles. The variables q and ϕ are defined by

$$q = (q^2)^{1/2}, \qquad \phi = \tan^{-1}(|q|/q^0). \tag{3.13}$$

Rotational invariance requires that

$$\Delta\Pi_{00} = \Delta\Pi_{00}(q, \phi \mid \mu),$$
$$\Delta\Pi_{\mu\mu} = \Delta\Pi_{\mu\mu}(q, \phi \mid \mu). \tag{3.14}$$

In Appendix A, we analyze the polarization tensor, $\Delta\Pi_{\mu\nu}$. We show that $\Delta\Pi_{00}$ and $\Delta\Pi_{\mu\mu}$ are invariant under $\phi \to -\phi$. Using this observation, and the definitions

$$\Lambda_1(q^2, \phi) \equiv e^2(1/\sin^2\phi)\Delta\Pi_{00}(q^2, \phi \mid \mu),$$
$$\Lambda_2(q^2, \phi) \equiv \tfrac{1}{2}e^2(\Delta\Pi_{\mu\mu}(q^2, \phi \mid \mu) - (1/\sin^2\phi)\Delta\Pi_{00}(q^2, \phi \mid \mu)), \tag{3.15}$$

Eq. (3.12) for $\Omega_I^{\rm pl}$ becomes

$$\Omega_I^{\rm pl} = \frac{1}{(2\pi)^3}\int_0^\infty q^2 dq^2 \int_0^{\pi/2} \sin^2\phi\, d\phi \left\{ \ln\left[1 + \frac{\Lambda_1(q^2, \phi)}{q^2}\right] + 2\ln\left[1 + \frac{\Lambda_2(q^2, \phi)}{q^2}\right] - \frac{\Lambda_1(q^2, \phi)}{q^2} - \frac{2\Lambda_2(q^2, \phi)}{q^2} \right\}. \tag{3.16}$$

Infrared divergences would arise in Eq. (3.16) if we were to expand the logarithms in powers of Λ because of the nonzero value of $\Lambda_i(q^2, \phi)$ at $q^2 = 0$. These potential infrared-divergent contributions to Eq. (3.16), may, however, be isolated by writing

$$\Omega_I^{\rm pl} = \Omega_1^{\rm pl} + \Omega_2^{\rm pl}, \tag{3.17}$$

where

$$\Omega_1^{\rm pl} = \frac{1}{(2\pi)^3}\int_0^\infty q^2 dq^2 \int_0^{\pi/2} \sin^2\phi\, d\phi \left\{ \ln\left[1 + \frac{1}{q^2}\Lambda_1(0, \phi)\right] + 2\ln\left[1 + \frac{1}{q^2}\Lambda_2(0, \phi)\right] \right.$$
$$\left. + \frac{1}{2}\frac{1}{q^2}\frac{1}{q^2 + 4\mu^2}[\Lambda_1^2(0, \phi) + 2\Lambda_2^2(0, \phi)] - \frac{1}{q^2}[\Lambda_1(0, \phi) + 2\Lambda_2(0, \phi)] \right\} \tag{3.18}$$

and

$$\Omega_2^{\rm pl} = \frac{1}{(2\pi)^3}\int_0^\infty q^2 dq^2 \int_0^{\pi/2} \sin^2\phi\, d\phi \left\{ \ln\left[1 + \frac{1}{q^2}\Lambda_1(q^2, \phi)\right] - \ln\left[1 + \frac{1}{q^2}\Lambda_1(0, \phi)\right] + 2\ln\left[1 + \frac{1}{q^2}\Lambda_2(q^2, \phi)\right] \right.$$
$$- 2\ln\left[1 + \frac{1}{q^2}\Lambda_2(0, \phi)\right] - \frac{1}{2}\frac{1}{q^2(q^2 + 4\mu^2)}[\Lambda_1^2(0, \phi) + 2\Lambda_2^2(0, \phi)]$$
$$\left. - \frac{1}{q^2}[\Lambda_1(q^2, \phi) - \Lambda_1(0, \phi) + 2\Lambda_2(q^2, \phi) - 2\Lambda_2(0, \phi)] \right\}. \tag{3.19}$$

The virtue of Eq. (3.18) is that the q^2 integration can be performed immediately, removing all potential divergences. In Eq. (3.19), effects to order e^4 may be found by expanding the logarithms to order Λ^2.

The result of the q^2 integration in Eq. (3.18) is

$$\Omega_1^{\rm pl} = \frac{1}{(2\pi)^3}\int_0^{\pi/2} \sin^2\phi\, d\phi \left[\tfrac{1}{2}\Lambda_1^2(0, \phi)\ln\frac{\Lambda_1(0, \phi)}{4\mu^2} + \Lambda_2^2(0, \phi)\ln\frac{\Lambda_2(0, \phi)}{4\mu^2} - \tfrac{1}{2}\Lambda_1^2(0, \phi) - \Lambda_2^2(0, \phi)\right]. \tag{3.20}$$

The remaining integration over ϕ in this equation can be performed in the zero-electron-mass limit. In Appendix A, we show that, as $m \to 0$,

$$\Lambda_1(0, \phi) = \frac{4\mu^2}{\sin^2\phi}\left(\frac{\alpha}{\pi}\right)(1 - \phi\cot\phi) \tag{3.21}$$

and

$$\Lambda_2(0, \phi) = \frac{2\alpha\mu^2}{\pi}\left[1 - \frac{1}{\sin^2\phi}(1 - \phi\cot\phi)\right]. \tag{3.22}$$

Using Eqs. (3.21) and (3.22) in Eq. (3.20), we find

$$\Omega_1^{\rm pl} = \frac{1}{3\pi^2}\frac{1}{4}\mu^4\left(\frac{\alpha}{\pi}\right)^2\left(a\ln\frac{\alpha}{\pi} + b\right), \tag{3.23}$$

where

$$a = \frac{12}{\pi}\int_0^{\pi/2} \sin^2\phi\, d\phi \left[\frac{(1 - \phi\cot\phi)^2}{\sin^4\phi} + \frac{1}{2}\left(1 - \frac{1 - \phi\cot\phi}{\sin^2\phi}\right)^2\right] \tag{3.24}$$

and

$$b=\frac{12}{\pi}\int_0^{\pi/2}\sin^2\phi\, d\phi\left\{\frac{(1-\phi\cot\phi)^2}{\sin^4\phi}\left(\ln\frac{1-\phi\cot\phi}{\sin^2\phi}-\tfrac{1}{2}\right)+\frac{1}{2}\left(1-\frac{1-\phi\cot\phi}{\sin^2\phi}\right)\left[\ln\frac{1}{2}\left(1-\frac{1-\phi\cot\phi}{\sin^2\phi}\right)-\tfrac{1}{2}\right]\right\}. \tag{3.25}$$

The integration over ϕ in Eq. (3.24) may be performed directly. We have performed a numerical evaluation for Eq. (3.25). The results of these calculations are

$$a=\tfrac{3}{2},\qquad b=-1.5813. \tag{3.26}$$

To evaluate Ω_2^{pl}, the logarithms in Eq. (3.19) are expanded to second order in Λ_i, with the result

$$\Omega_2^{\text{pl}}=-\frac{1}{2(2\pi)^3}\int_0^\infty q^2dq^2\int_0^{\pi/2}\sin^2\phi\, d\phi\left\{\frac{1}{q^4}[\Lambda_1^{\,2}(q^2,\phi)-\Lambda_1^{\,2}(0,\phi)+2\Lambda_2^{\,2}(q^2,\phi)-2\Lambda_2^{\,2}(0,\phi)]+\frac{1}{q^2(q^2+4\mu^2)}[\Lambda_1^{\,2}(0,\phi)+2\Lambda_2^{\,2}(0,\phi)]\right\}. \tag{3.27}$$

This integral is evaluated in Appendix B in the zero-electron-mass limit by using a contour-integral representation for the Λ_i. The result of the evaluation is

$$\Omega_2^{\text{pl}}=\frac{1}{3\pi^2}\tfrac{1}{4}\mu^4\left(\frac{\alpha}{\pi}\right)^2\left(\frac{\pi^2}{2}+4\ln 2-\tfrac{19}{4}\right). \tag{3.28}$$

Finally, we obtain Ω^{pl} by combining Ω_1^{pl} and Ω_2^{pl}, with the result

$$\Omega_I^{\text{pl}}=\frac{1}{3\pi^2}\tfrac{1}{4}\mu^4\left(\frac{\alpha}{\pi}\right)^2\left(\tfrac{3}{2}\ln\frac{\alpha}{\pi}+1.3761\right). \tag{3.29}$$

IV. A GRAPHICAL ANALYSIS OF Ω_I^{Σ} AND Ω_I^{Λ}

The remaining contributions to the thermodynamic potential in order e^4 are given by Ω_I^{Σ} and Ω_I^{Λ}. These quantities are shown graphically in Figs. 7 and 8. To evaluate Ω_I^{Σ} and Ω_I^{Λ}, we first perform a graphical analysis to relate these terms to phase-space integrals over Feynman amplitudes. (In Sec. V, these phase-space integrals will be evaluated.)

The contribution of Ω_I^{Σ} is shown in Fig. 13(a) with the energies of the loop integrals made explicit. As in Secs. II and III, the contour integral over p_0 is deformed down by $i\mu$ in the complex p_0 plane and a contribution associated with the energy density of the vacuum is discarded. The terms which remain after this deformation are shown in Fig. 13(b). These remaining contributions arise from a simple and a double pole of the fermion propagator at $p_0=iE_p$, $\mu>E_p>0$.

As was the case in the analysis of the exchange energy, the remaining contributions involve fermion self-mass insertions which would vanish for $p_0=iE_p$ if the photon energy integration contours split the positive- and negative-frequency poles of their integrands. Again, we deform the integrations over q_0 and q_0' from the Euclidean region to a contour which splits positive- and negative-frequency poles. In the process of the deformation, nonvanishing contributions are obtained as residues of negative-frequency poles in the upper half energy plane.

In the first term in Eq. 13(b), the only negative-frequency poles appearing in the upper half energy plane are at

$$\begin{aligned} q_0&=i(E_p-E_{p-q}),\quad \mu>E_p>E_{p-q}>0\\ q_0'&=i(E_p-E_{p-q'}),\quad \mu>E_p>E_{p-q'}>0. \end{aligned} \tag{4.1}$$

The result of encircling these poles is shown in Figs. 13(c) and 13(d).

