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Indefinite-metric fields and the renormalization group*

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We derive the renormalization-group equations for the Green's functions of an indefinite-metric field theory. In these equations we retain the mass dependence of the coefficient functions, since in the indefinite-metric theories the masses cannot be neglected. The behavior of the effective coupling constant in the asymptotic and infrared limits is analyzed. We illustrate the analysis by means of a simple model incorporating indefinite-metric fields. The model scales at first order, and at this order also the effective coupling constant has both ultraviolet and infrared fixed points, the former being the bare coupling constant.

I. INTRODUCTION

The phenomenon of scaling has been observed in the deep-inelastic scattering of electrons off proton targets.[1] Certain structure functions behave in a very special manner as the deep-inelastic limit is taken. It has been noted that one way to explain this scaling behavior is to consider the proton as made up of pointlike constituents—or partons.[2] In the deep-inelastic scattering of an electron off a single such parton the structure functions also exhibit scaling behavior. It has been possible to justify choosing these partons as spin-$\frac{1}{2}$ quarks. However, in deriving the scaling behavior of the structure functions for such a model one must make the extra assumption that the partons are free objects, or at least that they behave like free objects during the time it takes for the incident virtual photon to interact with the proton. This assumption is justified in the parton model by choosing to work only in a special infinite-momentum frame,[3] a frame which is mathematically not well defined and physically does not exist.

Many attempts have been made to construct a field theory to describe this model. In such a theory the basic fields would be the constituents and the gluons which are responsible for binding the constituents together to form a bound state—the proton.[4] With the advent of renormalizable non-Abelian gauge theories,[5] it was seen that one could construct a field-theory model in which the constituents were spin-$\frac{1}{2}$ fermions and the gluons were spin-1 non-Abelian vector bosons; moreover, the model would be renormalizable, but would it scale, and in what sense could the constituents be free?

Recently there was also a resurgence of interest in the renormalization-group work of Gell-Mann and Low.[6] These authors examined quantum electrodynamics to see how the theory behaved under a change of renormalization point. Their ideas were applied to other renormalizable models.[7] When the analysis was carried out for a massless Yang-Mills field theory a remarkable discovery was made[8]: The effective coupling constant, which was responsible for the behavior of many functions of the theory as the renormalization point was changed, tended to zero as the scale of the renormalization point approached infinity. (In models previously examined, it had been noticed, in lowest order, that the effective coupling constant increased as the momentum scale became large, though it was not clear whether it remained finite, or became infinite in the exact theory.) This phenomenon occurs in the Euclidean region of momentum space—the theory behaves as if it were a free theory in the deep Euclidean region (for many processes this limit corresponds to the deep-inelastic limit). However, this would not quite be a free theory, as a free theory would yield exact scaling in the deep-inelastic limit. Instead, the asymptotically free theories, as they became known, would predict calculable logarithm violations to the exact scaling results.[7] The present experimental data are not clear enough to distinguish, for once and for all, between exact scaling and logarithm violations; for models predicting inverse-power deviations in the prescaling region (characteristic of models which predict scaling in the limit) have been fitted to the data, as well as models with the logarithm modifications to scaling which are associated with asymptotically free theories.[9] Thus, it is not yet clear whether or not the property of asymptotic freedom is essential to our understanding of physics.

If one believes in the existence of quarks and gluons, then it may be only by means of asymptotic freedom that the experimental results can be explained. However, all known theories involving such quantities are beset with divergences, albeit they are renormalizable. On the other hand, if one is prepared to relinquish conventional ideas, it may be possible to obtain a theory which predicts scaling, and is finite. At least some theories are known to exist which are finite in four space-time dimensions—for example, the indefinite-metric field theories.[10]

It is expected that a finite theory will give good behavior, as such a theory will not suffer from undefined (and divergent) integrals in higher-order corrections. As a first step in the investigation of the scaling behavior of such theories, in this paper we apply the renormalization-group analysis to theories with indefinite-metric fields. The main changes when compared with other approaches[8,11] are that one must retain the mass dependence of the models to derive the renormalization-group equations. In general it seems that the effective coupling constants of such theories will have both ultraviolet and infrared fixed points, though they will be nonzero. To illustrate some points in the discussion we examine a simple model, in particular, an indefinite-metric scalar-field model. For this model we find that the stable ultraviolet fixed point of the effective coupling constant is, in fact, the bare (or unrenormalized) coupling constant. One might expect that this should be true for all finite theories.[7] However, we suggest that this is not obvious especially when there are many mass parameters at hand. For the infrared limit, for example, these extra masses play a major role in determining the fixed point.

II. RENORMALIZATION-GROUP EQUATIONS FOR INDEFINITE-METRIC FIELDS

The basic idea in constructing an indefinite-metric field theory is as follows: We begin with a conventional quantum-field-theory model which is divergent. We introduce into the model extra fields which are quantized with "opposite-sign" commutation relations. They are introduced in such a manner that the divergent contributions to scattering amplitudes cancel among themselves. The result is a finite theory. The extra parameters which enter the theory are the masses associated with these "shadow fields." In Sec. IV we consider an indefinite-metric scalar field theory as one simple example. More complicated models have been considered, for example, a finite model of QED[10] which has two shadow fermion fields as well as the electron and photon fields. The different results of such models depend upon various combinations (including ratios) of the many mass parameters involved. It is clear from the construction of these models that the leading, or mass-independent, terms cancel among the various diagrams which contribute to a scattering amplitude. Thus, working in such an approximation will result in a noninteracting theory—and this approximation in no way represents the full theory.

