Some properties of monopole harmonics

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(Received 6 April 1977)

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In this paper some properties of monopole harmonics[1] will be derived.

I. SIMPLE PROPERTIES

Theorem 1.

$$Y^*_{q,j,m}=(-1)^{q+m}Y_{-q,j,-m}\ .$$

This is easily proved explicitly. Notice that this theorem holds in both R_a and R_b. It is valid in the gauge of Ref. 1.

Theorem 2. In the explicit form defined in Ref. 1, the monopole harmonics satisfy

$$Y_{q,j,m}=Y_{m,j,q}\quad \text{in } R_a\ ,$$
$$Y_{q,j,m}=Y_{m,j,q}\exp[2i\phi(m-q)]\quad \text{in } R_b\ .$$

This is easily proved with the aid of (B7) of Ref. 1. Hou Pei-yu and Hsi Ting-chang of Sian and Peking have pointed out to us that this theorem can be simply understood[2] if one considers the e-g system as a spinning top.

Theorem 3. If $q+q'+q''=0$ and $m+m'+m''=0$, then

$$\int Y_{q,l,m}Y_{q',l',m'}Y_{q'',l'',m''}d\Omega = \left[\frac{(2l+1)(2l'+1)(2l''+1)}{4\pi}\right]^{1/2} \times \begin{pmatrix} l & l' & l'' \\ m & m' & m'' \end{pmatrix}\begin{pmatrix} l & l' & l'' \\ q & q' & q'' \end{pmatrix}(-1)^{l+l'+l''}\ , \tag{1}$$

where the round brackets are $3j$ symbols.[3]

Proof. We first choose the gauge of Ref. 1 in which the Y's are explicitly defined. It follows from (D2) of that paper that

$$Y_{q,l,m}Y_{q',l',m'}=\sum_{j,m_j}K(q,q',l,l',l'') \times\langle ll'jm_j|lml'm'\rangle Y_{-q'',j,m_j}\ . \tag{2}$$

Multiply by $Y_{q'',l'',m''}$ and integrate over $d\Omega$. Using theorem 1 above we find that the left-hand side of (1) is equal to

$$\begin{pmatrix} l & l' & l'' \\ m & m' & m'' \end{pmatrix} G(l,l',l'',q,q',q'')\ , \tag{3}$$

where G is independent of the m's. Now use the symmetry of theorem 2. The left-hand side of (1) is unchanged if we switch all the m's with the q's. Thus

$$\int Y_{q,l,m}Y_{q',l',m'}Y_{q'',l'',m''}d\Omega =\begin{pmatrix} l & l' & l'' \\ m & m' & m'' \end{pmatrix}\begin{pmatrix} l & l' & l'' \\ q & q' & q'' \end{pmatrix} f(l,l',l'')\ . \tag{4}$$

To evaluate f, take the case

$$q''=-m''=l'',\quad m'=l',\quad q=l\ .$$

The integral in (4) can be evaluated for such a case in a straightforward manner. $f(l,l',l'')$ can then be evaluated. We thus verify (1). If we now choose a different gauge, the integrand in (1) is invariant because of the condition $q+q'+q''=0$. Thus (1) is valid in general.

Theorem 4.

$$Y_{q,l,m}Y_{q',l',m'}=\sum_{l''}(-1)^{l+l'+l''+q''+m''}\left[\frac{(2l+1)(2l'+1)(2l''+1)}{4\pi}\right]^{1/2}\begin{pmatrix} l & l' & l'' \\ m & m' & m'' \end{pmatrix}\begin{pmatrix} l & l' & l'' \\ q & q' & q'' \end{pmatrix}Y_{-q'',l'',-m''}\ , \tag{5}$$

where $m''=-m-m',\ \ q''=-q-q'$.

Proof. Comparison of (1) and (2) allows an evaluation of K. Substitution of the result into (2) gives (5). Again (5) is valid in any gauge.