The analysis of the second term in Fig. 13(b) is slightly more complicated. In this term, the integrand possesses singularities of the photon and fermion propagators in the complex q_0 and q_0' planes at

$$\begin{aligned} q_0&=\pm i|\vec{q}|,\\ q_0&=\pm i|\vec{q}'|,\\ q_0&=i(E_p\pm E_{p-q}),\\ q_0+q_0'&=i(E_p\pm E_{p-q-q'}). \end{aligned} \tag{4.2}$$

The q_0 and q_0' integrations are along the real, Euclidean q_0 and q_0' axis. If the q_0 and q_0' contours of integration split the positive and negative poles of the integrand, then this contribution will vanish.

The finite contributions are given as the residues of the negative-frequency poles in the upper half q_0

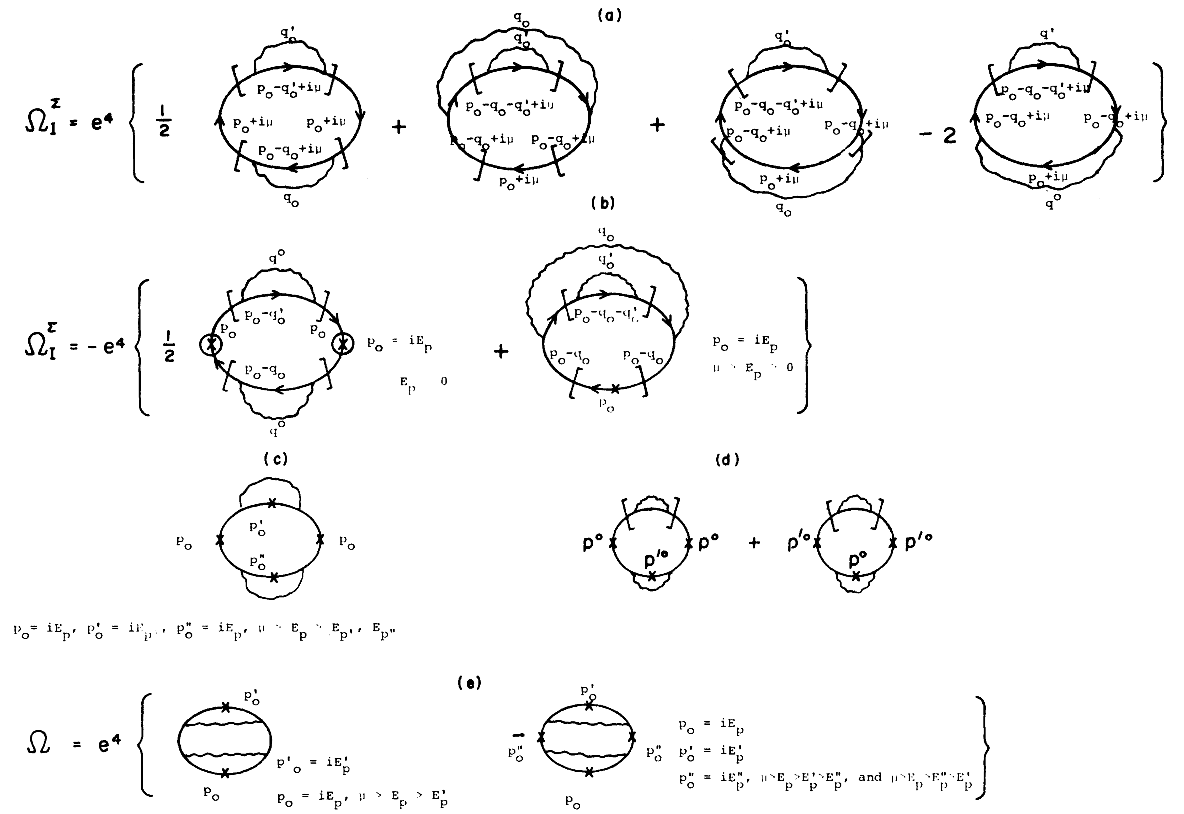

FIG. 13. (a) A graphical representation for Ω_I^Σ. (b) The contribution to Ω_I^Σ after deforming the p^0 contour. (c) The nonvanishing contribution to the first term in (b). (d) Self-energy insertion corrections to the exchange energy arising from the first and second terms of (b). (e) The finite contributions to the second term in (b).

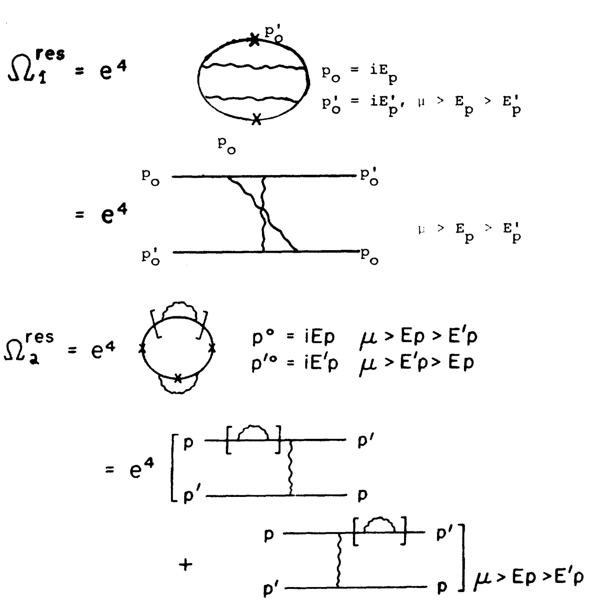

FIG. 14. (a) A rescattering correction to the exchange energy. (b) Self-energy insertion corrections to the exchange energy.

and q_0' planes, at the positions given by the last two terms in Eq. (4.2). These residues, and the conditions where the poles may appear in the upper half energy planes, are shown diagrammatically in Figs. 13(c), 13(d), and 13(e), and divide naturally into three separate contributions. The first of these is a two-body amplitude which gives a rescattering correction of the exchange energy. This contribution, $\Omega_1^{\rm res}$, is shown diagrammatically in Fig. 14(a). After a Wick rotation to Euclidean space, this term possesses the integral representation

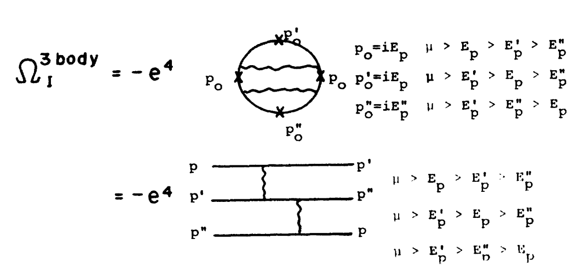

FIG. 15. The three-body contribution to Ω_I^Σ.

$$\Omega_1^{\text{res}} = (4\pi\alpha)^2 i \int \frac{d^4p}{(2\pi)^4}\frac{d^4p'}{(2\pi)^4}\theta(\mu - p_0)\theta(p_0 - p_0')\theta(p_0')2\pi\delta(p^2+m^2)2\pi\delta(p'^2+m^2)$$
$$\times \int \frac{d^4q}{(2\pi)^4}\,\mathrm{tr}(m-\not p)\gamma^\mu(m-\not p+\not q)\gamma^\nu(m-\not p')\gamma^\beta(m-\not p+\not q)\gamma^\alpha\left[g_{\mu\alpha}+(\omega-1)q_\mu q_\alpha/q^2\right]$$
$$\times\left[g_{\nu\beta}+(\omega-1)\frac{(p'-p+q)_\nu(p'-p+q)_\beta}{(p'-p+q)^2}\right]\frac{1}{q^2}\frac{1}{(p'-p+q)^2}\frac{1}{[(p-q)^2+m^2]^2}. \tag{4.3}$$

This result is written here in the Minkowski metric. It is not gauge invariant. Gauge invariance is attained only after a rescattering correction contained in Ω_I^{A} is added to this contribution.

Another contribution to Ω_I^{C} is a three-body graph, shown in Fig. 15. This term must be handled carefully because of the double pole at $p_0 = iE_p$. The double pole generates a singularity in the zero-temperature limit. The singular limit may be taken into account by replacing the restrictions $E_i > E_j$ with

$$\theta(E_i - E_j) \to \lim_{\beta\to\infty}\frac{1}{\exp\{\beta[(1/i)p^0 - E_j]\}+1}\bigg|_{p^0=iE_i}, \tag{4.4}$$

for all terms in which a double pole appears at $p_0 = iE_i$. With such replacement, contour integrals encircling double poles become

$$\oint \frac{dp_0}{2\pi}\frac{1}{(p^0 - iE)^2}\theta(E-E')f((1/i)p_0) \to \theta(E-E')f'(E) + \delta(E-E')f(E). \tag{4.5}$$

Using the transformation of Eq. (4.5), it can easily be seen that

$$\Omega_1^{3\text{ body}} = (4\pi\alpha)^2\int\frac{d^3p}{(2\pi)^3 2E_p}\frac{d^3p'}{(2\pi)^3 2E_p'}\frac{d^3p''}{(2\pi)^3 2E_p''}$$
$$\times\Big\{2E_p\delta(\mu - E_p)\theta(E_p - E_p')\theta(E_p' - E_p'')$$
$$-[\theta(\mu-E_p)\theta(E_p-E_p')\theta(E_p'-E_p'')+\theta(\mu-E_p')\theta(E_p'-E_p)\theta(E_p-E_p'')$$
$$+\theta(\mu-E_p')\theta(E_p'-E_p'')\theta(E_p''-E_p)]2iE_p\frac{d}{dp^0}\Big\}$$
$$\times\,\mathrm{tr}(m-\not p)\gamma^\mu(m-\not p')\gamma_\mu(m-\not p)\gamma^\nu(m-\not p'')\gamma_\nu\frac{1}{(p^0+iE)^2}\frac{1}{(p-p')^2}\frac{1}{(p-p'')^2}\bigg|_{p_0=iE_p,\ p_0'=iE_{p'},\ p_0''=iE_p''} \tag{4.6}$$

In this expression, all gauge-dependent terms have disappeared, since the electrons are on the mass shell. Here we have not converted from the Euclidean to the Minkowski metric.

