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Fermions and gauge vector mesons at finite temperature and density. I. Formal techniques*

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We construct the thermodynamic potential for quantum electrodynamics and quantum chromodynamics at finite temperature and density. We find the thermodynamic potential as a functional of the full propagators and vertices. This functional is stationary under variations of the full propagators and vertices by virtue of the Schwinger-Dyson equations, and may be interpreted as a classical action. We also discuss the renormalization of the propagators and vertices.

INTRODUCTION

Recent speculations about the structure of matter at very high density and/or temperature have led to the conjecture that, at some critical density and temperature, a phase transition from nuclear matter to quark matter takes place.[1-3] Such a phenomenon might occur in the core of a heavy neutron star[4-6] or in a heavy-ion collision in an accelerator, if the density of matter at which the phase transition takes place is not too far above the density of nuclear matter. In fact, the density of matter inside a proton, $\rho\sim\frac{1}{2}$ baryons/fm^3, sets a natural scale for the occurrence of a phase transition. If nuclear matter is compressed to a density such that all the protons and neutrons overlap, it would be natural to expect that the matter is directly described by the constituents of the protons and neutrons. However, if a phase transition occurs at this density, interactions of quarks are likely to be important. The MIT bag model suggests that the natural coupling scale for quarks inside a proton is $g^2/16\pi\sim\frac{1}{2}$.[7,8] Thus, to determine if a phase transition occurs, corrections to the energy density of interacting quark matter must be accurately taken into account. It is the purpose of this paper, the first of a series of three, to formulate the techniques needed to calculate such corrections. In a second paper, the methods developed here will be used to calculate the ground-state energy of a relativistic electron gas up to and including effects of order e^4. An electron gas allows a simple application of these techniques without the complications of the non-Abelian group structure of quark interactions. In the third paper of this series, the ground-state energy of a quark gas interacting by the exchange of non-Abelian, colored gauge mesons will be calculated up to, and including, effects of order g^4. Nonperturbative effects, such as the plasmon oscillation, will be determined, and the constraints of the renormalization group will be implemented.

The techniques developed in this paper are similar to those of DeDominicus and Martin for a nonrelativistic electron gas, and of Norton and Cornwall for a relativistic system of spinless mesons.[9,10] However, for fermions and gauge mesons at finite temperature and density, the calculation of the thermodynamic potential is technically more difficult. Even after all renormalizations have been performed, the thermodynamic potential possesses a Feynman-graph expansion which is naively quadratically divergent. At each order in perturbation theory, different graphs must be added together to obtain a finite expression. The advantage of the methods followed here is that the thermodynamic potential is expressed in terms of full, renormalized propagators and vertices. This result, and the diagrammatic techniques developed in the following papers, immensely reduce the technical difficulties related to naive divergences in the calculation of the thermodynamic potential.

The remainder of this paper is divided into six sections. In the first, the thermodynamic potential, $\Omega(\beta,\mu)$, is introduced, and its formal properties are briefly mentioned. The piece of $\Omega(\beta,\mu)$ due to interactions, $\Omega_I(\beta,\mu)$, is isolated. A trivial renormalization, due to the divergent energy density of the vacuum, is implemented.

In the second section, a functional integral representation for $\Omega_I(\beta,\mu)$ is introduced, and gauge invariance is discussed. A transformation to Euclidean field operators and γ matrices is performed. Explicit forms for the thermodynamic potential as a functional integral are given for quantum electrodynamics and quantum chromodynamics. These forms are used to introduce the mass, wave-function, and charge renormalizations.

In the third section, the full propagators and vertices of quantum electrodynamics are discussed by using Schwinger-Dyson equations. The propagators and vertices are renormalized, and the parts of the propagators and vertices associated with finite-temperature and -density effects are shown to be finite, and thus not in need of renormalization.

The fourth section extends the discussion of propagators and vertices, and of their renormalization, to quantum chromodynamics.

In the fifth section, an expression for the thermodynamic potential in terms of full propagators and vertices is obtained. The thermodynamic potential is shown to be stationary with respect to independent variations of full propagators and vertices. This expression allows for the interpretation of the thermodynamic potential as a classical action and opens the possibility of performing variational calculations. In the sixth section, the implications of the Sec. V are briefly discussed.

I. THE THERMODYNAMIC POTENTIAL

To specify the thermodynamic properties of a multiparticle system, the thermodynamic potential $\Omega(\beta, \vec{\mu})$, must be determined. Here β is the inverse temperature, and $\vec{\mu}$ is the set of chemical potentials corresponding to the conserved and mutually commuting charges which specify the system. With the Hamiltonain denoted by H and the charges by $\vec{\text{N}}$, the thermodynamic potential is given by

$$\beta V\Omega(\beta, \vec{\mu}) = -\ln \text{Tr} \exp[\beta(\vec{\mu}\cdot\vec{\text{N}} - H)] . \qquad (1.1)$$

In this expression, the volume of the system, V, is assumed to be large compared to the Compton wavelengths, $\lambda\!\!\!^{-}_C$, of the massive, elementary excitations described by H. This restriction allows the neglect of surface effects which, on dimensional grounds, should approach zero as $\lambda\!\!\!^{-}_C/V^{1/3}$ for an increasingly large V. The average energy and charge densities are easily found by differentiating Eq. (1) with respect to β and $\vec{\mu}$:

$$V\mathfrak{N}_i(\beta, \vec{\mu}) \equiv \frac{\text{Tr}\, N_i \exp[\beta(\vec{\mu}\cdot\vec{\text{N}} - H)]}{\text{Tr} \exp[\beta(\vec{\mu}\cdot\vec{\text{N}} - H)]} = -\frac{\partial}{\partial \mu_i} V\Omega(\beta, \vec{\mu}) \qquad (1.2)$$

and

$$V\mathcal{E}(\beta, \mu) \equiv \frac{\text{Tr} H \exp[\beta(\vec{\mu}\cdot\vec{\text{N}} - H)]}{\text{Tr} \exp[\beta(\vec{\mu}\cdot\vec{\text{N}} - H)]} = \left(\frac{\partial}{\partial\beta} + \frac{\vec{\mu}}{\beta}\cdot\frac{\partial}{\partial\mu}\right)\beta V\Omega(\beta, \vec{\mu}) . \qquad (1.3)$$

In constructing the thermodynamic potential by Eq. (1.1), it is tacitly assumed that the energy density and charge densities of the vacuum are zero. That is, at zero temperature and zero chemical potential, a thermodynamic system is in its ground state and should have vanishing energy and charge densities. In a quantum field theory, however, the Hamiltonian, given as a functional of the field operators, has a nontrivial, divergent vacuum expectation value. It is well known, however, that theories satisfying charge rotational invariance, without symmetry breaking, have zero vacuum expectation value of the charges. This condition is a consequence of the absence of a preferred direction in charge space. A nontrivial vacuum expectation value would single out a particular direction. Quantum electrodynamics and quantum chromodynamics possess charge rotational invariance, and only the problem of the divergent vacuum energy must be considered.

The vacuum energy density is found directly from Eq. (1.1). As the temperature and chemical potentials vanish, only the state of lowest energy, the vacuum, contributes to the trace. Thus,

$$\mathcal{E}_{\text{vac}} = \lim_{\substack{\beta\to\infty \\ \vec{\mu}\to 0}} \Omega(\beta, \vec{\mu}) . \qquad (1.4)$$

Since the vacuum energy density is divergent, and the physically measurable properties of a system are given by energy differences, it is useful to define a "renormalized" thermodynamic potential as

$$\Omega^R(\beta, \vec{\mu}) \equiv \Omega(\beta, \vec{\mu}) - \lim_{\substack{\beta\to\infty \\ \vec{\mu}\to 0}} \Omega(\beta, \vec{\mu}) . \qquad (1.5)$$

In the end, only the finite, renormalized thermodynamic potential, $\Omega^R(\beta, \vec{\mu})$, will be determined.

For purposes of calculation, it is convenient to break $\Omega(\beta, \vec{\mu})$ into two pieces. In general, the Hamiltonian has the structure

$$H = H_0 + H_I , \qquad (1.6)$$

where H_0 is a Hamiltonian describing free, noninteracting particles, and H_I contains all interaction terms. If H_I is ignored and the charge operators, $\vec{\text{N}}$, contain no explicit dependence on interactions, the thermodynamic potential, $\Omega(\beta, \vec{\mu})$, is given by the thermodynamic potential for an ideal gas, $\Omega_0(\beta, \vec{\mu})$. The remaining piece arising from interactions is

$$\Omega_I(\beta, \vec{\mu}) = \Omega(\beta, \vec{\mu}) - \Omega_0(\beta, \vec{\mu}) = \frac{-1}{\beta V} \ln \frac{\text{Tr} \exp[\beta(\vec{\mu}\cdot N - H)]}{\text{Tr} \exp[\beta(\vec{\mu}\cdot\vec{\text{N}} - H_0)]} . \qquad (1.7)$$

II. FUNCTIONAL INTEGRALS

In a quantum field theory, the Hamiltonian and charge operators are functionals of the independent momenta and coordinates $(\pi_\alpha(x_0, \vec{x}), \phi_\beta(x_0, \vec{y}))$ defined on an equal-time surface. Explicitly,

$$H = \int_V d^3x\, \mathcal{H}(\pi(x), \phi(x)) \tag{2.1}$$

and

$$\vec{N} = \int_V d^3x\, \vec{\mathfrak{N}}(\pi(x), \phi(x)) \,. \tag{2.2}$$

Since the independent momenta and coordinates form a complete set of operators, the trace in Eq. (1.7) for $\Omega_I(\beta, \vec{\mu})$ can be evaluated by inserting a complete set of eigenstates of the momenta and coordinates. This yields the Feynman path-integral[11] representation for $\Omega_I(\beta, \vec{\mu})$ as

$$\Omega_I(\beta, \vec{\mu}) = \frac{-1}{\beta V} \ln \frac{\int [d\pi][d\phi] \exp\{i \int_0^{-i\beta} dx_0 \int_V d^3x [\vec{\mu}\cdot\vec{\mathfrak{N}} + \pi\dot{\phi} - \mathcal{H}(\pi, \phi)]\}}{\int [d\pi][d\phi] \exp\{i \int_0^{-i\beta} dx_0 \int_V d^3x [\vec{\mu}\cdot\vec{\mathfrak{N}} + \pi\dot{\phi} - \mathcal{H}_0(\pi, \phi)]\}} \tag{2.3}$$

The path integral is over all classical paths in the (π, ϕ) function space which connect $(\pi_\alpha(0,\vec{x}), \varphi_\beta(0,\vec{y}))$ to $\pm(\pi_\alpha(-i\beta,\vec{x}), \varphi_\beta(-i\beta,\vec{y}))$. The (anti) periodicity of the paths in function space arises from the trace operation of Eq. (1.7). The positive sign is for Bose-Einstein fields and the minus sign is for Fermi-Dirac fields. The origin of the minus sign is in the anticommutation relations of the Fermi-Dirac fields.

To define the functional integral in Eq. (2.3), proper spatial boundary conditions must be imposed on the π and ϕ fields. For a thermodynamic system where the volume V is large, different choices of boundary conditions should yield differences in Ω_I which are proportional to the surface area. These differences approach zero as $1/V^{1/3}$ when V becomes large. Spatial boundary conditions may thus be freely specified. To maintain symmetry between spatial coordinates and the imaginary time, $-i\beta$, the volume V will be chosen as a large cube with Bose-Einstein fields satisfying the periodic and Fermi-Dirac fields, the antiperiodic boundary conditions.