Theorems 1, 3, and 4 are generalizations of theorems for the usual spherical harmonics to the case of monopole harmonics. Another theorem, the spherical-harmonics addition theorem[4] can be similarly generalized and will be given below in Sec. III.

II. RELATIONSHIP TO $D(\alpha,\beta,\gamma)$

We now use the function d and the matrix D:

$$d^{(j)}_{m'm}(\beta) \equiv \langle jm'|\exp(i\beta J_y)|jm\rangle \ , \tag{6}$$

$$D^{(j)}(\alpha,\beta,\gamma) \equiv \exp(i\alpha J_z)\exp(i\beta J_y)\exp(i\gamma J_z) \tag{7}$$

defined in Chap. 4 of Ref. 3. It is easy to explicitly prove[5] the following.

Theorem 5. In region a,

$$Y_{q,l,m}(\theta,\phi) = [(2l+1)/4\pi]^{1/2} e^{i(q+m)\phi} d^{(l)}_{-m,q}(\theta) \ . \tag{8}$$

Notice that

$$D(\alpha+2\pi,\beta,\gamma) = D(\alpha,\beta,\gamma+2\pi) = (-1)^{2j} D(\alpha,\beta,\gamma) \ . \tag{9}$$

It is convenient to remember that

$$d^{(j)}_{m'm} = d^{(j)}_{mm'} e^{-i\pi(m-m')} \ , \qquad d^{(j)}_{mm'}(\pi) = \delta(m+m')(-1)^{j-m} \ , \tag{10}$$

and that the matrix d satisfies

$$\tilde{d}d = 1 \ .$$

It follows from this and from theorem 2 that in region a,

$$(D^{-1})_{mm'} = [4\pi/(2l+1)]^{1/2} e^{-im'(\gamma+\alpha)} Y^*_{-m',l,m}(\beta,\gamma) \ . \tag{11}$$

III. MONOPOLE-HARMONICS ADDITION THEOREM

Consider a rotation of coordinate axis[6] by Euler's angles α,β,γ which changes the coordinates r,θ,ϕ of a point to r,θ',ϕ', where

$$\sin\theta\, e^{i\phi} = -\cos\theta' \sin\beta\, e^{-i\alpha} + \sin\theta'[\cos(\phi'-\gamma)\cos\beta\, e^{-i\alpha} + i\sin(\phi'-\gamma)e^{-i\alpha}] \ ,$$

$$\cos\theta = \cos\theta'\cos\beta + \sin\theta'\cos(\phi'-\gamma)\sin\beta \ , \tag{12}$$

$$0\le\beta\le\pi, \quad -2\pi\le\alpha\le 0, \quad 0\le\gamma\le 2\pi \ .$$

Under such a transformation the usual spherical harmonic Y_{lm} undergoes a linear transformation

$$Y_{lm}(\theta',\phi') = \sum_{m'} Y_{lm'}(\theta,\phi) D^{(l)}_{m'm}(\alpha,\beta,\gamma) \ . \tag{13}$$

These are the equations[6] given in Ref. 3. For monopole harmonics, as already discussed in Eq. (50) of Ref. 1, the same transformation gives

$$Z_{q,l,m}(\theta',\phi') = \sum_{m'} Y_{q,l,m'}(\theta,\phi) D^{(l)}_{m'm} \ , \tag{14}$$

where Z is $Y_{q,l,m}(\theta',\phi')$ but in a different gauge. To change Z into $Y(\theta',\phi')$ we need to multiply by the factor $T_{a'a}$ given by Eq. (46) of Ref. 1. Thus

$$Y_{q,l,m}(\theta',\phi') = B(\alpha,\beta,\gamma) e^{-iq\Omega} \sum_{m'} Y_{q,l,m'}(\theta,\phi) D^{(l)}_{m'm}(\alpha,\beta,\gamma) \ , \tag{15}$$

where Ω is defined to be the solid angle subtended at the point P by the shaded area in Fig. 1. In other words

Ω = solid angle at 0 between shaded area and the extension of the line $P0$
= area of spherical ΔABP on the unit sphere.

