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Supergravity and the S matrix*

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We discuss supergravity from an S-matrix point of view. Kinematical constraints on helicity amplitudes determine the spin-2 and spin-3/2 Born amplitudes almost uniquely, and force Born amplitudes involving spin-5/2 fermions to vanish. Global supersymmetry is then used to determine the Born amplitudes completely. We suggest that Lorentz invariance, presence of only one (dimensional) coupling constant κ, and global supersymmetry lead to a unique locally supergauge-invariant theory of spin-2 and spin-3/2 particles. We draw similar conclusions for a model uniting supergravity and the Wess-Zumino scalar multiplet.

I. INTRODUCTION

In a recent paper[1] Freedman, van Nieuwenhuizen, and Ferrara proposed a Lagrangian for supergravity, incorporating fields for gravitons and massless spin-$\frac{3}{2}$ fermions only. The action for this system is not only generally covariant but also invariant under local supergauge transformations. This Lagrangian is the sum of the usual Einstein Lagrangian, a minimal covariant generalization of the Rarita-Schwinger Lagrangian[2] for a massless spin-$\frac{3}{2}$ Majorana field and an additional four-fermion interaction term. Deser and Zumino[3] were able to show that the four-fermion interaction can be obtained from an action consisting only of an Einstein action with torsion and the Rarita-Schwinger action by eliminating the torsion field by means of the equations of motion.

In this paper we examine the supergauge-invariant system using an S-matrix viewpoint previously applied to gravitation.[4] We show that the Born approximation for the two-particle scattering amplitude in any local Lorentz-invariant theory describing gravitons and massless spin-$\frac{3}{2}$ fermions is uniquely determined by the following requirements: (1) There should be only one coupling constant κ, which should have the dimension of reciprocal mass and which should enter the cubic part of the Lagrangian linearly and the quartic part quadratically, and (2) the theory should be *globally* supersymmetric. We argue that the three- and four-point couplings are uniquely determined and are therefore those of the locally supersymmetric Lagrangian. This result is the supersymmetric analog of the theorems of Weinberg[5] relating particle content and Lorentz invariance to global charge conservation and local electromagnetic gauge invariance of the S matrix and to global energy-momentum conservation and local gravitational gauge invariance of the S matrix.

Another candidate for a supergravity theory would incorporate a spin-2–spin-$\frac{5}{2}$ system.[6] However, we show that fermion-fermion and fermion-graviton Born amplitudes must vanish in a theory similar to that described above satisfying condition (1), but with spin-$\frac{5}{2}$ rather than spin-$\frac{3}{2}$ fermions. Should the theory satisfy condition (2) as well the graviton-graviton Born amplitudes also vanish. The graviton-graviton amplitudes are already constrained to be those of conventional Einstein gravitation[4] since only gravitons can be exchanged in graviton-graviton scattering. We conclude that the gravitational constant κ must vanish. This result rules out the spin-2–spin-$\frac{5}{2}$ system as a candidate for an interesting supergauge-invariant theory.

The plan of this paper follows: In Sec. II we summarize some constraints that helicity amplitudes for massless particles satisfy in a Lorentz-invariant local theory. If the theory in addition satisfies condition (1) the constraints determine the Born amplitudes for the spin-2–spin-$\frac{3}{2}$ system uniquely, except for overall multiplicative constants. For the system containing spin-$\frac{5}{2}$ fermions and gravitons the $\frac{5}{2}$-$\frac{5}{2}$ and 2-$\frac{5}{2}$ amplitudes can satisfy the constraints only by vanishing identically.

In Sec. III we demonstrate that global supersymmetry, if it is a symmetry of the S matrix, relates the fermion and graviton amplitudes, removing the ambiguity associated with the multiplicative constants for the amplitudes of the 2-$\frac{3}{2}$ system, and forcing the graviton-graviton amplitudes to vanish for the 2-$\frac{5}{2}$ system.

In Sec. IV we discuss the form of the Born amplitudes for a system combining supergravity and a massless Wess-Zumino (WZ) scalar multiplet[7] in a globally supersymmetric way. The amplitudes involving two WZ particles and two supergravitons are again determined uniquely by the requirements imposed by kinematical constraints and global supersymmetry.

In Appendix A we give some details of the kinematical considerations used in Sec. II. In Appendix B we work out the action of global supergauge transformations on helicity states. The explicit form is not needed to establish most of the results of Sec. III, but it provides a useful check.

II. THE BORN AMPLITUDES

We shall consider a theory that may be described by the Lagrangian

$$L = L_2 + \kappa L_3 + \kappa^2 L_4 + \kappa^3 L_5 + \cdots , \tag{1}$$

where the free Lagrangian L_2 defines a Lorentz-covariant field theory of two particles, a spin-2 massless boson (the graviton) and a spin-$\frac{3}{2}$ massless fermion. The interaction terms κL_3 and $\kappa^2 L_4$ are cubic and quartic in the fields and their derivatives, and have the physical dimension of mass to the fourth power. The coupling constant κ has the dimension of reciprocal mass; all its appearances in L have been explicitly indicated in Eq. (1).

A graviton having four-momentum p and helicity ± 2 will be described by a tensor wave function

$$\epsilon_{\mu\nu}(p, \pm 2) = \epsilon_\mu(p, \pm 1)\epsilon_\nu(p, \pm 1) . \tag{2}$$

A fermion having four-momentum p and helicity $\pm\frac{3}{2}$ will be described by a spinor-vector wave function

$$u_{a\mu}(p, \pm\tfrac{3}{2}) = u_a(p, \pm\tfrac{1}{2})\epsilon_\mu(p, \pm 1) . \tag{3}$$

The functions $\epsilon_\mu(p, \pm 1)$ occurring in (2) and (3) are the usual polarization vectors for a photon, and the $u_a(p, \pm\frac{1}{2})$ are the usual spinors for a spin-$\frac{1}{2}$ fermion; these functions satisfy

$$p \cdot \epsilon(p, \pm 1) = 0, \quad \gamma \cdot p u(p, \pm\tfrac{1}{2}) = 0 , \tag{4a}$$

$$\epsilon(p, \lambda)^* \cdot \epsilon(p, \sigma) = \delta_{\lambda\sigma}, \quad u(p, \lambda)^* u(p, \sigma) = \delta_{\lambda\sigma} . \tag{4b}$$

It is convenient to associate the graviton with a local symmetric second-rank tensor field $h_{\mu\nu}$ and the fermion with a local Rarita-Schwinger Majorana field $\psi_{a\mu}$.

