Gauge vacuums and the conformal group

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Recent discussions on the degeneracy of classical gauge vacuums assume that spacelike hypersurfaces in an $(n+1)$-dimensional space-time have the topology of n-dimensional spheres. We point out a connection between this assumption and the requirement that the conformal group acts properly as a group of transformations on space-time. The latter is possible only if the Minkowski space $\tilde{M}^{n+1}$ is enlarged to a compact manifold M^{n+1} by the addition of suitable points at infinity. Spacelike hypersurfaces in M^{n+1} do indeed have the topology of S^n.

I. INTRODUCTION

In a recent work, Belavin *et al.*[1] constructed a solution of certain Euclidean gauge theories which was localized in both space and "time" (the "pseudoparticle"[2]). Subsequently, it was pointed out[3,4] that classically the vacuums of such theories in Minkowski space are degenerate, and that the pseudoparticle describes tunneling between these vacuums when these theories are quantized.[2-4] The nature of the true vacuum in quantum theory is thereby profoundly affected. The properties of this new vacuum in turn lead to important physical effects.

In the work hitherto, the demonstration that classical gauge vacuums are degenerate has required the critical assumption that spacelike hypersurfaces in Minkowski space are to be compactified in a suitable fashion. [This assumption comes about because of the imposition of boundary conditions at infinity on the gauge functions. Cf. Ref. 4, Eq. (5) and what follows.] Thus, in $1+1$ space-time the spacelike hypersurface R^1 is to be compactified to the circle S^1, while in $3+1$ space-time R^3 is to be compactified to the three-dimensional sphere S^3. (In both these examples, the compactification can be achieved by the addition of a single point at spatial infinity, say by stereographic projection.) It is easy to show that the preceding degeneracy of the vacuums is *not* present if spacelike hypersurfaces are (noncompact) Euclidean spaces. Thus the conclusions regarding the structure of the vacuums seem to rely heavily on an assumed topology of spacelike hypersurfaces. On the other hand, we are aware of no detailed argument in the literature on gauge vacuums which justifies such assumptions on the global topology of space-time.

In this paper, we suggest that the conformal group may provide a clue as to the appropriate topology of space-time. The free Yang-Mills Lagrangian is known to be formally invariant under conformal transformations. It is also well-known, however, that the conformal group does not act properly as a transformation group on the usual (noncompact) Minkowski space $\tilde{M}^{n+1}$. This is possible only if $\tilde{M}^{n+1}$ is conformally compactified to a manifold M^{n+1}.[5] Thus if we are to maintain the conformal invariance of Yang-Mills theories in a literal sense, they have to be formulated on M^{n+1} rather than on $\tilde{M}^{n+1}$. The importance of working with M^{n+1} when investigating the conformal properties of zero-mass fields has been most emphasided by Penrose.[5] We point out in this paper that the topology of M^{n+1} is such that spacelike hypersurfaces in M^{n+1} have the structure demanded by discussions on gauge vacuums.

The manifold M^{n+1} has the topology of $S^n \times S^1$.[5] This means that if the space- *and* time-dependent gauge group is G, the elements $\{g\}$ in G can in general be classified by *two* topological numbers ρ and σ. Here, if the points of $S^n \times S^1$ are denoted by (P,Q), where $P \in S^n$ and $Q \in S^1$, then ρ characterizes the degree of mapping (the topological number) of g when regarded as a function of P with fixed Q. Similarly σ characterizes its degree of mapping when regarded as a function of Q with fixed P. The classification of classical gauge vacuums in the literature utilizes only the number ρ. Thus it may seem that conformal compactification gives rise to an additional topological number σ with possible physical significance. We show, however, that the Hamiltonian formulation is consistent with the requirement that every physically meaningful variable is invariant under gauge transformations g with $\rho = 0$ and any value of σ. Hence a vacuum with $\sigma \neq 0$ is "gauge equivalent" to the vacuum with $\sigma = 0$. Therefore it seems difficult to give a physical meaning to σ without additional *ad hoc* assumptions about the nature of the Hamiltonian. The net result is that we are left with a classification of classical gauge vacuums only by the number ρ. This is of course what has been suggested in the literature.