By a well-known theorem,

$$\Omega = R + Z + Z' - \pi \ . \tag{16}$$

This function has the required discontinuity[1] of 4π. It is easy to see geometrically that

$$Z = \phi - (-\alpha-\pi), \quad Z' = \gamma - \phi' \ . \tag{17}$$

To determine $B(\alpha,\beta,\gamma)$ we put the point P at B, i.e.,

$$\theta' = 0, \quad \theta=\beta, \quad \phi = -\alpha-\pi, \quad \Omega = 0 \ .$$

The left-hand side of (15) becomes then, in region a,

$$[(2l+1)/4\pi]^{1/2} e^{i(q+m)\phi'} \delta_{q+m} = [(2l+1)/4\pi]^{1/2} \delta_{q+m} \ , \tag{18}$$

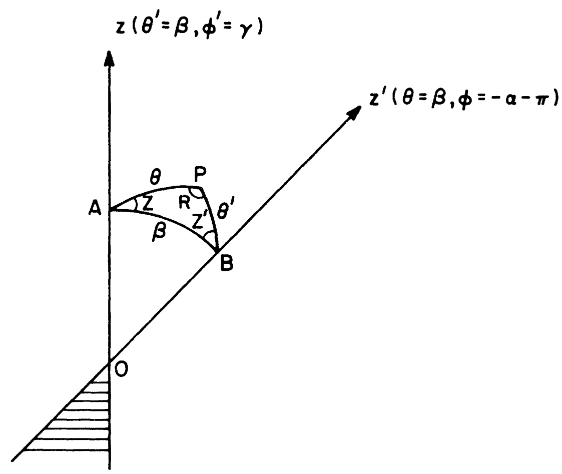

FIG. 1. Rotation of coordinate axes. The rotation changes the spherical coordinates of P from $(1,\theta,\phi)$ to $(1,\theta',\phi')$. The transformation is explicitly given in (12). The spherical coordinates (Ref. 6) of the z axis are $\theta'=\beta$, $\phi'=\gamma$. Those of the z' axis are $\theta=\beta$, $\phi=-\alpha-\pi$. Angles Z and Z' are given by (17). ABP is a spherical triangle on the unit sphere around the origin O.

while the right-hand side becomes

$$B(\alpha,\beta,\gamma)\sum_{m'} Y_{q,l,m'}(\beta,-\alpha-\pi)D^{(l)}_{m'm}(\alpha,\beta,\gamma)\ . \tag{19}$$

Equating (18) and (19), and using (8), (7), and (10), we obtain

$$B(\alpha,\beta,\gamma)=e^{iq(\alpha+\gamma)}\ . \tag{20}$$

Thus we have, using (16) and (17), the following theorem.

Theorem 6. Under the transformation (12), in region a,

$$Y_{q,l,m}(\theta',\phi') = e^{iq(\phi'-\phi-R)}\sum_{m'} Y_{q,l,m'}(\theta,\phi)D^{l}_{m'm}(\alpha,\beta,\gamma)\ , \tag{21}$$

where R is the angle defined in Fig. 1.

We can write this equation in a more convenient form by operating with D^{-1} on both sides, obtaining

$$\sum_m (D^{-1})^l_{mm'}Y_{q,l,m}(\theta',\phi')=e^{iq(\phi'-\phi-R)}Y_{q,l,m'}(\theta,\phi)\ . \tag{22}$$

Using (11) one reduces this to

$$\sum_m Y_{q,l,m}(\theta',\phi')Y^*_{q',l,m}(\beta,\gamma) = [(2l+1)/4\pi]^{1/2}Y_{q,l,-q'}(\theta,0)\times e^{i(q\phi'-q'\gamma)}e^{-i(qR+q'Z-q'\pi)}\ , \tag{23}$$

where we have put $q'=-m'$, and have used (17). Now in the $x'y'z'$ coordinate system, the angular coordinates of P and A are respectively θ',ϕ' and β,γ. Thus we change notation as follows:

$$\theta'=\beta'\ ,\quad \phi'=\gamma'\ ,\quad Z=R'\ , \tag{24}$$

and obtain Fig. 2. Equation (23) then becomes

Theorem 7.