Although we shall not need an explicit form of the Lagrangian the following result is useful: In Ref. 4 we have observed that the three-graviton vertex conserves helicity when two of the gravitons are on-shell and collinear. A similar result holds for the graviton-two-fermion vertex. Indeed, that vertex may be obtained from the interaction term

$$\kappa h_{\mu\nu}\bar{\psi}_{a\lambda} M_{ab}^{\mu\nu\lambda\sigma\rho} \partial_\rho \psi_{b\sigma} ,$$

where M is a numerical matrix constructed from γ matrices. The interaction must have exactly one differential operator in order to have physical dimension of mass to the fourth power. Now if the fermions are on-shell and collinear, any term in which ∂_ρ is contracted with $\psi_{a\lambda}$, $\psi_{b\sigma}$, or a matrix γ^α_{cd} in M will give zero. Note that our choice of helicity wave functions is consistent with the gauge for which

$$\partial^\mu \psi_{a\mu} = \gamma^\mu_{ab}\psi_{b\mu} = \gamma^\mu_{ab}\partial_\mu \psi_{b\lambda} = 0 \tag{5}$$

for fermions on the mass shell. We conclude that a nonvanishing vertex arises only from terms in which ∂_ρ is contracted with $h_{\mu\nu}$. Since the contraction of γ^α with $\bar{\psi}_{a\lambda}$ or $\psi_{b\sigma}$ vanishes, the Greek indices of $\bar{\psi}_{a\lambda}$ must be contracted with those of $\psi_{b\sigma}$, leading to a vertex proportional to

$$\epsilon(p', \lambda')^* \cdot \epsilon(p, \lambda) .$$

However, when $p = p'$ and $\lambda \neq \lambda'$ the expression above vanishes because of the orthogonality of the helicity vectors $\epsilon(p, \lambda')$ and $\epsilon(p, \lambda)$. It follows that a one-graviton-two-fermion vertex conserves helicity if the two fermions are on-shell.

We let $F(\lambda_3, \lambda_4; \lambda_1, \lambda_2; s, t, u)$ denote the helicity amplitude for the process $1 + 2 \to 3 + 4$. The square of the center-of-mass energy is called s; the scattering angle is called θ. Then

$$s = -(p_1 + p_2)^2, \quad t = -(p_1 - p_3)^2, \quad u = -(p_1 - p_4)^2.$$

We define λ_s, λ_t, and λ_u by

$$\begin{aligned} \lambda_s &= |\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4| , \\ \lambda_t &= |\lambda_1 - \lambda_2 - \lambda_3 + \lambda_4| , \\ \lambda_u &= |\lambda_1 - \lambda_2 + \lambda_3 - \lambda_4| . \end{aligned} \tag{6}$$

When all the particles are massless the helicity amplitude is a Lorentz-invariant function of s, t, and u. Ader, Capdeville, and Navelet (ACN)[8] have established a very useful theorem concerning the kinematical singularities of helicity amplitudes for processes involving massless particles only, processes for which

$$\cos(\theta/2) = (-u/s)^{1/2}, \quad \sin(\theta/2) = (-t/s)^{1/2} . \tag{7}$$

Define $\tilde{F}$ as follows:

$$F(\lambda_3, \lambda_4; \lambda_1, \lambda_2) = (\sqrt{s})^{\lambda_s}(\sqrt{-t})^{\lambda_t}(\sqrt{-u})^{\lambda_u}\tilde{F} . \tag{8}$$

Then according to ACN the function $\tilde{F}$ has dynamical singularities only.

Equation (8) puts strong constraints on the amplitude F if the particles have high spin. When we combine those constraints with our knowledge of the possible forms for the dynamical singularities of Born amplitudes, we are often led to a complete determination of such amplitudes.

Let us consider the spin-2–spin-$\frac{3}{2}$ system. If it can be described by the Lagrangian given in Eq. (1) the Born amplitudes will be proportional to κ^2, have the kinematical factors displayed in Eq. (8), and have the form implied by

$$\tilde{F}=\phi/(stu)\,, \tag{9}$$

with ϕ a polynomial in s, t, and u. The fact that the possible dynamical singularities are simple poles at $s=0$, $t=0$, or $u=0$ has been taken into account in a symmetrical fashion by inserting the denominator stu in Eq. (9). Furthermore, the amplitude F is dimensionless and satisfies some symmetry properties which we have described in Appendix A. Then, as is shown in more detail in Appendix A, for a given choice of λ's F is determined as a function of s, t, and u, up to a multiplicative constant.

We list below a basic set of amplitudes. Amplitudes not listed can be obtained from the listed ones by using the crossing operation, Bose or Fermi statistics, or discrete symmetry operations (space reflection, or time reversal). The appropriate relations are given in Appendix A in Eqs. (A1)–(A4).

We are assuming that the theory is invariant under the discrete symmetry operations. Otherwise, although the functional forms of the unlisted amplitudes are uniquely determined, the multiplicative constants are unrelated to the ones appearing in the listed amplitudes.

The graviton-graviton amplitudes are found to be

$$F(2,2;2,2)=c\kappa^2 s^3/(tu)\,, \tag{10a}$$

$$F(2,-2;2,-2)=c\kappa^2 u^3/(st)\,, \tag{10b}$$

$$F(2,2;2,-2)=0\,, \tag{10c}$$

$$F(2,2;-2,-2)=0\,. \tag{10d}$$

These amplitudes are the same as those[4] of the conventional Einstein theory without the cosmological term. The cosmological term is ruled out, among other reasons, because it does not allow the L_2, L_3, and L_4 of Eq. (1) to be independent of κ (or an equivalent parameter). The vanishing of $F(2,2;-2,-2)$ is established using the theorem mentioned earlier regarding helicity conservation at a three-graviton vertex with two gravitons on shell; for details of the argument see Ref. 4 or the analogous fermion argument in Appendix A.

If κ^2 is assigned its conventional value, namely $32\pi G$, where G is Newton's gravitational constant, then the usual Einstein theory has $c=i/4$. The $\frac{3}{2}$-$\frac{3}{2}$ amplitudes are the following:

$$F(\tfrac{3}{2},\tfrac{3}{2};\tfrac{3}{2},\tfrac{3}{2})=c'\kappa^2 s^3/(tu)\,, \tag{11a}$$

$$F(\tfrac{3}{2},-\tfrac{3}{2};\tfrac{3}{2},-\tfrac{3}{2})=c'\kappa^2 u^3/(st)\,, \tag{11b}$$

$$F(\tfrac{3}{2},\tfrac{3}{2};\tfrac{3}{2},-\tfrac{3}{2})=0\,, \tag{11c}$$

$$F(\tfrac{3}{2},\tfrac{3}{2};-\tfrac{3}{2},-\tfrac{3}{2})=0\,. \tag{11d}$$

The result given by Eq. (11d) is established using the helicity-conservation theorem proven immediately preceding Eq. (5); details are given in Appendix A.