The emphasis on the Hamiltonian approach in our discussion is due to the fact that we are not aware of any other systematic method for quantization of classical theories. It may be that the lack of an invariant meaning for σ is peculiar to this approach, and that there are alternative approaches to quantum theory where σ acquires significance.

There is no compelling physical reason why zero-mass field theories should be formulated in such a way that they are literally invariant under the conformal group. Thus our discussion only points out a connection between the structure of gauge vacuums and the topology of M^{n+1} and does not really justify the assumption in the literature on the topology of spacelike hypersurfaces. It is interesting, however, that the global topology of space-time seems to play such an important role in determining the properties of gauge theories. Perhaps the correct approach for the determination of this topology requires considerations based on general relativity, and the latter influences particle physics by affecting the topology of the gauge group and hence the nature of the true vacuum. Such an influence can be quite substantial although gravity is a weak force, as is evident from the dramatic impact on gauge theories that results from the compactification of spacelike surfaces.[2-4]

In Sec. II, we briefly review the method for the conformal compactification of the Minkowski space $\tilde{M}^{n+1}$. We also sketch the proofs of the results that the compactified manifold M^{n+1} is $S^n \times S^1$ and that the spacelike surfaces in M^{n+1} are S^n. In Sec. III, the gauge properties of physically meaningful variables are analyzed in the Hamiltonian framework. It is shown that only the topological number ρ can be given a well-defined physical meaning within this framework.

II. THE CONFORMAL COMPACTIFICATION OF MINKOWSKI SPACE

The conformal group $C(n,1)$ of the Minkowski space $\tilde{M}^{n+1}$ contains elements which map points of $\tilde{M}^{n+1}$ to infinity. A simple example is inversion I which acts as follows:

$$\tilde{M}^{n+1} \ni x \underset{I}{\rightarrow} Ix = \frac{x}{x^2} . \tag{2.1}$$

Thus I sends the entire light cone about the origin to infinity. It follows that $C(n,1)$ does not act properly as a transformation group on $\tilde{M}^{n+1}$, and that it is necessary to add points at infinity to $\tilde{M}^{n+1}$ if we want a well-defined action of $C(n,1)$. This enlarged manifold will be denoted by M^{n+1}.

The structure of M^{n+1} is most easily inferred by observing that $C(n,1)$ is locally isomorphic to the pseudo-orthogonal group $SO(n+1,2)$. The latter acts linearly on the $(n+3)$-dimensional real vector space V^{n+3} with the diagonal metric $(+--\cdots-+)$. Further, it maps null rays through the origin of V^{n+3} into other null rays through the origin. The manifold M^{n+1} can be identified with the (projective) space of these null rays.[5] The details of this identification are given below. The induced action of $SO(n+1,2)$ on $\tilde{M}^{n+1} \subset M^{n+1}$ is locally identical to the action of $C(n,1)$. The space of the null rays, however, is larger than $\tilde{M}^{n+1}$, and these additional points are what is needed to compactify $\tilde{M}^{n+1}$ to M^{n+1}.

We shall now make the preceding ideas precise. Let $\xi = (\xi_0, \xi_1, \ldots, \xi_{n+2})$ denote a vector in V^{n+3}. The null cone N through the origin is specified by

$$N = \{\xi \mid \xi^2 = \xi_0^2 - \xi_1^2 - \cdots - \xi_{n+1}^2 + \xi_{n+2}^2 = 0\} . \tag{2.2}$$

We exclude the origin $\xi = 0$ itself from N. A null ray $[\xi]$ through ξ in N is, by definition, the set

$$\{\lambda\xi \mid -\infty < \lambda < \infty, \lambda \neq 0\} . \tag{2.3}$$

Thus the space $\{[\xi]\}$ is nothing other than N when regarded as a projective space. Clearly

$$[\xi] = [\lambda\xi] \tag{2.4}$$

for any real nonzero λ.