The final contribution to Ω_I^{C} is a self-mass insertion correction to the exchange energy. This contribution is shown in Fig. 14(b) and possesses the integral representation

$$\Omega_3^{\text{res}} = -(4\pi\alpha)^2\int\frac{d^4p}{(2\pi)^4}\int\frac{d^4p'}{(2\pi)^4}2\pi\delta(p^2+m^2)2\pi\delta(p'^2+m^2)$$
$$\times\{\theta(\mu-p_0)\theta(p_0-p_0')\theta(p_0')+(p_0\leftrightarrow p_0')\}\frac{1}{(p'-p)^2}$$
$$\times\,\mathrm{tr}(m-\not p)\frac{d\Sigma(p)}{d\not p}\gamma^\mu(m-\not p')\gamma^\nu\left(g_{\mu\nu}+(\omega-1)\frac{(p'-p)_\mu(p'-p)_\nu}{(p'-p)^2}\right). \tag{4.7}$$

In this equation Σ is the renormalized self-mass kernel. If dimensional continuation to $4+\epsilon$ dimensions is used to regulate this insertion, this insertion is nonvanishing.[16]

The remaining contributions in order e^4 are given by Ω_I^{A}, which is shown in Fig. 8. These contributions can be rewritten in the form shown in Fig. 16(a) by using the identity

$$\Lambda_R = \Lambda + (Z_1 - 1)\gamma. \tag{4.8}$$

The first three terms in Fig. 16(a) give a renormalization counterterm contribution to the exchange energy, as shown in Fig. 16(b). In the remaining terms, the fermion-loop energy integration may be deformed downward by $i\mu$ in the complex energy plane to give the result shown in Fig. 16(c).

As was the case for Ω_I^{C}, this contribution would vanish if the photon integration contours were to split the positive- and negative-frequency poles of these integrands. This is not the case, however, as there are

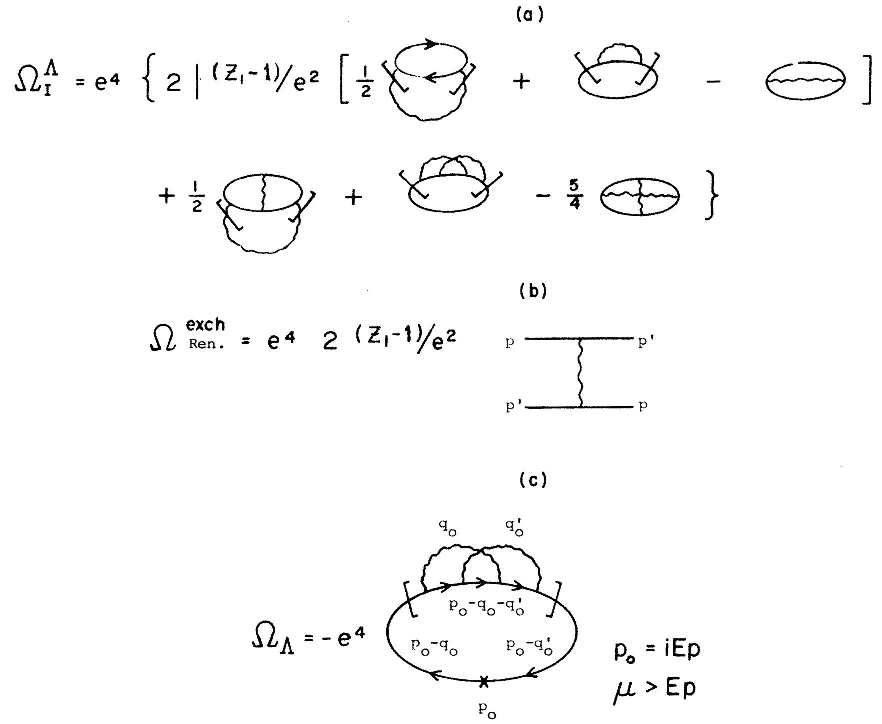

FIG. 16. (a) The contributions to Ω_I^Λ. (b) The first three terms of (a). (c) The result of deforming the fermion-body momenta down by $i\mu$ in the complex energy plane for the last three terms in (a).

negative-frequency poles in the upper half q_0 and q_0' planes. Using techniques entirely in analogy to the techniques employed in the calculation of Ω_I^Σ, we can verify that

$$\Omega_I^\Lambda = \Omega_3^{\text{res}} + \Omega_4^{\text{res}} + \Omega_2^{3\text{ body}}\,. \tag{4.9}$$

These contributions are shown diagrammatically in Figs. 17(a)–17(c).

The rescattering correction to the exchange energy associated with the uncrossed ladder graph, Ω_3^{res}, possesses the integral representation

$$\begin{aligned}\Omega_3^{\text{res}} = i(4\pi\alpha)^2 \int \frac{d^4p}{(2\pi)^4}\int\frac{d^4p'}{(2\pi)^4}\,&2\pi\delta(p^2+m^2)2\pi\delta(p'^2+m^2)\theta(\mu-p_0)\theta(p_0-p_0')\theta(p_0')\int\frac{d^4q}{(2\pi)^4}\\ &\times \mathrm{tr}(m-\not p)\gamma^\mu(m-\not p+\not q)\gamma^\nu(m-\not p')\gamma^\alpha(m-\not p'-\not q)\gamma^\beta\left[g_{\mu\alpha}+(\omega-1)\frac{q_\mu q_\alpha}{q^2}\right]\\ &\times\left[g_{\nu\beta}+(\omega-1)\frac{(p'-p+q)_\nu(p'-p+q)_\beta}{(p'-p+q)^2}\right]\\ &\times\frac{1}{(p-q)^2+m^2}\,\frac{1}{(p'+q)^2+m^2}\,\frac{1}{q^2}\,\frac{1}{(p'-p+q)^2}\,.\end{aligned} \tag{4.10}$$

The rescattering correction associated with a vertex insertion is

$$\begin{aligned}\Omega_4^{\text{res}} = (4\pi\alpha)^2\int\frac{d^4p}{(2\pi)^4}\int\frac{d^4p'}{(2\pi)^4}\,&2\pi\delta(p^2+m^2)2\pi\delta(p'^2+m^2)\\ &\times[\theta(\mu-p_0)\theta(p_0-p_0')\theta(p_0')+(p_0\leftrightarrow p_0')]\,\mathrm{tr}(m-\not p)\Lambda^\mu(p,p')(m-\not p')\gamma^\nu\\ &\times\frac{1}{(p'-p)^2}\left[g_{\mu\nu}+(\omega-1)\frac{(p'-p)_\mu(p'-p)_\nu}{(p'-p)^2}\right].\end{aligned} \tag{4.11}$$

The three-body contribution is

$$\Omega_2^{3\ \text{body}} = -(4\pi\alpha)^2 \int \frac{d^4p}{(2\pi)^4}\frac{d^4p'}{(2\pi)^4}\frac{d^4p''}{(2\pi)^4} 2\pi\delta(p^2+m^2)2\pi\delta(p'^2+m^2)2\pi\delta(p''^2+m^2)$$
$$\times[\theta(\mu-p_0)\theta(p_0-p_0')\theta(p_0'-p_0'')\theta(p_0'') + (p_0'\leftrightarrow p_0'') + (p_0\leftrightarrow p_0'') + (p_0\leftrightarrow p_0') + (p_0\leftrightarrow p_0', p_0'\leftrightarrow p_0'', p_0''\leftrightarrow p_0)$$
$$+ (p_0\leftrightarrow p_0'', p_0'\leftrightarrow p_0, p_0''\leftrightarrow p_0')]\,\mathrm{tr}(m-\not p)\gamma^\mu(m-\not p')\gamma^\nu(m-\not p'')$$
$$\times\gamma_\mu(m-\not p-\not p''+\not p')\gamma_\nu \frac{1}{(p+p''-p')^2+m^2}\frac{1}{(p-p'^2)}\frac{1}{(p'-p'')^2}. \qquad (4.12)$$

In Eqs. (4.10)–(4.12), the metric is Minkowski.

V. THE EVALUATION OF $\Omega_I^{\rm E}$ AND $\Omega_I^{\rm A}$ IN THE SMALL-ELECTRON-MASS LIMIT

In this section, the contributions to the thermodynamic potential discussed in Sec. IV will be evaluated in the small-electron-mass limit. This limit is appropriate to the description of a high-density electron gas. As we will see, it is possible to find analytic expressions for all but one contribution to $\Omega_I^{\rm E}$ and $\Sigma_I^{\rm A}$ in this limit.

As we discussed in Sec. IV, calculation of the individual contributions to $\Omega_I^{\rm E}$ and $\Omega_I^{\rm A}$ is complicated by infrared divergences and a singular massless-electron limit. However, the sum of these contributions is infrared finite and nonsingular in the massless-electron limit. To calculate these contributions avoiding the problems of superficial infrared divergences, it is convenient to work in $4+\epsilon$ dimensions. In $4+\epsilon$ dimensions, the massless-electron and infrared singularities are transformed into singularities of the $\epsilon\to 0$ limit. These singularities are easily canceled in $4+\epsilon$ dimensions,

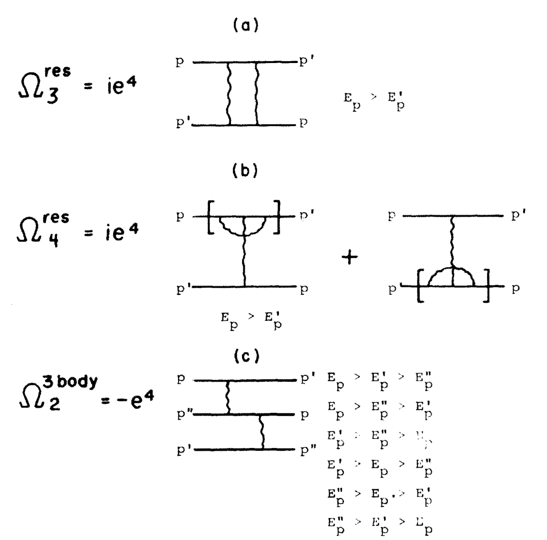

FIG. 17. (a) The uncrossed photon ladder contribution to Ω_I. (b) The vertex correction to the exchange energy. (c) The three-body scattering corrections, Ω_2^3body.

allowing for a smooth continuation to $\epsilon=0$.

