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Color confinement

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We argue that the effective charge $\bar{g}(\lambda)$ in a color SU(N) gauge theory becomes infinite in the infrared limit $\lambda \to 0$ in the Landau gauge, provided the number F of massless quark flavors satisfies $13N/4 < F < 11N/2$. To deduce this, we must assume that (i) the R transformation $\vec{A}_\mu(x) \to \vec{A}_\mu(x) + \vec{r}_\mu$, $\vec{A}_\mu$ = gauge fields, $\vec{r}_\mu$ = constant vectors, commutes with the gauge-invariant cutoff removal in the renormalized field equations and current definitions, (ii) functional methods can be used to derive the consequences of invariance under the R transformation, and (iii) if $\bar{g}(\lambda) \to g_\infty < \infty$ as $\lambda \to 0$, the vertex functions are well defined and nonvanishing at the value g_∞ of the renormalized coupling constant. With these assumptions, generalizations of one-particle irreducible vertex functions involving both gauge fields and color currents are shown to have an infrared behavior which is inconsistent with the existence of the infrared fixed point at g_∞. Therefore, $\bar{g}(0) = \infty$, suggesting that color is confined in these models. If we assume that the infrared decoupling theorem is valid for the exact theory, our result can be extended to both massive and massless quarks, for all $F < 11N/2$.

I. INTRODUCTION

It has become popular in recent years to assume that hadrons are described by an $SU(3)\otimes G(F)$ non-Abelian gauge theory (NAGT), where the "color" group SU(3) is appropriate to the strong interaction and the "flavor" group $G(F)$ is appropriate to the weak and electromagnetic interactions.[1] For generality, we will take the color group to be SU(N). We need not specify the flavor group except for the number F of flavors.[2] The popularity of this scheme is based on the success of the constituent colored quark model and the unified gauge models of weak interactions and on the observed symmetries of the strong interaction as partially broken by the weak and electromagnetic interactions. Furthermore, for

$$0 \leq F < \tfrac{11}{2}N, \tag{1.1}$$

a theory is asymptotically free (AF) and so may describe the scaling laws observed in electroproduction[3] and, more importantly, may confine color so that quarks and other colored particles cannot be observed. It is this latter hope, which must be at least approximately realized if the theory is to be consistent with experiment, to which we address ourselves in this paper.

The desired properties of the above models are most simply expressed in terms of the invariant charge $\bar{g}(\lambda)$. We define the "physical" coupling constant g by renormalizing the gauge-field four-point vertex at a typical hadronic mass (~1 GeV). One has

$$\bar{g}(\lambda) \underset{\lambda\to\infty}{\simeq} \text{const}\times(\ln\lambda)^{-1/2} + O((\ln\lambda)^{-3/2}), \tag{1.2}$$

and this is all that has been heretofore known for sure. This determines the behavior of the vertex functions in the "mathematical" ultraviolet (UV) limit of λ sufficiently large so that $(\ln\lambda)^{-1}$ is negligible compared to unity. To understand Bjorken scaling, we must hope that $\bar{g}(\lambda)$ remains small for λ not much larger than 1 (the "physical" UV region); to understand strong interactions, one must hope that $\bar{g}(\lambda)$ suddenly becomes of order one[4] for λ near 1 (the "resonance" region); and to understand confinement, one hopes that $\bar{g}(\lambda)$ becomes very large for λ near zero [the infrared (IR) region]. It has, unfortunately, been impossible to prove or disprove these hopes even heuristically because of the irrelevance of perturbation theory for large effective charge.

We study this problem using nonperturbative methods previously employed to deduce the "R invariance" of AFNAGT'S.[5,6] For AFNAGT'S with sufficiently many massless fermion multiplets,

$$\tfrac{13}{4}N < F < \tfrac{11}{2}N, \tag{1.3}$$

we conclude that $\bar{g}$ does indeed become infinite in the IR limit (at least in the Landau gauge) Our arguments involve some smoothness assumptions, but since heretofore there has been absolutely no indication that color confinement occurs in realistic theories, we feel that even a nonrigorous argument to this effect is encouraging. Our results extend to massive fermions and to all $F<\frac{11}{2}N$ if the IR decoupling theorem is true in the exact theory.

Our result is that

$$\bar{g}(\lambda)\xrightarrow[\lambda\to 0]{}\infty, \qquad (1.4)$$

but an an unspecified rate so that this at best establishes only "mathematical" confinement. "Physical" confinement would require $\bar{g}(\lambda)$ already to be very large for λ corresponding to observable distances. This, as well as the hoped-for behavior of $\bar{g}$ in the resonance and physical UV regions must still be taken on faith, but these desired behaviors are perhaps now more plausible.

We deduce (1.4) by showing that the contrary result $\bar{g}(0)=\text{const}$ (the only other possibility in the Landau gauge) is untenable because it implies an IR behavior which is inconsistent with that given by the Ward-Takahashi (WT) identities associated with R invariance. To obtain this contradiction, we must use generalizations of vertex functions involving color currents as well as gauge fields because the gauge-field vertex functions can have an IR behavior consistent with the WT identities if the gauge fields have a negative IR dimension, whereas the color currents are constrained to have canonical dimensions.

Even given (1.4), there is no rigorous proof of the intuitive expectation that confinement will obtain in the strong-coupling limit. There are, however, interesting indications that confinement does indeed occur. It has been shown that an NAGT of strong interactions becomes a color-confinement boundary condition on the weak and electromagnetic interactions in the strong-coupling limit.[7] It has also been argued that the dual string arises as a strong-coupling limit of an NAGT.[8]

It is crucial for us to work in the Landau gauge. The technical reasons for this are explained in Sec. II where the Landau-gauge formulation[9] of NAGT's is summarized. The renormalization group is exploited in order to establish the R invariance of the theory. The gauge-field vertex functions are studied in Sec. III, where the implications of renormalization invariance and R invariance are deduced. It is shown that (1.4) can only fail to occur if the gauge-field IR dimension is negative. In Sec. IV, generalizations of vertex functions involving both gauge fields and color currents are introduced. In Sec. IV A, it is shown that the currents change by a finite amount under R transformation only in models satisfying (1.3). In Sec. IV B, the mixed vertex-function generalizations are defined by a suitable functional Legendre transformation, and in Sec. IV C, their IR behavior is deduced from both the renormalization group and R invariance. These behaviors are not consistent and so (1.4) is at least formally established in the models (1.3). The extension of our results to models involving massive quarks is discussed in Sec. V. We conclude in Sec. VI with a discussion of our assumptions and the implications of our results.