The 2-$\frac{3}{2}$ amplitudes are the following:

$$F(2,\tfrac{3}{2};2,\tfrac{3}{2})=c''\kappa^2 s^2(-su)^{1/2}/(tu)\,, \tag{12a}$$

$$F(2,-\tfrac{3}{2};2,-\tfrac{3}{2})=c''\kappa^2 u^2(-su)^{1/2}(ts)\,, \tag{12b}$$

$$F(2,\tfrac{3}{2};2,-\tfrac{3}{2})=F(2,\tfrac{3}{2};-2,-\tfrac{3}{2})=0\,, \tag{12c}$$

$$F(2,-\tfrac{3}{2};-2,\tfrac{3}{2})=F(2,\tfrac{3}{2};-2,\tfrac{3}{2})=0\,. \tag{12d}$$

In the next section we shall use global supersymmetry to relate the constants c, c', and c'', and to check the functional forms given in (10), (11), and (12).

Let us briefly discuss a theory containing spin $\frac{5}{2}$ instead of spin $\frac{3}{2}$, which is, according to Salam and Strathdee[6] and Zumino, another candidate for a supergauge-invariant system. If we continue to assume that the Lagrangian is of the form given in Eq. (1) it is easy to verify that all the spin-2–spin-$\frac{5}{2}$ and spin-$\frac{5}{2}$–spin-$\frac{5}{2}$ Born amplitudes must vanish; Eq. (8) requires more powers of s in the numerator than is consistent with the amplitudes being dimensionless, proportional to κ^2, and containing only the factor stu in the denominator. Since, as we shall see, global supersymmetry relates these amplitudes to those for graviton-graviton scattering, the latter must vanish as well. But the graviton-graviton amplitudes cannot have been affected by the introduction of fermions into a system consisting of gravitons only, since no fermions may be exchanged internally (in the Born approximation), and so the amplitudes must still be those of Einstein theory. The only way the Einstein amplitudes can vanish is for κ itself to vanish; gravitation must be switched off (and not just in the Born approximation). This conclusion is consistent with the assertion of Deser and Zumino[3] concerning the spin-2–spin-$\frac{5}{2}$ system.

III. IMPLICATIONS OF GLOBAL SUPERSYMMETRY

We assume in this section the existence of a set of Hermitian supersymmetry generators Q_a which transform like spinors under the action of the homogeneous Lorentz group (the index a is a spinor index) and each of which commutes with the S matrix.[9] We contract the generators Q_a with the anticommuting c-number components η_a of a Majorana spinor; the operators $Q(\eta)=\bar{\eta}_a Q_a$ will then satisfy simple *commutation* relations with each other and with other operators for our physical system (fermions and gravitons). Let $a(p,\sigma)$ be the "in" annihilation operator for gravitons of momentum p and helicity $\lambda=2\sigma$; let $c(p,\sigma)$ be the "in" annihilation operator for fermions of momentum p and helicity $\lambda=\frac{3}{2}\sigma$. We expect the action of $Q(\eta)$ on a graviton to produce a fermion with the same four-momentum and handedness σ. Handedness is conserved by $Q(\eta)$ since it can change spin by at most $\frac{1}{2}$. So

$$[Q(\eta), a(p,\sigma)] = \Gamma(\eta, p, \sigma) c(p,\sigma) , \tag{13a}$$

$$[Q(\eta), c(p,\sigma)] = \Delta(\eta, p, \sigma) a(p,\sigma) . \tag{13b}$$

The coefficients Γ and Δ are linear in the anticommuting c-number variables η_a.

Letting P_μ be the components of the free-particle "in" four-momentum operator and S the scattering operator, we require that

$$[Q(\eta), S] = 0 , \quad [Q(\eta), P_\mu] = 0 ,$$
$$P_\mu = \sum_\sigma \int d^3p \, p_\mu [a(p,\sigma)^* a(p,\sigma) + c(p,\sigma)^* c(p,\sigma)] . \tag{14}$$

We also assume that the vacuum state is invariant under the action of $Q(\eta)$;

$$Q(\eta) |\text{vac}\rangle = 0 , \tag{15}$$

that is, we exclude the case of spontaneous breaking of supersymmetry and of the attendant Goldstone fermions.

The supersymmetry algebra forces the commutator of two Q generators to produce a translation; with our normalization convention

$$[Q(\eta), Q(\zeta)] = -2i\bar{\eta}\gamma \cdot P\zeta . \tag{16}$$

The vanishing of the commutator of $Q(\eta)$ with P_μ forces

$$\Gamma(\eta, p, \sigma)^* = \Delta(\eta, p, \sigma) , \tag{17}$$

while Eq. (16) and the Jacobi identity imply that

$$\Gamma(\eta, p, \sigma)\Delta(\zeta, p, \sigma) - \Gamma(\zeta, p, \sigma)\Delta(\eta, p, \sigma) = -2i\bar{\eta}\gamma \cdot p\zeta . \tag{18}$$

We complete the determination of Γ in Appendix B, finding that

$$\Gamma(\eta, p, \sigma) = i[2E(p)]^{1/2}\bar{\eta}\gamma \cdot \epsilon(p,\sigma)^* u(p,\sigma) . \tag{19}$$

We are now in a position to relate helicity amplitudes using global supersymmetry. We illustrate the method for amplitudes in which all four particles have positive handedness. Consider the equation

$$\langle \text{vac} | c(q',+) a(p',+) [Q(\eta), S] a(p,+)^* a(q,+)^* | \text{vac}\rangle = 0 . \tag{20}$$

Working this out, we find that

$$\Delta(q')F(2,2;2,2) - \Gamma(p')F(\tfrac{3}{2},\tfrac{3}{2};2,2)$$
$$= \Gamma(p)^* F(2,\tfrac{3}{2};\tfrac{3}{2},2) + \Gamma(q)^* F(2,\tfrac{3}{2};2,\tfrac{3}{2}) , \tag{21}$$

where $\Gamma(p') = \Gamma(\eta, p', +)$, and so on. We have taken care to observe the minus sign that results when a Γ or Δ is moved past a Fermi operator.