It is usual in applications of the Gell-Mann–Low approach to the renormalization group to work only with the mass-independent (i.e., leading) terms.[8] This may be a valid approximation in simple one-coupling-constant one-mass-parameter models. However, in models with more than one mass parameter we should not ignore these parameters by setting them to zero, for the reasons given above. Clearly then, we should not use the Callan-Symanzik equations either, as in that approach the masses are relegated to a minor role, while the major results are derived in a mass-independent limit. Similarly, the more recent mass-independent approaches to the renormalization group are not applicable here.[11] *For these reasons we follow the Gell-Mann–Low approach, but retain throughout the mass dependence of the various functions of the theory.* It has come to our attention that the retention of the mass dependence in this approach has previously been seen.[12] However, since we are not examining conventional models in which masses play a minor role, but rather indefinite-metric models in which the mass parameters have an important role to play, we feel that it is both interesting and instructive to work through the approach and see the results develop; and this we now do.

We restrict our attention to renormalizable models which are characterized by one renormalized coupling constant g_R and (at least) two renormalized masses m and M. We denote the unrenormalized parameters of the model by g_0, m_0, and M_0, respectively. The renormalization scheme we use is of the Gell-Mann–Low type. The coupling constant g_R is defined in terms of an n-point ($n=3$ or 4) Green's function at a momentum point (possibly Euclidean) which is determined by an arbitrary mass parameter μ. The wave-function and vertex renormalization constants, which we collectively denote by Z, are defined in terms of the propagators and the proper vertex functions of the theory, respectively, at momentum points determined by the mass parameter μ. However, the renormalized masses m and M are defined as the positions of the poles in the momentum-space representation of the propagators. We also restrict our attention to theories with dimensionless coupling constants. If we include a momentum cutoff Λ, for generality although for our purposes it is not necessary, then the renormalized parameters and renormalization constants have the following functional dependence on the bare parameters of the theory:

$$
\begin{aligned}
g_R &= g_R\left(g_0, \frac{m_0}{\mu}, \frac{M_0}{\mu}, \frac{\Lambda}{\mu}\right),\\
Z &= Z\left(g_0, \frac{m_0}{\mu}, \frac{M_0}{\mu}, \frac{\Lambda}{\mu}\right),\\
m &= m(g_0, m_0, M_0, \Lambda),\\
M &= M(g_0, m_0, M_0, \Lambda).
\end{aligned} \tag{1}
$$

$\Gamma_U(p; g_0, m_0, M_0, \Lambda)$ is an unrenormalized n-point one-particle-irreducible (1PI) Green's function, where p stands for a set of external momenta $(p_1, \ldots, p_n)$. The corresponding renormalized Green's function we denote by $\Gamma_R(p; g_R, m, M, \mu)$, and by virtue of the renormalizability of the theory we have

$$
\Gamma_R(p; g_R, m, M, \mu) = Z_\Gamma\left(g_0, \frac{m_0}{\mu}, \frac{M_0}{\mu}, \frac{\Lambda}{\mu}\right)\Gamma_U(p; g, m_0, M_0, \Lambda), \tag{2}
$$

where Z_Γ is the relevant combination of renormalization constants. For theories which are divergent this equation is to be understood to mean that if we regroup all the arguments on the right-hand side in terms of renormalized parameters, then the divergent Λ dependence is hidden within the definition of g_R, m, and M. Then we can take the limit $\Lambda \to \infty$ keeping g_R, m, and M fixed to find the left-hand side of Eq. (2). For finite theories we do not have to worry about the Λ dependence as it has already canceled out among the various contributions to Z_Γ and Γ_U.

The renormalization-group equations tell us how the renormalized functions (Green's and otherwise) of the theory behave as we change the mass parameter μ. So, differentiating Eq. (2) with respect to μ yields these equations, and they can be written in the form

$$
\left[\mu \frac{\partial}{\partial \mu} + \beta\left(g_R, \frac{m}{\mu}, \frac{M}{\mu}\right)\frac{\partial}{\partial g_R} - \gamma_\Gamma\left(g_R, \frac{m}{\mu}, \frac{M}{\mu}\right)\right] \times \Gamma_R(p; g_R, m, M, \mu) = 0 . \tag{3}
$$

The coefficient functions in this equation are defined by

$$
\beta\left(g_R, \frac{m}{\mu}, \frac{M}{\mu}\right) = \mu \frac{\partial}{\partial \mu} g_R\left(g_0, \frac{m_0}{\mu}, \frac{M_0}{\mu}, \frac{\Lambda}{\mu}\right)
$$

and (4)

$$
\gamma_\Gamma\left(g_R, \frac{m}{\mu}, \frac{M}{\mu}\right) = \mu \frac{\partial}{\partial \mu} \ln Z_\Gamma\left(g_0, \frac{m_0}{\mu}, \frac{M_0}{\mu}, \frac{\Lambda}{\mu}\right),
$$

where on the right-hand side after differentiation we reexpress everything in terms of renormalized quantities. We have used dimensional arguments in writing down the arguments of these functions.

Introducing dimensional analysis we can turn (3) into a scaling equation for Γ_R, and this will be more useful. We scale all the momenta uniformly, i.e., $p = \kappa p_0$ or $(p_1, \ldots, p_n) = \kappa(p_{10}, \ldots, p_{n0})$. Then if D_Γ is the mass dimension of Γ_R, dimensional analysis gives us

$$
\left(\kappa \frac{\partial}{\partial \kappa} + m \frac{\partial}{\partial m} + M \frac{\partial}{\partial M} + \mu \frac{\partial}{\partial \mu} - D_\Gamma\right) \times \Gamma_R(\kappa p_0; g_R, m, M, \mu) = 0 , \tag{5}
$$

and combining Eqs. (3) and (5) gives us the scaling equation for Γ_R,

$$
\left[\kappa \frac{\partial}{\partial \kappa} - \beta\left(g_R, \frac{m}{\mu}, \frac{M}{\mu}\right)\frac{\partial}{\partial g_R} + m \frac{\partial}{\partial m} + M \frac{\partial}{\partial M} - D_\Gamma + \gamma_\Gamma\left(g_R, \frac{m}{\mu}, \frac{M}{\mu}\right)\right]\Gamma_R(\kappa p_0; g_R, m, M, \mu) = 0 . \tag{6}
$$