To calculate the contributions to $\Omega_I^{\rm A}$, we must perform charge renormalization. This renormalization is easily carried out by working in a Landau gauge generalized to $4+\epsilon$ dimensions, where the photon propagator takes the form

$$D_0{}^{-1}{}_{\mu\nu}(q) = \left[g_{\mu\nu} - \frac{(1+\frac{1}{2}\epsilon)^2}{1+\frac{1}{4}\epsilon}\frac{q_\mu q_\nu}{q^2}\right]\frac{1}{q^2}. \qquad (5.1)$$

Then, as $\epsilon\to 0$, the renormalization constants $Z_1 = Z_2$ are finite in the massless-electron limit and are given by

$$Z_1 = Z_2 = 1 - \frac{3\alpha}{8\pi}. \qquad (5.2)$$

In $4+\epsilon$ dimensions, the following identities are useful in the calculation of $\Omega_I^{\rm E}$ and $\Omega_I^{\rm A}$ (see Ref. 9):

$$\gamma_\mu \not a \gamma^\mu = 2(1+\tfrac{1}{2}\epsilon)\not a,$$
$$\gamma_\mu \not a \not b \gamma^\mu = 4a\cdot b - \epsilon \not a \not b, \qquad (5.3)$$
$$\gamma_\mu \not a \not b \not c \gamma^\mu = 2\not c \not b \not a + \epsilon \not a \not b \not c,$$

$$\int \frac{d\Omega^{4+\epsilon}}{(2\pi)^{4+\epsilon}} = \frac{1}{2\pi}\int_0^\pi d\theta_1 \sin^{2+\epsilon}\theta_1 \int \frac{d\Omega^{3+\epsilon}}{(2\pi)^{3+\epsilon}}$$
$$= \frac{1}{(2\pi)^2}\int_0^\pi d\theta_1 \sin^{2+\epsilon}\theta_1\, d\theta_2 \sin^{1+\epsilon}\theta_2 \int \frac{d\Omega^{2+\epsilon}}{(2\pi)^{2+\epsilon}}$$
$$= \frac{1}{2^{1+\epsilon/2}}\frac{1}{(2\pi)^{2+\epsilon/2}}\frac{1}{\Gamma(2+\frac{1}{2}\epsilon)}, \qquad (5.4)$$

$$\int_0^\infty dq\, q^{3+\epsilon+a}(q^2+M^2)^{-b}$$
$$= \frac{1}{2}\frac{\Gamma(2+\frac{1}{2}(\epsilon+a))\,\Gamma(b-2-\frac{1}{2}(\epsilon+a))}{\Gamma(b)(M^2)^{b-2-(\epsilon+a)/2}}. \qquad (5.5)$$

We begin our analysis by calculating $\Omega_I^{\rm E}$. As shown in Sec. IV, there are three contributions to $\Omega_I^{\rm E}$. One contribution is a rescattering correction to the exchange energy given in Eq. (4.3), and another contribution is a three-body scattering graph given by Eq. (4.6). In the massless-electron limit, the trace algebra may be performed directly, obtaining for $\Omega^{\rm res}$

$$\Omega_1^{\rm res}=(4\pi\alpha)^2\int\frac{d^{4+\epsilon}p}{(2\pi)^{4+\epsilon}}\frac{d^{4+\epsilon}p'}{(2\pi)^{4+\epsilon}}\theta(\mu-p_0)\theta(p_0-p_0')\theta(p_0')2\pi\delta(p^2)2\pi\delta(p'^2)M_1^{\rm res}(p,p')\,, \tag{5.6}$$

where

$$M_1^{\rm res}=\overline{M}_1^{\rm res}+M_1^{\rm gauge}\,, \tag{5.7}$$

$$\overline{M}_1^{\rm res}(p,p')=32i(1+\tfrac{1}{2}\epsilon)^2\int\frac{d^{4+\epsilon}q}{(2\pi)^{4+\epsilon}}(p'\cdot q\,p\cdot q-\tfrac{1}{2}p\cdot p'q^2)\frac{1}{q^2}\frac{1}{(p'-p+q)^2}\frac{1}{(p-q)^4} \tag{5.8}$$

and

$$M_1^{\rm gauge}=16i\frac{(1+\frac{1}{2}\epsilon)^3}{(1+\frac{1}{4}\epsilon)}\int\frac{d^{4+\epsilon}q}{(2\pi)^{4+\epsilon}}\times\left[p'\cdot p\frac{1}{q^4}\frac{1}{(p'-p+q)^2}+\frac{1}{2}\frac{(1+\frac{1}{2}\epsilon)}{(1+\frac{1}{4}\epsilon)}(p'\cdot q\,p\cdot q-\tfrac{1}{2}p'\cdot p\,q^2)\frac{1}{q^4}\frac{1}{(p'-p+q)^4}\right]. \tag{5.9}$$

The three-body graph is given by

$$\Omega_1^{3\ \rm body}=-4(1+\tfrac{1}{2}\epsilon)^2(4\pi\alpha)^2\int\frac{d^{3+\epsilon}p}{(2\pi)^{3+\epsilon}2E}\frac{d^{3+\epsilon}p'}{(2\pi)^{3+\epsilon}2E'}\frac{d^{3+\epsilon}p''}{(2\pi)^{3+\epsilon}2E''}\times\Big[\delta(\mu-E)\theta(\mu-E')\theta(E'-E'')1/E+\theta(\mu-E)\theta(E-E')\theta(E'-E'')\times\left(\frac{1}{E^2}+\frac{1}{p\cdot p''}+\frac{1}{p\cdot p'}-\frac{p'\cdot p''}{p'\cdot p\,p''\cdot p}+(p\leftrightarrow p')+(p\leftrightarrow p'')\right)\Big]. \tag{5.10}$$

The integrations in Eqs. (5.8) and (5.9) are easily carried out by standard Feynman-parameter techniques with the result

$$\overline{M}_1=4(-p'\cdot p)^{\epsilon/2}(2\pi)^{-2-\epsilon/2}\frac{\pi}{\epsilon\sin(\pi\epsilon/2)}\frac{\Gamma(1+\frac{1}{2}\epsilon)}{\Gamma(1+\epsilon)}\frac{(1+\frac{1}{2}\epsilon)^3}{(1+\epsilon)} \tag{5.11}$$

and

$$M_1^{\rm gauge}=-2(2\pi)^{-2-\epsilon/2}(-p'\cdot p)^{\epsilon/2}\frac{\pi}{\sin(\pi\epsilon/2)}\frac{\Gamma(1+\frac{1}{2}\epsilon)}{\Gamma(1+\epsilon)}\frac{(1+\frac{1}{2}\epsilon)^3}{(1+\frac{1}{4}\epsilon)}(1-\tfrac{1}{8}\epsilon)\,. \tag{5.12}$$

In the derivation of these expressions, many terms which vanish as $\epsilon\to 0$ have been ignored. Now, using the identity

$$\int\frac{d^{4+\epsilon}p}{(2\pi)^{4+\epsilon}}\int\frac{d^{4+\epsilon}p'}{(2\pi)^{4+\epsilon}}\theta(\mu-p_0)\theta(p_0-p_0')\theta(p_0')\,2\pi\delta(p^2)2\pi\delta(p'^2)(-p'\cdot p)^{\epsilon/2}=\frac{\Gamma(1+\frac{1}{2}\epsilon)}{\Gamma(1+\frac{3}{2}\epsilon)}\,\tfrac{1}{4}\mu^{4+3\epsilon}(2\pi)^{-4-\epsilon}2^{(3/2)\epsilon-1}\frac{1}{(1+\frac{3}{4}\epsilon)^2(1+\frac{3}{2}\epsilon)(1+\epsilon)}, \tag{5.13}$$

we find $\Omega_1^{\rm res}$ as

$$\Omega_1^{\rm res}=\alpha^2\tfrac{1}{4}\mu^{4+3\epsilon}(2\pi)^{-4-(3/2)\epsilon}2^{3+(3/2)\epsilon}\frac{\pi}{\epsilon\sin(\pi\epsilon/2)}\frac{\Gamma^2(1+\frac{1}{2}\epsilon)}{\Gamma(1+\frac{3}{2}\epsilon)\Gamma(1+\epsilon)}\frac{(1+\frac{1}{2}\epsilon)^3}{(1+\frac{3}{4}\epsilon)^2(1+\epsilon)^2(1+\frac{3}{2}\epsilon)}(1-\tfrac{1}{2}\epsilon-\tfrac{5}{16}\epsilon^2)\,. \tag{5.14}$$

Similarly, the three-body scattering graph is found as

$$\Omega_1^{3\ \rm body}=-\alpha^2\tfrac{1}{4}\mu^{4+3\epsilon}2^{4+(3/2)\epsilon}(2\pi)^{-4-(3/2)\epsilon}\frac{\Gamma^3(1+\frac{1}{2}\epsilon)}{\Gamma^3(1+\epsilon)}\frac{1}{\epsilon^2}\frac{(1+\frac{1}{2}\epsilon)^2}{(1+\frac{3}{4}\epsilon)(1+\epsilon)^2}(1-\tfrac{9}{4}\epsilon+4\epsilon^2)\,. \tag{5.15}$$

The final contribution to $\Omega_I^{\rm D}$ is given by $\Omega_2^{\rm res}$. In the limit of zero electron mass in $4+\epsilon$ dimensions, this contribution is simply proportional to $1-Z_2$ times the exchange energy,

$$\Omega_2^{\rm res}=\frac{1}{3\pi^2}\tfrac{1}{4}\mu^4\left(\frac{\alpha}{\pi}\right)^2(\tfrac{9}{8})\,. \tag{5.16}$$

The sum of $\Omega^{\rm res}$ and $\Omega^{3\ \rm body}$ is finite in the $\epsilon\to 0$ limit. The sum of these terms yields

$$\Omega_I^{\rm D}=-\frac{1}{3\pi^2}\tfrac{1}{4}\mu^4\left(\frac{\alpha}{\pi}\right)^2\tfrac{3}{4}\,. \tag{5.17}$$

This finite result is, of course, gauge dependent, and had a gauge other than the Landau gauge been used, a different, infrared-divergent result would have been obtained.