For gauge theories, the constraint of intergration in the functional integral of Eq. (2.3) to independent momenta and coordinates must be handled carefully. Nevertheless, Bernard (for systems at finite temperature), Faddeev, and Faddeev and Popov have shown for gauge theories that the integration over momenta may be performed yielding a functional integral over unconstrained fields weighted by the exponential of an effective action.[12-14] For quantum electrodynamics in a covariant gauge, the effective action results in only a trivial modification of the longitudinal part of the free photon propagator. For non-Abelian gauge theories, Faddeev-Popov ghosts arise. In what follows, we will assume that S, the effective action, is given in a covariant gauge as a Lorentz-invariant functional of the fields. That is,

$$S(\beta, \vec{\mu}; V; \phi) = i \int_0^{-i\beta} dx_0 \int_V d^3x\, \mathcal{L}(\vec{\mu}; \phi(x), \partial^\mu \phi(x)) \,. \tag{2.4}$$

Because of the Lorentz-invariant structure of Eq. (2.4) and the presence of the imaginary time, $-i\beta$, it is simple to rewrite Eqs. (2.3) and (2.4) in forms which expose the underlying Euclidean invariance of the finite-temperature and -density theory. To do this, we first observe that in Eq. (2.4) for S, all Lorentz indices are contracted and summed over. If the zeroth components of all Lorentz vectors are scaled by $-i$, then the metric, $g^{\mu\nu}$, will become the four-dimensional Euclidean metric, $\delta^{\mu\nu}$. For example, x^0 is scaled as $x^0 \to -ix^0$. Zeroth components of vector fields transform as $A^0 \to iA^0$, and zeroth components of γ matrices as $\gamma^0 \to -i\gamma^0$. The Euclidean γ matrices satisfy the anticommutation relations

$$\{\gamma^\mu, \gamma^\nu\} = -2\delta^{\mu\nu} \,. \tag{2.5}$$

The action, S, is now a manifestly real, Euclidean-invariant functional of the fields

$$S[\beta, \vec{\mu}; V; \phi] = \int_0^\beta dx_0 \int_V d^3x\, \mathcal{L}(\vec{\mu}, \phi(x), \partial^\mu \phi(x)) \,. \tag{2.6}$$

The fields satisfy (anti) periodic boundary conditions on the surface of the four-volume (β, V). Under these substitutions, the functional-integral representation of Eq. (2.3) becomes

$$\Omega_I(\beta, \vec{\mu}) = \frac{-1}{\beta V} \ln \frac{\int [d\phi] \exp(S[\beta, \vec{\mu}; V; \phi])}{\int [d\phi] \exp(S_0[\beta, \vec{\mu}; V; \phi])} \,, \tag{2.7}$$

where S_0 is the limit of the action S with all interaction terms set equal to zero.

For quantum electrodynamics, the preceding formal manipulations must be modified. In order for an electron gas to come into thermodynamic equilibrium, the system must be electrically neutral. This condition requires the introduction of a neutralizing background charge density. Thus, the action must be generalized to include an extra $J\cdot A$ term, where J is a classical, static external field. The requirement of electrical neutrality demands that the average electromagnetic field,

$A^{\mu}(x)$, in the presence of the current, $J^{\mu}(x)$, vanish, that is,

$$\begin{aligned}\mathcal{A}^{\mu}(x) &\equiv \langle A^{\mu}(x)\rangle \\ &\equiv \frac{\mathrm{Tr} A^{\mu}(x)\exp[\beta(\vec{\mu}\cdot\vec{\mathrm{N}}-H+J\cdot A)]}{\mathrm{Tr}\exp[\beta(\vec{\mu}\cdot\vec{\mathrm{N}}-H_0+J\cdot A)]} \\ &=0 .\end{aligned} \tag{2.8}$$

With this modification, the action for quantum electrodynamics is

$$\begin{aligned}S[\beta,\vec{\mu},V;\psi,\bar{\psi},A] = \int_0^{\beta} dx_0 \int_V d^3x\{&-\bar{\psi}(x)[(1/i)\partial\!\!\!/_x - m_0]\psi(x) \\ &-\tfrac{1}{4}[\partial_\mu A_\nu(x)-\partial_\nu A(x)][\partial_\mu A_\nu(x)-\partial_\nu A_\mu(x)]-(1/2d_0)(\partial_\mu A_\mu(x))^2 \\ &+e_0 A_\mu(x)\bar{\psi}(x)\gamma_\mu\psi(x) - i\mu\bar{\psi}(x)\gamma_0\psi(x)+A_\mu(x)J_\mu(x)\} .\end{aligned} \tag{2.9}$$

Here the fields are unrenormalized, the bare charge is denoted by e_0, and the bare electron mass by m_0. The bare gauge-fixing parameter, which modifies the longitudinal part of the free photon propagator, is denoted by d_0. The action in the limit of no interactions is given by Eq. (2.9) when e_0 is set equal to zero. In this limit, the $A_\mu J_\mu$ coupling vanishes, and m_0 is replaced by the physical electron mass m. The free action is

$$\begin{aligned}S_0[\beta,\vec{\mu};V,\psi,\bar{\psi},A] = \int_0^{\beta} dx_0 \int_V d^3x\{&-\bar{\psi}(x)[(1/i)\partial\!\!\!/_x - m]\psi(x) - \tfrac{1}{4}[\partial_\mu A_\nu(x)-\partial_\nu A_\mu(x)][\partial_\mu A_\nu(x)-\partial_\nu A_\mu(x)] \\ &-(1/2d_0)(\partial_\mu A_\mu(x))^2 - i\mu\bar{\psi}(x)\gamma_0\psi(x)\} .\end{aligned} \tag{2.10}$$

In Eq. (2.9), the fields and the parameters e_0, m_0, and d_0 are unrenormalized. To implement the renormalization program, we use Eq. (2.7) and conclude that the difference of Ω_I at finite temperature and density and Ω_I at zero temperature and density is invariant under the rescalings in S of

$$(\psi,\bar{\psi}) \to Z_2^{1/2}(\psi,\bar{\psi}) \tag{2.11a}$$

and

$$A_\mu \to Z_3^{1/2} A_\mu . \tag{2.11b}$$

Defining the physical charge as

$$e \equiv Z_1^{-1} Z_3^{1/2} Z_2 e_0 , \tag{2.12}$$

the physical electron mass as

$$Z_2 m \equiv Z_2 m_0 - \delta m , \tag{2.13}$$

and the renormalized gauge-fixing parameter as

$$d \equiv Z_3^{-1} d_0 , \tag{2.14}$$

the action becomes

$$\begin{aligned}S[\beta,\mu;V;\psi,\bar{\psi},A] = \int_0^{\beta} dx_0 \int_V d^3x\{&-Z_2\bar{\psi}(x)[(1/i)\partial\!\!\!/_x + i\mu\gamma_0 + m]\psi(x) \\ &-\delta m\bar{\psi}(x)\psi(x) - \tfrac{1}{4}Z_3[\partial_\mu A_\nu(x)-\partial_\nu A_\mu(x)][\partial_\mu A_\nu(x)-\partial_\nu A_\mu(x)] \\ &-(1/2d)(\partial_\mu A_\mu(x))^2 + Z_1 e A_\mu(x)\bar{\psi}(x)\gamma_\mu\psi(x) \\ &+Z_3^{1/2} A_\mu(x) J_\mu(x)\} .\end{aligned} \tag{2.15}$$

Moreover, since the difference of the thermodynamic potential at finite temperature and density and zero temperature and density is invariant under $d_0 \to d$ in S_0, the free action becomes

$$\begin{aligned}S_0[\beta,\vec{\mu};V,\psi,\bar{\psi},A] = \int_0^{\beta} dx_0 \int_V d^3x\{&-\bar{\psi}(x)[(1/i)\partial\!\!\!/_x + i\mu\gamma_0 + m]\psi(x) \\ &-\tfrac{1}{4}[\partial_\mu A_\nu(x)-\partial_\nu A_\mu(x)][\partial_\mu A_\nu(x)-\partial_\nu A_\mu(x)] - (1/2d)(\partial_\mu A_\mu(x))^2\} .\end{aligned} \tag{2.16}$$

The advantage of using Eqs. (2.15) and (2.16) to calculate the thermodynamic potential is that Ω_I possesses a Feynman-graph expansion, and the quantities Z_1, Z_2, Z_3, and δm regularize divergent subintegrations within the graphs. In the next section, the renormalization procedure will be discussed in more detail. For now, we note that Z_1, Z_2, Z_3, and δm are functions of e, satisfying

$$Z_1, Z_2, Z_3 \,|_{e=0} = 1\,, \qquad \delta m\,|_{e=0} = 0\,. \tag{2.17}$$

The free action, S_0, is thus the limit of S as $e \to 0$, so that in this limit Ω_I vanishes.

In quantum chromodynamics, that is, fermions of different colors and flavors coupled to colored gauge vector bosons, we are interested in describing thermodynamic systems at finite temperature and flavor charge density. However, we confine our attention to color-singlet systems where the average color charges are all zero. Such systems are parametrized by a set of flavor chemical potentials with the color chemical potentials set equal to zero because of rotational invariance in color charge space. If there are no chemical potentials to break the color rotational invariance (there is no preferred direction in color charge space), the average color charges must vanish. Moreover, since the color gauge bosons couple only to color charge, if the average color charge is zero then the average color gauge boson fields must also vanish. Thus, for quantum chromodynamics, it is not necessary to introduce a neutralizing background charge density, as was the case for quantum electrodynamics.

The action for quantum chromodynamics is specified by a set of flavor and color fermion fields, $(\psi_{a,b}, \bar\psi_{a,b})$, with color indices a and flavor indices b, an octet of color gauge boson fields, A^μ_α, and an octet of anticommuting ghost fields, $(\omega_\alpha, \bar\omega_\alpha)$. The color index is denoted by α for both the gauge vector boson and ghost fields. The flavor charge densities are given by the unrenormalized fermion fields as

$$\mathfrak{N}_b(x) = -i\sum_a \bar\psi_{a,b}(x)\gamma_0\psi_{a,b}(x)\,. \tag{2.18}$$

The action, in terms of unrenormalized fields, bare masses, charges, and a gauge-fixing term, is

$$\begin{aligned} S[\beta,\mu;V;\psi,\bar\psi,\omega,\bar\omega,A] = \int_0^\beta dx_0 \int_V d^3x\{ &-\bar\psi_{a,b}(x)[(1/i)\,\not\partial_x + m_b^0]\psi_{a,b}(x) - (\partial_\mu\bar\omega_\alpha(x))(\partial_\mu\omega_\alpha(x)) \\ &-\tfrac14[\partial_\mu A^\alpha_\nu(x) - \partial_\nu A^\alpha_\mu(x) + g_0 c^{\alpha\beta\gamma}A^\beta_\mu(x)A^\gamma_\nu(x)]^2 \\ &-(1/2d_0)(\partial_\mu A^\alpha_\mu(x))^2 - g_0 c^{\alpha\beta\gamma}\bar\omega_\alpha(x)\partial_\mu A^\gamma_\mu(x)\omega_\beta(x) \\ &+g_0 A^\alpha_\mu(x)\bar\psi_{a,b}(x)\tau^\alpha_{a,a'}\gamma_\mu\psi_{a',b}(x) \\ &-i\mu_b\bar\psi_{a,b}(x)\gamma_0\psi_{a,b}(x)\}\,. \end{aligned} \tag{2.19}$$

Here all indices are summed over. The τ matrices are the generators of the color gauge symmetry group and satisfy the commutation relations

$$[\tau^\alpha, \tau^\beta] = i\,c^{\alpha\beta\gamma}\tau_\gamma\,. \tag{2.20}$$

As in quantum electrodynamics, the renormalization procedure is implemented by the scalings

$$(\psi_{a,b};\bar\psi_{a,b}) \to Z_{2,b}{}^{1/2}(\psi_{a,b};\bar\psi_{a,b})\,, \tag{2.21a}$$

$$A^\alpha_\mu \to Z_3{}^{1/2}A^\alpha_\mu\,, \tag{2.21b}$$

$$(\omega^\alpha;\bar\omega^\alpha) \to \tilde Z_3{}^{1/2}(\omega^\alpha;\bar\omega^\alpha)\,. \tag{2.21c}$$