Substituting the values given in Eqs. (10) and (12) or values obtained from those by crossing or using particle interchanges, we find that

$$\Gamma(q')^* c\kappa^2 \frac{s^3}{tu}$$
$$= -\Gamma(p)^* c'' \kappa^2 \frac{s^2(-st)^{1/2}}{tu} + \Gamma(q)^* c'' \kappa^2 \frac{s^2(-su)^{1/2}}{tu} . \tag{22}$$

The three-momenta of all four of our particles lie in the x-z plane (the scattering plane). The four-momentum of such a massless particle can be written as

$$p = (E, E\sin\tilde{\theta}, 0, E\cos\tilde{\theta}), \quad 0 \le \tilde{\theta} \le 2\pi . \tag{23}$$

According to Eq. (B22a), derived in Appendix B,

$$\Gamma(\eta, p, +) = (2E)^{1/2}(\eta_1 \cos\tfrac{1}{2}\tilde{\theta} + \eta_2 \sin\tfrac{1}{2}\tilde{\theta}) , \tag{24}$$

where the four components of the Majorana spinor η have the following forms when expressed in terms of the two anticommuting quantities η_1 and η_2:

$$\eta = \tfrac{1}{2}(\eta_1 + \eta_2^*, -\eta_1^* + \eta_2, -\eta_1 + \eta_2^*, -\eta_1^* - \eta_2) . \tag{25}$$

Letting $\tilde{\theta}_i$ be the angle associated with the ith particle, we have $\tilde{\theta}_1 = 0$, $\tilde{\theta}_2 = \pi$, $\tilde{\theta}_3 = \theta$, the scattering angle, and $\tilde{\theta}_4 = \theta + \pi$. Then

$$\Gamma(p) = (2E)^{1/2}\eta_1 , \tag{26a}$$

$$\Gamma(q) = (2E)^{1/2}\eta_2 , \tag{26b}$$

$$\Gamma(q') = (2E)^{1/2}\left[-\eta_1\left(\frac{-t}{s}\right)^{1/2} + \eta_2\left(\frac{-u}{s}\right)^{1/2}\right], \tag{26c}$$

for arbitrary η_1 and η_2. The nontrivial dependence of $\Gamma(\eta, p, \sigma)$ on the momentum four-vector, illustrated by Eqs. (26), is just right to allow the coefficients of η_1 *and* η_2 in Eq. (22) to vanish separately, provided that

$$c'' = c .$$

In a similar fashion, starting with

$$\langle \text{vac} | c(q',+) a(p',+) [Q, S] c(p,+)^* c(q,+)^* | \text{vac}\rangle = 0 ,$$

we find that

$$c' = c'' .$$

We have therefore established the following theorem: Except for an overall multiplicative constant, which can be absorbed in the definition of κ anyway, the Born amplitudes for the spin-2–spin-$\frac{3}{2}$ system are completely determined by Lorentz invariance, invariance under global supersymmetry transformations, and the form of the Lagrangian specified in Eq. (1).

A similar method applied to the spin-2–spin-$\frac{5}{2}$

amplitudes immediately leads to the vanishing of the graviton-graviton amplitudes, showing that such a theory is trivial.

We can now draw some conclusions about the uniqueness of the effective Lagrangian for supergravity. The uniqueness of the graviton-graviton amplitudes implies that the relative strengths of the three-point and the four-point graviton interaction terms is fixed. Similarly, the uniqueness of the fermion-fermion amplitudes, which contain both graviton exchange terms and four-fermion contact terms, implies that the relative strengths of the four-fermion contact terms and the one-graviton–two-fermion interaction terms are fixed. The requirement of global supersymmetry then fixes the relative strengths of all three- and four-point couplings.

There are two main ingredients (beside global supersymmetry) that have been used in establishing these results: Lorentz invariance and minimal coupling. Lorentz invariance of the helicity amplitudes in a theory of massless particles with high spin is the S-matrix equivalent of gauge invariance. We take as our definition of minimal coupling the requirement that the n-point term in the Lagrangian contain κ to the power n-2 as a multiplicative constant, and that otherwise L_n depend only on the fields and their derivatives, with no other dimensional factors. This requirement of minimal coupling is very powerful. In a theory having gravitons only, it excludes cosmological interactions as well as generalized Weyl interactions (ones with terms proportional to R^2, $R_{\mu\nu}R^{\mu\nu}$, and so forth). We conjecture that the considerations here applied to the four-point Born amplitudes, when suitably extended to the n-point tree amplitudes, will establish the uniqueness of the full Einstein effective Lagrangian for pure gravity, a result previously established in a slightly different way by Boulware and Deser.[10]

When our considerations are extended from four-point supergraviton tree amplitudes to the general n-point supergraviton tree amplitudes they should lead to the uniqueness of the generally covariant supergauge-invariant effective Lagrangian up to field transformations which do not affect the S matrix.

IV. SUPERGRAVITY AND SUPERMATTER

In this section we shall investigate some properties of tree amplitudes in a globally supersymmetric theory incorporating supergravitons and matter particles of low spin. We shall assume that a successful marriage of supergravity and the Wess-Zumino scalar supermultiplet[7] is possible. For simplicity we shall take the WZ particles to be massless, though an extension to massive WZ particles is relatively straightforward. The resulting model is presumed to contain two coupling constants, the supergravity constant κ and the Wess-Zumino dimensionless constant, which we call β.

In order to present the form of the Lagrangian we assume, we introduce some convenient abbreviations for the local fields involved. Any linear combination of Wess-Zumino fields and their derivatives will be called W, and W can be a different combination everyplace it occurs. For example, the two W's in W^2 need not be the same WZ fields. Similarly any linear combination of supergraviton fields and their derivatives will be called G. The free Lagrangian then has the simple form

$$L_2 = W^2 + G^2 . \tag{27}$$

We define the following two replacement operations:

$$W \to W + \beta W^2 + \kappa WG + \kappa W^2 , \tag{28a}$$

$$G \to G + \kappa G^2 . \tag{28b}$$

The most general effective Lagrangian we will allow can be obtained from the free Lagrangian by repeated application of the two replacement operations (28a),(28b). The three-point and four-point couplings we will allow are then of the form

$$L^{(3)} = \beta W^3 + \kappa G^3 + \kappa W^2 G + \kappa W^3 , \tag{29a}$$

$$L^{(4)} = \beta^2 W^4 + \kappa^2 G^4 + \kappa^2 W^2 G^2 + \beta\kappa W^3 G + \kappa^2 W^3 G + \beta\kappa W^4 + \kappa^2 W^4 . \tag{29b}$$

An expression such as βW^3 stands for a sum of terms of that form, with each term in the sum having the physical dimension of mass to the fourth power. A similar remark applies to each coupling indicated in Eqs. (29), with the proviso that κ is understood to be a reciprocal mass, and that β is dimensionless.