II. LANDAU GAUGE

The models of interest here involve N^2-1 renormalized (Z_3 = renormalization constant) non-Abelian gauge fields A_μ^a ($a=1,\ldots,N^2-1$) transforming according to the adjoint representation of SU(N), NF massless[10] renormalized (Z_2 = renormalization constant) fermions Ψ [F fermion N-tuplets, each transforming according to the fundamental representation of SU(N)], $2(N^2-1)$ renormalized ghosts C_1^a, C_2^a ($\tilde{Z}_3$ = renormalization constant), and a renormalized ($Z_1=Z_3\tilde{Z}_1/\tilde{Z}_3$ = renormalization constant) coupling constant g. The appropriate SU(N) structure constants f^{abc} and fermion representation matrices T^a are used to define covariant derivatives

$$\mathfrak{D}_\mu^{ab}=\delta^{ab}\partial_\mu+(Z_1/Z_3)gf^{abc}A_\mu^c, \qquad (2.1)$$

$$D_\mu=\partial_\mu-igT^cA_\mu^c(Z_1/Z_3), \qquad (2.2)$$

and field strength

$$F_{\mu\nu}^a=\partial_\mu A_\nu^a-\partial_\nu A_\mu^a+(Z_1/Z_3)gf^{abc}A_\mu^bA_\nu^c, \qquad (2.3)$$

or more concisely,

$$\vec{F}_{\mu\nu}=\partial_\mu\vec{A}_\nu-\partial_\nu\vec{A}_\mu+(Z_1/Z_3)g\vec{A}_\mu\times\vec{A}_\nu. \qquad (2.4)$$

In addition, one has the normalization conditions

$$f^{acd}f^{bcd}=N\delta^{ab}, \qquad (2.5)$$

$$\mathrm{Tr}(T^aT^b)=\tfrac{1}{2}F\delta^{ab}, \qquad (2.6)$$

$$(T^aT^a)_{ij}=[(N^2-1)/2N]\delta_{ij}. \qquad (2.7)$$

The generalized Lorentz (non-Landau) gauges are defined by the Lagrangian density[11]

$$L=L_{\rm cl}+L_{\rm gh}+(1/\alpha)(\partial\cdot\vec{A})^2, \quad \alpha\neq 0, \qquad (2.8)$$

the sum of the classical, ghost, and gauge-fixing pieces. The latter piece involves the renormalized gauge parameter α and provides a momentum canonically conjugate to A_0^a:

$$[A_0^a(x),(1/\alpha)\partial\cdot A^b(y)]\delta(x_0-y_0)=-i\delta^{ab}\delta^4(x-y). \qquad (2.9)$$

This equal-time commutator is finite in each order of perturbation theory. L gives rise to the renormalized field equations

$$\partial^\nu(\partial_\mu \vec{A}_\nu - \partial_\nu \vec{A}_\mu) = -(Z_3\alpha)^{-1}\partial_\mu \partial\cdot\vec{A} - \vec{K}_\mu, \quad (2.10)$$

$$\partial\cdot\mathfrak{D}\vec{C}_1 = 0, \quad (2.11)$$

$$\mathfrak{D}\cdot\partial\vec{C}_2 = 0, \quad (2.12)$$

and

$$\not{D}\Psi = 0, \quad (2.13)$$

where

$$\vec{K}_\mu = g(Z_1/Z_3)[\vec{F}_{\mu\nu}\times\vec{A}^\nu + \partial^\nu(\vec{A}_\mu\times\vec{A}_\nu) + (1/Z_3)(Z_2\bar{\Psi}\gamma_\mu\vec{T}\Psi + \tilde{Z}_3\partial_\mu\vec{C}_2\times\vec{C}_1)]. \quad (2.14)$$

In addition to depending on momenta, g, and α, the vertex functions of the theory are functions of the renormalization mass μ. One has, for example, the renormalization condition

$$\Gamma^{(4)}_{\alpha_1\alpha_2\alpha_3\alpha_4}(p_1,p_2,p_3,p_4;g,\alpha;\mu)|_{S_\mu} = g \quad (2.15)$$

on the gauge-field one-particle-irreducible (1PI) 4-point vertex at the mass-μ symmetry point S_μ:

$$p_i\cdot p_j = \tfrac{1}{3}\mu^2(4\delta_{ij}-1). \quad (2.16)$$

The arbitrariness of μ is then expressed by the renormalization group (RG) equation[12]

$$\left[\mu\frac{\partial}{\partial\mu} + \beta(g,\alpha)\frac{\partial}{\partial g} + \delta(g,\alpha)\frac{\partial}{\partial\alpha} - n\gamma(g,\alpha)\right] \times\Gamma^{(n)}(p_1,\ldots,p_n;g,\alpha;\mu) = 0 \quad (2.17)$$

for the n-point gauge-field vertex functions.

Gauge invariance gives the relation

$$\delta(g,\alpha) = -\alpha\gamma(g,\alpha) \quad (2.18)$$

between δ and γ. Because of the presence of two coupling constants (g and α), the solutions to (2.17) involve two effective coupling constants $\bar{g}(\lambda)$ and $\bar{\alpha}(\lambda)$ defined by

$$\lambda\frac{\partial\bar{g}(\lambda;g,\alpha)}{\partial\lambda} = \beta(\bar{g},\bar{\alpha}), \quad \bar{g}(1;g,\alpha) = g, \quad (2.19)$$

$$\lambda\frac{\partial\bar{\alpha}(\lambda;g,\alpha)}{\partial\lambda} = \delta(\bar{g},\bar{\alpha}), \quad \bar{\alpha}(1;g,\alpha) = \alpha. \quad (2.20)$$

If (1.1) is satisfied, one has the exact[13] asymptotic behaviors

$$\bar{g}(\lambda;g,\alpha)\xrightarrow[\lambda\to\infty]{} 0,$$
$$\bar{\alpha}(\lambda;g,\alpha)\xrightarrow[\lambda\to\infty]{} 0 \quad \text{if } F/2N > \tfrac{13}{8}, \quad (2.21)$$
$$\bar{\alpha}(\lambda;g,\alpha)\xrightarrow[\lambda\to\infty]{} \tfrac{13}{3} - 4F/3N \quad \text{if } F/2N < \tfrac{13}{8},$$

and the vertex functions are exactly computable in the deep Euclidean limit. One can, in particular, calculate the exact behavior of the renormalization constants $Z_i(\Lambda/\mu)$ for $\Lambda\to\infty$ (Ref. 13) and the behavior of field products such as $A^a_\mu(x)A^b_\nu(y)$ for $x\to y$.[6] It follows from these calculations that[5]

$$Z_1/Z_3 \to 0 \quad \text{always}, \quad (2.22)$$

and[6]

$$(Z_1/Z_3)^2 A^a_\mu(x)A^b_\nu(x) \to 0 \quad \text{if and only if } F > N. \quad (2.23)$$