Despite the nontrivial mass dependence of the functions, it is possible to solve this equation using the same method as in other approaches, where now we treat m/μ and M/μ as dimensionless constants. The solution takes the form

$$
\Gamma_R(\kappa p_0; g_R, m, M, \mu) = \kappa^{D_\Gamma} \exp\left[-\int_1^\kappa \gamma_\Gamma\left(g(\kappa'), \frac{m(\kappa')}{\mu}, \frac{M(\kappa')}{\mu}\right)\frac{d\kappa'}{\kappa'}\right]\Gamma_R(p_0; g(\kappa), m(\kappa), M(\kappa), \mu) \tag{7}
$$

in terms of the effective coupling constant $g(\kappa)$ and two effective masses $m(\kappa)$ and $M(\kappa)$. The effective coupling constant is the renormalization constant which results from subtracting at a momentum point specified by $\kappa\mu$. It satisfies the differential equation

$$
\kappa \frac{d}{d\kappa} g(\kappa) = \beta\left(g(k), \frac{m(\kappa)}{\mu}, \frac{M(\kappa)}{\mu}\right) \tag{8}
$$

subject to the initial condition $g(1) = g_R$. The effective masses are defined analogously by $\kappa dm(\kappa)/d\kappa = -m(\kappa)$, $m(1) = m$, and $\kappa dM(\kappa)/d\kappa = -M(\kappa)$, $M(1) = M$, with solution the naively scaled masses m/κ and M/κ, respectively. So Eq. (8) becomes

$$
\kappa \frac{d}{d\kappa} g(\kappa) = \beta\left(g(\kappa), \frac{m}{\mu\kappa}, \frac{M}{\mu\kappa}\right). \tag{9}
$$

It is obvious from (7) that the scaling behavior of the Green's function Γ_R is governed by the behavior of $g(\kappa)$ when the scale κ is varied. A detailed examination of the effective coupling constant is the next priority.

So far in this section we have considered only the behavior of the one-particle-irreducible Green's functions. To apply the scaling analysis to the structure functions which are important from the experimental point of view, it is necessary to examine the short-distance behavior of products of currents. These quantities are analyzed by means of Wilson's operator-product expansion.[7] The renormalization-group equations for the coefficients of this expansion are of the same form as those of Eq. (3), except for the γ factor. However, as far as the coupling-constant part of the analysis is concerned the equations, and the subsequent analysis, are the same as given here for the one-particle-irreducible Green's function.[9] As we do not examine any models with physical application in this paper, we do not proceed with the analysis of the operator-product expansion.

III. BEHAVIOR OF THE EFFECTIVE COUPLING CONSTANT

We have seen, in Sec. II, the behavior of the 1PI Green's functions of the models is dependent on the behavior of the effective coupling constant $g(\kappa)$ as the scale of the external momenta κ is allowed to vary. The regions of interest in the range of κ are the ultraviolet or asymptotic ($\kappa\to\infty$) and the infrared ($\kappa\to 0$) limits.

In the asymptotic region, there may exist a fixed point of the Gell-Mann–Low equation. For this to occur we must have

$$\lim_{\kappa\to\infty}\beta\left(g(\kappa),\ \frac{m}{\mu\kappa},\ \frac{M}{\mu\kappa}\right)\equiv\beta(g^*,0,0)=0\ ,$$

so that $\kappa dg(\kappa)/d\kappa\to 0$ as $\kappa\to\infty$ and $g(\kappa)\to g^*$. In the case $g^*=0$, e.g., in the case of many non-Abelian gauge theories,[8,13] the theory is said to be asymptotically free. For most known theories, however, this does not occur. In fact, they do not even have fixed points. For such theories the effective coupling constant $g(\kappa)\to\infty$ either for finite κ or as $\kappa\to\infty$, in first-order perturbation theory.

In the infrared limit, also fixed points may exist. In this case we must have

$$\lim_{\kappa\to 0}\beta\left(g(\kappa),\ \frac{m}{\mu\kappa},\ \frac{M}{\mu\kappa}\right)\equiv\beta(g_1,\infty,\infty)=0\ ,$$

so that here too

$$\kappa\frac{d}{d\kappa}g(\kappa)\to 0 \text{ as } \kappa\to 0 \text{ and } g(\kappa)\to g_1\ .$$

However, for those theories which are asymptotically free it is usually argued that in the infrared limit they will be subject to slavery, that is,

$g(\kappa)\to\infty$ as κ tends to zero,
or some small finite number.

We will now see, to first order, that for finite theories it is most likely that there exist finite nonzero fixed points in both limits.

The β function, defined by Eq. (4), has the following structure:

$$\beta\left(g_R\left(g_0,\frac{m}{\mu},\frac{M}{\mu}\right),\frac{m}{\mu},\frac{M}{\mu}\right)=g_0\mu\frac{d}{d\mu}Z_g\left(g_0,\ \frac{m}{\mu},\frac{M}{\mu},\frac{\Lambda}{\mu}\right), \tag{10}$$

where

$$g_R=g_0Z_g\left(g_0,\frac{m}{\mu},\frac{M}{\mu},\frac{\Lambda}{\mu}\right)$$

$$\text{(e.g., in } \phi^4 \text{ theory } Z_g=(Z_\phi)^2/Z_{\phi^4} \text{ or in QED } Z_g=Z_2Z_3{}^{1/2}/Z_1). \tag{11}$$

However, to have the β function in a usable form we must express it in terms of renormalized quantities only. If we had an exactly solvable model, then we could analyze the effective coupling constant in an exact manner. However, exactly solvable models have little or no application, as yet. The models in which we are interested can be solved in perturbation theory, at best. For this reason, the analysis and results of this section are valid only in perturbation treatments, and at that only in first order.