We shall look at the Born amplitudes for processes involving two external supergravitons and two external WZ particles. Our assumption, Eqs. (29), on the form of the interaction terms in the effective Lagrangian implies that these amplitudes will be proportional to κ^2.

The Wess-Zumino scalar supermultiplet contains three particles: a scalar boson, a pseudoscalar boson, and a spin-$\frac{1}{2}$ Majorana fermion described by fields A, B, and ψ_a. In Appendix B we demonstrate the utility of introducing chiral fields

$$\mathcal{A}_+ = (\tfrac{1}{2})^{1/2}(A + iB) , \quad \mathcal{A}_- = (\tfrac{1}{2})^{1/2}(A - iB) . \tag{30}$$

We now list a basic set of amplitudes describing the scattering process $W + G \to W + G$, using helicity values as labels for the particle types. We let $+0$

be associated with $\mathcal{A}_+$ particles, and -0 be associated with $\mathcal{A}_-$ particles. This assignment is consistent with the fact that space reflection sends $\mathcal{A}_\pm$ into $\mathcal{A}_\mp$. [The space-reflection operation is still described by Eq. (A1) of Appendix A.] The amplitudes not listed can still be obtained by crossing, particle-interchange, and discrete symmetry operations, just as in the pure-supergravity case. Using kinematical considerations analogous to those of Sec. II and Appendix A, we find that

$$F(2,0;2,0)=c_1\kappa^2\frac{su}{t}\,, \tag{31a}$$

$$F(2,-0;2,-0)=c_1\kappa^2\frac{su}{t}\,, \tag{31b}$$

$$F(2,0;-2,0)=f_1\kappa^2\frac{t^3}{su}+f_2\kappa^2 t\,, \tag{31c}$$

$$F(2,0;2,-0)=f_3\kappa^2\frac{su}{t}\,, \tag{31d}$$

$$F(2,0;-2,-0)=f_4\kappa^2\frac{t^3}{su}+f_5\kappa^2 t\,, \tag{31e}$$

$$F(2,-0;-2,0)=f_6\kappa^2\frac{t^3}{su}+f_7\kappa^2 t\,, \tag{31f}$$

$$F(\tfrac{3}{2},0;\tfrac{3}{2},0)=c_1'\kappa^2\frac{s(-su)^{1/2}}{t}+f_8\kappa^2(-su)^{1/2}\,, \tag{32a}$$

$$F(\tfrac{3}{2},-0;\tfrac{3}{2},-0)=c_1'\kappa^2\frac{u(-su)^{1/2}}{t}+f_8\kappa^2(-su)^{1/2}\,, \tag{32b}$$

$$F(\tfrac{3}{2},0;-\tfrac{3}{2},0)=0\,, \tag{32c}$$

$$F(\tfrac{3}{2},0;\tfrac{3}{2},-0)=f_9(-su)^{1/2}\,, \tag{32d}$$

$$F(\tfrac{3}{2},0;-\tfrac{3}{2},-0)=0\,, \tag{32e}$$

$$F(\tfrac{3}{2},-0;-\tfrac{3}{2},0)=0\,, \tag{32f}$$

and that

$$F(2,\tfrac{1}{2};2,\tfrac{1}{2})=c_1''\kappa^2\frac{s(-su)^{1/2}}{t}\,, \tag{33a}$$

$$F(2,\tfrac{1}{2};-2,\tfrac{1}{2})=f_{10}\kappa^2\frac{t^2(-su)^{1/2}}{su}\,, \tag{33b}$$

$$F(2,-\tfrac{1}{2};2,-\tfrac{1}{2})=c_1''\kappa^2\frac{u(-su)^{1/2}}{t}\,, \tag{33c}$$

$$F(2,\tfrac{1}{2};2,-\tfrac{1}{2})=0\,, \tag{33d}$$

$$F(2,\tfrac{1}{2};-2,-\tfrac{1}{2})=0\,, \tag{33e}$$

$$F(2,-\tfrac{1}{2};-2,\tfrac{1}{2})=0\,, \tag{33f}$$

$$F(\tfrac{3}{2},\tfrac{1}{2};\tfrac{3}{2},\tfrac{1}{2})=\tilde{c}_1\kappa^2\frac{su}{t}+f_{11}\kappa^2 s\,, \tag{34a}$$

$$F(\tfrac{3}{2},-\tfrac{1}{2};\tfrac{3}{2},-\tfrac{1}{2})=\tilde{c}_1\kappa^2\frac{su}{t}+f_{11}\kappa^2 u\,, \tag{34b}$$

$$F(\tfrac{3}{2},\tfrac{1}{2};-\tfrac{3}{2},-\tfrac{1}{2})=f_{12}\kappa^2\frac{t^3}{su}+f_{13}\kappa^2 t\,, \tag{34c}$$

$$F(\tfrac{3}{2},-\tfrac{1}{2};-\tfrac{3}{2},\tfrac{1}{2})=f_{14}\kappa^2\frac{t^3}{su}+f_{15}\kappa^2 t\,, \tag{34d}$$

$$F(\tfrac{3}{2},\tfrac{1}{2};-\tfrac{3}{2},\tfrac{1}{2})=0\,, \tag{34e}$$

$$F(\tfrac{3}{2},\tfrac{1}{2};\tfrac{3}{2},-\tfrac{1}{2})=0\,, \tag{34f}$$

and that

$$F(2,-\tfrac{1}{2};\tfrac{3}{2},-0)=\hat{c}_1\kappa^2\frac{u(-st)^{1/2}}{t}\,, \tag{35a}$$

$$F(2,-\tfrac{1}{2};\tfrac{3}{2},0)=f_{16}\kappa^2\frac{u(-st)^{1/2}}{t}\,, \tag{35b}$$

$$F(2,\tfrac{1}{2};-\tfrac{3}{2},\pm 0)=f_{17}^{(\pm)}\kappa^2\frac{t^2(-st)^{1/2}}{su}+f_{18}^{(\pm)}\kappa^2(-st)^{1/2}+f_{19}^{(\pm)}\kappa^2\frac{t(s-u)(-st)^{1/2}}{su}\,, \tag{35c}$$

$$F(2,\tfrac{1}{2};\tfrac{3}{2},\pm 0)=0\,, \tag{35d}$$

$$F(2,-\tfrac{1}{2};-\tfrac{3}{2},\pm 0)=0\,. \tag{35e}$$

We now require that the system exhibit global supersymmetry, and deduce relations between the constants c_1, c', c'', $\tilde{c}_1$, and $\hat{c}_1$, and the f_i. We write the action of the supersymmetry operations on the annihilation operators $b(p,\pm)$ and $d(p,\pm)$ associated with $\mathcal{A}_\pm$ and ψ, respectively, using results derived in Appendix B:

$$[Q(\eta),b(p,\pm)]=i\Gamma(\eta,p,\pm)^*d(p,\pm)\,, \tag{36a}$$

$$[Q(\eta),d(p,\pm)]=-i\Gamma(\eta,p,\pm)b(p,\pm)\,, \tag{36b}$$

with the Γ still given by Eqs. (B22). Note that handedness, or chirality, is conserved by the supersymmetry operation, although spinless particles clearly have no helicity. Our choice of the chiral spin-0 fields $\mathcal{A}_\pm$ was designed to "diagonalize" the action of supersymmetry operations in the Wess-Zumino supermultiplet. Note also that the Γ and Γ^* have been interchanged in Eqs. (36) as compared to Eqs. (13).