The physical gauge-fixing parameter, masses, and charge are defined by

$$d = Z_3^{-1}d_0\,, \tag{2.22}$$

$$Z_{2,b}m_b = Z_{2,b}m_b^0 - \delta m_b\,, \tag{2.23}$$

and

$$g = Z_3{}^{3/2}Z_1^{-1}g_0\,. \tag{2.24}$$

The constants Z_1, $Z_{2,b}$, Z_3, $\tilde Z_3$, and δm_b will be specified later. They satisfy

$$Z_1, Z_{2,b}, Z_3, \tilde Z_3\,|_{g=0} = 1\,, \qquad \delta m_b\,|_{g=0} = 0\,. \tag{2.25}$$

Under the scalings of Eqs. (2.21a)–(2.21c) and the definitions of Eqs. (2.22)–(2.24), the action becomes

$$S[\beta, \mu; V; \psi, \bar{\psi}, \omega, \bar{\omega}, A]$$
$$= \int_0^\beta dx_0 \int_V d^3x\{-Z_{2b}\bar{\psi}_{a,b}(x)[(1/i)\not\partial_x + i\mu_b\gamma_0 + m_b]\psi_{a,b}(x)$$
$$-\delta m_b\bar{\psi}_{a,b}(x)\psi_{a,b}(x) - \tfrac{1}{4}Z_3[\partial_\mu A_\nu^\alpha(x) - \partial_\nu A_\mu^\alpha(x) + Z_1Z_3^{-1}gc^{\alpha\beta\gamma}A_\mu^\beta(x)A_\nu^\gamma(x)]^2$$
$$-\tilde{Z}_3(\partial_\mu\bar{\omega}^\alpha(x))(\partial_\mu\omega^\alpha(x)) - (1/2d)(\partial_\mu A_\mu^\alpha(x))^2 - Z_1Z_3^{-1}\tilde{Z}_3gc^{\alpha\beta\gamma}\bar{\omega}^\alpha(x)\partial_\mu A_\mu^\gamma(x)\omega^\beta(x)$$
$$+ Z_1Z_{2,b}Z_3^{-1}gA_\mu^\alpha(x)\bar{\psi}_{a,b}(x)\tau_{a,a'}^\alpha\gamma_\mu\psi_{a',b}(x)\} . \tag{2.26}$$

The action in the $g\to 0$ limit is

$$S_0[\beta, \mu; V; \psi, \bar{\psi}, \omega, \omega, A] = \int_0^\beta dx_0 \int_V d^3x\{-\bar{\psi}_{a,b}(x)[(1/i)\not\partial_x + i\mu_b\gamma_0 + m_b]\psi_{a,b}(x) - (\partial_\mu\bar{\omega}^\alpha(x))(\partial_\mu\omega^\alpha(x))$$
$$-\tfrac{1}{4}[\partial_\mu A_\nu^\alpha(x) - \partial_\nu A_\mu^\alpha(x)]^2 - (1/2d)(\partial_\mu A_\mu^\alpha(x))^2\} . \tag{2.27}$$

III. SCHWINGER-DYSON EQUATIONS FOR QUANTUM ELECTRODYNAMICS

In this section, the finite-temperature and -density Schwinger-Dyson equations for quantum electrodynamics will be discussed.[15,16] The wave-function, mass, and charge renormalization prescriptions will be outlined, and finite equations for the full propagators and vertices obtained.

The quantities of interest in quantum electrodynamics are the average electromagnetic field,

$$\mathcal{A}_1 \equiv \mathcal{A}_{\mu_1}(x_1) \equiv \langle A_{\mu_1}(x_1)\rangle , \tag{3.1}$$

the full, connected photon propagator,

$$D_{12} \equiv D_{\mu_1\mu_2}(x_1, x_2) \equiv \langle T(A_{\mu_1}(x_1)A_{\mu_2}(x_2))\rangle - \mathcal{A}_{\mu_1}(x_1)\mathcal{A}_{\mu_2}(x_2) , \tag{3.2}$$

the full, connected electron propagator

$$S_{12} \equiv S_{\alpha_1\alpha_2}(x_1, x_2) \equiv \langle T(\psi_{\alpha_1}(x_1)\bar{\psi}_{\alpha_2}(x_2))\rangle , \tag{3.3}$$

and the full, connected photon-fermion vertex,

$$e\Gamma_{12;3} \equiv e\Gamma_{\alpha_1\alpha_2;\mu_3}(x_1, x_2; x_3)$$
$$\equiv D_{\mu_3\bar{\mu}_3}^{-1}(x_3, \bar{x}_3)S_{\alpha_1\bar{\alpha}_1}^{-1}(x_1, \bar{x}_1)\langle T(\psi_{\bar{\alpha}_1}(\bar{x}_1)\psi_{\bar{\alpha}_2}(\bar{x}_2)A_{\bar{\mu}_3}(\bar{x}_3))\rangle S_{\bar{\alpha}_2\alpha_2}^{-1}(\bar{x}_2, x_2)$$
$$-D_{\mu_3\bar{\mu}_3}^{-1}(x_3, \bar{x}_3)\mathcal{A}_{\bar{\mu}_3}(\bar{x}_3)S_{\alpha_1\alpha_2}^{-1}(x_1, x_2) . \tag{3.4}$$

Here T is the time-ordering operator generalized to Euclidean time. All repeated indices are summed over, and repeated coordinates are integrated over. The brackets, $\langle\ \rangle$, indicate the averaging procedure,

$$\langle T(\psi_{a_1}\cdots\psi_{a_N}\bar{\psi}_{b_1}\cdots\bar{\psi}_{b_M}A_{c_1}\cdots A_{c_P})\rangle$$
$$\equiv \frac{\mathrm{Tr}\,T(\psi_{a_1}\cdots\psi_{a_N}\bar{\psi}_{b_1}\cdots\bar{\psi}_{b_M}A_{c_1}\cdots A_{c_P}\exp[\beta(\vec{\mu}\cdot\vec{N} - H + J\cdot A)])}{\mathrm{Tr}\,T\{\exp[\beta(\vec{\mu}\cdot\vec{N} - H + J\cdot A)]\}}$$
$$\equiv \frac{\int[d\bar{\psi}][d\psi][dA]\,(\psi_{a_1}\cdots\psi_{a_N}\bar{\psi}_{b_1}\cdots\bar{\psi}_{b_M}A_{c_1}\cdots A_{c_P})\exp(S[\beta, \vec{\mu}; V; \psi, \bar{\psi}, A])}{\int[d\bar{\psi}][d\psi][dA]\exp(S[\beta, \vec{\mu}; V; \psi, \bar{\psi}, A])} . \tag{3.5}$$

We are using a notation where all spin labels and coordinate labels are denoted by a collective index.

In Eqs. (3.1)–(3.4), translational invariance has not been used to express the vertices and propagators in terms of relative coordinates, since in Sec. V, variations will be performed on bare propagators and vertices. The variational parameters do not in general yield a translationally invariant theory. Only after all the variations have been performed, and an expression for the thermodynamic potential found, can translational invariance be used.

To make this clear, we write the action for quantum electrodynamics in a form that will be useful later. We define

$$S_0^{-1}{}_{12} \equiv [(1/i)\not\partial_x + i\mu\gamma_0 + m]_{\alpha_1\alpha_2}\delta^{(4)}(x_1 - x_2) , \tag{3.6}$$

$$\tilde{S}_0^{-1}{}_{12} \equiv \{Z_2[(1/i)\not\partial_x + i\mu\gamma_0 + m] + \delta m\}_{\alpha_1\alpha_2}$$
$$\times\,\delta^{(4)}(x_1 - x_2) , \tag{3.7}$$

$$D_0{}^{-1}{}_{12} \equiv [-\Box\,\delta_{\mu_1\mu_2} + (1 - 1/d_0)\,\partial_{\mu_1}\partial_{\mu_2}] \times \delta^{(4)}(x_1 - x_2)\,, \tag{3.8}$$

$$\tilde{D}_0{}^{-1}{}_{12} \equiv [Z_3(-\Box\,\delta_{\mu_1\mu_2} + \partial_{\mu_1}\partial_{\mu_2}) - (1/d)\partial_{\mu_1}\partial_{\mu_2}] \times \delta^{(4)}(x_1 - x_2)\,, \tag{3.9}$$

$$\Gamma_{0\,12,3} \equiv \gamma_{\alpha_1\alpha_2;\,\mu_3}\,\delta^{(4)}(x_1 - x_2)\,\delta^{(4)}(x_1 - x_3)\,, \tag{3.10}$$

and

$$\tilde{\Gamma}_{0\,12,3} \equiv Z_1\gamma_{\alpha_1\alpha_2\,;\,\mu_3}\,\delta^{(4)}(x_1 - x_2)\,\delta^{(4)}(x_1 - x_3)\,. \tag{3.11}$$

In this notation, Eqs. (2.15) and (2.16) become

$$S \equiv -\bar{\psi}_1\tilde{S}_0{}^{-1}{}_{12}\psi_2 - \tfrac{1}{2}A_1\tilde{D}_0{}^{-1}{}_{12}A_2 + e\tilde{\Gamma}_{0\,12,3}\bar{\psi}_1\psi_2A_3 + A_1\tilde{J}_1 \tag{3.12}$$

and

$$S_0 = -\bar{\psi}_1S_0{}^{-1}{}_{12}\psi_2 - \tfrac{1}{2}A_1D_0{}^{-1}{}_{12}A_2\,. \tag{3.13}$$

The current, $\tilde{J}_1$, is the ordinary current, J_1, scaled by $Z_3{}^{1/2}$. In what follows, $\tilde{J}$, $\tilde{S}$, $\tilde{D}$, and $\tilde{\Gamma}$ will be considered as variational parameters. That is, we will perform small variations around the values given in Eqs. (3.6)–(3.11). Only in the final form for the thermodynamic potential will these quantities return to their physical values.

Using Eq. (3.12) for the action, the definitions of the average photon field, propagators, and vertex given by Eqs. (3.1)–(3.4), and the functional integral representation of Eq. (3.5), it is possible to derive the Schwinger-Dyson equations. The results are

$$\tilde{D}_0{}^{-1}\mathfrak{a} = -e\tilde{\Gamma}_0S + \tilde{J}\,, \tag{3.14}$$

$$D^{-1} = \tilde{D}_0{}^{-1} + \Pi\,, \tag{3.15}$$

$$S^{-1} = \tilde{S}_0{}^{-1} - e\tilde{\Gamma}_0\,\mathfrak{a} + \Sigma\,, \tag{3.16}$$

and

$$\Gamma = \tilde{\Gamma}_0 + \Lambda\,. \tag{3.17}$$

These equations are represented graphically in Figs. 1 and 2. Summation over indices has been suppressed, since Figs. 1 and 2 make explicit the suppressed summations. The photon polarization tensor, Π, is given in terms of $\tilde{\Gamma}_0$, S, D, and Γ by

$$\Pi = e^2\tilde{\Gamma}_0S\Gamma S\,. \tag{3.18}$$

The fermion self-mass kernel, Σ, also involves $\tilde{\Gamma}_0$, S, D, and Γ and is

$$\Sigma = -e^2\tilde{\Gamma}_0S\Gamma D\,. \tag{3.19}$$

The photon polarization tensor and the fermion self-mass kernel are shown in Figs. 3 and 4. The part of the vertex function arising from interactions, Λ, does not involve $\tilde{\Gamma}_0$ and is a function

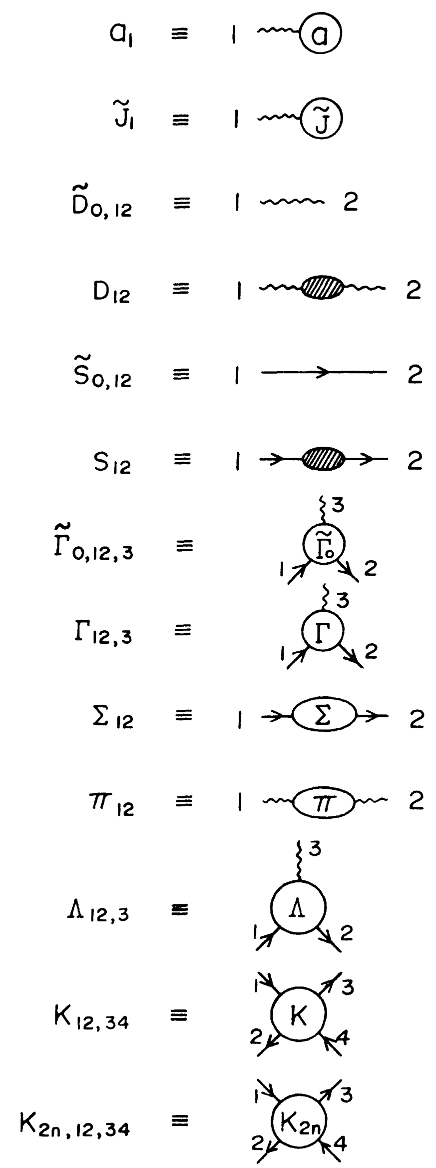

FIG. 1. Definitions of bare and full propagators and vertices for quantum electrodynamics.