These results imply that the exact field equations (2.10)–(2.14) are invariant to the R transformation[14]

$$\vec{A}_\mu\to\vec{A}_\mu + \vec{r}_\mu, \quad \Psi\to\Psi, \quad \vec{C}_1\to\vec{C}_1, \quad \vec{C}_2\to\vec{C}_2 \quad (2.24)$$

for arbitrary constants r^a_μ, if

$$N < F < \tfrac{11}{2}N. \quad (2.25)$$

To investigate which models confine color, one wants to know the behavior of $\bar{g}$ for $\lambda\to 0$. Because of the presence of two coupling constants and the associated two effective charges, this can be an extremely complicated problem. The coupling between $\bar{g}$ and $\bar{\alpha}$ in (2.19), (2.20) allows for many different types of asymptotic behavior in the IR limit, e.g., fixed point, limit cycle, etc.[15] The enormous simplification of this situation in the Landau gauge ($\alpha = 0$) indicates that we should specialize to this gauge. In the Landau gauge, because of (2.18), the α dependence of the RG equations completely disappears. One has $\bar{\alpha} = 0$, and β, γ, and $\bar{g}(\lambda)$ are functions of g only. Then there are only two possible types of IR behavior for $\bar{g}(\lambda;g)$, depending on the behavior of $\beta(g)$. In the AF models (1.1), $\beta(g)$ vanishes at $g=0$ with a negative slope, and the issue is whether or not $\beta(g)$ has a second zero at some finite $g = g_\infty$ (Ref. 16):

$$\beta(g_\infty) = 0, \quad g_\infty > 0. \quad (2.26)$$

The slope of β is then necessarily positive at $g = g_\infty$ so that g_∞ is an IR-stable fixed point and

$$\bar{g}(\lambda;g)\xrightarrow[\lambda\to 0]{} g_\infty. \quad (2.27)$$

If, on the contrary, $\beta(g)$ has no second zero, then

$$\bar{g}(\lambda;g)\xrightarrow[\lambda\to 0]{} \infty \quad \text{for } g\neq 0. \quad (2.28)$$

We will argue that the second zero of β (2.26) cannot exist in the models (1.3).

Our arguments involve the use of an operator formulation of the Landau gauge, to which we now turn.[9] The formalism is slightly more complicated than for $\alpha\neq 0$ in that there exists an additional set of scalar fields $\vec{B}$ (formally $\lim_{\alpha\to 0}\alpha^{-1}\partial\cdot\vec{A}$) transforming according to the adjoint representation of SU(N). The Lagrangian density becomes

$$L = L_{cl} + L_{gh} + \vec{B}\cdot(\partial\cdot\vec{A}), \tag{2.29}$$

so that $\vec{B}$ is canonically conjugate to $\vec{A}_0$:

$$[A_0^a(x), B^b(y)]\,\delta(x_0 - y_0) = -i\,\delta^{ab}\delta^4(x-y). \tag{2.30}$$

This relation is finite in each order of perturbation theory. The only field equation in (2.10)–(2.14) which changes is (2.10), which becomes

$$\partial^\nu(\partial_\mu\vec{A}_\nu - \partial_\nu\vec{A}_\mu) = -Z_3^{-1}\partial_\mu\vec{B} - \vec{K}_\mu, \tag{2.31}$$

and there is the new equation

$$\partial^\mu\vec{A}_\mu = 0, \tag{2.32}$$

characteristic of the Landau gauge.

The renormalization conditions and RG equations (2.15)–(2.20), etc, remain valid as before but with $\alpha = \delta = \bar{\alpha} = 0$. The renormalization constants may again be calculated exactly and are given by[13]

$$Z_i(\Lambda) \underset{\Lambda\to\infty}{\simeq} (\ln\Lambda)^{-a_i/b}, \quad i = 1, 2, 3, \tilde{3}, \tag{2.33}$$

with

$$b = (1/24\pi^2)(11N - 2F), \tag{2.34a}$$

$$a_2 = 0, \tag{2.34b}$$

$$a_3 = -(1/48\pi^2)(13N - 4F), \tag{2.34c}$$

$$a_{\tilde{3}} < 0, \tag{2.34d}$$

$$a_1 = \tfrac{1}{2}(3a_3 + b). \tag{2.34e}$$

We thus always have

$$Z_2 = \text{const and } \tilde{Z}_3 \to \infty \quad [\text{in any theory (1.1)}], \tag{2.35}$$

whereas Z_1 and Z_3 depend on N and F.

The behavior of the renormalization constants divides AF theories into two distinct categories. For $\frac{11}{2}N > F > \frac{13}{4}N$, we obtain theories which are perturbation-theory violating (PV) in that all renormalizations of fundamental fields are finite: $Z_3 \to 0$, $Z_2 \to$ const, $Z_1 \to 0$. For the standard SU(3) color group, one obtains PV theories when $16 \geq F \geq 10$. The remaining AF theories are fermion deficient (FD) with respect to PV theories. FD theories have $Z_3 \to \infty$, $Z_1 \to 0$ for $F > \frac{17}{8}N$, $Z_1 \to \infty$ for $F < \frac{17}{8}N$, $Z_2 \to$ const. The fact that the exact gluon wave-function renormalization is infinite for FD theories indicates that the exact theory itself resembles its perturbation expansion to some extent, and for $F < \frac{17}{8}N$ this resemblance is even more pronounced. In summary,

$$Z_1 = 0,\ Z_3 = 0 \quad \text{for } \tfrac{11}{2}N > F > \tfrac{13}{4}N \text{ (PV)}, \tag{2.36a}$$

$$Z_1 = 0,\ Z_3 = \infty \quad \text{for } \tfrac{13}{4}N > F > \tfrac{17}{8}N \text{ (FD)}, \tag{2.36b}$$

$$Z_1 = \infty,\ Z_3 = \infty \quad \text{for } 0 \leq F < \tfrac{17}{8}N \text{ (FD)}. \tag{2.36c}$$

The above renormalization constant results together with the results of Ref. 6 on the behavior of $A_\mu^a(x)A_\nu^b(y)$ for $x \to y$ imply the following behavior in the AF models:

$$Z_1/Z_3(\Lambda) \underset{\Lambda\to\infty}{\longrightarrow} 0, \tag{2.37}$$

$$(Z_1/Z_3)A_\mu^a A_\nu^b(x,\Lambda) \underset{\Lambda\to\infty}{\longrightarrow} \text{finite operator}, \tag{2.38}$$

$$(Z_1/Z_3)^2 A_\mu^a A_\nu^b(x,\Lambda) \underset{\Lambda\to\infty}{\longrightarrow} 0. \tag{2.39}$$

Because of these relations, the exact formal field equations of an AF theory in the Landau gauge are invariant under the R transformation

$$\vec{A}_\mu(x) \to \vec{A}_\mu(x) + \vec{r}_\mu, \quad \vec{B}, \Psi, \vec{C}_1, \vec{C}_2 \text{ unchanged}. \tag{2.40}$$