Bearing this in mind, let us suppose that, to first order,

$$Z_g=1+g_0X_1\ . \tag{12}$$

Then we can invert Eq. (11) to read

$$g_0=g_R(1-g_RX_1)=g_R\,Z_g{}^{-1}\left(g_R,\frac{m}{\mu},\frac{M}{\mu},\frac{\Lambda}{\mu}\right)\ . \tag{13}$$

So, to this order

$$\begin{aligned}\beta\left(g_R,\frac{m}{\mu},\frac{M}{\mu}\right)&=g_0\mu\frac{\partial}{\partial\mu}(1+g_0X_1)\\&\cong g_R{}^2\frac{\partial}{\partial\mu}X_1\\&=g_R\mu\left[\frac{\partial}{\partial\mu}(1+g_RX_1)\right]_{g_R\,\text{fixed}}\ ,\end{aligned} \tag{14}$$

where it is important to note that in the final expression the partial derivative is $(\partial/\partial\mu)_{g_R\,\text{fixed}}$, and not $(\partial/\partial\mu)_{g_0\,\text{fixed}}$ as is usually understood.

We are now in a position to solve the Gell-Mann–Low equation at this order. The equation can be rewritten as

$$\kappa \frac{d}{d\kappa} g(\kappa) = g(\kappa)(\mu\kappa)\frac{\partial}{\partial(\mu\kappa)} \times [1+g(\kappa)X_1(\kappa)]_{g(\kappa)\text{fixed}} = g^2(\kappa)\left(\kappa \frac{d}{d\kappa}X_1(\kappa)\right) , \quad (15)$$

where $X_1(\kappa)$ is obtained from $X_1(m/\mu, M/\mu, \Lambda/\mu)$ by the replacement $\mu \to \mu\kappa$. The solution to this equation is

$$g(\kappa) = \frac{g_R}{(1+g_R X_1) - g_R X_1(\kappa)} . \quad (16)$$

Clearly all the κ dependence is contained within the function $X_1(\kappa) = X(m/\mu\kappa, M/\mu\kappa, \Lambda/\mu\kappa)$. For those theories which are finite, the Λ dependence is absent, so that we have then $X_1(m/\mu\kappa, M/\mu\kappa)$. The $\kappa \to \infty$ limit is, in this case, the same as the limit $m, M \to 0$ *in constant ratio*. It is not at all obvious that $X_1(\kappa) \to 0$ in this limit, and indeed one would expect the limit to depend on the ratio M/m somehow. However, in Sec. IV we will show that for a simple model $X_1(\kappa) \to 0$ in this limit. If $g(\kappa)$ remains finite in taking the $\kappa \to \infty$ limit, and $X_1(\kappa)$ vanishes in this limit, then there exists a stable ultraviolet fixed point

$$g^* = \frac{g_R}{1+g_R X_1}$$

which, by virtue of Eq. (13) is just the bare coupling constant, i.e.,

$$g^* = g_0 . \quad (17)$$

For the particular model it is gratifying that this result follows, as it agrees with results obtained in other models[14] (see Sec. V).

For divergent theories the Λ is very much present. Since we are restricting our attention to renormalizable theories, and we are working within the context of perturbation theory, the Λ dependence is at most logarithmic. X_1 can directly depend on Λ through Λ/μ, Λ/m, or Λ/M. In the first case, $X_1(\kappa)$ will depend on $\Lambda/\mu\kappa$ through the factor $\ln(\Lambda/\mu\kappa) = \ln(\Lambda/\mu) - \ln\kappa$. Thus, in the limit $\kappa \to \infty$ this term will approach $-\infty$. This could give rise to an asymptotically free theory if the overall coefficient of $\ln\kappa$ in the denominator of $g(\kappa)$ is positive. If, however, this coefficient is negative, then $g(\kappa)$ will approach infinity for a finite value of κ (e.g., a regular ϕ^4 theory). In either of the latter cases above, $X_1(\kappa)$ can at most depend on κ through $m/\mu\kappa$ and $M/\mu\kappa$. Thus, $X_1(\kappa)$ cannot become infinite unless the theory is infinite in the zero-mass limit. If this were to occur, then again asymptotic freedom could result. Otherwise, the $\kappa \to \infty$ limit will be equivalent to $m, M, \Lambda \to 0$ *in constant ratio* (so Λ/m, Λ/M, and M/m remain fixed), and a finite limit for $X_1(\kappa)$ could result.

In the infrared region ($\kappa \ll 1$) the situation is quite analogous to the above. For finite theories the limit is equivalent to *the constant-ratio* infinite-mass limit $m, M \to \infty$. We will see in Sec. IV that for a simple model this limit depends upon the ratio M/m and is nonzero. For divergent theories, again we can have the Λ dependence through $\Lambda/\mu, \Lambda/m$, or Λ/M. In the former case the factor $\ln\kappa$ will blow up in the limit $\kappa \to 0$. Thus, if the sign of the coefficient of $\ln\kappa$ in Eq. (16) is positive, we will have infrared freedom, while if it is negative we will have infrared slavery. It is usually argued, in the absence of mass dependence, that an asymptotically free theory will have infrared slavery. In the latter two cases above, the $\kappa \to \infty$ limit corresponds to the limit $m, M, \Lambda \to \infty$ *in constant ratio* (so that M/m and Λ/m remain constant).