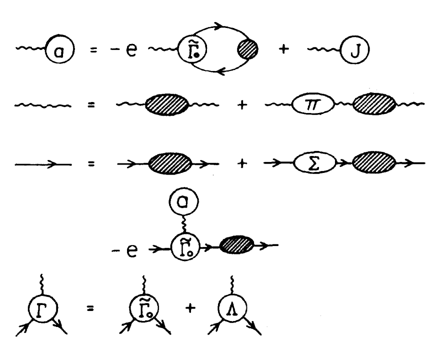

FIG. 2. Schwinger-Dyson equations for quantum electrodynamics.

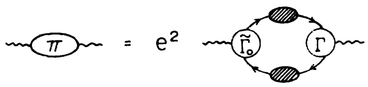

FIG. 3. The photon polarization tensor.

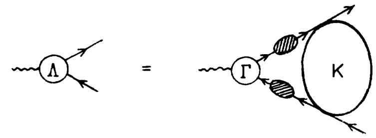

FIG. 5. Equation for the part of the vertex function due to interactions, Λ.

only of S, D, and 1 That is, there is a skeleton graph expansion for Λ. The skeleton graph expansion may be derived graphically by (a) drawing all graphs for Λ in terms of bare propagators and vertices, (b) removing all graphs which, upon breaking two or three particle lines, separate into pieces which involve vertex insertions on the bare vertex $\tilde{\Gamma}_0$ (that is, radiative corrections to the bare vertex), and (c) replacing in the remaining graphs all bare propagators and vertices by full propagators and vertices. The kernel of the Bethe-Salpeter equation, K, can be used to construct explicitly Λ.[17] This kernel is the sum of all graphs, for the electron-positron scattering amplitude, which cannot be broken into two pieces by cutting one photon line or two lines associated with an electron-positron pair in the direct electron-positron channel. With this definition,

$$\Lambda = S\Gamma S K\,, \tag{3.20}$$

where

$$K=\sum_{n=1}^{\infty} e^{2n}K_{2n}\,. \tag{3.21}$$

The $2n$th moment of K, K_{2n}, in Eq. (3.21) is the sum of those skeleton graph contributions to K which contain $2n$ full vertices, n full photon propagators, and $2n-2$ full fermion propagators. The equation for Λ, Eq. (3.20), is represented in Fig. 5. The skeleton graph expansion of the Bethe-Salpeter kernel, Eq. (3.21), is shown in Fig. 6. Examples of skeleton graphs not included in K are given in Fig. 7.

Although we will need the Schwinger-Dyson equations for a nontranslationally invariant theory to derive an expression for the thermodynamic potential, in the end the thermodynamic potential will be written as a functional of full propagators and vertices for a translationally invariant theory. Thus, it is essential to discuss renormalization for the translationally invariant theory. We will show that with an appropriate choice of the counterterms given in the preceding section, the full propagators and vertices are finite. Moreover, the values of these counterterms are independent of β and μ. That is, all effects due to finite temperature and density lead to finite corrections to the propagators and vertices.

To discuss renormalization for a translationally invariant theory, it is useful to Fourier-transform to momentum space. Since the thermodynamic system is enclosed in a very large volume, V, the spatial momenta are continuous. However, at finite temperature energies are discrete, becoming continuous in the zero temperature limit ($\beta\to\infty$). The boundary conditions discussed in Sec. II require that fermion energies be odd-integer multiples of π/β and boson energies be even-integer multiples of π/β. That is,

$$\begin{aligned}(p_0,\vec{p}) &= \left(\frac{(2n+1)\pi}{\beta}\,,\ \vec{p}\right), \quad \text{fermions}\\ (q_0,\vec{q}) &= \left(\frac{2n\pi}{\beta}\,,\vec{q}\right), \quad \text{bosons}\,.\end{aligned} \tag{3.22}$$

Since we are interested in systems in which the average photon field is zero, it is sufficient to discuss the equations for S, D, and Γ. A similar discussion for the average photon field would follow directly from the discussion of these equations. In momentum space, the photon propagator satisfies

$$D^{-1}(q\,|\,\beta,\mu)=\tilde{D}_0^{-1}(q)+\Pi(q\,|\,\beta,\mu)\,. \tag{3.23}$$

This equation makes use of the fact that $\tilde{D}_0^{-1}$ has no explicit dependence on β or μ. To renormalize the

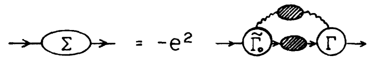

FIG. 4. The fermion self-mass kernel.

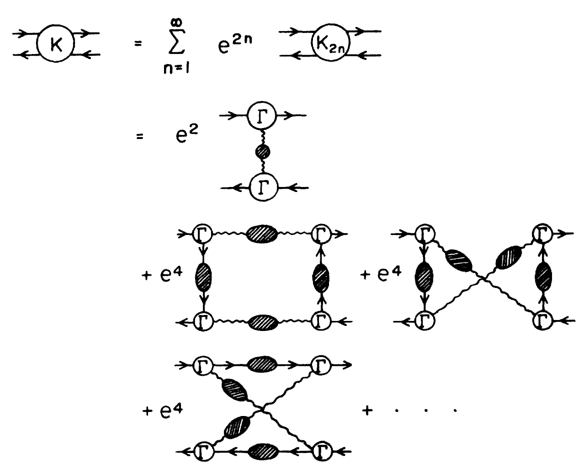

FIG. 6. Skeleton-graph expansion for the Bethe-Salpeter kernel, K.

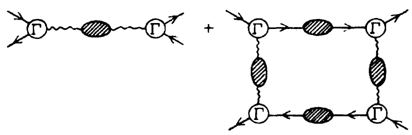

FIG. 7. Graphs not included in the skeleton-graph expansion for K.

equation, we isolate finite-temperature and -density effects by writing

$$\Delta\Pi(q\,|\,\beta,\mu)=\Pi(q\,|\,\beta,\mu)-\Pi(q)\,, \tag{3.24}$$

where

$$\Pi(q)=\lim_{\substack{\beta\to\infty\\ \mu\to 0}}\Pi(q\,|\,\beta,\mu)\,. \tag{3.25}$$

Equation (3.23) now becomes

$$D^{-1}(q\,|\,\beta,\mu)=\tilde{D}_0^{-1}(q)+\Delta\Pi(q\,|\,\beta,\mu)+\Pi(q)\,. \tag{3.26}$$

The first and third terms on the right-hand side of Eq. (3.26) define the inverse propagator in the vacuum. We now use gauge invariance to write

$$\begin{aligned}D^{-1}(q)&\equiv\tilde{D}_0^{-1}(q)+\Pi(q)\\ &=Z_3(q^2\tilde{I}-\tilde{q}\tilde{q})+(1/d)\tilde{q}\tilde{q}+(q^2\tilde{I}-\tilde{q}\tilde{q})\Pi(q^2)\\ &=(q^2\tilde{I}-\tilde{q}\tilde{q})[1+\overline{\Pi}_R(q^2)]+(1/d)\tilde{q}\tilde{q}\,,\end{aligned} \tag{3.27}$$

in which we use

$$\overline{\Pi}_R(q)\equiv\Pi(q)-\Pi(0)\,, \tag{3.28}$$

and in which Z_3 has been chosen to satisfy

$$\Pi(0)\equiv 1-Z_3\,. \tag{3.29}$$

Here, the photon wave-function renormalization has been performed on the mass shell, $q^2=0$. This subtraction procedure makes finite the vacuum part of the photon propagator. Moreover, the difference, $\Delta\Pi$, between the finite-temperature and -density propagator and the vacuum propagator is the difference of two terms which individually possess integral representations, logarithmically divergent in the ultraviolet. In the ultraviolet limit, β and μ are effectively scaled to zero so that the divergences of both integrals are identical. The difference, $\Delta\Pi$, is thus finite and not in need of renormalization. The renormalized photon propagator is given, therefore, by

$$D^{-1}(q\,|\,\beta,\mu)=D_0^{-1}(q)+\Pi_R(q)+\Delta\Pi(q\,|\,\beta,\mu)\,. \tag{3.30}$$

The renormalization of the fermion propagator is performed somewhat differently from that of the photon propagator. The difference is dictated by the explicit dependence on the chemical potential, μ, in the free fermion propagator. This dependence is trivial, however, since

$$\tilde{S}_0^{-1}(p\,|\,\mu)=\tilde{S}_0^{-1}(p+i\mu\,|\,0)\,. \tag{3.31}$$

The equation for the fermion propagator is

$$S^{-1}(p\,|\,\beta,\mu)=\tilde{S}_0^{-1}(p+i\mu\,|\,0)+\Sigma(p\,|\,\beta,\mu)\,. \tag{3.32}$$

The effects of finite temperature are isolated by writing

$$\Sigma(p\,|\,\mu)=\lim_{\beta\to\infty}\Sigma(p\,|\,\beta,\mu) \tag{3.33}$$

and

$$\Delta\Sigma(p\,|\,\beta,\mu)=\Sigma(p\,|\,\beta,\mu)-\Sigma(p\,|\,\mu)\,. \tag{3.34}$$

The difference $\Delta\Sigma$, like the difference $\Delta\Pi$, is finite and not in need of renormalization.