This invariance is present in the Landau gauge for any number of fermions as long as the AF condition (1.1) is satisfied. However, implicit in this result is our first smoothness assumption—that the R transformation commutes with the gauge-invariant cutoff-removal limit implicit in the field equations. The R invariance of AF models could be rigorously established by showing that the point-separated regularized finite local field equations are explicitly invariant to (2.40). This has already been accomplished for models without fermions.[17]

III. GAUGE-FIELD VERTEX FUNCTIONS

Properties of the Landau-gauge Green's functions and vertex functions in the models can be conveniently summarized in terms of their generating functionals. In this section, we will study the connected Green's functions $G_{\alpha_1\cdots\alpha_n}^{(n)a_1\cdots a_n}(p_1,\ldots,p_n;g,\mu)$ and 1PI vertex functions $\Gamma_{\alpha_1\cdots\alpha_n}^{(n)a_1\cdots a_n}(p_1,\ldots,p_n;g,\mu)$ involving only the gauge fields A_μ^a. Consideration of these functions will not be sufficient to deduce (2.28), but will illustrate our approach in the simplest possible context. The derivation of (2.28) will be given in the following section.

The generating functional Z of connected gauge-field Green's functions is given by

$$\exp[Z(\vec{M}_A)] = D\int(d\sigma)\exp\left\{i\int d^4x[L(x) + \vec{M}_{A\mu}\cdot\vec{A}^\mu]\right\}, \tag{3.2}$$

where $L(x)$ is the Lagrangian density (2.29), $(d\sigma)$ is a gauge-invariant measure on the fields, $\vec{M}_{A\mu}$ is a classical source, and D divides out vacuum bubbles. The generating functional Γ of one-particle-irreducible vertices is obtained from Z by Legendre-transforming the source M_A:

$$\Gamma(\vec{S}_A) = Z(\vec{M}_A) - i\int d^4x\,\vec{S}_{A\mu}\cdot\vec{M}_A^\mu, \tag{3.3}$$

where

$$\delta\Gamma/\delta\vec{S}_A = i\vec{M}_A, \quad \delta Z/\delta\vec{M}_A = -i\vec{S}_A. \tag{3.4}$$

Functional differentiation of (3.3) with respect to S_A and then setting S_A equal to zero gives the nonlinear equations defining the 1PI vertices in terms of the Green's functions of the theory. The simplest of these are (symbolically)

$$\Gamma^{(2)} = -(\Gamma^{(2)})^2 G^{(2)}, \tag{3.5}$$

$$G^{(3)} = (G^{(2)})^3 \Gamma^{(3)}, \tag{3.6}$$

$$G^{(4)} - G^{(2)}(\Gamma^{(3)})^2 = (G^{(2)})^4 \Gamma^{(4)}, \tag{3.7}$$

where $\Gamma^{(n)}$ is a 1PI vertex for n gauge fields and $G^{(n)}$ is the corresponding Green's function. Once (3.5) is solved for the inverse propagator

$$\Gamma^{(2)} = -(G^{(2)})^{-1}, \tag{3.8}$$

all other vertices may be obtained recursively. To calculate the inverse of (3.8), the mixing of $\vec{A}_\mu$ and $\vec{B}$ must be taken into account. The details are given in Sec. IV, where a more complicated problem arises. The Γ vertices have interpretations in terms of Feynman diagrams; they correspond to those connected diagrams with no one-particle intermediate states and with external propagators removed.

The R-symmetry WT identity is derived from (3.2) by application of an infinitesimal R transformation. The measure in (3.2) is R-invariant, and so we obtain

$$\int d^4x\, \vec{M}_A^\mu \cdot \vec{r}_\mu Z = 0. \tag{3.9}$$

In terms of the 1PI generating functional, this functional WT identity becomes

$$\int d^4x\, \delta\Gamma/\delta\vec{S}_A = 0. \tag{3.10}$$

Functional differentiation and Fourier transformation give the low-energy theorems

$$\Gamma^{(n)}(p_1, \ldots, p_n; g, \mu) = 0 \quad \text{when any } p_i = 0, \tag{3.11}$$

for any $n > 1$. A case of particular interest is $n = 2$, which, by (3.8), implies that the gauge-field propagator has a singularity, but not necessarily a pole, at $q = 0$.

The 1PI vertices satisfy the familiar renormalization group equation

$$[\lambda\partial/\partial\lambda - \beta(g)\partial/\partial g + 4 - nd(g)]\Gamma^{(n)}(\lambda P; g, \mu) = 0 \tag{3.12}$$

in the Landau gauge, since the fermions are massless. $P = (p_1, \ldots, p_n)$, $d(g) = 1 + \gamma(g)$ is the full dimension of the gauge field, and μ is the subtraction mass.

We now assume that $\beta(g)$ has a second zero for $g = g_\infty \neq 0$. We make the standard assumptions that $\Gamma^{(n)}(P; g, \mu)$ and $\gamma(g)$ are well behaved at the fixed point so that the $\Gamma^{(n)}(P; g_\infty; \mu)$ describe a scale-invariant theory in which $\vec{A}_\mu$ has the scale dimension $d = 1 + \gamma(g_\infty)$. This conventional assumption is our second smoothness assumption. Then one has the infrared behavior

$$\Gamma^{(n)}(\lambda P; g, \mu) \underset{\lambda\to 0}{\sim} \lambda^{4-nd}\, \Gamma^{(n)}(P; g_\infty, \mu) f^{(n)}(\lambda) + \text{less singular in } \lambda, \tag{3.13}$$

for all nonexceptional P, where $f^{(n)}(\lambda)$ may be zero or infinite as $\lambda \to 0$ but not as a power of λ.

Equation (3.13) is consistent with (3.11) only if, for each n, either

$$4 - nd \geq 0 \tag{3.14}$$

or

$$\Gamma^{(n)}(P; g_\infty, \mu) = 0 \quad \text{for all } P. \tag{3.15}$$

By considering large n, we deduce that either

$$d \leq 0 \tag{3.16}$$

or

$$\Gamma^{(n)}(P; g, \mu) = 0 \quad \text{for all } P, \tag{3.17}$$

for $n > 4/d$. The behavior (3.17) is presumably impossible for $g \neq 0$. We are then left with (3.16); but, since we see no reason to exclude a negative dimension,[18] we have not produced a contradiction.

The possibility that $d \leq 0$ thwarts our attempt to deduce a contradiction. This suggests that we should try to employ fields of known positive dimension in our analysis. The logical candidates are the conserved color currents $\vec{J}_\mu$ of dimension three. These will be considered in this context in the following section.