The null ray $[\xi]$ can be labeled by a suitable point $\eta = \lambda\xi$ on this ray. When $\gamma = \xi_{n+1} + \xi_{n+2}$ is not zero, one such choice of η can be obtained by noticing that owing to (2.2),

$$\xi = \left(\xi_0, \xi_1, \ldots, \frac{\gamma}{2}\left(1 + \frac{\xi^\alpha \xi_\alpha}{\gamma^2}\right), \frac{\gamma}{2}\left(1 - \frac{\xi^\alpha \xi_\alpha}{\gamma^2}\right)\right) . \tag{2.5}$$

Here the index α runs from 0 to n and $\xi^\alpha \xi_\alpha$ is the Minkowskian scalar product $\xi_0^2 - \xi_1^2 - \cdots - \xi_n^2$. With the choice $\lambda = 1/\gamma$,

$$[\xi] = \left[\left(\xi_0/\gamma, \xi_1/\gamma, \ldots, \xi_n/\gamma, \frac{1}{2}\left(1 + \frac{\xi^\alpha \xi_\alpha}{\gamma^2}\right), \frac{1}{2}\left(1 - \frac{\xi^\alpha \xi_\alpha}{\gamma^2}\right)\right)\right] . \tag{2.6}$$

for $\gamma \neq 0$, $[\xi]$ is thus uniquely labeled by $(1/\gamma)(\xi_0, \xi_1, \ldots, \xi_n)$. The correspondence between a point $x \in \tilde{M}^{n+1}$ and ξ is given by

$$x_\alpha = \xi_\alpha/\gamma, \quad \gamma \neq 0 . \tag{2.7}$$

The map (2.7) from the rays $[\xi]$ with $\gamma \neq 0$ to the points $x \in \tilde{M}^{n+1}$ is clearly one to one and onto. We see from (2.7) that the rays $[\xi]$ with $\gamma = 0$ can be thought of as additional points at infinity of the Minkowski space. The inclusion of these points changes the latter to M^{n+1}.

The topology of M^{n+1} can be determined by introducing new representative points for the null rays as follows: From (2.2),

$$\begin{aligned}\xi_0{}^2 + \xi_{n+2}{}^2 &= \xi_1{}^2 + \xi_2{}^2 + \cdots + \xi_{n+1}{}^2 \\ &= r^2 \end{aligned} \tag{2.8}$$

say. Here r cannot be zero since we have excluded the point $\xi = 0$ from N. Thus

$$[\xi] = [(\hat{\xi}_0, \hat{\xi}_1, \ldots, \hat{\xi}_{n+1}, \hat{\xi}_{n+2})] , \tag{2.9}$$

where

$$\hat{\xi}_1{}^2 + \hat{\xi}_2{}^2 + \cdots + \hat{\xi}_{n+1}{}^2 = \hat{\xi}_0{}^2 + \hat{\xi}_{n+2}{}^2 = 1 . \tag{2.10}$$

The $\hat{\xi}$'s thus constitute an $S^n \times S^1$. We cannot yet conclude that M^{n+1} is also $S^n \times S^1$ since both $\hat{\xi}$ and $-\hat{\xi}$ give the same null ray:

$$[\hat{\xi}] = [-\hat{\xi}] . \tag{2.11}$$

The topology of the manifold (2.10) with the identification of $\hat{\xi}$ and $-\hat{\xi}$ is still $S^n \times S^1$ (see Ref. 5) so that

$$M^{n+1} = S^n \times S^1 . \tag{2.12}$$

The parametrization of x in terms of $\hat{\xi}$ can be obtained by writing an arbitrary vector ξ in two ways:

$$\xi = \gamma(x, \tfrac{1}{2}(1+x^2), \tfrac{1}{2}(1-x^2)) \tag{2.13}$$

$$= r\hat{\xi} . \tag{2.14}$$

Hence

$$\begin{aligned} x &= \frac{r}{\gamma}(\hat{\xi}_0, \hat{\xi}_1, \ldots, \hat{\xi}_n) \\ &= \frac{1}{\hat{\xi}_{n+1} + \hat{\xi}_{n+2}} (\hat{\xi}_0, \hat{\xi}_1, \ldots, \hat{\xi}_n) . \end{aligned} \tag{2.15}$$