To renormalize $\Sigma(p\,|\,\mu)$ we write, in analogy to Eq. (3.31),

$$\Sigma(p\,|\,\mu)=\Delta\Sigma(p\,|\,\mu)+\Sigma(p+i\mu\,|\,0)\,. \tag{3.35}$$

The difference, $\Delta\Sigma(p\,|\,\mu)$, between the zero-temperature–finite-density self-mass kernel, $\Sigma(p\,|\,\mu)$, and the analytic continuation of the zero-density–temperature self-mass kernel, $\Sigma(p\,|\,0)$, from real Euclidean energy, p^0, to the complex energy $p^0+i\mu$, is finite and not in need of renormalization. The piece of the self-mass kernel, $\Sigma(p+i\mu\,|\,0)$, is renormalized by defining

$$\delta m\equiv-\Sigma(p\,|\,0)\big|_{\not p=-m} \tag{3.36}$$

and

$$1-Z_2=(\partial/\partial\not p)\Sigma(\not p\,|\,0)\big|_{\not p=-m}\,, \tag{3.37}$$

so that

$$\begin{aligned}\Sigma_R(p+i\mu\,|\,0)=\Sigma(p+i\mu\,|\,0)+\delta m&\\ -(1-Z_2)[(p+i\mu)\cdot\gamma+m)]&\end{aligned} \tag{3.38}$$

The renormalization has been performed here at the non-Euclidean point $\not p=-m$, that is, on the fermion mass shell. Finally, the finite, renormalized equation for $S^{-1}(p\,|\,\beta,\mu)$ is

$$\begin{aligned}S^{-1}(p\,|\,\beta,\mu)=S_0^{-1}(p+i\mu\,|\,0)+\Sigma_R(p+i\mu\,|\,0)&\\ +\Delta\Sigma(p\,|\,\mu)+\Delta\Sigma(p\,|\,\beta,\mu)\,.&\end{aligned} \tag{3.39}$$

The vertex function, Γ, is renormalized entirely in analogy to the renormalization of S^{-1}. To isolate the effects of finite temperature, we define

$$\Gamma(p,p+q;q\,|\,\mu)\equiv\lim_{\beta\to\infty}\Gamma(p,p+q;q\,|\,\beta,\mu) \tag{3.40}$$

and

$$\begin{aligned}\Delta\Gamma(p,p+q;q\,|\,\beta,\mu)\equiv\Gamma(p,p+q;q\,|\,\beta,\mu)&\\ -\Gamma(p,p+q;q\,|\,\mu)\,.&\end{aligned} \tag{3.41}$$

Then, as in the case of the fermion self-mass kernal, the difference, $\Delta\Gamma(p,p+q;q\,|\,\mu)$, is

$$\Delta\Gamma(p,p+q;q\,|\,\mu)\equiv\Gamma(p,p+q;q\,|\,\mu)-\Gamma(p+i\mu,p+q+i\mu;q\,|\,0)\,. \quad (3.42)$$

The analytic continuation to complex energy of Γ is the only term in need of renormalization. Writing

$$\Gamma(p+i\mu,p+q+i\mu;q\,|\,0)=Z_1\gamma+\Lambda(p+i\mu,p+q+i\mu;q\,|\,0)\,, \quad (3.43)$$

and defining

$$(1-Z_1)\gamma=\Lambda(p,p+q;q\,|\,0)\,|_{\not p=-m,\,\not p+\not q=-m,\,q^2=0} \quad (3.44)$$

and

$$\Lambda_R(p+i\mu,p+q+i\mu;q\,|\,0)=-(1-Z_1)\gamma+\Lambda(p+i\mu,p+q+i\mu;q\,|\,0)\,, \quad (3.45)$$

the renormalization prescription is complete, since $\Gamma(p,p+q;q\,|\,\beta,\mu)$ satisfies the finite equation

$$\Gamma(p,p+q;q\,|\,\beta,\mu)=\gamma+\Lambda_R(p+i\mu,p+q+i\mu;q\,|\,0)+\Delta\Gamma(p,p+q;q\,|\,\mu)+\Delta\Gamma(p,p+q;q\,|\,\beta,\mu)\,. \quad (3.46)$$

IV. SCHWINGER-DYSON EQUATIONS FOR QUANTUM CHROMODYNAMICS

In this section, the Schwinger-Dyson equations for quantum chromodynamics at finite density and temperature will be discussed, and the renormalization prescriptions outlined. Modifications of the renormalization procedure to avoid infrared divergences which would arise were the renormalizations performed on the mass shell will be noted.

In quantum chromodynamics, for color-singlet systems where the average gluon field vanishes, the quantities of interest are the gluon propagator

$$D_{12}\equiv\langle T(A_1A_2)\rangle\,, \quad (4.1)$$

the fermion propagator

$$S_{12}\equiv\langle T(\psi_1\bar\psi_2)\rangle\,, \quad (4.2)$$

the ghost propagator

$$W_{12}\equiv\langle T(\omega_1\bar\omega_2)\rangle\,, \quad (4.3)$$

the fermion-gluon vertex

$$g\Gamma^F_{12,3}\equiv D_{33'}{}^{-1}S_{11'}{}^{-1}\langle T(\psi_{1'}\bar\psi_{2'}A_{3'})\rangle S_{2'2}{}^{-1}\,, \quad (4.4)$$

the ghost-gluon vertex

$$g\Gamma^G_{12,3}\equiv D_{33'}{}^{-1}W_{11'}{}^{-1}\langle T(\omega_{1'}\bar\omega_{2'}A_{3'})\rangle W_{2'2}{}^{-1}\,, \quad (4.5)$$

the three-gluon vertex

$$g\,\Gamma^V_{123}\equiv D_{11'}{}^{-1}D_{22'}{}^{-1}D_{33'}{}^{-1}\langle T(A_{1'}A_{2'}A_{3'})\rangle\,, \quad (4.6)$$

and the four-gluon vertex

$$g^2\Gamma^V_{1234}\equiv D_{11'}{}^{-1}D_{22'}{}^{-1}D_{33'}{}^{-1}D_{44'}{}^{-1}\langle T(A_{1'}A_{2'}A_{3'}A_{4'})\rangle-D_{12}{}^{-1}D_{34}{}^{-1}-D_{13}{}^{-1}D_{24}{}^{-1}-D_{14}{}^{-1}D_{23}{}^{-1}-\Gamma^V_{125}D_{55'}\Gamma^V_{5'34}-\Gamma^V_{135}D_{55'}\Gamma^V_{5'24}-\Gamma^V_{145}D_{55'}\Gamma^V_{5'23}\,. \quad (4.7)$$

As in quantum electrodynamics, it is useful to rewrite the action, S, of Eq. (2.26) in a notationally compact form. To do this, we define

$$\tilde D_0{}^{-1}{}_{12}\equiv[Z_3(-\Box\delta_{\mu_1\mu_2}+\partial_{\mu_1}\partial_{\mu_2})-(1/d)\partial_{\mu_1}\partial_{\mu_2}]\delta_{\alpha_1\alpha_2}\delta^{(4)}(x_1-x_2)\,, \quad (4.8)$$

$$\tilde S_0{}^{-1}{}_{12}\equiv\{Z_{2b_1}[(1/i)\not\partial+i\mu_b\gamma_0+m_{b_1}]+\delta m_{b_1}\}\delta_{a_1a_2}\delta_{b_1b_2}\delta^{(4)}(x_1-x_2)\,, \quad (4.9)$$

$$\tilde W_0{}^{-1}{}_{12}\equiv\tilde Z_3(-\Box)\delta_{\alpha_1\alpha_2}\delta^{(4)}(x_1-x_2)\,, \quad (4.10)$$

$$\tilde\Gamma_0{}^F{}_{12,3}\equiv Z_1Z_{2b_1}Z_3{}^{-1}\tau^{a_1a_2}_{\alpha_1}\gamma_{\mu_3}\delta_{b_1b_2}\delta^{(4)}(x_1-x_2)\delta^{(4)}(x_1-x_3)\,, \quad (4.11\text{a})$$

$$\tilde\Gamma_0{}^G{}_{12,3}\equiv Z_1Z_3{}^{-1}\tilde Z_3c_{\alpha_1\alpha_2\alpha_3}\partial^{x_1}_{\mu_3}\delta^{(4)}(x_1-x_3)\delta^{(4)}(x_2-x_3)\,, \quad (4.11\text{b})$$

$$\begin{aligned}\tilde\Gamma_0{}^V{}_{123}\equiv Z_1c_{\alpha_1\alpha_2\alpha_3}\{&\delta_{\mu_1\mu_2}[\delta^{(4)}(x_2-x_3)\partial^{x_1}_{\mu_3}\delta^{(4)}(x_1-x_2)-\delta^{(4)}(x_1-x_3)\partial^{x_2}_{\mu_3}\delta^{(4)}(x_2-x_1)]\\&+\delta_{\mu_1\mu_3}[\delta^{(4)}(x_1-x_2)\partial^{x_3}_{\mu_2}\delta^{(4)}(x_3-x_1)-\delta^{(4)}(x_2-x_3)\partial^{x_1}_{\mu_2}\delta^{(4)}(x_1-x_3)]\\&+\delta_{\mu_2\mu_3}[\delta^{(4)}(x_1-x_3)\partial^{x_2}_{\mu_1}\delta^{(4)}(x_2-x_1)-\delta^{(4)}(x_1-x_2)\partial^{x_3}_{\mu_1}\delta^{(4)}(x_3-x_1)]\}\,,\end{aligned} \quad (4.12)$$

and

$$\begin{aligned}\tilde\Gamma_0{}^V{}_{1234}\equiv-Z_1{}^2Z_3{}^{-1}\delta^{(4)}(x_1-x_2)\delta^{(4)}(x_1-x_3)\delta^{(4)}(x_1-x_4)[&c_{\alpha_1\alpha_2\rho}c_{\alpha_3\alpha_4\rho}(\delta_{\mu_1\mu_3}\delta_{\mu_2\mu_4}-\delta_{\mu_1\mu_4}\delta_{\mu_2\mu_3})\\&+c_{\alpha_1\alpha_3\rho}c_{\alpha_2\alpha_4\rho}(\delta_{\mu_1\mu_2}\delta_{\mu_3\mu_4}-\delta_{\mu_1\mu_4}\delta_{\mu_2\mu_3})\\&+c_{\alpha_1\alpha_4\rho}c_{\alpha_3\alpha_2\rho}(\delta_{\mu_1\mu_3}\delta_{\mu_4\mu_2}-\delta_{\mu_1\mu_2}\delta_{\mu_3\mu_4})]\,.\end{aligned} \quad (4.13)$$

The vertices and propagators listed above will be written without the tilde whenever a free propagator or vertex is needed, that is, with all Z_i's set equal to one with $\delta m = 0$. With this notation, the action takes the form

$$S = -\bar\psi_1 \tilde S_0^{-1}{}_{12}\psi_2 - \bar\omega_1 \tilde W_0^{-1}{}_{12}\omega_2 - \tfrac12 A_1 \tilde D_0^{-1}{}_{12} A_2 + g\tilde\Gamma_0^F{}_{12,3}\bar\psi_1\psi_2 A_3 + g\tilde\Gamma_0^G{}_{12,3}\bar\omega_1\omega_2 A_3 + \tfrac16 g\tilde\Gamma_0^V{}_{123}A_1A_2A_3 + \tfrac1{24}g^2\tilde\Gamma_0^V{}_{1234}A_1A_2A_3A_4 . \quad (4.14)$$

In the limit $g \to 0$, S becomes

$$S_0 = -\bar\psi_1 S_0^{-1}{}_{12}\psi_2 - \bar\omega_1 W_0^{-1}{}_{12}\omega_2 - \tfrac12 A_1 D_0^{-1}{}_{12} A_2 . \quad (4.15)$$

As in quantum electrodynamics, it is possible to derive Schwinger-Dyson equations for the propagators and vertices of quantum chromodynamics. The equations are

$$D^{-1} = \tilde D_0^{-1} + \Pi , \quad (4.16)$$

$$S^{-1} = \tilde S_0^{-1} + \Sigma^F , \quad (4.17)$$

$$W^{-1} = \tilde W_0^{-1} + \Sigma^G , \quad (4.18)$$

$$\Gamma^F = \tilde\Gamma_0^F + \Lambda^F , \quad (4.19)$$

$$\Gamma^G = \tilde\Gamma_0^G + \Lambda^G , \quad (4.20)$$

$$\Gamma^V_{(3)} = \tilde\Gamma_0^V{}_{(3)} + \Lambda^V_{(3)} , \quad (4.21)$$

and

$$\Gamma^V_{(4)} = \tilde\Gamma_0^V{}_{(4)} + \Lambda^V_{(4)} . \quad (4.22)$$

These equations are represented graphically in Figs. 8 and 9. The gluon polarization tensor is a function of the full propagators and bare and full vertices It is

$$\Pi = g^2\tilde\Gamma_0^V S\Gamma^V S + g^2\tilde\Gamma_0^G W\Gamma^G W - \tfrac12 g^2\tilde\Gamma_0^V{}_{(3)}DD\Gamma^V_{(3)} - \tfrac16 g^4\tilde\Gamma_0^V{}_{(4)}DDD\Gamma^V_{(4)} - \tfrac12 g^4\tilde\Gamma_0^V{}_{(4)}(D\Gamma^V_{(3)}D)(D\Gamma^V_{(3)}D) - \tfrac12 g^2\tilde\Gamma_0^V{}_{(4)}D . \quad (4.23)$$

This equation is represented graphically in Fig. 10. The fermion self-mass kernel is simpler in form than the gluon polarization tensor. It is a function of only $\tilde\Gamma_0^F$, S, and D,

$$\Sigma^F = -g^2\tilde\Gamma_0^F S\Gamma^F D , \quad (4.24)$$

and is shown in Fig. 11. The ghost self-mass kernel is similar in structure to the fermion self-

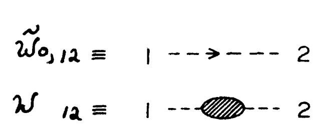

FIG. 8. Definition of bare and full ghost propagator.