To calculate the remaining contributions, those given by Ω^{Λ}, we use the same techniques as in the case of $\Omega_I^{\rm E}$. In the massless-electron limit, these contributions are of the form

$$\Omega_I^{\rm res}=\int\frac{d^{4+\epsilon}p}{(2\pi)^{4+\epsilon}}\frac{d^{4+\epsilon}p'}{(2\pi)^{4+\epsilon}}\theta(\mu-p_0)\theta(p_0-p_0')\theta(p_0')2\pi\delta(p^2)2\pi\delta(p'^2)M_i^{\rm res}(p,p'), \tag{5.18}$$

for the rescattering corrections to the exchange energy, and

$$\begin{aligned}\Omega_2^{3\ \rm body}=8(4\pi\alpha)^2(1+\tfrac{1}{2}\epsilon)\int\frac{d^{3+\epsilon}p}{(2\pi)^{3+\epsilon}2E}\frac{d^{3+\epsilon}p'}{(2\pi)^{3+\epsilon}2E'}\frac{d^{3+\epsilon}p''}{(2\pi)^{3+\epsilon}2E''}\theta(\mu-E)\theta(E-E')\theta(E'-E'')\\ \times\bigg[\frac{p\cdot p''}{p\cdot p''-p'\cdot p-p'\cdot p''}\left(\frac{1}{p\cdot p'}+\frac{1}{p'\cdot p''}\right)+\frac{p'\cdot p}{p'\cdot p-p'\cdot p''-p\cdot p''}\left(\frac{1}{p'\cdot p''}+\frac{1}{p\cdot p''}\right)\\ +\frac{p'\cdot p''}{p'\cdot p''-p\cdot p''-p\cdot p'}\left(\frac{1}{p\cdot p''}+\frac{1}{p\cdot p'}\right)\bigg].\end{aligned} \tag{5.19}$$

The matrix elements $M_i^{\rm res}(p,p')$ are given by

$$M_3=\bar{M}_3-M_1^{\rm gauge}, \tag{5.20}$$

where

$$\begin{aligned}\bar{M}_3=-32i(1+\tfrac{1}{2}\epsilon)\int\frac{d^{4+\epsilon}q}{(2\pi)^{4+\epsilon}}[p\cdot p'(p-q)\cdot(p'+q)-\tfrac{1}{2}\epsilon(p'\cdot q\,p\cdot q-\tfrac{1}{2}p'\cdot p\,q^2]\\ \times\frac{1}{q^2}\frac{1}{(p'-p+q)^2}\frac{1}{q^2-2p'\cdot q}\frac{1}{q^2-2p\cdot q},\end{aligned} \tag{5.21}$$

and $M^{\rm gauge}$ is given by Eq. (5.9). The matrix element M_4 can be written as

$$M_4=\bar{M}_4+M_4^{\rm ren}, \tag{5.22}$$

where

$$\begin{aligned}\bar{M}_4=32i(1+\tfrac{1}{2}\epsilon)\frac{1}{p\cdot p'}\int\frac{d^{4+\epsilon}q}{(2\pi)^{4+\epsilon}}\\ \times\left\{[p'\cdot(p-q)p\cdot(p'-q)+\tfrac{1}{4}\epsilon\,p'\cdot p\,q^2]\frac{1}{q^2}\frac{1}{q^2-2p\cdot q}\frac{1}{q^2-2p'\cdot q}-\frac{1}{4}\frac{(1+\frac{1}{2}\epsilon)^2}{(1+\frac{1}{4}\epsilon)}p'\cdot p\frac{1}{q^4}\right\}\end{aligned} \tag{5.23}$$

and $M_4^{\rm ren}$ is the finite renormalization of the electromagnetic vertex proportional to $(1-Z_1)$.

Evaluating the rescattering corrections to the exchange energy, we find

$$\Omega_3^{\rm res}=\alpha^2\tfrac{1}{4}\mu^{4+3\epsilon}(2\pi)^{-4-(3/2)\epsilon}2^{(3/2)\epsilon+4}\frac{\Gamma^2(1+\frac{1}{2}\epsilon)}{\Gamma(1+\frac{3}{2}\epsilon)\Gamma(1+\epsilon)}\frac{\pi}{\epsilon\sin(\pi\epsilon/2)}\frac{(1+\frac{1}{2}\epsilon)e^{i\pi\epsilon/2}}{(1+\frac{3}{4}\epsilon)^2(1+\epsilon)^2(1+\frac{3}{2}\epsilon)}(1+\tfrac{5}{4}\epsilon+\tfrac{13}{32}\epsilon^2), \tag{5.24}$$

$$\Omega_4^{\rm res}=-\alpha^2\tfrac{1}{4}\mu^{4+3\epsilon}(2\pi)^{-4-(3/2)\epsilon}2^{(3/2)\epsilon+4}\frac{\Gamma^2(1+\frac{1}{2}\epsilon)}{\Gamma(1+\frac{3}{2}\epsilon)\Gamma(1+\epsilon)}\frac{\pi}{\epsilon\sin(\pi\epsilon/2)}\frac{(1+\frac{1}{2}\epsilon)}{(1+\frac{3}{4}\epsilon)^2(1+\epsilon)^2(1+\frac{3}{2}\epsilon)}(1+\tfrac{1}{4}\epsilon+\tfrac{7}{16}\epsilon^2) \tag{5.25}$$

It should be noted that $\Omega_3^{\rm res}$ acquires an imaginary part. This imaginary part cancels against an imaginary part of the three-body contribution, $\Omega_2^{3\ \rm body}$.

The calculation of $\Omega_2^{3\ \rm body}$ is complicated. However, the real part of $\Omega_2^{3\ \rm body}$ is easily divided into one piece which is singular in the $\epsilon\to 0$ limit and is analytically calculable, and another piece which must be calculated numerically and is finite as $\epsilon\to 0$. This division is achieved by writing

$${\rm Re}\Omega_2^{3\ \rm body}={\rm Re}\bar{\Omega}_2^{3\ \rm body}+{\rm Re}\tilde{\Omega}_2^{3\ \rm body}, \tag{5.26}$$

where

$$\begin{aligned}{\rm Re}\bar{\Omega}_2^{3\ \rm body}=8(4\pi\alpha)^2(1+\tfrac{1}{2}\epsilon)\int\frac{d^{4+\epsilon}p}{(2\pi)^{4+\epsilon}}\frac{d^{4+\epsilon}p'}{(2\pi)^{4+\epsilon}}\frac{d^{4+\epsilon}p''}{(2\pi)^{4+\epsilon}}2\pi\delta(p^2)2\pi\delta(p'^2)2\pi\delta(p''^2)\theta(\mu-p_0)\theta(p_0-p_0')\theta(p_0'-p_0'')\\ \times\theta(p_0'')\left(\frac{1}{p\cdot p'}+\frac{1}{p'\cdot p''}+\frac{1}{p\cdot p''}\right)\end{aligned} \tag{5.27}$$

and

$$\tilde{\Omega}_2^{3\ \text{body}} = 8(4\pi\alpha)^2 \,\text{Re} \int \frac{d^4p}{(2\pi)^4} \frac{d^4p'}{(2\pi)^4} \frac{d^4p''}{(2\pi)^4}$$
$$\times 2\pi\delta(p^2) 2\pi\delta(p'^2) 2\pi\delta(p''^2) \theta(\mu - p_0)\theta(p_0 - p_0')\theta(p_0' - p_0'')$$
$$\times \theta(p_0'')\left[\frac{p\cdot p''}{p\cdot p'' - p\cdot p' - p'\cdot p''}\left(\frac{1}{p'\cdot p} + \frac{1}{p'\cdot p''}\right) + \frac{p'\cdot p}{p'\cdot p - p'\cdot p'' - p\cdot p''}\left(\frac{1}{p''\cdot p'} + \frac{1}{p''\cdot p}\right)\right.$$
$$\left. + \frac{p''\cdot p'}{p''\cdot p' - p''\cdot p - p\cdot p'}\left(\frac{1}{p\cdot p''} + \frac{1}{p\cdot p'}\right) - \frac{1}{p\cdot p'} - \frac{1}{p'\cdot p''} - \frac{1}{p\cdot p''}\right]. \tag{5.28}$$

Using Eqs. (5.26), (5.27), and (5.28), we find that

$$\text{Re}\Omega_2^{3\ \text{body}} = -\alpha^2 \tfrac{1}{4} \mu^{4+3\epsilon} 2^{5+(3/2)\epsilon} (2\pi)^{-4-(3/2)\epsilon} \frac{\Gamma^3(1+\frac{1}{2}\epsilon)}{\Gamma^3(1+\epsilon)} \frac{1}{\epsilon} \frac{(1+\frac{1}{2}\epsilon)}{(1+\frac{3}{4}\epsilon)(1+\epsilon)^2}\left[1 - \epsilon(\tfrac{7}{4} + \delta)\right], \tag{5.29}$$

where δ is given by

$$\delta = \frac{2\pi^4}{\alpha^2\mu^4} \,\text{Re}\tilde{\Omega}_2^{3\ \text{body}}. \tag{5.30}$$

We have evaluated δ numerically by Sheppey[17] and find

$$\delta = 1.36 \pm 0.02. \tag{5.31}$$

Finally, we obtain Ω_I^Λ as

$$\Omega_I^\Lambda = \Omega_3^{\text{res}} + \Omega_4^{\text{res}} + \Omega_2^{3\ \text{body}}$$
$$= \frac{1}{3\pi^2} \tfrac{1}{4}\mu^4 \left(\frac{\alpha}{\pi}\right)^2 6(\delta - \pi^2/8 - \tfrac{17}{32})$$
$$= \frac{1}{3\pi^2} \tfrac{1}{4}\mu^4 \left(\frac{\alpha}{\pi}\right)^2 (-2.43 \pm 0.12). \tag{5.32}$$