Using methods similar to those displayed in Eqs. (20)–(26), we find that

$$c_1=c_1'=c_1''=\tilde{c}_1=\hat{c}_1\,,\quad f_i=0,\ i=1,2,\ldots,19\,. \tag{37}$$

We conclude that all the $W+G\to W+G$ amplitudes are determined up to one overall constant c_1.

We note that these amplitudes have additive contributions from three kinds of graph: one with two WWG vertices, one with one GGG vertex and one WWG vertex, and one with a single $WWGG$

contact vertex. Unless each of these graphs vanishes separately (which can happen only in the trivial case, with the *WWG* coupling vanishing), we observe that the individual graphs do not possess the form required by the ACN theorem. Since the sum does, the relative strengths of the different graphs are fixed, which fixes the relative strength of the *WWG* and *GGG* vertices. But the *GGG* vertex must be that of pure Einstein theory, so the constant c_1 can only be zero, or have the value obtained from the Einstein coupling of gravitation to other matter,[4] that is,

$$c_1 = c\,. \tag{38}$$

APPENDIX A

We give in this appendix some details of the computation of the helicity amplitudes of Sec. II. Helicity amplitudes satisfy the following relations as a consequence of space-reflection symmetry, time-reversal invariance, and particle interchange. (We do not follow the Jacob-Wick second-particle phase convention,[11] but instead follow the conventions of Ader, Capdeville, and Navelet.[8])

Space-reflection symmetry implies that

$$F(\lambda_3,\lambda_4;\lambda_1,\lambda_2;s,t,u) = \eta_1\eta_2\eta_3\eta_4(-1)^{\lambda_s}(-1)^{\lambda_r} \times F(-\lambda_3,-\lambda_4;-\lambda_1,-\lambda_2;s,t,u)\,, \tag{A1}$$

with λ_s as in Eq. (6), and λ_r the sum of the spins of the four particles involved. The η_i are intrinsic parity factors associated with each of the particles. Time-reversal invariance implies that

$$F(\lambda_3,\lambda_4;\lambda_1,\lambda_2;s,t,u) = (-1)^{\lambda_t}F(\lambda_1,\lambda_2;\lambda_3,\lambda_4;s,t,u)\,, \tag{A2}$$

with λ_t as in Eq. (6). Interchanging particles 1 and 2 in the amplitude yields

$$F(\lambda_3,\lambda_4;\lambda_1,\lambda_2;s,t,u) = (-1)^{\lambda_t}(-1)^{2s_2}(-1)^{\sigma_{12}}F(\lambda_3,\lambda_4;\lambda_2,\lambda_1;s,u,t)\,, \tag{A3}$$

where s_2 is the spin of particle 2, and σ_{12} is unity when particles 1 and 2 are both fermions, and is zero otherwise.

Note that in the case of massless particles, the relative parity of the states of different handedness is not determined by invariance under proper orthochronous Lorentz transformations, and must be determined by examining the interactions the associated local fields enjoy, or by convention.

For massless particles the following crossing relation holds:

$$F(\lambda_3,\lambda_4;\lambda_1,\lambda_2;s,t,u) = (-1)^{\sigma_{14}}(-1)^{2s_2}e^{i\pi(\lambda_2-\lambda_3)}(-1)^{s_1-\lambda_1}(-1)^{s_4-\lambda_4} \times F(\lambda_3,-\lambda_1;-\lambda_4,\lambda_2;t,s,u)\,, \tag{A4}$$

where s_i is the spin of particle i, and σ_{14} is zero unless both particle 1 and particle 4 are fermions, in which case σ_{14} is one. Usually, crossing relates the amplitudes for particles to those for antiparticles. However, since we deal only with Majorana fermions, and our bosons are their own antiparticles, in our case crossing relates particle amplitudes to each other.

In Ref. 4 we have discussed in detail the determination of graviton-graviton amplitudes. Here we shall discuss the spin-$\frac{3}{2}$-spin-$\frac{3}{2}$ and spin-2-spin-$\frac{3}{2}$ amplitudes. For the $\frac{3}{2}$-$\frac{3}{2}$ amplitudes the procedure parallels that for the graviton-graviton amplitudes given in Ref. 4. We can restrict ourselves to four amplitudes, and using Eq. (8) we find that

$$F(\tfrac{3}{2},\tfrac{3}{2};\tfrac{3}{2},\tfrac{3}{2}) = s^3\tilde{F}(++++)\,, \tag{A5a}$$

$$F(\tfrac{3}{2},\tfrac{3}{2};\tfrac{3}{2},-\tfrac{3}{2}) = (stu)^{3/2}\tilde{F}(+++-)\,, \tag{A5b}$$

$$F(\tfrac{3}{2},-\tfrac{3}{2};\tfrac{3}{2},-\tfrac{3}{2}) = u^3\tilde{F}(+-+-)\,, \tag{A5c}$$

$$F(\tfrac{3}{2},\tfrac{3}{2};-\tfrac{3}{2},-\tfrac{3}{2}) = \tilde{F}(++--)\,. \tag{A5d}$$

In the Born approximation $\tilde{F}$ can only have simple dynamical poles at $s=0$ and $t=0$ and $u=0$; furthermore $\tilde{F}$ is proportional to κ^2, so we can write that

$$\tilde{F} = \frac{\kappa^2}{stu}\,\phi(s,t,u)\,, \tag{A6}$$

where ϕ must be analytic in s, t, and u according to the theorem of ACN.[8] The simplicity of Born amplitudes then forces ϕ to be a polynomial in s, t, and u.