IV. CURRENT-FIELD VERTEX FUNCTIONS

A. Color currents

The global color transformations

$$\begin{aligned} \delta\vec{A}_\mu &= g\vec{A}_\mu \times \vec{L}, \\ \delta\Psi &= g\vec{T}\cdot\vec{L}\Psi, \\ \delta\vec{C}_1 &= g\vec{C}_1 \times \vec{L}, \end{aligned} \tag{4.1}$$

where $\vec{L}$ is a constant vector, are generated by the spatial integral of the renormalized "color" current[11,19]

$$\vec{J}_\mu = g\{(Z_3\vec{F}_{\mu\nu} + g_{\mu\nu}\vec{B})\times\vec{A}^\nu + Z_2\bar{\Psi}\vec{T}\gamma_\mu\Psi + \tilde{Z}_3[\vec{C}_2 \times D_\mu\vec{C}_1 + (\partial_\mu\vec{C}_2)\times\vec{C}_1]\}. \tag{4.2}$$

Under the R transformation, the change in $\vec{J}_\mu$ (again assuming interchange of symmetry transformation and cutoff removal) is

$$\Delta\vec{J}_\mu = g\left(Z_3(\partial_\mu\vec{A}_\nu - \partial_\nu\vec{A}_\mu)\times\vec{r}^\nu + gZ_1\{(\vec{A}_\mu\times\vec{A}_\nu)\times\vec{r}^\nu + \text{perm} + (\vec{A}_\mu\times\vec{r}_\nu)\times\vec{r}^\nu + \text{perm} + (\tilde{Z}_3/Z_3)[\vec{C}_2\times(\vec{r}_\mu\times\vec{C}_1)]\} + \vec{B}\times\vec{r}_\mu\right), \tag{4.3}$$

while the change in this operator is

$$\Delta(\Delta\vec{J}_\mu) = g^2 Z_1(\vec{A}_\mu\times\vec{r}_\nu)\times\vec{r}^\nu + \text{perm}. \tag{4.4}$$

These equations have profoundly different forms depending on whether a theory is PV or FD.

For PV theories, both Z_1 and Z_3 are zero. This together with (2.38) implies that all terms in (4.3) vanish except

$$\Delta\vec{J}_\mu = g\{\vec{B}\times\vec{r}_\mu + (Z_1\tilde{Z}_3/Z_3)[\vec{C}_2\times(\vec{r}_\mu\times\vec{C}_1)]\}. \tag{4.5}$$

This may be simplified further by writing down the "short-distance" expansion for C_2C_1 (Ref. 20)

$$(Z_1\tilde{Z}_3/Z_3)(\Lambda)C_2^a C_1^b(x,\Lambda) \to Z_1(\Lambda)\!:\!A_\nu^a A^{b\nu}(x)\!:, \tag{4.6}$$

where the colons indicate free-field Wick ordering. Both (4.6) and (4.4) are zero for PV theories, and so the final results for PV theories are

$$\Delta\vec{J}_\mu = g\vec{B}\times\vec{r}_\mu \quad \text{(PV theories)}, \tag{4.7}$$

$$\Delta(\Delta\vec{J}_\mu) = 0. \tag{4.8}$$

In contrast, we find that in no FD theory, is $\Delta\vec{J}_\mu$ finite. This arises because Z_3 is infinite for these theories, implying that $\Delta\vec{J}_\mu$ picks up an infinite $Z_3(\partial_\mu\vec{A}_\nu - \partial_\nu\vec{A}_\mu)$ piece. This can never be canceled by an infinity in the C_2C_1 term. The C_2C_1 term may have a $\partial_\mu\vec{A}_\nu - \partial_\nu\vec{A}_\mu$ piece, but it is not multiplied by Z_3. For $\frac{13}{4}N > F > \frac{17}{8}N$, $\Delta(\Delta\vec{J}_\mu)$ is zero as it is in PV theories; however, for $F < \frac{17}{8}N$, not even $\Delta(\Delta\vec{J}_\mu)$ exists since $Z_1 \to \infty$.

These results mean that we cannot define Green's functions with $\Delta\vec{J}_\mu$ insertions for any FD theory. The bad behavior of the current will prevent us from applying our methods to FD theories. This striking contrast between PV and FD theories indicates that these categories may have very different types of color-confining behavior. Nonperturbatively all AF field theories do not necessarily have the same type of behavior.

B. Vertex functions

The generating functional Z of connected Green's functions of A fields with insertions of color currents $\vec{J}_\mu$ and $\vec{B}$ fields is given by[21]

$$\exp[Z(\vec{M}_A, \vec{M}_J, \vec{M}_B)] = D\int(d\sigma)\exp\left\{i\int d^4x[L(x) + \vec{M}_{A\mu}\cdot\vec{A}^\mu + \vec{M}_{J\mu}\cdot\vec{J}^\mu + \vec{M}_B\cdot\vec{B}]\right\}, \tag{4.9}$$

where $\vec{M}_J$ is the classical source of the current $\vec{J}_\mu$ and $\vec{M}_B$ is the classical source of $\vec{B}$.

R invariance implies the functional WT identity

$$\int d^4x[\vec{r}_\mu\cdot\vec{M}_A^\mu - ig\vec{M}_J^\mu\cdot(\vec{r}_\mu\times\delta/\delta\vec{M}_B)]Z = 0 \tag{4.10}$$

for color-current-inserted Green's functions in PV theories. Equation (4.10) follows from (4.9) when (4.7), (4.8) are taken into account. The integrated form is necessary because the R transformation is not local. The variation with respect to $\vec{M}_B$ enters because of the non-R-invariance of $\vec{J}_\mu$. The reason that our methods fail for FD theories is now apparent: We cannot write down R-symmetry WT identities for current-inserted Green's functions in these theories.

To obtain simple low-energy theorems from (4.10), we must define a new set of vertices in the following manner. Introduce a $3k$-component ($k = N^2 - 1$) "field" F_μ^i, where $F_\mu^i = A_\mu^i$, $i = 1,\ldots,k$; $F_\mu^i = J_\mu^{i-k}$, $i = k+1,\ldots,2k$; $F_\mu^i = g_{\mu 0}B^{i-2k}$, $i = 2k+1,\ldots,3k$. Introduce also the source M_μ^i of this field by $M_\mu^i = M_{A\mu}^i$, $i = 1,\ldots,k$; $M_\mu^i = M_{J\mu}^{i-k}$, $i = k+1,\ldots,2k$; $M_\mu^i = g_{\mu 0}M_B^{i-2k}$, $i = 2k+1,\ldots,3k$. Define the Legendre transform S_μ^i of M_μ^i as $S_\mu^i = S_{A\mu}^i$, $i = 1,\ldots,k$; $S_\mu^i = S_{J\mu}^i$, $i = k+1,\ldots,2k$; $S_\mu^i = g_{\mu 0}S_B^i$, $i = 2k+1,\ldots,3k$. S and M are related by $S_\mu^i = -i\delta Z/\delta M_\mu^i$, $M_\mu^i = i\delta\Lambda/\delta S_\mu^i$. Λ is the generating functional of a new type of vertex obtained from Z by Legendre-transforming the sources of $\vec{J}_\mu$ and $\vec{B}$ in addition to the source of $\vec{A}_\mu$. The Legendre transform is