VI. CONCLUSIONS

In the preceding sections, the thermodynamic potential for a relativistic electron gas was evaluated in the small-electron-mass limit. The results of this evaluation are

$$\Omega_I = \Omega_I^{(e)} + \Omega_I^{\text{pl}} + \Omega_I^{\text{pol}} + \Omega_I^\Sigma + \Omega_I^\Lambda, \tag{6.1}$$

where

$$\Omega_I^{(e)} = \frac{1}{3\pi^2} \tfrac{1}{4}\mu^4 \left(\frac{\alpha}{\pi}\right)(\tfrac{3}{2}), \tag{6.2}$$

$$\Omega_I^{\text{pl}} = \frac{1}{3\pi^2} \tfrac{1}{4}\mu^4 \left(\frac{\alpha}{\pi}\right)^2 \left(\tfrac{3}{2}\ln\frac{\alpha}{\pi} + 1.3761\right), \tag{6.3}$$

$$\Omega_I^{\text{pol}} = \frac{1}{3\pi^2} \tfrac{1}{4}\mu^4 \left(\frac{\alpha}{\pi}\right)^2 \left(\tfrac{1}{2}\ln\frac{\mu^2}{m^2} - 1.1402\right), \tag{6.4}$$

$$\Omega_I^\Sigma = \frac{1}{3\pi^2} \tfrac{1}{4}\mu^4 \left(\frac{\alpha}{\pi}\right)^2 (-\tfrac{3}{4}), \tag{6.5}$$

and

$$\Omega_I^\Lambda = \frac{1}{3\pi^2} \tfrac{1}{4}\mu^4 \left(\frac{\alpha}{\pi}\right)^2 (-2.43 \pm 0.12). \tag{6.6}$$

In these equations, Ω_I^Σ and Ω_I^Λ are not individually gauge invariant, although their sum is. This observation was exploited in Sec. V, where a gauge was employed which made Ω_I^Σ and Ω_I^Λ individually infrared finite. In a later paper on the thermodynamic potential of a zero-temperature relativistic quark gas, the infrared finiteness of Ω_I^Σ and Ω_I^Λ will be used to demonstrate the infrared finiteness of the quark thermodynamic potential.

In Eq. (6.4), a logarithmic singularity would appear if the zero-electron-mass limit were to be taken. This singularity is artificial, and arises from defining the photon wave-function renormalization constant, Z_3, at zero momentum. The singularity may be removed by defining Z_3 at a Euclidean subtraction point, ${\mu_0}^2$. Of course, the change in the definition of Z_3 cannot lead to any differences in the physical results of the theory. This invariance of the theory will be used in a later paper to write the thermodynamic potential in a form which makes the invariance manifest, that is, the renormalization group will be applied to Eqs. (6.1)–(6.6). Use of the renormalization vacuum polarization tensor subtracted at $q^2 = {\mu_0}^2$ by

$$\Pi_R(q^2; {\mu_0}^2) \equiv \Pi_R(q^2; 0) - \Pi_R(q^2 = {\mu_0}^2; 0). \tag{6.7}$$

In the evaluation of the thermodynamic potential, $\Pi_R(q^2)$ appears in phase-space integrals only where the momentum integrals peak the integrand at large q^2. For our purposes, Eq. (6.7) is needed, therefore, only for $q^2 \gg m^2$. If we choose $q^2 \gtrsim {\mu_0}^2 \gg m^2$, Eq. (6.7) becomes

$$\Pi_R(q^2; {\mu_0}^2) = -\frac{1}{4\pi^2}\frac{1}{3}\ln\frac{q^2}{{\mu_0}^2}. \tag{6.8}$$

With this modification, Eq. (3.10) for $\Omega_I^{\text{pol}}(\mu, \mu_0)$ becomes

$$\Omega_I^{\text{pol}}(\mu; \mu_0) = \frac{1}{3\pi^2} \tfrac{1}{4}\mu^4 \left(\frac{\alpha'}{\pi}\right)^2 \left[\tfrac{1}{2}\ln\frac{\mu^2}{{\mu_0}^2} - (1 - \ln 2)\right]. \tag{6.9}$$

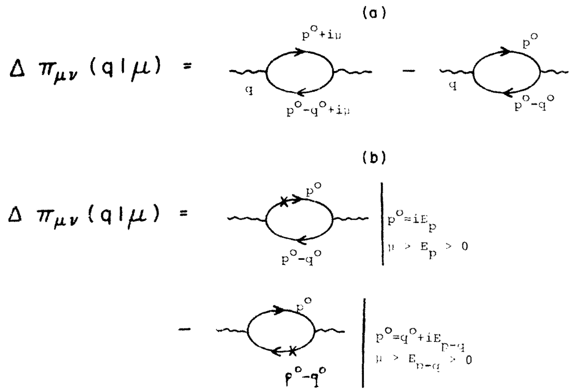

FIG. 18. (a) The polarization tensor $\Delta\pi_{\mu\nu}$. (b) The finite contributions to $\Delta\Pi_{\mu\nu}$.

We should note that α' is not the same α which appears in Eq. (3.10), since changing the definition of Z_3 also changes the definition of α. This fact will be exploited in a later paper where the renormalization group is considered in detail.

Using Eqs. (6.1)–(6.6) and Eq. (6.9), we find

$$\Omega_I(\mu^2, \mu_0{}^2=0)=\frac{1}{3\pi^2}\,\tfrac{1}{4}\mu^4\left[\frac{3}{2}\,\frac{\alpha}{\pi}+\frac{3}{2}\left(\frac{\alpha}{\pi}\right)^2\ln\frac{\alpha}{\pi}+\frac{1}{2}\left(\frac{\alpha}{\pi}\right)^2\ln\frac{\mu^2}{m^2}-\left(\frac{\alpha}{\pi}\right)^2(2.94\pm0.12)\right] \tag{6.10}$$

for Z_3 defined conventionally, and

$$\Omega_I(\mu^2;\mu_0{}^2)=\frac{1}{3\pi^2}\,\tfrac{1}{4}\mu^4\left[\frac{3}{2}\,\frac{\alpha'}{\pi}+\frac{3}{2}\left(\frac{\alpha'}{\pi}\right)^2\ln\frac{\alpha'}{\pi}+\frac{1}{2}\left(\frac{\alpha'}{\pi}\right)^2\ln\frac{\mu^2}{\mu_0{}^2}-\left(\frac{\alpha'}{\pi}\right)^2(2.11\pm0.12)\right] \tag{6.11}$$

for Z_3 defined at an Euclidean subtraction point, $\mu_0{}^2\gg m^2$.

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APPENDIX A: THE POLARIZATION TENSOR, $\Delta\Pi^{(2)}_{\mu\nu}$

In this appendix, we shall derive integral representations for $\Delta\Pi^{(2)}_{\mu\nu}$. We recall that $\Delta\Pi^{(2)}_{\mu\nu}$ is the difference between the photon polarization tensor at finite chemical potential and at zero chemical potential. This difference is finite and not in need of renormalization.

The polarization tensor, $\Delta\Pi^{(2)}_{\mu\nu}$ is represented graphically in Fig. 18(a). To evaluate $\Delta\Pi^{(2)}_{\mu\nu}$, we use the techniques of Sec. II. Deforming the p^0 integration contour down by $i\mu$ in the complex p^0 plane, we obtain from the first graph of Fig. 18(a) two finite pieces and a piece which cancels the second graph of Fig. 18(a). The two finite pieces are the residues of the poles at

$$\begin{aligned}&p^0=iE_p,\quad \mu>E_p>0\\&p^0=q^0+iE_{p-q},\quad \mu>E_{p-q}>0\,.\end{aligned} \tag{A1}$$

These finite pieces are represented graphically in Fig. 18(b). An overall minus sign is indicated for Fig. 18(b), since the poles are encircled in a clockwise direction as the contours are distorted.

The graph of Fig. 18(b) corresponds to the integral representation

$$\begin{aligned}\Delta\Pi^{(2)}_{\mu\nu}(q\,|\,\mu)=-\int\frac{d^3p}{2E_p(2\pi)^3}\,\theta(\mu-E_p)\bigg[&\frac{1}{(p-q)^2+m^2}\,\mathrm{tr}(m-\not p)\gamma_\mu(m-\not p+\not q)\gamma_\nu\\&+\frac{1}{(p+q)^2+m^2}\,\mathrm{tr}(m-\not p-\not q)\gamma_\mu(m-\not p)\gamma_\nu\bigg]\,.\end{aligned} \tag{A2}$$

The trace algebra is easily performed with the result

$$\begin{aligned}\Delta\Pi^{(2)}_{\mu\nu}(q\,|\,\mu)=-8\,\mathrm{Re}\int&\frac{d^3p}{2E_p(2\pi)^3}\,\theta(\mu-E_p)\\&\times\{p_\mu(p-q)_\nu+(p-q)_\mu p_\nu-\delta_{\mu\nu}[m^2+p\cdot(p-q)]\}\frac{1}{(p-q)^2+m^2}\bigg|_{p_0=iE_p}\,,\end{aligned} \tag{A3}$$

This integral representation may be used to show that $\Delta\Pi^{(2)}_{\mu\nu}$ satisfies the constraint of current conservation

$$q^{\mu}\Delta\Pi^{(2)}_{\mu\nu}(q\,|\,\mu)=0\,. \tag{A4}$$

This constraint, together with rotational invariance, imply that

$$\Delta\Pi^{(2)}_{\mu\nu}(q\,|\,\mu)=(q^2\delta_{\mu\nu}-q_\mu q_\nu)\frac{1}{\vec{q}^{\,2}}\,\Delta\Pi^{(2)}_{00}(q\,|\,\mu)+\delta_{\mu k}(\vec{q}^{\,2}\delta_{kl}-q_k q_l)\delta_{l\nu}\frac{1}{2\vec{q}^{\,2}}\left(\Delta\Pi^{(2)}_{\mu\mu}(q\,|\,\mu)-\frac{3q^2}{\vec{q}^{\,2}}\,\Delta\Pi_{00}(q\,|\,\mu)\right), \tag{A5}$$

where from Eq. (A3)

$$\Delta\Pi^{(2)}_{00}(q\,|\,\mu)=-8\,\mathrm{Re}\int\frac{d^3p}{2E_p(2\pi)^3}\,\theta(\mu-E_p)[2p^0(p^0-q^0)-m^2-p\cdot(p-q)]\,\frac{1}{(p-q)^2+m^2}\,\bigg|_{p_0=iE_p} \tag{A6}$$

and

$$\Delta\Pi^{(2)}_{\mu\mu}(q\,|\,\mu)=16\,\mathrm{Re}\int\frac{d^3p}{2E_p(2\pi)^3}\,\theta(\mu-E_p)\,[2m^2+p\cdot(p-q)]\frac{1}{(p-q)^2+m^2}\,\bigg|_{p_0=iE_p} \tag{A7}$$

These equations make manifest the invariance of $\Delta\Pi_{00}$ and $\Delta\Pi_{\mu\nu}$ under $q^0\to -q^0$.