Furthermore, the original helicity amplitudes F have been defined in such a way that they are dimensionless. We find that

$$F(\tfrac{3}{2},\tfrac{3}{2};\tfrac{3}{2},\tfrac{3}{2}) = \frac{\kappa^2 s^3}{stu}\,\phi_1(++++)\,, \tag{A7a}$$

$$F(\tfrac{3}{2},\tfrac{3}{2};\tfrac{3}{2},-\tfrac{3}{2}) = \frac{\kappa^2[(stu)^{1/2}]^3}{stu}\,\phi_{5/2}(+++-)\,, \tag{A7b}$$

$$F(\tfrac{3}{2},-\tfrac{3}{2};\tfrac{3}{2},-\tfrac{3}{2}) = \frac{\kappa^2 u^3}{stu}\,\phi_1(+-+-)\,, \tag{A7c}$$

$$F(\tfrac{3}{2},\tfrac{3}{2};-\tfrac{3}{2},-\tfrac{3}{2}) = \frac{\kappa^2}{stu}\,\phi_4(++--)\,, \tag{A7d}$$

where the subscript on the function ϕ indicates its polynomial degree.

The particle-interchange relation given by Eq. (A3) implies that $F(\frac{3}{2},\frac{3}{2};\frac{3}{2},\frac{3}{2})$ must be symmetric when u and t are interchanged holding s fixed. That is, expressing F (or ϕ) as a function of s and t alone, since $s+t+u=0$ for massless particles, we have that F (or ϕ) should be invariant under the operation

$$t \to -(s+t)\,. \tag{A8}$$

The only first-order polynomial in s and t invariant under the operation of Eq. (A8) is $c's$, where c' is a dimensionless constant. Then $\phi_1(++++)=c's$.

Clearly there is no polynomial of degree $\frac{5}{2}$, so $F(\frac{3}{2},\frac{3}{2};\frac{3}{2},-\frac{3}{2})$ vanishes. It is interesting to note that quite generally, for processes involving external massless particles, the coefficient of square roots in Eq. (8), namely $\lambda_s+\lambda_t+\lambda_u$, will always give an odd power if the sum of the helicity flips is an odd integer. For example, if the same particles come out of the reaction as went in, an odd power results when exactly one fermion flips its helicity, and an even power results otherwise.

By crossing operations $F(\frac{3}{2},-\frac{3}{2};\frac{3}{2},-\frac{3}{2};s,t,u)=F(\frac{3}{2},\frac{3}{2};\frac{3}{2},\frac{3}{2};u,t,s)$. Finally, by using crossing operations and particle interchanges we deduce that $F(\frac{3}{2},\frac{3}{2};-\frac{3}{2},-\frac{3}{2})$ is totally symmetric in s, t, and u, and that therefore $\phi_4(++--)$ must be a totally symmetric fourth-degree polynomial in s, t, and u. There is only one polynomial in s and t simultaneously symmetric under the interchange $s\to t$, $t\to s$, and the operation of Eq. (A8), up to an overall constant factor, so $\phi_4(++--)=\bar{c}'(s^4+t^4+u^4)$.

We now use an argument that refers to the specific form of the Lagrangian given by Eq. (1). The pole at $t=0$ in the $F(\frac{3}{2},\frac{3}{2};-\frac{3}{2},-\frac{3}{2})$ given by Eq. (A7d) is produced by a diagram in which a graviton is exchanged in the t channel. We have argued in Sec. II that at $t=0$ (for forward scattering, that is) the two-fermion–graviton vertex must vanish when the fermion helicity is flipped. Since that vertex appears as a factor in the residue of the pole at $t=0$, the residue must vanish. But inspection of the term

$$F(\tfrac{3}{2},\tfrac{3}{2};-\tfrac{3}{2},-\tfrac{3}{2})=\frac{\bar{c}'\kappa^2}{stu}(s^4+t^4+u^4)$$

shows that it vanishes only if $\bar{c}'=0$.

We may restrict ourselves to a consideration of the 2-$\frac{3}{2}$ amplitudes displayed in Eqs. (12), since the others can be obtained by the discrete operations of Eqs. (A1)–(A4). The odd-powers theorem for reactions with the same particles coming out as going in enables us to conclude immediately that three of the six amplitudes vanish: $F(2,\frac{3}{2};2,-\frac{3}{2})$, $F(2,\frac{3}{2};-2,-\frac{3}{2})$, and $F(2,-\frac{3}{2};-2,\frac{3}{2})$.

Application of the ACN theorem to $F(2,\frac{3}{2};2,\frac{3}{2})$ leads to the expression

$$F(2,\tfrac{3}{2};2,\tfrac{3}{2})=\frac{\kappa^2 s^3(-su)^{1/2}}{stu}\,\phi_0(++++)\,, \tag{A9}$$

so the polynomial of the zero degree can only be the dimensionless constant called c'' in Eq. (12a).

Application of the crossing operation of the fermions in Eq. (A9) implies that

$$F(2,-\tfrac{3}{2};2,-\tfrac{3}{2};s,t,u)=F(2,\tfrac{3}{2};2,\tfrac{3}{2};u,t,s)\,. \tag{A10}$$

Finally, applying the ACN theorem to $F(2,\frac{3}{2};-2,\frac{3}{2})$ leads to the expression

$$F(2,\tfrac{3}{2};-2,\tfrac{3}{2})=\frac{\kappa^2 su(-su)^{1/2}t^2}{stu}\,\phi_{-1}(++-+)\,, \tag{A11}$$

and there is no polynomial of degree minus one, so that amplitude must vanish.

APPENDIX B

In this appendix we compute the transformation coefficients Γ of Eqs. (13), (19), and (24) explicitly. While their form follows almost uniquely from general considerations, such as those embodied in Eqs. (17) and (18) and the spin-$\frac{1}{2}$ transformation law for $Q(\eta)$ under Lorentz transformations, it is useful to derive them from the explicit supergauge transformations of Ref. 1 (and also from the explicit supersymmetry transformations of Wess and Zumino).