$$\Lambda(S_\mu^i) = Z(M_\mu^i) - i\int d^4x\, S_\mu^i M^{i\mu}. \tag{4.11}$$

The Λ vertices obtained by functional differentiation with respect to S are nonlinear combinations of Green's functions which do not have simple interpretations in terms of Feynman diagrams. They satisfy equations which are simple generalizations of the relations satisfied by the 1PI vertices. The simplest of these are (symbolically)

$$\Lambda(F^iF^j) = -\Lambda(F^iF^k)\Lambda(F^jF^m)G(F^kF^m), \tag{4.12}$$

$$G(F^iF^jF^k) = G(F^iF^l)G(F^jF^m)G(F^kF^n)\Lambda(F^lF^mF^n), \tag{4.13}$$

$$G(F^iF^jF^kF^l) = G(F^iF^n)G(F^jF^m)G(F^kF^r)G(F^lF^s)[\Lambda(F^nF^mF^rF^s) + \Lambda(F^nF^mF^t)G(F^tF^u)\Lambda(F^uF^rF^s)]. \tag{4.14}$$

Once the $G(F^iF^j)$ matrix is inverted so that

$$\Lambda(F^iF^j) = -G(F^iF^j)^{-1}, \tag{4.15}$$

the Λ vertices may be solved for recursively in complete analogy to the 1PI vertices.

Let us write down explicitly the two-point Λ vertices. In momentum space, the relevant Green's functions are, in the Landau gauge,[9]

$$G(A^a_\mu B^b)(q) = (q_\mu/q^2)\delta^{ab}, \tag{4.16}$$

$$G(B^aB^b)(q) = 0, \tag{4.17}$$

$$G(J^a_\mu B^b)(q) = 0, \tag{4.18}$$

$$G(A^a_\mu A^b_\nu)(q) = (g_{\mu\nu} - q_\mu q_\nu/q^2)C^{ab}_{AA}(q^2), \tag{4.19a}$$

$$G(A^a_\mu J^b_\nu)(q) = (g_{\mu\nu} - q_\mu q_\nu/q^2)C^{ab}_{AJ}(q^2), \tag{4.19b}$$

$$G(J^a_\mu J^b_\nu)(q) = (g_{\mu\nu} - q_\mu q_\nu/q^2)C^{ab}_{JJ}(q^2) + (q_\mu q_\nu/q^2)D^{ab}(q^2). \tag{4.19c}$$

Upon inverting the matrix defined by these expressions, we obtain the Λ vertices

$$\Lambda(B^aB^b) = 0, \tag{4.20}$$

$$\Lambda(A^a_\mu B^b) = -q_\mu\delta^{ab}, \tag{4.21}$$

$$\Lambda(J^a_\mu B^b) = 0, \tag{4.22}$$

$$\Lambda(A^a_\mu A^b_\nu) = -(g_{\mu\nu} - q_\mu q_\nu/q^2)(C^{-1})^{ab}_{AA}, \tag{4.23a}$$

$$\Lambda(A^a_\mu J^b_\nu) = -(g_{\mu\nu} - q_\mu q_\nu/q^2)(C^{-1})^{ab}_{AJ}, \tag{4.23b}$$

$$\Lambda(J^a_\mu J^b_\nu) = -(g_{\mu\nu} - q_\mu q_\nu/q^2)(C^{-1})^{ab}_{JJ} - (q_\mu q_\nu/q^2)(D^{-1})^{ab}. \tag{4.23c}$$

Once (4.20)–(4.23) have been written down, all one has to do is calculate Green's functions and use (4.12)–(4.14) and their generalizations to higher-point vertices to calculate Λ vertices. In particular, perturbation-theory expansions for the Λ vertices can be directly obtained from the perturbation-theory expansions of the Green's functions.

C. Infrared behavior

Replacing the sources in (4.9) by their Legendre transforms gives the functional R-symmetry WT identity for Λ vertices

$$\int d^4x\,[\vec{r}^{\,\mu}\cdot(\delta\Lambda/\delta\vec{S}_{A\mu})Z - ig(\delta\Lambda/\delta\vec{S}_{J\mu})\cdot(\vec{r}^{\,\mu}\times\vec{S}_B)] = 0. \tag{4.24}$$

For Λ vertices with no $\vec{B}$ insertions, (4.24) is effectively

$$\int d^4x\,\vec{r}^{\,\mu}\cdot\delta\Lambda/\delta\vec{S}_{A\mu} = 0, \tag{4.25}$$

which is formally the same as the 1PI identity. Taking functional derivatives with respect to $\vec{S}_A$ and $\vec{S}_J$ and Fourier transformation gives the low-energy theorems

$$\Lambda^{(n,m)}(p_1,\ldots,p_n;q_1,\ldots,q_m;g,\mu) = 0 \quad \text{for any } p_i = 0. \tag{4.26}$$

$\Lambda^{(n,m)}$ is a Λ vertex for n gauge fields and m current insertions. Equation (4.26) is true for arbitrary m, provided $n \geq 1$.

The renormalization group equation for current-inserted Λ vertices follows immediately once one knows how to renormalize current-inserted Green's functions. The canonical current algebra

$$[j^a_0(x), a^b_\mu(y)]\delta(x_0 - y_0) = g_0 f^{abc}a^c_\mu(x)\delta^4(x-y), \tag{4.27}$$

$$[j^a_0(x), j^b_\mu(y)]\delta(x_0 - y_0) = g_0 f^{abc}j^c_\mu(x)\delta^4(x-y) \tag{4.28}$$

(Schwinger terms are removed by introducing a covariant T^* product for Green's functions) implies the WT identities

$$\partial^\mu_x G_0^{(n,m)}[j^a_\mu(x)j^{a_2}_{\mu_2}(x_1)\cdots j^{a_m}_{\mu_m}(x_m)a^{a_{m+1}}_{\nu_1}(x_{m+1})\cdots a^{a_{n+m}}_{\nu_n}(x_{n+m})] = \sum_{\text{perm}}\delta^4(x-x_j)g_0 f^{aa_jc}\hat{G}_0^{(n,m-1)}[j^{a_2}_{\mu_2}(x_1)\cdots j^{a_m}_{\mu_m}(x_m)a^{a_{m+1}}_{\nu_1}(x_{m+1})\cdots a^{a_{n+m}}_{\nu_n}(x_{n+m})], \tag{4.29}$$

where the caret indicates that the field (current) with argument x_j is replaced by $a^c_j(x)$ $[j^c_j(x)]$. In the above formulas, lower-case letters indicate unrenormalized fields and currents, while a zero subscript indicates an unrenormalized parameter. The recursion relations (4.29) in addition to our knowledge of the renormalizations of g_0 and Green's functions for gauge fields tell us how to define the renormalized Green's functions with current insertions:

$$G^{(n,m)} = \lim_{\Lambda\to\infty} Z_J^{-m/2}(\Lambda)Z_3^{-n/2}(\Lambda)G_0^{(n,m)}(\Lambda), \tag{4.30}$$

where Z_J renormalizes g_0.