The result that the ultraviolet fixed point can be the bare coupling constant (and is for a model) has been derived at first order only. It would be preferable to see this result occur in higher orders. As a first step in this direction we consider the second-order renormalized coupling constant

$$g_R = g_0 Z_g = g_0(1+g_0 X_1 + g_0{}^2 X_2) . \quad (18)$$

If we invert this equation to express the unrenormalized constant in terms of the renormalized constant, we find

$$g_0 = \frac{g_R}{1+g_R X_1 + g_R{}^2(X_2 - X_1{}^2)} = \frac{g_R}{F\left(g_R, \frac{m}{\mu}, \frac{M}{\mu}, \frac{\Lambda}{\mu}\right)} . \quad (19)$$

In terms of the renormalization constant F the β function takes a particularly nice form. We have

$$\beta\left(g_R, \frac{m}{\mu}, \frac{M}{\mu}\right) = g_0 \mu \frac{\partial}{\partial\mu}(1+g_0 X_1 + g_0{}^2 X_2) ,$$

and reexpressing each g_0 on the right-hand side in terms of renormalized quantities we find, to this order,

$$\beta\left(g_R, \frac{m}{\mu}, \frac{M}{\mu}\right) = g_R{}^2\left(\mu\frac{\partial}{\partial\mu}X_1\right) + g_R{}^3\left(\mu\frac{\partial}{\partial\mu}X_2 - 2X_1\mu\frac{\partial}{\partial\mu}X_1\right) ,$$

and this is clearly equal to

$$g_R \mu \left\{\frac{\partial}{\partial\mu}[1+g_R X_1 + g_R{}^2(X_2 - X_1{}^2)]\right\}_{g_R\text{ fixed}} ,$$

i.e.,

$$\beta\left(g_R, \frac{m}{\mu}, \frac{M}{\mu}\right) = g_R \mu \left[\frac{\partial}{\partial\mu} F\left(g_R, \frac{m}{\mu}, \frac{M}{\mu}, \frac{\Lambda}{\mu}\right)\right]_{g_R \text{ fixed}} . \tag{20}$$

However, it is not a simple problem to solve the corresponding Gell-Mann–Low equation; this takes the form

$$\kappa \frac{d}{d\kappa} g(\kappa) = g^2(\kappa) A(\kappa) + g^3(\kappa) B(\kappa) ,$$

where $A(\kappa)$ and $B(\kappa)$ are known but extremely complicated functions of κ. Here, $B(\kappa)$ corresponds to the two-loop corrections to the coupling constant. In fact, even when A and B are taken to be independent of κ, as in the usual approaches, an explicit solution to this equation is not known.

IV. ILLUSTRATION USING A SIMPLE INDEFINITE-METRIC FIELD THEORY

To illustrate the analysis presented in Sec. II and III it is of benefit to examine a model in detail. The model we consider is probably the simplest example—an indefinite-metric version of a self-interacting scalar-field theory. The general idea is as follows: We begin with a self-interacting scalar-field theory, described by the Lagrangian

$$\mathcal{L} = \tfrac{1}{2}[(\partial_\mu \phi_1)^2 - m_0^2 \phi_1^2] - \frac{1}{4!} \lambda \phi_1^4 . \tag{21}$$

This theory is divergent but renormalizable. To make the model finite we add extra scalar fields. These fields are quantized with the opposite sign to ϕ_1, i.e., if the ϕ_1 field obeys commutation rules

$$[\phi_1(x), \phi_1(y)] = \alpha\Delta(x-y) , \tag{22}$$

then the extra field ϕ_2 will satisfy

$$[\phi_2(x), \phi_2(y)] = -\alpha\Delta(x-y) . \tag{23}$$

For this model it is sufficient to add just one extra field. In other, more complicated models, for example when treating quantum electrodynamics more than one such field may be required. Actually, in the finite model of QED, the extra fields satisfying the opposite-sign quantization conditions are Fermi fields, so they obey opposite-sign anticommutation rules.

We add ϕ_2 to the model as shown in the Lagrangian function

$$\mathcal{L} = \tfrac{1}{2}[(\partial_\mu \phi_1)^2 - m_0^2 \phi_1^2] - \tfrac{1}{2}[(\partial_\mu \phi_2)^2 - M_0^2 \phi_2^2] - \frac{1}{4!} \lambda(\phi_1 + \phi_2)^4 . \tag{24}$$

We see here the manifestation of the negative-metric quantization in the sign of the free ϕ_2 Lagrangian. The effect of this is to give the ϕ_2 Feynman propagator an extra minus sign. It is this mechanism which will cause the cancellation of divergences.

We can think of the extra field ϕ_2 as a type of regularization of the original model. The self-interaction term in (24) is such that we can view the effect of the ϕ_2 field as giving us an effective propagator[10]

$$\Delta_{\text{eff}}(k) = i\left(\frac{1}{k^2 - m_0^2} - \frac{1}{k^2 - M_0^2}\right) = i \frac{m_0^2 - M_0^2}{(k^2 - m_0^2)(k^2 - M_0^2)} . \tag{25}$$

This modification will be sufficient to make (most of) the integrals of the theory well-defined, so that no momentum-space cutoff is required. An alternative approach is to treat the fields separately, the ϕ_2 being a shadow field; in this case we need to retain the cutoff Λ to make separate integrals well-defined, but in summing up the diagram contributions the Λ dependence will be seen to cancel out.

Unfortunately, as it stands the model described by (24) is not totally free of divergences. The lowest-order mass-shell corrections to the propagators remain, i.e., those diagrams with at most one internal line. It is clear, however, that any diagram of the original theory with more than one internal line will be made finite. This is easily seen because the divergent (primitive) diagrams in the original theory, besides mass renormalizations, were those in which each integration loop had two propagators, and Eq. (25) is sufficient to make these finite. Thus, only the mass renormalizations remain divergent. This will not affect our analysis anywhere; however, we note that if we restrict the interaction term in (24) to be normal ordered then

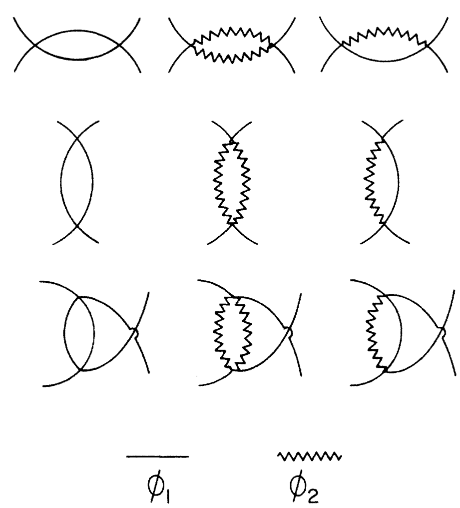

FIG. 1. All diagrams contributing to coupling-constant renormalization at second order.

such divergences no longer appear, i.e., the interaction is $(-1/4!)\lambda{:}(\phi_1+\phi_2)^4{:}$.