The topology of spacelike surfaces in M^{n+1} follows from (2.15). Consider, for instance, the surface $x_0 = 0$. By (2.15) and (2.10), $x_0 = 0$ implies $\hat{\xi}_0 = 0$, $\hat{\xi}_{n+2} = \pm 1$. It is sufficient to consider the solution $\hat{\xi}_{n+2} = +1$ in view of the equivalence

$$[(0, \hat{\xi}_1, \ldots, \hat{\xi}_{n+1}, -1)] = [(0, -\hat{\xi}_1, \ldots, -\hat{\xi}_{n+1}, +1)] .$$

[Note that the right-hand side of (2.15) is invariant under $\hat{\xi} \to -\hat{\xi}$ as it should be.] Thus,

$$(0, x_1, \ldots, x_n) = \frac{1}{\hat{\xi}_{n+1} + 1} (0, \hat{\xi}_1, \ldots, \hat{\xi}_n) . \tag{2.16}$$

Such a surface is therefore parametrized by $(\hat{\xi}_1, \hat{\xi}_2, \ldots, \hat{\xi}_{n+1})$, which spans an S^n. It follows that the time-zero surface in M^{n+1} is an S^n. It is not difficult to show, by using a diagrammatic representation of M^{n+1} for example,[5] that every spacelike surface in M^{n+1} is also an S^n.

The compactification of spacelike surfaces of Minkowski space to S^n is achieved here by the addition of a single point at spatial infinity as in the literature on gauge vacuums. This can be seen from (2.16), which shows that spatial infinity corresponds to $\hat{\xi}_{n+1} = -1$. By (2.10), the latter implies that $\hat{\xi}_1 = \hat{\xi}_2 = \cdots = \hat{\xi}_n = 0$. Thus spatial infinity is represented by one point of S^n.

The circle S^1 in the $S^n \times S^1$ decomposition of M^{n+1} describes null rays in Minkowski space. The points at infinity of each null ray in the past and future are to be identified. We refer to the next section and to the literature[5] for a discussion of this point.

III. HAMILTONIAN DYNAMICS AND GAUGE VACUUMS

In this section, we first make a few remarks concerning the arbitrariness in the time evolution of canonical variables in gauge theories. We then discuss what we mean by physically meaningful variables (or physical variables for short) in any theory. Using this discussion, we finally show that the topological number ρ introduced in Sec. I can be given a physical meaning while the number σ does not have such a meaning.

In a gauge theory, not every variable has a well-defined time evolution. Suppose that $S(x_0)$ denotes a possible set of values of the variables of the theory at time x_0 when the boundary conditions $S(0)$ are specified at time zero. Then we have the freedom to perform a gauge transformation at time x_0 with a gauge function $g(x_0, \vec{x})$ to get another possible set of values $S^g(x_0)$ for the variables. Clearly the variables, which are not invariant under g, will fail to have a well-defined time evolution. Note, however, that such gauge transformations g are not entirely arbitrary, since the transformed variables $S^g(x_0)$ must reduce to the given boundary conditions $S(0)$ when $x_0 \to 0$. To analyze the consequences of this restriction, consider the transformation laws of the Yang-Mills vector fields A_μ^α $(\mu = 0, 1, 2, 3)$ and the corresponding conjugate momenta π_i^α $(i = 1, 2, 3)$. (The canonical variables cannot be chosen arbitrarily on the physical submanifold of the full phase space. In particular, $\pi_0^\alpha = 0$ on this submanifold.[6,7] This is the reason why we ignore it in the discussion.) When the global group is semisimple, these laws are given by[8]