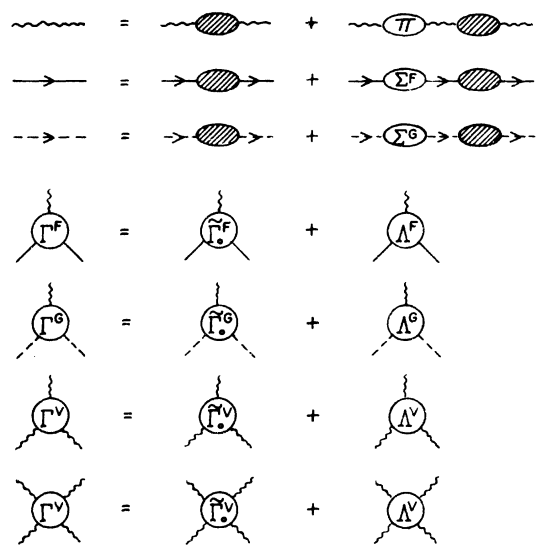

FIG. 9. Schwinger-Dyson equations for quantum chromodynamics.

mass kernel, satisfying

$$\Sigma^G = -g^2\tilde\Gamma_0^G W\Gamma^G D \quad (4.25)$$

(see Fig. 12).

The vertices of quantum chromodynamics satisfy more complicated equations than the vertices of quantum electrodynamics. An explicit graphical procedure can be given for the construction of three-particle vertices: (a) All Feynman graphs

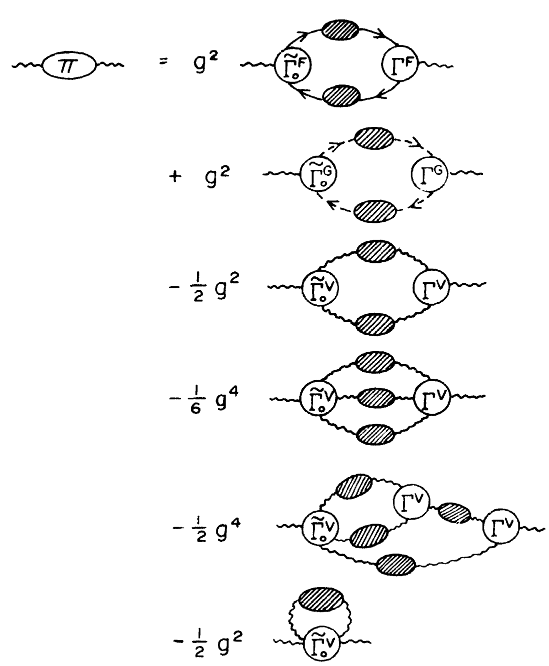

FIG. 10. The gluon polarization tensor.

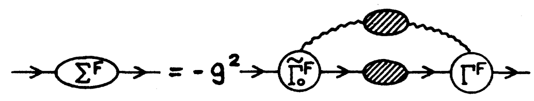

FIG. 11. The fermion self-mass kernel.

are written in terms of bare propagators and vertices for the three-particle vertices. (b) Any graph is eliminated which upon cutting two- or three-particle lines yields a disconnected bare three-particle vertex with vertex insertions. (c) In the remaining graphs, all bare propagators and three-particle vertices are replaced by full propagators and vertices. This procedure defines the skeleton graphs which must be added to the elementary, bare vertex to give the full vertex. This procedure yields equations for the three-particle vertices as

$$\Gamma^V_{(3)} = \tilde{\Gamma}_0{}^V{}_{(3)} + \Lambda^V_{(3)}(\tilde{\Gamma}_0{}^V{}_{(4)}; S, W, D, \Gamma^F, \Gamma^G, \Gamma^V_{(3)}) , \tag{4.26}$$

$$\Gamma^F = \tilde{\Gamma}_0^F + \Lambda^F(\tilde{\Gamma}_0{}^V{}_{(4)}; S, W, D, \Gamma^F, \Gamma^G, \Gamma^V_{(3)}) , \tag{4.27}$$

and

$$\Gamma^G = \tilde{\Gamma}_0^G + \Lambda^G(\tilde{\Gamma}_0{}^V{}_{(4)}; S, W, D, \Gamma^F, \Gamma^G, \Gamma^V_{(3)}) . \tag{4.28}$$

Further, as described by Dahmen and Jona-Lasinio,[18,19] it is possible to write the four-gluon vertex as a bare vertex and a functional of the full propagators and vertices, that is,

$$\Gamma^V_{(4)} = \tilde{\Gamma}_0{}^V{}_{(4)} + \Lambda^V_{(4)}(S, W, D, \Gamma^F, \Gamma^G, \Gamma^V_{(3)}, \Gamma^V_{(4)}) . \tag{4.29}$$

Equations (4.26)–(4.29) are represented graphically in Figs. 13–15.

The renormalization in quantum chromodynamics is almost identical to the renormalization in quantum electrodynamics. However, there is one technical difference. Infrared problems in the perturbation expansion require that all wave-function and charge renormalizations be performed off the mass shell at some Euclidean momenta,

$$p^2 = q^2 = \rho^2 , \tag{4.30}$$

where ρ^2 is a positive, real number. The mass renormalizations can be done on the mass shell. It should be noted that once the wave-function renormalizations have been completed, there remains only one renormalization constant, Z_1, to make all of the three- and four-particle vertices finite. This circumstance, as discussed by Lee and Zinn-Justin,[20] is a consequence of gauge invariance. However, implementation of the charge renormalization is difficult, and in calculations it is convenient to use the dimensional regularization procedure of 't Hooft and Veltman.[21] This procedure will be discussed in more detail in a later paper.

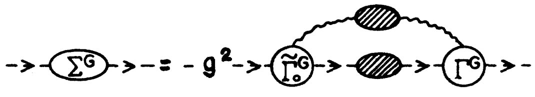

FIG. 12. The ghost self-mass kernel.

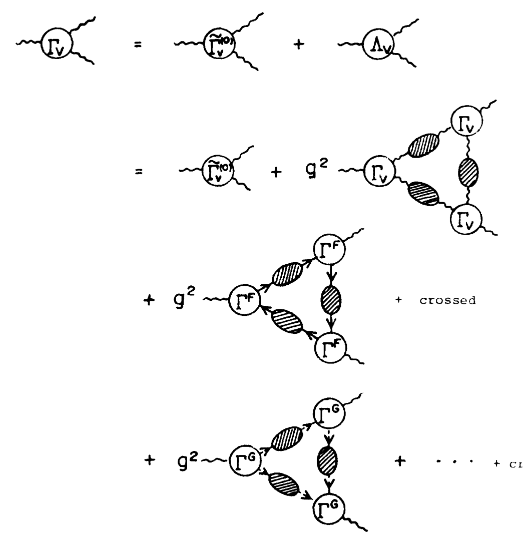

FIG. 13. Equation for the three-gluon vertex.

V. THE THERMODYNAMIC POTENTIAL AS A FUNCTION OF FULL PROPAGATORS AND VERTICES

To calculate efficiently the thermodynamic potential while taking into account the renormalizations discussed in Secs. III and IV, it is convenient

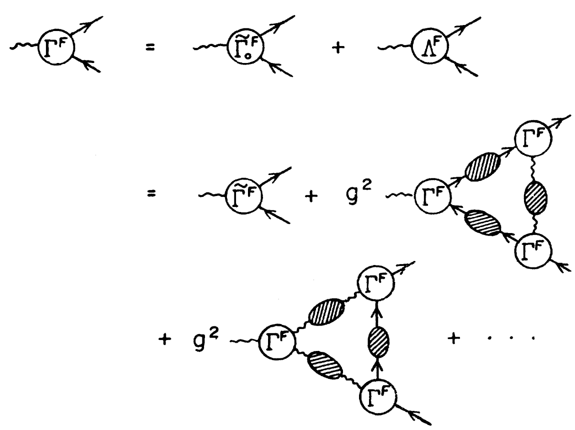

FIG. 14. Equation for the fermion-gluon vertex.

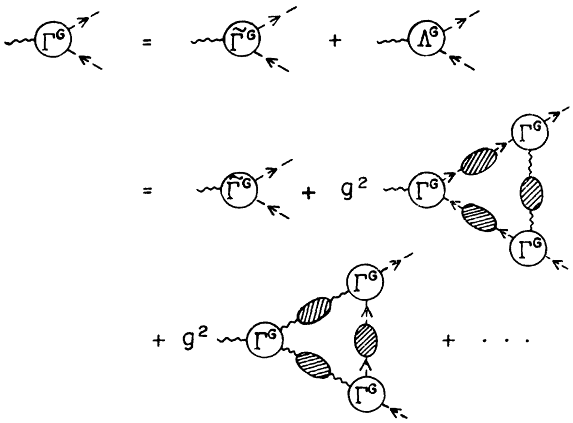

FIG. 15. Equation for the ghost-gluon vertex.

to have an expression written in terms of full propagators and vertices. Unfortunately, the functional-integral representation, Eq. (2.7), is a functional of the bare propagators and vertices. For quantum electrodynamics, this representation yields an equation of form

$$\beta V \Omega_I(\beta, \mu) = W[\tilde{J}, \tilde{D}_0^{-1}, \tilde{S}_0^{-1}, \tilde{\Gamma}_0] , \tag{5.1}$$

with implicit dependence on D_0^{-1} and S_0^{-1} suppressed. Now we would like, as much as possible to remove dependence on bare quantities in favor of full quantities. This removal is accomplished by performing a Legendre transformation[20,22,23,24]

$$\Gamma[\mathcal{A}, D, S, \Gamma] = W[\tilde{J}, \tilde{D}_0^{-1}, \tilde{S}_0^{-1}, \tilde{\Gamma}_0] - \tilde{J} \cdot \frac{\delta W}{\delta \tilde{J}} - (\tilde{D}_0^{-1} - D^{-1}) \frac{\delta W}{\delta \tilde{D}_0^{-1}} - (\tilde{S}_0^{-1} - S^{-1}) \frac{\delta W}{\delta \tilde{S}_0^{-1}} - \tilde{\Gamma}_0 \frac{\delta W}{\delta \tilde{\Gamma}_0} . \tag{5.2}$$

For the case of physical interest, $\mathcal{A}$ is zero, so that upon using Eq. (3.12) for the action and Eq. (2.7) for W, the derivatives of W become

$$\delta W / \delta \tilde{J} = 0 , \tag{5.3}$$

$$\delta W / \delta \tilde{D}_0^{-1} = \tfrac{1}{2} D , \tag{5.4}$$

$$\delta W / \delta \tilde{S}_0^{-1} = -S , \tag{5.5}$$

$$\delta W / \delta \tilde{\Gamma}_0 = e^2 D S \Gamma S . \tag{5.6}$$

Notice that in these expressions the bare propagators and vertices are treated as variational parameters. In fact, Eqs. (5.2)–(5.6) allow for a variational construction of the thermodynamic potential. Suppose that the bare and full propagators and vertices are considered as independent parameters. If this is done, then Γ and W satisfy the stationarity conditions

$$\partial \Gamma / \partial \tilde{J} = \partial \Gamma / \partial \tilde{S}_0^{-1} = \partial \Gamma / \partial \tilde{D}_0^{-1} = \partial \Gamma / \partial \tilde{\Gamma}_0 = 0 \tag{5.7}$$

and

$$\partial W / \partial \mathcal{A} = \partial W / \partial S = \partial W / \partial D = \partial W / \partial \Gamma = 0 . \tag{5.8}$$

The stationarity condition on W, which is effectively the thermodynamic potential, has an interesting interpretation. If the thermodynamic potential is constructed as a functional of full and bare propagators and vertices, then independent variations of the full propagators and vertices will show that the thermodynamic potential is at a stationary point. This situation is a consequence of the Schwinger-Dyson equations satisfied by the full propagators and vertices. Thus, the thermodynamic potential is an effective classical action considered as a functional of full propagators and vertices.