In Sec. III we have seen that to determine the behavior of the effective coupling constant it is not necessary to know the β function; all we really need is the renormalization of the coupling constant. The first-order corrections come solely from the proper-vertex corrections—as in this model there are no wave-function renormalizations until at least second-order corrections are calculated. In Fig. 1 we list the different diagrams which contribute in this order. Basically, however, there are only three different diagrams to be considered; the remaining ones have the same structure.

The three basic diagrams, with momentum conventions, are shown in Fig. 2. We denote the contributions from Figs. 2(a), 2(b), and 2(c) by I_a, I_b, and I_c, respectively, and they are, taking account of symmetry factors, as follows:

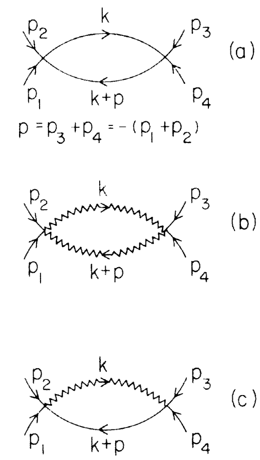

FIG. 2. Momentum conventions for coupling-constant renormalizations.

$$I_a=\frac{i\lambda^2}{32\pi^2}\left[1+\lim_{\Lambda\to\infty}\ln\frac{\Lambda^2}{m^2}-\left(1-\frac{4m^2}{p^2}\right)^{1/2}\ln\left|\frac{1+(1-4m^2/p^2)^{1/2}}{1-(1-4m^2/p^2)^{1/2}}\right|\right], \tag{26}$$

$$I_b=\frac{i\lambda^2}{32\pi^2}\left[1+\lim_{\Lambda\to\infty}\ln\frac{\Lambda^2}{M^2}-\left(1-\frac{4M^2}{p^2}\right)^{1/2}\ln\left|\frac{1+(1-4M^2/p^2)^{1/2}}{1-(1-4M^2/p^2)^{1/2}}\right|\right], \tag{27}$$

$$\begin{aligned}I_c=\frac{-i\lambda^2}{16\pi^2}\Bigg\{&1+\lim_{\Lambda\to\infty}\ln\frac{\Lambda^2}{p^2}-\frac{1}{2}\left(1+\frac{m^2-M^2}{p^2}-A\right)\ln\left[\frac{1}{2}\left(1+\frac{m^2-M^2}{p^2}-A\right)\right]\\&-\frac{1}{2}\left(1+\frac{m^2-M^2}{p^2}+A\right)\ln\left[\frac{1}{2}\left(1+\frac{m^2-M^2}{p^2}+A\right)\right]-\frac{1}{2}\left(1-\frac{m^2-M^2}{p^2}-A\right)\ln\left[\frac{1}{2}\left(1-\frac{m^2-M^2}{p^2}-A\right)\right]\\&-\frac{1}{2}\left(1-\frac{m^2-M^2}{p^2}+A\right)\ln\left[\frac{1}{2}\left(1-\frac{m^2-M^2}{p^2}+A\right)\right]\Bigg\},\end{aligned} \tag{28}$$

where

$$A=\left[\left(1+\frac{m^2-M^2}{p^2}\right)^2-\frac{4m^2}{p^2}\right]^{1/2}.$$

If we let $p'=p_1+p_4$ and $p''=p_1+p_3$, the contributions of the remaining diagrams on the second and third rows of Fig. 1 are got by making the replacements $p\to p'$ and $p\to p''$ in I_a, I_b, and I_c. We see above that the Λ dependence cancels in summing up the various contributions, yielding finite coupling-constant renormalizations.

Following the renormalization procedure outlined in Sec. II, we define the function X_1, where $Z_g=(1+g_0X_1)$ to lowest order, in terms of the corrections I_a, I_b, and I_c at a symmetric renormalization point

$$p_1^{\,2}=p_2^{\,2}=p_3^{\,2}=p_4^{\,2}=\mu^2,\quad p_i\cdot p_j=\tfrac{1}{3}\mu^2,\quad i\neq j\ . \tag{29}$$

Summing up all contributions at this momentum point, we can then display the κ dependence of $X_1(\kappa)$, which governs the asymptotic behavior of the effective coupling constant [see Eqs. (16)], as follows:

$$\begin{aligned}X_1(\kappa)=\frac{3}{32\pi^2}\Bigg[&\left(1+\frac{\alpha}{\kappa^2}\right)^{1/2}\ln\left|\frac{1+(1+\alpha/\kappa^2)^{1/2}}{1-(1+\alpha/\kappa^2)^{1/2}}\right|+\left(1+\frac{\alpha'}{\kappa^2}\right)^{1/2}\ln\left|\frac{1+(1+\alpha'/\kappa^2)^{1/2}}{1-(1+\alpha'/\kappa^2)^{1/2}}\right|\\&+\frac{1}{2}\,\frac{\Delta^2}{\kappa^2}\ln\frac{M}{m}-B(\kappa)\ln\left|\frac{1+s^2/4\kappa^2+B(\kappa)}{1+s^2/4\kappa^2-B(\kappa)}\right|\Bigg],\end{aligned} \tag{30}$$

with $B(\kappa)=(1+s^2/2\kappa^2+\Delta^4/16\kappa^4)^{1/2}$ and $\alpha=3m^2/\mu^2$, $\alpha'=3M^2/\mu^2$, $\Delta^2=\alpha'-\alpha$, $s^2=\alpha'+\alpha$. (Note: $X_1(\kappa)$ is got from X_1 by the replacement $\mu\to\mu\kappa$.) It is not obvious from this expression that $X_1(\kappa)$ even remains finite in the limit $\kappa\to\infty$; however, by rewriting $X_1(\kappa)$ as