From (4.30) follows a functional RG equation for Z:

$$\left[\mu\frac{\partial}{\partial\mu} + \beta\frac{\partial}{\partial g} + \gamma(g)\int d^4x\,\vec{M}^\mu_A\cdot\frac{\delta}{\delta\vec{M}_{A\mu}} + \gamma_J(g)\int d^4x\,\vec{M}^\mu_J\cdot\frac{\delta}{\delta\vec{M}_{J\mu}}\right]Z(\vec{M}_A,\vec{M}_J;g,\mu) = 0. \tag{4.31}$$

Taking the Legendre transform and rewriting $\vec{M}_A, \vec{M}_J$ in terms of the new independent sources $\vec{S}_A, \vec{S}_J$, we obtain a functional RG equation for Λ. Taking note of the extra parametric differentiations which arise because of the shift in independent variables, this reads

$$\left[\mu\frac{\partial}{\partial\mu}+\beta\frac{\partial}{\partial g}-\gamma(g)\int d^4x\,\vec{S}_{A\mu}\cdot\frac{\delta}{\delta\vec{S}_A^{\mu}}-\gamma_J(g)\int d^4x\,\vec{S}_J^{\mu}\cdot\frac{\delta}{\delta\vec{S}_{J\mu}}\right]\Lambda(\vec{S}_A,\vec{S}_J;g,\mu)=0\ . \tag{4.32}$$

Functional differentiation and the scaling equation

$$\left(\lambda\frac{\partial}{\partial\lambda}+\mu\frac{\partial}{\partial\mu}\right)\Lambda^{(n,m)}(\lambda P,\lambda Q;g,\mu)=(-4+n+3m)\Lambda^{(n,m)}(\lambda P,\lambda Q;g,\mu) \tag{4.33}$$

gives the RG equation satisfied by the Λ vertices:

$$\left(\lambda\frac{\partial}{\partial\lambda}-\beta\frac{\partial}{\partial g}+4-nd-md_J\right)\Lambda^{(n,m)}(\lambda P,\lambda Q;g,\mu)=0\ . \tag{4.34}$$

(The usual 1PI vertices with current insertions have a plus sign in front of d_J.)

Assuming again the existence of a second zero of $\beta(g)$ at $g=g_\infty$, as $\lambda\to 0$ we approach a scale-invariant theory in which the currents must have canonical dimension $d_J(g_\infty)=3$. We then obtain the scaling behavior

$$\Lambda^{(n,m)}(\lambda P,\lambda Q;g,\mu)\underset{\lambda\to 0}{\sim}\lambda^{4-nd(g_\infty)-3m}[\Lambda^{(n,m)}(P,Q;g_\infty,\mu)g^{(n,m)}(\lambda)+\text{terms less singular in }\lambda], \tag{4.35}$$

with $g^{(n,m)}(\lambda)$ the analog of $f^{(n)}(\lambda)$ in (3.13).

When the leading behavior in (4.35) is nonzero, we automatically obtain a contradiction with (4.26). For fixed n, we may always select an m large enough, $m\geq\frac{1}{3}[4-nd(g_\infty)]$, so that the leading λ power in (4.35) is negative. The only hope for (4.35) to be consistent with (4.26) is

$$\Lambda^{(n,m)}(P,Q;g_\infty,\mu)=0\ , \tag{4.36}$$

for all $P=(p_1,\ldots,p_n)$, $Q=(q_1,\ldots,q_m)$ for $m\geq\frac{1}{3}[4-nd(g_\infty)]$. This behavior may be excluded by invoking another mild smoothness assumption. One might, for example, use the unitarity equations[22] for a conformal-invariant field theory, or use functional methods.[23]

Accepting our smoothness assumptions, we have shown that the behavior (2.27) cannot occur in models satisfying (1.3). This leaves the desired behavior (2.28) as the only possibility in these models.

V. EXTENSION TO MASSIVE QUARKS

Our previous discussion in the framework of a quark-gluon model with non-Abelian gauge coupling has implicitly assumed that particle masses are zero. We take the underlying color gauge symmetry as unbroken, and so the gauge gluons always stay massless. In the popular schemes of chiral-symmetry breaking, the quarks have mass, and these mass terms are postulated in such a way as to account for the broken chiral symmetry as observed. We shall now review our previous argument, noting modifications where necessary to accommodate the more realistic models with quark-mass terms.

Our argument relies on the confrontation of the information from the low-energy theorem and that from the consequences of an assumed IR fixed point for the RG equation. The low-energy theorems (3.11) and (4.26) are derived using purely the short-distance or UV behavior of Wilson coefficients. The UV behavior is always independent of the presence of masses. In the language of the RG, the mass-insertion term in the inhomogeneous RG equation can always be neglected in the UV region.[12] This can be equivalently stated as the UV vanishing of the Weinberg RG-equation effective mass.[24] The conditions for the validity of the low-energy theorem therefore remain unchanged whether the quarks have mass or not.

At this point we must discuss the predictions of the decoupling theorem[25] (DT). In any IR limit, the DT says that the behavior of a theory with massive particles is the same (up to finite renormalizations) as any other theory with any *additional* number of massive particles of the same type. Thus a pure Yang-Mills theory is predicted to have the same IR behavior as a Yang-Mills theory containing an arbitrary number of massive quarks. The DT is a statement true in finite orders of perturbation theory, and its validity in the exact theory has never been established. Since low-energy theorems are derived for models with sufficiently many fermions, irrespective of their masses, the theorems are still valid if all quarks acquire masses. If we now apply the DT, we get the result that the theory without any massive quarks must obey the same low-energy theorems. Thus even though these low-energy theorems as originally derived need the presence of fermions, by application of the DT to the exact theory, we get the low-energy theorems for any theory with fewer massive fermions, including pure Yang-Mills theories.

Next we consider the part of the argument concerning the consequences of an IR fixed point. The behaviors (3.13) and (4.35) of the appropriate vertex functions are predicted by relying on the neglibibility of all masses in the IR limit. This negligibility is justifiable in either of two circumstances.