$$\mathfrak{A}_\mu \equiv A^\alpha_\mu T(\alpha) \underset{g}{\to} g\mathfrak{A}_\mu g^{-1} + \frac{i}{e} g\partial_\mu g^{-1} , \tag{3.1}$$

$$P_i \equiv \pi^\alpha_i T(\alpha) \to gP_i g^{-1} . \tag{3.2}$$

Here $T(\alpha)$ are generators of the Lie algebra of the global symmetry group. Since both sides of these equations must coincide at time 0, we find the following restrictions on g in the semisimple case:

$$g(0, \vec{x}) = 1 , \tag{3.3}$$

$$\partial_0 g(0, \vec{x}) = 0 . \tag{3.4}$$

For an Abelian group, the canonical momenta are gauge-invariant since the adjoint representation is trivial. Further, the vector field A_μ for the group U(1) for example transforms as $A_\mu \to A_\mu + (i/e)g\partial_\mu g^{-1}$. Thus, in the Abelian case, the transformation laws of the gauge fields require only the weaker condition $\partial_i g(0, \vec{x}) = 0$ instead of (3.3). However, it is evident that one recovers (3.3) in this case as well by enlarging the system to include an appropriate "charged" field which transforms according to a nontrivial representation [for instance, a field φ which transforms as $\varphi(x) \to e^{i\alpha(x)}\varphi(x)$ under $g(x) = e^{i\alpha(x)}$]. We shall therefore assume the validity of (3.3) and (3.4) for all groups.

Equations (3.3) and (3.4) specify a certain normal subgroup G_0 of the full (space and time dependent) gauge group G with which we can transform the variables of the theory at any time t in a fashion consistent with time evolution. Let us next define the physical variables of any theory as those which have a well-defined time evolution; their values at any time t are thus uniquely determined by appropriate boundary conditions at time 0. (We find such a definition reasonable.) It follows that the physical variables of a gauge theory are invariant under the action of G_0. The group G_0 is not the full group G. In particular, by (3.3) and (3.4), $G_0 \cap \mathcal{G} = \{e\}$, where $\mathcal{G}$ is the usual space- and time-independent global internal-symmetry group. Thus $G/G_0 \supset \mathcal{G}$. Elements of G/G_0 can map solutions involving physical variables alone into inequivalent solutions. Hence the theory will retain a nontrivial symmetry group G/G_0 even after we eliminate the unphysical variables or gauge degrees of freedom. In cases where G/G_0 is larger than $\mathcal{G}$, the nontrivial symmetry group of the theory is larger than the global group $\mathcal{G}$. The degeneracy of the classical gauge vacuums found in the literature is due to this circumstance and the fact that G/G_0 has a nontrivial action on the vacuum sector.

Physical variables as we have defined them are what are called first-class variables in the theory of Hamiltonian systems with constraints.[6,7] In the latter, the Hamiltonian contains Lagrange multiplier terms involving first-class constraints.[9] The arbitrariness in the choice of the Lagrange multipliers means that the physical variables must have (weakly) zero Poisson (or rather Dirac) brackets with these constraints in order to have a well-defined time evolution. Since first-class variables are also defined by the same requirements, we see that physical and first-class variables are the same.

The connection between this property of physical variables and their invariance under G_0 comes about from the fact that the Lagrange multiplier terms are the generators of an infinitesimal gauge transformation.[10] Therefore when we solve Hamilton's equations, the effect of these terms is to perform a gauge transformation with an element $g \in G$ at each time x_0. Further, when $x_0 \to 0$, the variables at x_0 will necessarily reduce to the boundary conditions $S(0)$. Thus $g \in G_0$. It is also easy to show that we can perform such a gauge transformation with any element $g \in G_0$ by a suitable choice of Lagrange multipliers.[10] It follows that a variable will have a well-defined time evolution if it is a first-class variable, or equivalently if it is invariant under G_0 (on the constrained hypersurface in phase space).