It is possible to calculate Γ by writing and solving first-order differential equations. The definition of Eq. (5.2) and the equations of motion, Eqs. (5.3)–(5.6) show that

$$\frac{\delta \Gamma}{\delta S} (S, D, \Gamma) = \tilde{S}_0^{-1} - 2e^2 (\tilde{\Gamma}_0 S \Gamma D) , \tag{5.9}$$

$$\frac{\delta \Gamma}{\delta D} (S, D, \Gamma) = - \tfrac{1}{2} \tilde{D}_0^{-1} - e^2 (\tilde{\Gamma}_0 S \Gamma S) , \tag{5.10}$$

and

$$\frac{\delta \Gamma}{\delta \Gamma} (S, D, \Gamma) = - e^2 (S \tilde{\Gamma}_0 S D) . \tag{5.11}$$

These equations can be solved by reexpressing bare quantities in terms of full quantities by the Schwinger-Dyson equations. The boundary condition is

$$\Gamma[S_0, D_0, \Gamma = 0] = W[\tilde{J} = 0, \tilde{S}_0^{-1}, \tilde{D}_0^{-1}, \tilde{\Gamma}_0 = 0] = 0 . \tag{5.12}$$

Using the Schwinger-Dyson equations to remove all dependence on bare propagators and vertices, Eqs. (5.9)–(5.11) become

$$\frac{\delta \Gamma}{\delta S} (S, D, \Gamma) = S^{-1} - e^2 \Gamma S \Gamma D + e^2 \Gamma S K \Gamma S D , \tag{5.13}$$

$$\frac{\delta \Gamma}{\delta D} (S, D, \Gamma) = - \tfrac{1}{2} D^{-1} - \tfrac{1}{2} e^2 \Gamma S \Gamma S + \tfrac{1}{2} e^2 S \Gamma S K S \Gamma S , \tag{5.14}$$

and

$$\frac{\delta \Gamma}{\delta \Gamma} (S, D, \Gamma) = - e^2 S \Gamma S D + e^2 S \Gamma S K S S D . \tag{5.15}$$

The solution to Eqs. (5.13)–(5.15) can be found by using the skeleton-graph expansion properties of K given by Eq. (3.21),

$$K = \sum_{n=1}^{\infty} e^{2n} K_{2n} .$$

Recalling that the 2nth moment of K, K_{2n}, is a functional of S, D, and Γ involving $2n-2$ fermion propagators S, n photon propagators D, and $2n$ vertices Γ, we can write the following useful identities:

$$\frac{\delta}{\delta S} S\Gamma S K_{2n} S\Gamma S D = (2n+2)\Gamma S K_{2n} S\Gamma S D\,, \qquad (5.16)$$

$$\frac{\delta}{\delta D} S\Gamma S K_{2n} S\Gamma S D = (n+1) S\Gamma S K_{2n} S\Gamma S\,, \qquad (5.17)$$

and

$$\frac{\delta}{\delta \Gamma} S\Gamma S K_{2n} S\Gamma S D = (2n+2) S\Gamma S K_{2n} S S D\,. \qquad (5.18)$$

The solution to Eqs. (5.13)–(5.15) is, therefore

$$\Gamma(S,D,\Gamma) = \mathrm{Tr}\,\ln S_0^{-1}S - \tfrac{1}{2}\mathrm{Tr}\,\ln D_0^{-1}D - \tfrac{1}{2}e^2\Gamma S\Gamma S D + \sum_{n=1}^{\infty}\frac{e^{2n+2}}{2n+2} S\Gamma S K_{2n} S\Gamma S D\,. \qquad (5.19)$$

Finally, we use Eq. (5.2) to find

$$W[\tilde J, \tilde D_0^{-1}, \tilde S_0^{-1}, \tilde\Gamma_0] = \mathrm{Tr}\,\ln S_0^{-1}S - \tfrac{1}{2}\mathrm{Tr}\,\ln D_0^{-1}D + \tfrac{1}{2}(\tilde D_0^{-1} - D^{-1})D - (\tilde S_0^{-1} - S^{-1})S + e^2\tilde\Gamma_0 S\Gamma S D - \tfrac{1}{2}e^2\Gamma S\Gamma S D + \sum_{n=1}^{\infty}\frac{e^{2n+2}}{2n+2} S\Gamma S K_{2n} S\Gamma S D\,. \qquad (5.20)$$

In this expression, the variational principle of Eq. (5.8) is satisfied by virtue of the Schwinger-Dyson equations. We now use these equations to obtain the thermodynamic potential as a functional of the full propagators and vertices. The result is

$$\beta V\Omega_I(\beta,\mu) = \mathrm{Tr}\,\ln S_0^{-1}S - \tfrac{1}{2}\mathrm{Tr}\,\ln D_0^{-1}D - e^2\Gamma S\Gamma S D + \tfrac{1}{2}\sum_{n=1}^{\infty} e^{2n+2}\left(1+\frac{1}{n+1}\right) S\Gamma S K_{2n} S\Gamma S D\,. \qquad (5.21)$$

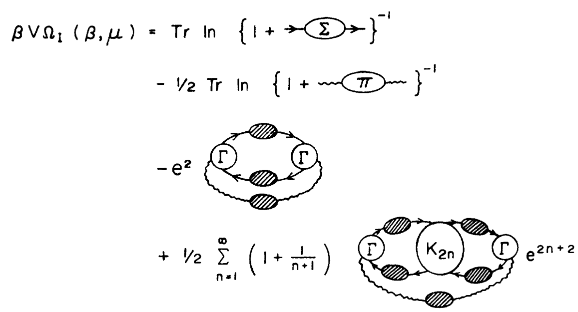

FIG. 16. The thermodynamic potential as a sum of vacuum graphs for quantum electrodynamics.

This equation is represented graphically in Fig. 16. Of course, Eq. (5.21) is not finite until the vacuum energy density has been subtracted. However, all propagators and vertices in Eq. (5.21) are renormalized so that a large number of potential infinities have been removed. The major obstacle to performing calculations with Eq. (5.21) is that even after all renormalizations have been performed and the vacuum energy density subtracted, the Feynman graphs for the thermodynamic potential are naively quadratically divergent. To obtain a finite result, efficient methods of handling overlapping divergences must be developed. Such methods are discussed in subsequent papers.

In quantum chromodynamics, the thermodynamic potential has an expression in terms of bare propagators and vertices of the functional form

$$\beta V\Omega_I(\beta,\mu) = W[\tilde S_0^{-1}, \tilde W_0^{-1}, \tilde D_0^{-1}, \tilde\Gamma_0^F, \tilde\Gamma_0^G, \tilde\Gamma_0^{V}{}_{(3)}, \tilde\Gamma_0^{V}{}_{(4)}]\,. \qquad (5.22)$$

Tc remove dependence on bare propagators and vertices in a favor of full propagators and vertices, we perform the Legendre transformation

$$\Gamma[S,W,D,\Gamma^F,\Gamma^G,\Gamma^V_{(3)},\Gamma^V_{(4)}] = W[\tilde S_0^{-1}, \tilde W_0^{-1}, \tilde D_0^{-1}, \tilde\Gamma_0^F, \tilde\Gamma_0^G, \tilde\Gamma_0^{V}{}_{(3)}, \tilde\Gamma_0^{V}{}_{(4)}] - (\tilde D_0^{-1} - D)\delta W/\delta\tilde D_0^{-1} - (\tilde S_0^{-1} - S^{-1})\delta W/\delta\tilde S_0^{-1} - (\tilde W_0^{-1} - W^{-1})\delta W/\delta\tilde W_0^{-1} - \tilde\Gamma_0^F\frac{\delta W}{\delta\tilde\Gamma_0^F} - \tilde\Gamma_0^G\frac{\delta W}{\delta\tilde\Gamma_0^G} - \tilde\Gamma_0^{V}{}_{(3)}\frac{\delta W}{\delta\tilde\Gamma_0^{V}{}_{(3)}} - \tilde\Gamma_0^{V}{}_{(4)}\frac{\delta W}{\delta\tilde\Gamma_0^{V}{}_{(4)}}\,. \qquad (5.23)$$

The derivatives of W satisfy

$$\frac{\delta W}{\delta\tilde D_0^{-1}} = \tfrac{1}{2}D\,, \qquad (5.24)$$

$$\frac{\delta W}{\delta\tilde S_0^{-1}} = -S\,, \qquad (5.25)$$

$$\frac{\delta W}{\delta\tilde W_0^{-1}} = -W\,, \qquad (5.26)$$

$$\frac{\delta W}{\delta \tilde{\Gamma}_0^F} = g^2 S \Gamma^F S D\,, \tag{5.27}$$

$$\frac{\delta W}{\delta \tilde{\Gamma}_0^G} = g^2 W \Gamma^G W D\,, \tag{5.28}$$

$$\frac{\delta W}{\delta \tilde{\Gamma}_{0(3)}^V} = -\tfrac{1}{6} g^2 DDD \Gamma^V_{(3)}\,, \tag{5.29}$$

$$\begin{aligned}\frac{\delta W}{\delta \tilde{\Gamma}_{0(4)}^V} = -\tfrac{1}{24}\,[&g^2(D_{12}D_{34}+D_{13}D_{24}+D_{14}D_{23}) + g^4(D_{11'}D_{22'}D_{33'}D_{44'})\Gamma^V_{(4)1'2'3'4'}\\ &+g^4(D_{11'}D_{22'}\Gamma^V_{(3)1'2'5'}D_{5'6'}D_{33'}D_{44'}\Gamma^V_{(3)3'4'6'} + D_{11'}D_{33'}\Gamma^V_{(3)1',3',5'}D_{5',6'}D_{22'}D_{44'}\Gamma^V_{(3)2'4'6}\\ &+D_{11'}D_{44'}\Gamma^V_{(3)1'4'5'}D_{5'6'}D_{22'}D_{33'}\Gamma^V_{2'3'6'})]\,.\end{aligned} \tag{5.30}$$