First, we consider again the inhomogeneous RG equation

$$\left(\lambda \frac{\partial}{\partial\lambda} - \beta\frac{\partial}{\partial g} - nd + 4\right)\Gamma^{(n)}(\lambda P, g, m; \mu) = \Delta\Gamma^{(n)} . \tag{5.1}$$

The mass-inserted right-hand side can be neglected if its power behavior is less singular than that of $\Gamma^{(n)}$ in the IR limit. If mass insertions can indeed be neglected, we have the IR behavior

$$\Gamma^{(n)}(\lambda) \underset{\lambda\to 0}{\sim} \lambda^{4-nd} \tag{5.2}$$

and

$$\Delta\Gamma^{(n)}(\lambda) \underset{\lambda\to 0}{\sim} \lambda^{d_\theta - nd} , \tag{5.3}$$

where d_θ is the IR dimension of the mass operator $\theta = \bar{\Psi} m \Psi$. This gives the condition

$$d_\theta > 4 , \tag{5.4}$$

which appears to be rather unlikely. Equivalently, one may demand the vanishing of the Weinberg effective mass, and this again gives the condition (5.4). If the mass-inserted Green's function is not negligible compared to $\Gamma^{(n)}$ in the IR limit, the true IR behavior cannot be determined from (5.1) even assuming an IR fixed point. The behavior can be more complex than a power, and the concept of IR anomalous dimension may not make any sense at all.

The second, actually simpler, circumstance which justifies neglecting masses is the validity of the DT for the exact theory. Neither of (3.13) and (4.35) makes reference to the number of fermions present, and so application of the DT results in these behaviors being valid with no modification necessary even in the presence of massive quarks.

To summarize, the extension to massive quarks leans heavily on the validity of the DT in the exact theory. Making that assumption, we can extend the validity of our low-energy theorems to *any* AFNAGT of Yang-Mills gluons and quarks of arbitrary nonzero mass. The DT says that the consequences of the IR fixed point are not changed by adding masses. Hence, we obtain the previous contradiction in every case and predict infinite IR effective coupling for all AFNAGT's.

This result remains correct even if there are some massless quarks present. This can be seen in the following manner. Suppose we are interested in a theory containing $K < \frac{13}{4} N$ massless quarks. Construct a PV theory containing $F = M + K$ flavors of quark, where M is a number of massive quarks such that F satisfies (1.3). This theory has the IR behavior (3.11) and (4.26). An application of the DT now implies that this IR behavior is identical to that in a theory with the M massive fermions deleted; i.e., the low-energy theorems must also be true for FD theories of massless quarks.

The DT is thus seen to lead to a considerable extension of our results. However, it is conceivable that the DT is not applicable for our purposes. This is true not because we believe that massive fields may generate long-range forces in AFNAGT's (the IR behavior should be determined solely by the behavior of the massless quanta), but because it is not obvious that the finite renormalizations induced by the presence of massive particles can be neglected in the exact theory where the existence of R symmetry is a consequence of renormalizations. The finite renormalizations could conceivably add up to zero or infinity in the exact theory and ruin our analysis. We must assume that they do not.

VI. DISCUSSION

Our arguments in several places depend on the assumption that the R transformation commutes with the gauge-invariant cutoff-removal limit. In symbols, we had to assume that if

$$F(A) = \lim_{\Lambda\to\infty} F_\Lambda(A) , \tag{6.1}$$

where $F(A)$ is a renormalized operator and $F_\Lambda(A)$ is the corresponding regulated operator with Λ the cutoff, then

$$F(A+r) = \lim_{\Lambda\to\infty} F_\Lambda(A+r) . \tag{6.2}$$

Because of the implicit nonoperator formulation of these regularizations, it is difficult to access the validity of such interchanges. However, it is possible to avoid having to make this assumption by deriving a (non-gauge-invariant) point-separated regularized expression for $F(A)$. With this regularization, F becomes an explicit operator function of A so that (6.2) manifestly follows from (6.1). The effect of the R transformation could then be rigorously established, just as was previously done for quantum electrodynamics.

Another nonrigorous aspect of our work has been our use of functional methods to deduce the consequences of R invariance. It would again be straightforward but tedious to check these results using rigorous structural methods. This has already been done for the gauge-field vertex functions, but remains to be done for current-field vertex functions.

A final possibly troublesome point is our neglect of terms of order $(\ln\Lambda)^{-1}$ in deducing the symmetry relations

$$F(A)=F(A+r)\,, \qquad (6.3)$$

which really read something like

$$F(A)=F(A+r)+\lim_{\Lambda\to\infty}(\ln\Lambda)^{-1}G(A)\,. \qquad (6.4)$$

It is conceivable that, even though (6.3) is formally true, the slowly vanishing term in (6.4) cannot be neglected in the field equations.

At this point, we re-emphasize the nonperturbative nature of the R-symmetry relations (6.3). The symmetry is not present classically or in any finite order of perturbation theory. It and the consequent WT identities [(3.11), (4.26)] are properties of only the exact theory. Likewise, the consequences [(3.13), (4.35)] of the RG equation with an IR fixed point are valid only for the exact theory. Our conclusion is thus also valid only for the exact theory. This result is perfectly consistent with the absence of IR singularities[26] in inclusive cross sections in finite orders of perturbation theory since results obtained in finite orders of perturbation theory are irrelevant to the confinement issue.[27]

Even if the IR strong-coupling limit were rigorously established by our methods, the conclusion that color would then be confined would remain to be definitely established. However, confinement is physically reasonable and is suggested by the works[7,8] mentioned in Sec. I.

We have stressed that (2.28) is derived only in the Landau gauge. If this does indeed imply confinement, then the obvious gauge invariance of the confinement concept implies confinement in all gauges. This need not mean that $\bar{g}(0)=\infty$ in all gauges. The strong-coupling limit is hopefully a sufficient condition for confinement, but not necessarily a necessary condition.

In spite of the greater complexity of the RG equations in non-Landau gauges, our methods still lead to some interesting conclusions. Consider an IR-stable fixed point at $g=g_\infty$, $\alpha=\alpha_\infty$. The conditions for the existence of such a fixed point are

$$\beta(g_\infty,\alpha_\infty)=0\,, \qquad (6.5)$$

$$\delta(g_\infty,\alpha_\infty)=-\alpha_\infty\gamma(g_\infty,\alpha_\infty)=0\,. \qquad (6.6)$$

The RG again gives the IR behavior (3.11) with

$$d(g_\infty,\alpha_\infty)=1+\gamma(g_\infty,\alpha_\infty)\,, \qquad (6.7)$$

and the R symmetry (present only if $F>N$) again gives (3.13) so that (6.7) must be negative. Comparison of (6.6) and (6.7) requires that $\alpha_\infty=0$, and we conclude that an IR fixed point can only occur at the Landau gauge. Proceeding as in Sec. IV again disproves the existence of such a fixed point. However, we cannot conclude that $\bar{g}(0)=\infty$, because of the other possible types of behavior such as limit cycles.

Returning to the Landau gauge, we may conclude that all AFNAGT's confine color, if all of our assumptions are valid. The importance of this conclusion would seem to justify further investigations of these assumptions. Work in this direction is in progress.

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