$$X_1(\kappa)=\frac{3}{32\pi^2}\left\{\left[\left(1+\frac{\alpha}{\kappa^2}\right)^{1/2}+1\right]\ln\left[\left(1+\frac{\alpha}{\kappa^2}\right)^{1/2}+1\right]-\left[\left(1+\frac{\alpha}{\kappa^2}\right)^{1/2}-1\right]\ln\left[\left(1+\frac{\alpha}{\kappa^2}\right)^{1/2}-1\right]\right.$$
$$+\left[\left(1+\frac{\alpha'}{\kappa^2}\right)^{1/2}+1\right]\ln\left[\left(1+\frac{\alpha'}{\kappa^2}\right)^{1/2}+1\right]-\left[\left(1+\frac{\alpha'}{\kappa^2}\right)^{1/2}-1\right]\ln\left[\left(1+\frac{\alpha'}{\kappa^2}\right)^{1/2}-1\right]$$
$$-\left(1+\frac{s^2}{4\kappa^2}+B(\kappa)\right)\ln\left(1+\frac{s^2}{4\kappa^2}+B(\kappa)\right)-\left(1+\frac{s^2}{4\kappa^2}-B(\kappa)\right)\ln\left(1+\frac{s^2}{4\kappa^2}-B(\kappa)\right)$$
$$\left.-2\ln 2+\frac{\alpha'\ln\alpha'+\alpha\ln\alpha}{2\kappa^2}-\frac{\alpha'+\alpha}{2\kappa^2}\ln 2\kappa^2\right\}, \tag{31}$$

we see in fact that, as $\kappa\to\infty$, $X_1(\kappa)$ has a vanishing limit. A computer analysis of the function X_1 yields the following properties: (1) $X_1(\kappa)<0$ for all finite κ if $M\neq m$, (2) $X_1(\kappa)\to 0^-$ monotonically as $\kappa\to\infty$. The effective coupling constant is $g(\kappa)=1/[1/g_R+X_1-X_1(\kappa)]$. If $X_1<0$ is such that $1/g_R+X_1>0$, then $g(\kappa)$ decreases from g_R (a positive number) to $g_R/(1+g_RX_1)$, the stable ultraviolet fixed point which is the bare coupling constant. On the other hand, if $X_1<0$ is such that $1/g_R+X_1<0$, then the numerator of $g(\kappa)$ will vanish for a finite value of κ — that value such that $X_1(\kappa)=1/g_R+X_1$. In this case there does not exist a fixed point, since $g(\kappa)$ becomes infinite for finite κ.

We can check whether or not this latter case can occur. Using Eq. (12) we have

$$g_R=g_0(1+g_0X_1),$$

and if we have both $g_0>0$ and $g_R>0$ then the function X_1 must satisfy $X_1>-1/g_0$. We have seen that to ensure the existence of the fixed point we must have $X_1>-1/g_R$. This condition can be rewritten as

$$g_0{}^2X_1{}^2+X_1+1/g_0>0\,. \tag{32}$$

Writing $f(X)=g_0{}^2X^2+X+1/g_0$ we see that $f(X)$ attains a *minimum* value at $X=-1/2g_0$, and the minimum value is $\frac{3}{4}g_R$. Thus, we see that the condition (32) is always satisfied, given that both g_0 and g_R are positive. In fact this gives us a restriction on the allowed values of M to ensure the existence of the fixed point—only those values which give $g_R>0$ are allowed.

In the infrared limit, however, $X_1(\kappa)$ remains nonvanishing, and we have

$$\lim_{\kappa\to 0}X_1(\kappa)=2\left(1-\frac{M^2+m^2}{M^2-m^2}\ln\frac{M}{m}\right)\equiv X_1(0)\,. \tag{33}$$

In fact, in this limit $X_1-X_1(\kappa)>0$ for all values of κ and $X_1-X_1(\kappa)\to X_1-X_1(0)$ as $\kappa\to 0$. Thus, a stable infrared fixed point exists also for this model and is given by

$$\frac{g_R}{1+g_RX_1-g_RX_1(0)},$$

where, as before, $X_1=X_1(\kappa)|_{\kappa=1}$.

To understand the scaling behavior of the 1PI Green's functions we need to examine the γ_Γ functions defined in Eq. (4). Since to first order there are no wave-function renormalizations (either infinite or finite), then at this level of perturbation theory $\gamma_\Gamma=0$. This tells us that to lowest order the model scales. To have more confidence in this result we should examine the next order of perturbation theory; however, we have seen that at that level we are not able to predict the behavior of $g(\kappa)$. For this reason we do not proceed further with this analysis.

V. DISCUSSION AND REMARKS

The existence of finite fixed points for the effective coupling constant in simple (one-coupling-constant) indefinite-metric field theories constitutes the main result of this paper. To derive this result we examined the renormalization-group equations which were applicable to such models. These equations and their solutions, for the one-particle-irreducible Green's functions, differ from the usual cases examined, especially in the importance attached to the mass dependence. We saw that a possible value for the ultraviolet fixed point is the bare charge of the theory, and this value was attained for the model of Sec. IV. That this result should be true has previously been argued.[7] Recently, it has been shown to be true also for a solvable model—the Zachariasen model.[14] In deriving his result the author used the exact theory, not a perturbative approximation as in this paper. He found that, prior to taking the infinite cutoff limit ($\Lambda\to\infty$), the effective coupling constant has as limit the bare coupling constant as κ was allowed to go to infinity. However, in that model, in taking the limit $\Lambda\to\infty$ afterwards the bare coupling went to zero, thus exhibiting asymptotic freedom. The present author wishes to emphasize that in the perturbation-theory approach of this paper the result $g(\kappa)\to g_0$ as $\kappa\to\infty$ does not necessarily follow. One must examine the different models to see whether or not the result holds.