After some algebra, Eqs. (A6) and (A7) may be rewritten as

$$\Delta\Pi^{(2)}_{00}(q\,|\,\mu)=16\int\frac{d^2p}{2E_p(2\pi)^3}\,\theta(\mu-E_p)[E_p{}^2|\vec{q}|^2-(\vec{p}\cdot\vec{q})^2]\,\frac{1}{(p-q)^2+m^2}\,\frac{1}{(p+q)^2+m^2}\,\bigg|_{p_0=iE_p} \tag{A8}$$

and

$$\Delta\Pi^{(2)}_{\mu\mu}=16\int\frac{d^3p}{2E_p(2\pi)^3}\,\theta(\mu-E_p)[m^2q^2-2(p\cdot q)^2]\,\frac{1}{(p-q)^2+m^2}\,\frac{1}{(p+q)^2+m^2}\,\bigg|_{p_0=iE_p} \tag{A9}$$

These equations are useful forms for taking the massless-electron limit. In this limit, Eqs. (A8) and (A9) become

$$\Delta\Pi^{(2)}_{00}(q^2;\phi|\mu)=\frac{\mu^2}{\pi^2}\,\frac{\sin^2\phi}{q^2}\int_0^1\eta\,d\eta\int_{-1}^{+1}dz\,(1-z^2)/[1-(4\mu^2/q^2)\sin^2\phi(z+i\cot\phi)^2\eta] \tag{A10}$$

and

$$\Delta\Pi^{(2)}_{\mu\mu}(q^2;\phi|\mu)=-\frac{2\mu^2}{\pi^2}\,\frac{\sin^2\phi}{q^2}\int_0^1\eta\,d\eta\int_{-1}^{+1}dz\,(z+i\cot\phi)^2/\,[1-(4\mu^2/q^2)(z+i\cot\phi)^2\sin^2\phi\eta]\,. \tag{A11}$$

Here ϕ is the angular variable $\tan\phi=|q|/q^0$. In the small-q limit, these equations simplify, becoming for $0<\phi<\pi/2$

$$\Delta\Pi^{(2)}_{00}(0;\phi\,|\,\mu)=(1/\pi^2)\mu^2(1-\phi\cot\phi) \tag{A12}$$

and

$$\Delta\Pi^{(2)}_{\mu\mu}(0;\phi\,|\mu)=(1/\pi^2)\mu^2\,. \tag{A13}$$

Using the definitions of Λ_1 and Λ_2 in Eq. (2.13), we find

$$\Lambda_1(0;\phi)=\frac{4\alpha}{\pi}\,\mu^2\,\frac{1-\phi\cot\phi}{\sin^2\phi}\,, \tag{A14}$$

$$\Lambda_2(0;\phi)=\frac{2\alpha}{\pi}\,\mu^2\left(1-\frac{1-\phi\cot\phi}{\sin^2\phi}\right). \tag{A15}$$

For arbitrary q^2, a Sommerfeld-Watson integral representation for $\Delta\Pi_{00}$ and $\Delta\Pi_{\mu\mu}$ can be constructed. From Eqs. (A10) and (A11), we find

$$\Delta\Pi^{(2)}_{00}(q^2;\phi\,|\,\mu)=\frac{\mu^2}{4\pi^2}\int_C\frac{d\lambda}{2\pi i}\,\frac{\pi}{\sin\pi\lambda}\int_0^1 d\eta\int_{-1}^{+1}dz\,(1-z^2)\,\frac{1}{(z+i\cot\phi)^2}\left[\frac{-q^2}{4\mu^2\sin^2\phi(z+i\cot\phi)^2\eta}\right]^{\lambda} \tag{A16}$$

and

$$\Delta\Pi^{(2)}_{\mu\mu}(q^2;\phi\,|\,\mu)=-\frac{\mu^2}{2\pi^2}\int_C\frac{d\lambda}{2\pi i}\,\frac{\pi}{\sin\pi\lambda}\int_0^1 d\eta\int_{-1}^{1}dz\left[\frac{-q^2}{4\mu^2\sin^2\phi(z+i\cot\phi)^2\eta}\right]^{\lambda}. \tag{A17}$$

Here the contour C lies to the left of the origin in the complex λ plane and is shown in Fig. 19.

The integrations over η and η' are performed for $0<\phi<\pi/2$ with the result

$$\Delta\Pi^{(2)}_{00} = -\frac{\mu^2}{\pi^2}\int_C \frac{d\lambda}{2\pi i}\,\Gamma_1(\lambda, \phi)(q^2/4\mu^2)^\lambda \qquad \text{(A18)}$$

and

$$\Delta\Pi^{(2)}_{\mu\mu} = -\frac{\mu^2}{\pi^2}\int_C \frac{d\lambda}{2\pi i}\,\Gamma_2(\lambda, \phi)(q^2/4\mu^2)^\lambda , \qquad \text{(A19)}$$

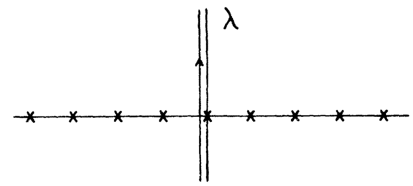

FIG. 19. The contour, <, in the complex λ plane.

where

$$\Gamma_1(\lambda, \phi) = \frac{\pi}{\sin\pi\lambda}\,\frac{1}{1-\lambda}\,\frac{1}{1-4\lambda^2}\left(\cos 2\lambda\phi - \frac{1}{2\lambda}\sin 2\lambda\phi\cot\phi\right) \qquad \text{(A20)}$$

and

$$\Gamma_2(\lambda, \phi) = \frac{\pi}{\sin\pi\lambda}\,\frac{1}{1-\lambda}\,\frac{1}{1-2\lambda}\,(\cos 2\lambda\phi - \sin 2\lambda\phi\cot\phi)\,. \qquad \text{(A21)}$$

Using Eq. (2.13) for Λ_1 and Λ_2, we obtain

$$\Lambda_1(q^2; \phi) = -\frac{4\alpha\mu^2}{\pi}\int_C \frac{d\lambda}{2\pi i}\,\frac{1}{\sin^2\phi}\,\Gamma_1(\lambda, \phi)(q^2/4\mu^2)^\lambda \qquad \text{(A22)}$$

and

$$\Lambda_2(q^2; \phi) = -\frac{2\alpha\mu^2}{\pi}\int_C \frac{d\lambda}{2\pi i}\,[\Gamma_2(\lambda, \phi) - (1/\sin^2\phi)\Gamma_1(\lambda, \phi)]\,(q^2/4\mu^2)^\lambda\,. \qquad \text{(A23)}$$

APPENDIX B: EVALUATION OF Ω^{pl}

In this appendix, the contour-integral representations for Λ_1 and Λ_2 given in Eqs. (A22) and (A23) will be used to evaluate Ω_2^{pl}. Using Eq. (3.27) for Ω_2^{pl}, together with the contour-integral representations for Λ_1 and Λ_2, we find that

$$\Omega_2^{\mathrm{pl}} = -\frac{1}{2(2\pi)^3}\int_0^\infty q^2 dq^2 \int_0^{\pi/2} \sin^2\phi\, d\phi$$
$$\times\left[\frac{1}{q^4}\int_C \frac{d\lambda}{2\pi i}\int_{C'} \frac{d\lambda'}{2\pi i}\left(\frac{q^2}{4\mu^2}\right)^{\lambda+\lambda'}\Gamma(\lambda, \lambda', \phi) - \frac{1}{q^2}\left(\frac{1}{q^2} - \frac{1}{q^2+4\mu^2}\right)\mathrm{Res}\Gamma(\lambda, \lambda', \phi)\Big|_{\lambda,\lambda'=0}\right]. \qquad \text{(B1)}$$

Here, C and C' are contours running to the left of the origin in the λ and λ' planes. The function $\Gamma(\lambda, \lambda', \phi)$ is given in terms of $\Gamma_1(\lambda, \phi)$ and $\Gamma_2(\lambda, \phi)$ by

$$\Gamma(\lambda, \lambda', \phi) = \left(\frac{4\alpha\mu^2}{\pi}\right)^2\left\{\frac{1}{\sin^4\phi}\,\Gamma_1(\lambda, \phi)\Gamma_1(\lambda', \phi)\right.$$
$$\left. + \frac{1}{2}\left[\Gamma_2(\lambda, \phi) - \frac{1}{\sin^2\phi}\,\Gamma_1(\lambda, \phi)\right]\left[\Gamma_2(\lambda', \phi) - \frac{1}{\sin^2\phi}\,\Gamma_1(\lambda', \phi)\right]\right\}. \qquad \text{(B2)}$$

The notation $\mathrm{Res}\Gamma(\lambda, \lambda', \phi)|_{\lambda,\lambda'=0}$ represents the result of the evaluation of the residue of the poles at $\lambda, \lambda', = 0$ of $\Gamma(\lambda, \lambda', \phi)$.