$$\sum_m Y_{q,l,m}(\beta',\gamma')Y^*_{q',l,m}(\beta,\gamma) = [(2l+1)/(4\pi)]^{1/2}Y_{q,l,-q'}(\theta,0)\times e^{i(q\gamma'-q'\gamma)}e^{-i(qR+q'R'-q'\pi)}\ , \tag{25}$$

where θ, R, and R' are defined in Fig. 2, and all Y's are evaluated in R_a. [If $Y_{q,l,m}(\beta',\gamma')$ is evaluated in R_a, and $Y_{q',l,m}(\beta,\gamma)$ in R_b, then one should replace the factor $e^{i(q\gamma'-q'\gamma)}$ in (25) by $e^{i(q\gamma'+q'\gamma)}$. Similar changes are necessary for the other two combinations of regions.]

IV. ROTATION AROUND z AXIS BY 360°

A spin-$\frac{1}{2}$ system acquires a phase factor of -1 when rotated by 360°. How does the e-g system in a state $l=\frac{1}{2}$ acquire such a factor? This problem

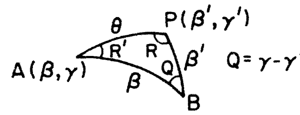

FIG. 2. ΔABP in the new notation after substitution (24). In the $x'y'z'$ coordinate system the spherical coordinates of P and A are as shown.

appears at first very puzzling since the monopole harmonics $Y_{q,l,m}$ are functions of θ,ϕ (see Table 1 of Ref. 1) and a rotation by 360° around the z axis,

$$\theta'=\theta,\quad \phi'=\phi+\Delta\ , \tag{26}$$

where $\Delta=2\pi$, seems to leave all $Y_{q,l,m}$ unchanged. The resolution of this difficulty lies in the fact that under a rotation (26) a gauge transformation is necessary. The inclusion of the gauge transformation leads naturally to the phase factor of -1.

To see this point more clearly, take $q=\frac{1}{2}$ and $l=\frac{1}{2}$. In this case, the harmonics $\sqrt{4\pi}\,Y_{1/2,1/2,m}$ are[1]

$m=\frac{1}{2}$	$m=-\frac{1}{2}$
$\psi_+^{(a)}=-e^{i\phi}(1-\cos\theta)^{1/2}$	$\psi_-^{(a)}=(1+\cos\theta)^{1/2}$
$\psi_+^{(b)}=-(1-\cos\theta)^{1/2}$	$\psi_-^{(b)}=e^{-i\phi}(1+\cos\theta)^{1/2}$,

(27)

while the transition function $S=\psi^{(a)}/\psi^{(b)}$ is $e^{i\phi}$. If ϕ is replaced by $\phi'=\phi+\Delta$ of (26), the corresponding table is

$m=\frac{1}{2}$	$m=-\frac{1}{2}$
$\psi_+^{(a)\prime}=-e^{i\phi}e^{i\Delta}(1-\cos\theta)^{1/2}$	$\psi_-^{(a)\prime}=(1+\cos\theta)^{1/2}$
$\psi_+^{(b)\prime}=-(1-\cos\theta)^{1/2}$	$\psi_-^{(b)\prime}=e^{-i\phi-i\Delta}(1+\cos\theta)^{1/2}$,