By working throughout in a region of momentum (κp_0) space such that $m/\mu\kappa$ and $M/\mu\kappa$ were not separately set to zero, we seem to have restricted the analysis of this paper to the prescaling region of momentum space. However, in going to the large-κ limit we pass over smoothly into the scaling region. Thus, this approach will also yield information about the approach to scaling behavior for such finite models. It is interesting to note that if we had naively ignored the masses of the theory and applied the conventional analysis,[8] the renormalization-group equations would have reduced to

$$\mu \frac{\partial}{\partial\mu} \Gamma_R(p; g_R, m, M, \mu) = 0$$

in place of Eq. (3), since in that case the β and γ_Γ functions vanish trivially. However, we also see in this approximation that for the indefinite-metric theory all corrections to the coupling constant vanish owing to the cancellations caused by the shadow fields. Thus, the coupling constant will remain the bare coupling constant; there will be no renormalization. The earlier sections of this paper show that this cannot be true in general, especially if the ratios of masses have a role to play. To say this another way, this naive result can be valid only if: (1) the (masses$\to$0 independently) limit of the massive theory, and (2) the zero-mass version of the theory are exactly equivalent.

An interesting comment on the results obtained here is that after all the renormalizations have been carried out, we should reproduce the known properties of the ϕ^4-scalar-field model in the limit $M_{\text{unphysical}}$ going to infinity—namely, that in lowest order the Gell-Mann–Low equation has an unstable ultraviolet, but stable infrared, fixed point at the origin of coupling-constant space. The ultraviolet fixed point is

$$g^* = \frac{g_R}{1 + g_R X_1} , \tag{34}$$

with X_1 given by Eq. (30) with $\kappa = 1$. The infrared fixed point is

$$g_1 = \frac{g_R}{1 + g_R [X_1 - X_1(0)]} , \tag{35}$$

with $X_1(0)$ given by Eq. (33). Examining both X_1 and $X_1 - X_1(0)$ in this limit we find

$$X_1 \sim \frac{3}{32\pi^2}\left(-\ln\alpha' + \sqrt{1+\alpha}\ \ln\frac{\sqrt{1+\alpha}+1}{\sqrt{1+\alpha}-1} + 2 - \frac{3\alpha}{64} + \ln\alpha\right) + \text{terms which vanish as } \alpha' \to \infty ,$$

where, as before, $\alpha = 3m^2/\mu^2$, $\alpha' = 3M^2/\mu^2$,

i.e., $X_1 \sim -\ln\alpha'$ as $\alpha' \to \infty$.

Clearly then X_1 tends to $-\infty$ as $M \to \infty$. We have seen that $X_1 > -1/g_R$, provided that both g_0 and g_R are positive. Then for some finite value of M, $X_1 + 1/g_R$ will vanish. Thus, in taking the limit $M_{\text{unphysical}} \to \infty$ we have destroyed the stable ultraviolet fixed point.

Turning to the infrared fixed point given by Eq. (35), we note that $X_1(0)$ has the correct structure, in terms of α', to cancel the parts of X_1 which are divergent when $M_{\text{unphysical}} \to \infty$. So in this limit

$$X_1 - X_1(0) \to \frac{3}{32\pi^2}\left(\sqrt{1+\alpha}\ln\frac{\sqrt{1+\alpha}+1}{\sqrt{1+\alpha}-1} - \frac{3\alpha}{64}\right).$$

This gives us a mass-dependent nonzero infrared-stable fixed point. Remembering that in the conventional treatments of the ϕ^4 theory the mass m is neglected, we consider the $m \to 0$ limit of this expression. We find

$$X_1 - X_1(0) \sim -\ln\alpha \text{ as } \alpha \to 0 ,$$

$$\text{i.e., } X_1 - X_1(0) \to +\infty \text{ as } m \to 0 ,$$

which gives us a vanishing infrared fixed point in the zero-mass limit. If we were to allow $g_R < 0$, then it would be possible, for certain values of m, that the denominator of (35) would vanish—thereby destroying the infrared fixed point, however, such a choice for g_R would not be of physical interest.

Of course, it is not clear what is the physical significance of these first-order perturbation-theory fixed points. In taking account of higher orders they may well be destroyed. On the other hand, the fact that the ultraviolet fixed point is the *bare coupling constant*, and a similar result has been derived *exactly* in another model,[14] leads us to believe that the ultraviolet fixed point, at any rate, exists independently of the perturbation approximation.

To draw any conclusions from the analysis of the effective coupling constant which would have physical consequences, we need to examine the question of scaling. We have seen already that in the model of Sec. IV, at first order the function γ_Γ, which gives the anomalous behavior of the Green's functions, vanished identically. We cannot conclude much from this—at most that the theory gives naive scaling *at this order*. It is necessary to go to higher orders to learn the nontrivial structure of the functions γ_Γ. However, there are indefinite-metric theories in which the lowest-order calculations give nonvanishing γ_Γ functions in the renormalization-group equations. For example, in the finite model of QED[10] we can explicitly check the anomalous dimensions of the renormalized fermion propagator. Since all the integrals in the model are well-defined, we can simply scale the momentum in the expression for the renormalized propagator. It is necessary, however, to renormalize the fields at off-mass-shell points, and not on mass shell as done in Ref. 10. We find that the renormalized fermion propagator has the same scaling behavior as the bare propagator, i.e., the anomalous dimension of the fermion field vanishes

$$\gamma(g^*, 0) = 0$$

even though the ultraviolet fixed point g^* may not vanish. Thus this model appears to scale; though one must further check the operator-product expansion.

It is interesting to compare this conclusion with the result of Callan and Gross.[15] These authors show for a wide class of field theories that the existence of Bjorken scaling must imply that the origin of coupling-constant space is an ultraviolet-stable fixed point of the renormalization-group equations. Essential to the proof is that the theories have positive-definite metric. So their result is not applicable to the models considered in this paper, as our models have indefinite metric. Thus, indefinite-metric theories may have the property of Bjorken scaling, and have nonzero stable ultraviolet fixed points of the Gell-Mann–Low equation.

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