Since the definition of G_0 depends on the time 0 through (3.3) and (3.4), it may seem that the definition of physical variables depends on this time. However, this is not the case since the set of all constraints is invariant under time evolution. (Dirac in fact constructs the full set of constraints by demanding that the time derivative of a constraint also be a constraint.[6,7]

We shall now characterize the elements of G_0 in terms of the topological numbers ρ and σ which were defined in the introduction. The number ρ is the degree of mapping of $g(x_0, \vec{x})$ when we restrict it to a spacelike surface, for instance when we regard it as a function of $\vec{x}$ for fixed x_0. Condition (3.3) implies that the elements of G_0 should have $\rho = 0$, since they can be continuously deformed to the identity mapping by changing the parameter x_0 to 0. Next we show that there is no restriction on the elements of G_0 as regards the topological number σ. As remarked earlier, the circle S^1 associated with the number σ may be thought of as a light ray whose past and future end points are identified. Let us write

$$(x_0, \vec{x}) = \left(\frac{u-v}{2}, \frac{u+v}{2}\hat{n}\right) , \tag{3.5}$$

$$u = \tan\tfrac{1}{2}p , \tag{3.6}$$

$$v = \tan\tfrac{1}{2}q , \tag{3.7}$$

where $\hat{n}$ is a spatial unit vector. Then as p is varied for fixed q, we clearly generate a light ray. We can regard p as the coordinate of S^1 with the identification of points p and $p+2\pi$. [Equation (3.7) shows that q and $q+2\pi$ are also to be identified.] Let $g_\sigma(p)$ be a gauge transformation which depends only on p and has topological number σ. It is required to fulfill the periodicity condition $g_\sigma(p) = g_\sigma(p+2\pi)$ since it is a function on S^1. We can assume that $g_\sigma(0)=1$ by replacing $g_\sigma(p)$ by $g_\sigma(p)g_\sigma(0)^{-1}$ if necessary. This replacement leaves the number σ unaltered. Next consider $h_\sigma(p) = g_\sigma(p-q-\sin(p-q))$. Since $p-q-\sin(p-q)$ covers S^1 exactly once as p increases by 2π, $h_0(p)$ is also characterized by the number σ. Further, since

$$\frac{\partial}{\partial x_0}[p-q-\sin(p-q)]\bigg|_{p=q} = [1-\cos(p-q)]\left(\frac{\partial p}{\partial x_0}-\frac{\partial q}{\partial x_0}\right)\bigg|_{p=q} = 0\,, \tag{3.8}$$

we find

$$h_\sigma(p)|_{p=q}=1\,, \tag{3.9}$$

$$\frac{\partial}{\partial x_0}h_\sigma(p)\bigg|_{p=q}=0\,. \tag{3.10}$$

Since $x_0=0$ corresponds to $p=q$, it follows that $h_\sigma \in G_0$. We can thus conclude that while $\rho=0$ for all the elements of G_0, there is no such restriction on σ. The classical gauge vacuums can be characterized by the value of ρ, but not by the value of σ since σ is not invariant under the action of G_0. The elements of the factor group G/G_0 are associated only with the number ρ.

The argument given above seems to generalize even to situations where the topology of space-time is not $S^n \times S^1$. It suggests that the only topological numbers of gauge functions which can be given a physical meaning are those associated with a spacelike hypersurface (or more generally with a hypersurface like the null plane on which the canonical formalism is set up).

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[8]These can be derived from the Poisson brackets of A_μ^α and π_i^α with the generators of gauge transformations. See, for example, Ref. 7, Chap. 6, Sec. B.

[9]We assume in this discussion that second-class constraints, if any, have been eliminated after replacing Poisson brackets by Dirac brackets (see Refs. 6 and 7).

[10]The Lagrange multiplier terms generate an infinitesimal gauge transformation with gauge functions proportional to A_0^α (cf. Ref. 7, Chap. 6, Sec. B). Since Hamilton's equations do not determine the time dependence of A_0^α (see Refs. 6 and 7) we are free to choose these functions at will so long as the choice is consistent with their specified boundary values $A_0^\alpha(0,\vec{x})$ at time 0.