These equations yield

$$\begin{aligned}\Gamma(S,W,D,\Gamma^F,\Gamma^G,\Gamma^V_{(3)},\Gamma^V_{(4)}) = {}& W[\tilde{S}_0^{-1},\tilde{W}_0^{-1},\tilde{D}_0^{-1},\tilde{\Gamma}_0^F,\tilde{\Gamma}_0^G,\tilde{\Gamma}^V_{0(3)},\tilde{\Gamma}^V_{0(4)}]\\ &+(\tilde{S}_0^{-1}-S^{-1})S+(\tilde{W}_0^{-1}-W^{-1})W-\tfrac{1}{2}(\tilde{D}_0^{-1}-D^{-1})D-g^2\tilde{\Gamma}_0^F S\Gamma^F SD\\ &-g^2\tilde{\Gamma}_0^G W\,\Gamma^G WD+\tfrac{1}{6}g^2\tilde{\Gamma}^V_{0(3)}DDD\Gamma^V_{(3)}+\tfrac{1}{8}g^2\tilde{\Gamma}^V_{0(4)}DD\\ &+\tfrac{1}{24}g^4\tilde{\Gamma}^V_{0(4)}DDDD\Gamma^V_{(4)}+\tfrac{1}{8}g^4\tilde{\Gamma}^V_{0(4)}(DD\Gamma^V_{(3)})D(DD\Gamma^V_{(3)})\,.\end{aligned} \tag{5.31}$$

Upon defining Γ' by

$$\begin{aligned}\Gamma[S,W,D,\Gamma^F,\Gamma^G,\Gamma^V_{(3)},\Gamma^V_{(4)}] \equiv {}& \Gamma'[S,W,D,\Gamma^F,\Gamma^G,\Gamma^V_{(3)},\Gamma^V_{(4)}]+\mathrm{Tr}\ln S_0^{-1}S+\mathrm{Tr}\ln W_0^{-1}W-\tfrac{1}{2}\mathrm{Tr}\ln D_0^{-1}D\\ &-\tfrac{1}{2}g^2\Gamma^F S\Gamma^F SD-\tfrac{1}{2}g^2\Gamma^G W\Gamma^G WD+\tfrac{1}{12}g^2\Gamma^V_{(3)}DDD\Gamma^V_{(3)}\,,\end{aligned} \tag{5.32}$$

we find that Γ' satisfies the functional differential equations

$$\frac{\delta\Gamma'}{\delta S} = g^2\Gamma^F S\Lambda^F D\,, \tag{5.33}$$

$$\frac{\delta\Gamma'}{\delta W} = g^2\Gamma^G W\Lambda^G D\,, \tag{5.34}$$

$$\begin{aligned}\delta\Gamma'/\delta D = {}&\tfrac{1}{2}g^2\Gamma^F S\Lambda^F S+\tfrac{1}{2}g^2\Gamma^G W\Lambda^G W-\tfrac{1}{4}g^2\Gamma^V_{(3)}DD\Lambda^V_{(3)}+\tfrac{1}{12}g^4\tilde{\Gamma}^V_{0(4)}DDD\Gamma^V_{(4)}\\ &+\tfrac{1}{4}g^4\tilde{\Gamma}^V_{0(4)}(D\Gamma^V_{(3)}D)D(D\Gamma^V_{(3)})+\tfrac{1}{8}g^4\Gamma^V_{(3)}DD\Gamma^V_{(4)}DD\Gamma^V_{(3)}\,,\end{aligned} \tag{5.35}$$

$$\frac{\delta\Gamma'}{\delta\Gamma^F} = +g^2 S\Lambda^F SD\,, \tag{5.36}$$

$$\frac{\delta\Gamma'}{\delta\Gamma^G} = +g^2 W\Lambda^G WD\,, \tag{5.37}$$

$$\frac{\delta\Gamma'}{\delta\Gamma^V_{(3)}} = \tfrac{1}{6}g^2\Lambda^V_{(3)}DDD+\tfrac{1}{4}g^4\Gamma^V_{0(4)}(DD\Gamma^V_{(3)})D\,, \tag{5.38}$$

and

$$\frac{\delta\Gamma'}{\delta\Gamma^V_{(4)}} = \tfrac{1}{24}g^4\tilde{\Gamma}^V_{0(4)}DDDD\,, \tag{5.39}$$

subject to the boundary conditions

$$\Gamma'[S_0,W_0,D_0,0,0,0,0]=0\,. \tag{5.40}$$

In quantum chromodynamics, it is difficult to find a closed-form expression for Γ'. However, a pertubative construction is possible. This construction is accomplished by drawing all skeleton vacuum graphs, with undetermined coefficients, to a given order in the skeleton-graph expansion. The combinatoric coefficients are determined by solving Eqs. (5.33)–(5.40), with all bare vertices and propagators considered as functionals of full propagators and vertices.

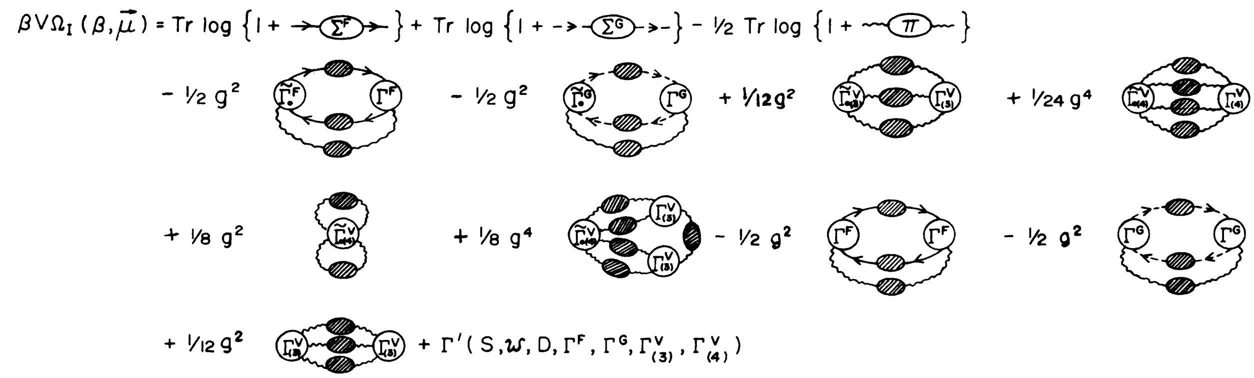

FIG. 17. The thermodynamic potential for quantum chromodynamics as a sum of vacuum graphs.

The thermodynamic potential is given directly by Eqs. (5.23) and (5.32). After some algebra, we find

$$\begin{aligned}\beta V\Omega_I(\beta,\mu) = {} & \mathrm{Tr}\ln S_0^{-1}S + \mathrm{Tr}\ln W_0^{-1}W \\ & -\tfrac{1}{2}\mathrm{Tr}\ln D_0^{-1}D - \tfrac{1}{2}g^2\tilde{\Gamma}_0^F S\Gamma^F SD - \tfrac{1}{2}g^2\tilde{\Gamma}_0^G W\Gamma^G WD + \tfrac{1}{12}g^2\tilde{\Gamma}^V_{0(3)}DDD\Gamma^V_{(3)} \\ & + \tfrac{1}{24}g^4\tilde{\Gamma}^V_{0(4)}DDDD\Gamma^V_{(4)} + \tfrac{1}{8}g^2\tilde{\Gamma}^V_{0(4)}DD + \tfrac{1}{8}g^4\tilde{\Gamma}^V_{0(4)}(DD\Gamma^V_{(3)})D(DD\Gamma^V_{(3)}) - \tfrac{1}{2}g^2\Gamma^F S\Gamma^F SD - \tfrac{1}{2}g^2\Gamma^G W\Gamma^G WD \\ & + \tfrac{1}{12}g^2\Gamma^{(V)}_{(3)}DDDD\Gamma^V_{(3)} + \Gamma'[S, W, D, \Gamma^F, \Gamma^G, \Gamma^V_{(3)}, \Gamma^V_{(4)}]\,. \end{aligned} \tag{5.41}$$

This equation is represented diagrammatically in Fig. 17.

VI. CONCLUSIONS

In Eqs. (5.21) and (5.41), the thermodynamic potential is given as a functional of full propagators and vertices. However, these expressions are naively quartically divergent. Subtraction of the vacuum energy density only reduces the degree of divergence to quadratic. Clearly, a large number of cancellations must occur if we are to obtain a finite result. In subsequent papers, graphical techniques will be exploited to obtain finite expressions for the thermodynamic potential to the three-loop level, that is, fourth order in the coupling.

There is also a problem of naive infrared divergences in the perturbative calculation of the thermodynamic potential. This problem is associated with the plasmon effect, whereby the massless gauge bosons acquire masses through interactions. To calculate this effect, nonperturbative methods must be employed. Fortunately, the terms which yield naive infrared divergent contributions are free of naive ultraviolet divergences, allowing a perturbative analysis of the naive ultraviolet-divergent contributions.

It should also be noted that the thermodynamic potential satisfied renormalization-group equations.[25-27] The difference between the thermodynamic potential at finite temperature and density and at zero temperature and density was shown in Sec. II to be invariant under a change in the choice of wave-function renormalization constants. Thus, this difference satisfies a Callan-Symanzik equation with zero anomalous dimension.[27,28] This observation will be used in subsequent papers to extend the results of the three-loop calculation so as to include effects beyond the three-loop level.

ACKNOWLEDGMENTS

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[1]J. C. Collins and M. J. Perry, Phys. Rev. Lett. 34, 1353 (1975).
[2]B. A. Freedman and L. D. McLerran, MIT Report No. 541, 1976 (unpublished).
[3]M. Kislinger and P. Morley, Phys. Lett. 67B, 371 (1977); Univ. of Chicago Report No. EFI 77/4 (unpublished).
[4]K. Brecher and G. Caporaso, Nature 259, 377 (1976).
[5]G. Baym and S. A. Chin, Phys. Lett. 62B, 241 (1976).
[6]G. Chapline and M. Nauenberg, Nature 259, 377 (1976).
[7]A. Chodos, R. L. Jaffe, K. Johnson, C. B. Thorn, and V. F. Weisskopf, Phys. Rev. D 9, 3471 (1974).
[8]T. DeGrand, R. L. Jaffe, K. Johnson, and J. Kiskis, Phys. Rev. D 12, 2060 (1975).

[9]C. DeDominicus and P. C. Martin, J. Math. Phys. 5, 14 (1964); 5, 31 (1964).
[10]R. E. Norton and J. N. Cornwall, Ann. Phys. (N.Y.) 91, 106 (1975).
[11]R. P. Feynman, Phys. Rev. 91, 1291 (1953).
[12]C. Bernard, Phys. Rev. D 9, 3312 (1974).
[13]L. D. Faddeev, Teor. Math. Fiz. 1, 3 (1970) [Theor. Math. Phys. 1, 1 (1970)].
[14]L. D. Faddeev and V. N. Popov, Phys. Lett. 25B, 29 (1967).
[15]J. Schwinger, Proc. Natl. Acad. Sci. 37, 452 (1951).
[16]F. J. Dyson, Phys. Rev. 75, 486 (1949); 75, 1736 (1949).
[17]E. Salpeter and H. Bethe, Phys. Rev. 84, 1232 (1951).
[18]G. Jona-Lasinio, Nuovo Cimento 34, 1790 (1964).
[19]G. Jona-Lasinio and H. D. Dahmen, Nuovo Cimento 10A, 513 (1969).
[20]B. W. Lee and J. Zinn-Justin, Phys. Rev. D 5, 3121 (1972); 5, 3137 (1972); 5, 3155 (1972).
[21]G. 't Hooft and M. Veltman, Nucl. Phys. B44, 189 (1972).
[22]R. Jackiw, Phys. Rev. D 9, 1686 (1974).
[23]J. M. Cornwall, R. Jackiw, and E. Tomboulis, Phys. Rev. D 10, 2428 (1974).
[24]L. Dolan and R. Jackiw, Phys. Rev. D 9, 3320 (1974).
[25]D. J. Gross and F. Wilczek, Phys. Rev. D 8, 3633 (1973); 9, 980 (1974).
[26]H. D. Politzer, Phys. Rev. Lett. 30, 1346 (1973).
[27]C. G. Callan, Phys. Rev. D 2, 1541 (1970).
[28]K. Symanzik, Commun. Math. Phys. 18, 227 (1970).