The integration over q^2 in Eq. (B1) may be evaluated by introducing a parameter, η, and allowing the integration over the range $0 < q^2 < \infty$ to be replaced by an integration over $4\mu^2\eta < q^2 < \infty$. (In the end, the limit $\eta \to 0$ will be taken.) This replacement allows the interchange of the q^2 integration with the λ and λ' integrations. Performing this interchange and evaluating the q^2 integral, we find

$$\Omega_2^{\mathrm{pl}} = \frac{1}{2(2\pi)^3}\int_0^{\pi/2} \sin^2\phi\, d\phi \lim_{\eta\to 0}\left[\int_C \frac{d\lambda}{2\pi i}\int_{C'} \frac{d\lambda'}{2\pi i}\,\frac{1}{\lambda+\lambda'}\,\eta^{\lambda+\lambda'}\,\Gamma(\lambda, \lambda', \phi) - \ln\eta\,\mathrm{Res}\Gamma(\lambda, \lambda', \phi)|_{\lambda,\lambda'=0}\right]. \qquad \text{(B3)}$$

The limit $\eta \to 0$ cannot be taken immediately in Eq. (B3) as the contours C and C' require $\mathrm{Re}\lambda + \lambda' < 0$. The contours can, however, be deformed. We first deform the C' contour to the right in the λ' plane to obtain a contour, $\overline{C}'$, encircling a singularity at $\lambda = -\lambda'$, a contour, C_0', encircling a singularity at $\lambda' = 0$, and a third contour which vanishes since $\mathrm{Re}\lambda+\lambda'>0$. The nonvanishing contours give

$$\Omega_2^{\mathrm{pl}}=\frac{1}{2(2\pi)^3}\int_0^{\pi/2}\sin^2\phi\,d\phi$$
$$\times\lim_{\eta\to 0}\left[-\int_C\frac{d\lambda}{2\pi i}\Gamma(\lambda,-\lambda,\phi)-\int_C\frac{d\lambda}{2\pi i}\eta^\lambda\frac{1}{\lambda}\mathrm{Res}\Gamma(\lambda,\lambda',\phi)\Big|_{\lambda'=0}-\ln\eta\,\mathrm{Res}\Gamma(\lambda,\lambda',\phi)\Big|_{\lambda,\lambda'=0}\right]. \tag{B4}$$

In the term with remaining η dependence, we deform the λ contour to the right. A contribution is given by a double pole at $\lambda=0$. Writing

$$\eta^\lambda\simeq 1+\lambda\ln\eta\,, \tag{B5}$$

we find that the $\ln\eta$ terms in Eq. (B4) cancel, giving

$$\Omega_2^{\mathrm{pl}}=\frac{1}{2(2\pi)^3}\int_0^{\pi/2}\sin^2\phi\,d\phi\left[\mathrm{Res}\frac{1}{\lambda}\Gamma(\lambda,\lambda',\phi)\Big|_{\lambda,\lambda'=0}-\int_C\frac{d\lambda}{2\pi i}\Gamma(\lambda,-\lambda,\phi)\right]. \tag{B6}$$

The evaluation of the residue is straightforward and yields

$$\mathrm{Res}\frac{1}{\lambda}\Gamma(\lambda,\lambda;\phi)\Big|_{\lambda,\lambda'=0}=\left(\frac{4\alpha\mu^2}{\pi}\right)^2\left[\frac{(1-\phi\cot\phi)^2}{\sin^4\phi}+\frac{1}{2}\left(1-\frac{1-\phi\cot\phi}{\sin^2\phi}\right)^2+(1-\phi\cot\phi)\left(1-\frac{1-\phi\cot\phi}{\sin^2\phi}\right)\right]. \tag{B7}$$

The integration over ϕ is direct, with the result

$$\bar{\Omega}_2^{\mathrm{pl}}=\left(\frac{\alpha}{\pi}\right)^2\tfrac{1}{4}\mu^4\frac{1}{\pi^2}\left(\frac{\pi^2}{6}-1\right). \tag{B8}$$

The remaining term is a contour integral over λ and an integral over ϕ. Using Eqs. (A20), (A21), (B2), and (B6), we find

$$\tilde{\Omega}_2^{\mathrm{pl}}=\frac{1}{\pi^3}\left(\frac{\alpha}{\pi}\right)^2\mu^4\int_0^{\pi/2}\sin^2\phi\,d\phi\int_C\frac{d\lambda}{2\pi i}$$
$$\times\frac{\pi^2}{\sin^2\pi\lambda}\frac{1}{1-\lambda^2}\frac{1}{1-4\lambda^2}\left[\frac{3}{2}\frac{1}{1-4\lambda^2}\frac{1}{\sin^4\phi}\left(\cos 2\lambda\phi-\frac{1}{2\lambda}\sin 2\lambda\phi\cot\phi\right)^2\right.$$
$$-\frac{1}{1-4\lambda^2}\frac{1}{\sin^2\phi}\left(\cos 2\lambda\phi-\frac{1}{2\lambda}\sin 2\lambda\phi\cot\phi\right)$$
$$\times(\cos 2\lambda\phi-2\lambda\sin 2\lambda\phi\cot\phi)$$
$$\left.+\frac{1}{2}\frac{1}{\sin^2\phi}(\sin^2\phi-\sin^2 2\lambda\phi)\right]. \tag{B9}$$

The first term in Eq. (B9) may be shown to cancel the second term upon using the identity

$$\frac{1}{\sin^2\phi}=-\frac{d}{d\phi}\cot\phi \tag{B10}$$

and integrating by parts. The remaining term is easily integrated with the result

$$\tilde{\Omega}_2^{\mathrm{pl}}=\frac{1}{2\pi^2}\left(\frac{\alpha}{\pi}\right)^2\tfrac{1}{4}\mu^4\int_C\frac{d\lambda}{2\pi i}\frac{\pi}{\sin\pi\lambda}\cos\pi\lambda\left[\frac{1}{\lambda(\lambda^2-1)(4\lambda^2-1)}\right]. \tag{B11}$$

The contour integral over λ may be closed to the left giving

$$\tilde{\Omega}_2^{\mathrm{pl}}=\frac{1}{2\pi^2}\left(\frac{\alpha}{\pi}\right)^2\tfrac{1}{4}\mu^4\left[\tfrac{25}{36}-\sum_{k=2}^{\infty}\frac{1}{k(k^2-1)(4k^2-1)}\right]. \tag{B12}$$

The sum over k in Eq. (B12) can be rewritten as

$$\sum_{k=2}^{\infty}\frac{1}{k(k^2-1)(4k^2-1)}=\sum_{k=2}^{\infty}\left(\frac{1}{k}+\frac{1}{6}\frac{1}{k-1}+\frac{1}{6}\frac{1}{k+1}-\frac{2}{3}\frac{1}{k-\frac{1}{2}}-\frac{2}{3}\frac{1}{k+\frac{1}{2}}\right). \tag{B13}$$

Using the definition of the ψ function

$$\psi(x) = -C - \sum_{k=0}^{\infty}\left(\frac{1}{x+k} - \frac{1}{k+1}\right), \tag{B14}$$

where C is Euler's constant, Eq. (B13) becomes

$$\sum_{k=2}^{\infty}\frac{1}{k(k^2-1)(4k^2-1)} = -\left[\psi(2) + \tfrac{1}{6}\,\psi(1) + \tfrac{1}{6}\,\psi(3) - \tfrac{2}{3}\,\psi(\tfrac{3}{2}) - \tfrac{2}{3}\psi(\tfrac{5}{2})\right]. \tag{B15}$$

The identities

$$\psi(x+1) = \psi(x) + \frac{1}{x}, \tag{B16}$$

and

$$\psi(\tfrac{1}{2}) = -C - 2\ln 2, \tag{B17}$$

$$\psi(1) = -C \tag{B18}$$

show that Eq. (B15) is

$$\sum_{k=2}^{\infty}\frac{1}{k(k^2-1)(4k^2-1)} = \tfrac{67}{36} - \tfrac{8}{3}\ln 2. \tag{B19}$$

Thus, $\tilde{\Omega}_2^{\text{pl}}$ is given by

$$\tilde{\Omega}_2^{\text{pl}} = \frac{1}{\pi^2}\left(\frac{\alpha}{\pi}\right)^2 \tfrac{1}{4}\mu^4(\tfrac{4}{3}\ln 2 - \tfrac{7}{12}) \tag{B20}$$

Combining together $\bar{\Omega}_2^{\text{pl}}$ and $\tilde{\Omega}_2^{\text{pl}}$, we finally find that

$$\Omega_2^{\text{pl}} = \frac{1}{3\pi^2}\,\tfrac{1}{4}\mu^4\left(\frac{\alpha}{\pi}\right)^2\left(\frac{\pi^2}{2} + 4\ln 2 - \tfrac{19}{4}\right). \tag{B21}$$

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[1]B. A. Freedman and L. D. McLerran, preceding paper, Phys. Rev. D 16, 1130 (1977).
[2]M. Gell-Mann and F. Low, Phys. Rev. 95, 1300 (1954).
[3]D. J. Gross and F. Wilczek, Phys. Rev. D 8, 3633 (1973); 9, 980 (1974).
[4]H. D. Politzer, Phys. Rev. Lett. 30, 1346 (1973).
[5]J. C. Collins and M. J. Perry, Phys. Rev. Lett. 34, 1353 (1975).
[6]M. B. Kislinger and P. D. Morley, Phys. Rev. D 13, 2771 (1976).
[7]M. Gell-Mann and K. A. Brueckner, Phys. Rev. 106, 364 (1957).
[8]I. A. Akhiezer and S. V. Peletminskii, Zh. Eksp. Teor. Fiz. 38, 1829 (1960) [Sov. Phys.—JETP 11, 1316 (1960)].
[9]G. 't Hooft and M. Veltman, Nucl. Phys. B44, 189 (1972).
[10]R. Gastmans and R. Meuldermans, Nucl. Phys. B63, 277 (1973).
[11]W. J. Marciano and A. Sirlin, Nucl. Phys. B88, 86 (1975).
[12]G. Marques and N. Papanicolaou, Phys. Rev. D 12, 1052 (1975); W. J. Marciano, P. Marques, and N. Papanicolaou, Nucl. Phys. B96, 237 (1975).
[13]A gauge similar to the gauge used in this paper is used to handle infrared divergences by W. J. Marciano, Phys. Rev. D 12, 3861 (1975).
[14]J. Schwinger, Proc. Natl. Acad. Sci. 37, 452 (1951).
[15]F. J. Dyson, Phys. Rev. 75, 486 (1949); 75, 1736 (1949).
[16]V. Baluni, MIT Report No. 591 (unpublished). In this paper, the ground-state energy of a relativistic Fermi gas is calculated by different techniques.
[17]G. L. Sheppey (unpublished).