(28)

while the transition function is now $S'=e^{i\phi}e^{i\Delta}$. To compare (28) with (27) we first perform a gauge transformation on ψ'_+ and ψ'_-. In R_a we multiply the wave functions by $e^{-i\Delta/2}$ and in R_b we multiply the wave functions by $e^{i\Delta/2}$:

$$\psi_\pm^{(a)\prime\prime}=e^{-i\Delta/2}\psi_\pm^{(a)\prime}\ ,\quad \psi_\pm^{(b)\prime\prime}=e^{i\Delta/2}\psi_\pm^{(b)}\ , \tag{29}$$

obtaining

$$\begin{aligned}\psi_+^{(a)\prime\prime}&=-e^{i\phi}e^{i\Delta/2}(1-\cos\theta)^{1/2},\\ \psi_-^{(a)\prime\prime}&=e^{-i\Delta/2}(1+\cos\theta)^{1/2}\ ,\\ \psi_+^{(b)\prime\prime}&=-e^{i\Delta/2}(1-\cos\theta)^{1/2}\ ,\\ \psi_-^{(b)\prime\prime}&=e^{i\phi-i\Delta/2}(1+\cos\theta)^{1/2}\ .\end{aligned} \tag{30}$$

Now we find the transition function $S''=\psi''^{(a)}/\psi''^{(b)}=e^{i\phi}$ which is the same as S. The wave functions ψ''_+ and ψ''_- can thus be properly compared with ψ_+ and ψ_-. Indeed we find (in both R_a and R_b)

$$\psi''_+=e^{i\Delta/2}\psi_+,\quad \psi''_-=e^{-i\Delta/2}\psi_-\ . \tag{31}$$

Thus under a rotation (26), the $m=\frac{1}{2}$ state acquires a phase factor $e^{i\Delta/2}$, and the $m=-\frac{1}{2}$ state acquires a phase factor $e^{-i\Delta/2}$. When $\Delta=2\pi$, both phase factors are -1.

We notice that the gauge transformation in R_a is, by (29),

$$\psi_{\pm}^{(a)\prime}=e^{i\Delta/2}\psi_{\pm}^{(a)\prime\prime} ,$$

which agrees with (21) above (as it should), since $R=0$, $\phi'-\phi=\Delta$ for the rotation (26).

*Work supported in part by the U. S. ERDA under Contract No. (11-1)-3227.

†Work supported in part by the National Science Foundation under Grant No. PHY7615328.

[1]Tai Tsun Wu and Chen Ning Yang, Nucl. Phys. B107, 365 (1976).

[2]Hou Pei-yu and Hsi Ting-chang, Scientia Sinica (to be published).

[3]A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton Univ. Press, Princeton, N.J., 1960).

[4]Cf. E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis* (Cambridge Univ. Press, London, 1946), p. 328; Ref. 3, p. 63.

[5]This theorem in region *a* is already known in the literature. See I. G. Tamm, Z. Phys. 71, 141 (1931); D. G. Boulware, L. S. Brown, R. N. Cahn, S. D. Ellis, and C. Lee, Phys. Rev. D 14, 2708 (1976).

[6]Equation (12) is also the definition of the transformation $\theta,\phi\to\theta',\phi'$ for Chap. 4 of Ref. 3. To show this, we take $l=1$ in (4.1.4) of Ref. 3 and readily verify that it is equivalent to (12). (The definition given in Ref. 3 at the bottom of p. 53 is incorrect.) Equation (12) can also be written in Cartesian coordinates:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} \cos\alpha\cos\beta\cos\gamma-\sin\alpha\sin\gamma & \cos\alpha\cos\beta\sin\gamma+\sin\alpha\cos\gamma & -\cos\alpha\sin\beta \\ -\sin\alpha\cos\beta\cos\gamma-\cos\alpha\sin\gamma & -\sin\alpha\cos\beta\sin\gamma+\cos\alpha\cos\gamma & \sin\alpha\sin\beta \\ \sin\beta\cos\gamma & \sin\beta\sin\gamma & \cos\beta \end{bmatrix}\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} .$$

Thus the z' axis is along the direction $\theta=\beta$, $\phi=-\alpha-\pi$, and the z axis is along the direction $\theta'=\beta$, $\phi'=\gamma$. These are indicated in Fig. 1. The identity transformation corresponds to $\alpha=-\pi$, $\beta=0$, $\gamma=\pi$.