The Lagrangian is invariant under the following local variation of the fields:

$$\delta\psi_{a\mu}(x)=\frac{4}{\kappa}D_\mu(x)\eta_a(x)+\frac{\kappa}{8}(2\bar{\psi}_\mu\gamma_\alpha\psi_\beta+\bar{\psi}_\alpha\gamma_\mu\psi_\beta)\sigma^{\alpha\beta}_{ab}\eta_b(x)\,, \tag{B1a}$$

$$\delta V^\alpha_\mu(x)=\frac{\kappa}{2}\bar{\eta}_a(x)\gamma^\alpha_{ac}\psi_{c\mu}(x)\,, \tag{B1b}$$

$$\delta g_{\mu\nu}(x)=\frac{\kappa}{2}\bar{\eta}_a(x)[\gamma_{\mu ab}\psi_{b\nu}(x)+\gamma_{\nu ab}\psi_{b\mu}(x)]\,. \tag{B1c}$$

Here $\sigma^{\alpha\beta}=\frac{1}{4}[\gamma^\alpha,\gamma^\beta]$, V^α_μ is the vierbein field, $g_{\mu\nu}$ the metric field, and

$$D_\mu(x)\eta_a(x)=\partial_\mu\eta_a(x)+\tfrac{1}{2}\omega_{\mu\alpha\beta}(x)\sigma^{\alpha\beta}_{ab}\eta_b(x)\,, \tag{B2a}$$

with

$$\omega_{\mu\alpha\beta}=\tfrac{1}{2}[V^\nu_\alpha(\partial_\mu V_{\beta\nu}-\partial_\nu V_{\beta\mu})+V^\rho_\alpha V^\sigma_\beta(\partial_\sigma V_{\gamma\rho})V^\gamma_\mu]-[\alpha\leftrightarrow\beta]. \tag{B2b}$$

The coefficients $\eta_a(x)$ anticommute with anything having a spinor (Latin) index; they are otherwise arbitrary, except that they satisfy the Majorana condition, which with our conventions for representations of the Dirac algebra is given by Eq. (25). Note that we have adjusted our normalization condition for the supergauge transformations to yield, for the iterated transformations, the spacetime translations specified by Eq. (16), which represents a trivial deviation from the conventions of Ref. 1.

We shall extract from Eqs. (B1) the transformation of the "in" fields under a global supersymmetry operation (η_a a constant) induced by a supergenerator $Q(\eta)$, with $i\delta\psi=[Q(\eta),\psi]$, and so on. We assume that the asymptotic "in" fields and "out" fields satisfy commutation relations with $Q(\eta)$ obtained by taking the asymptotic weak limit of the commutation relations for the interacting Heisenberg fields. In this limit only terms linear in the quantum fields survive. (We realize that we are being rather cavalier about asymptotic limits for a theory of massless particles, but we proceed undaunted.) It is therefore sufficient to expand the gravitational fields $g_{\mu\nu}$ and V^α_μ in powers of the quantum fields $h_{\mu\nu}$ and c^α_μ and to retain only the linear part. Then

$$g_{\mu\nu}=\eta_{\mu\nu}+\kappa h_{\mu\nu},\quad V^\alpha_\mu=\delta^\alpha_\mu+\kappa c^\alpha_\mu\,, \tag{B3}$$

with

$$c^\alpha_\mu=\tfrac{1}{2}\eta^{\alpha\nu}h_{\nu\mu}\,. \tag{B4}$$

We find, for the "in" fields,

$$[Q(\eta),\psi_{a\mu}(x)]=i[\partial_\beta h_{\mu\alpha}(x)-\partial_\alpha h_{\mu\beta}(x)]\sigma^{\alpha\beta}_{ab}\eta_b\,, \tag{B5a}$$

$$[Q(\eta),h_{\mu\nu}(x)]=\tfrac{1}{2}i\bar\eta[\gamma_\mu\psi_\nu(x)+\gamma_\nu\psi_\mu(x)]. \tag{B5b}$$

While these commutation relations were derived from the local supergauge relations given by Eqs. (B1), let us emphasize that Lorentz invariance, locality, linearity in η and the fields, and the fact that the spinor supergenerator has dimension $\frac{1}{2}$ are sufficient to establish the form of the right-hand sides of Eqs. (B5a) and (B5b). The relative normalization of those right-hand sides is determined by the condition that the free Hamiltonian be invariant, and the overall normalization by convention.

We now expand the "in" fields ψ and h in terms of particle creation and destruction operators in order to obtain the commutators of $Q(\eta)$ with such operators. We write

$$h_{\mu\nu}(x)=\sum_\sigma\int dm(p)[\epsilon_{\mu\nu}(p,\sigma)e^{ip\cdot x}a(p,\sigma)+\epsilon_{\mu\nu}(p,\sigma)^*e^{-ip\cdot x}a(p,\sigma)^*], \tag{B6a}$$

$$\psi_{a\mu}(x)=\sum_\sigma\int dm(p)[(2E)^{1/2}u_{a\mu}(p,\sigma)e^{ip\cdot x}c(p,\sigma)+(2E)^{1/2}D_{ab}u_{b\mu}(p,\sigma)^*e^{-ip\cdot x}c(p,\sigma)^*], \tag{B6b}$$

with

$$dm(p)=\frac{d^3p}{[(2\pi)^3 2E]^{1/2}}\,,\quad E=|\vec p|,\quad p\cdot x=\vec p\cdot\vec x-Et\,, \tag{B7}$$

and the matrix $D=C\gamma_4$, with C the charge-conjugation matrix. In our representation

$$C=\begin{pmatrix}0 & i\sigma_2\\ i\sigma_2 & 0\end{pmatrix},\quad \gamma^0=-i\begin{pmatrix}1&0\\0&-1\end{pmatrix},\quad \gamma_4=i\gamma^0. \tag{B8}$$

We have replaced the helicity label λ on the wave functions u and ϵ by the handedness label σ in the obvious way.

Then

$$a(p,\sigma)=\frac{i}{[(2\pi)^3 2E(p)]^{1/2}}\left[\int d^3x\, e^{-ip\cdot x}\overleftrightarrow{\partial}_t h_{\mu\nu}(x)\right]\times\epsilon^{\mu\nu}(p,\sigma)^*\,, \tag{B9a}$$

$$c(p,\sigma)=\frac{u^\mu_a(p,\sigma)^*}{[(2\pi)^3]^{1/2}}\left[\int d^3x\, e^{-ip\cdot x}\psi_{a\mu}(x)\right]. \tag{B9b}$$

To calculate $[Q(\eta),a(p,\sigma)]$ we replace $a(p,\sigma)$ using Eq. (B9a) and use Eq. (B5b) to eliminate the resulting commutator $[Q(\eta),h_{\mu\nu}]$. That results in terms proportional to ψ, which we replace using Eq. (B6b). So we end with $[Q(\eta),a(p,\sigma)]$ as a linear combination of $c(p',\sigma')$ and $c(p',\sigma')^*$, summed over p' and σ', with coefficients proportional to $\epsilon^{\mu\nu}(p,\sigma)^*$, $\bar\eta$, and u's or u^*'s. That is,

$$[Q(\eta),a(p,\sigma)]=\sum_\sigma\int d^3p'[K(p,\sigma;p',\sigma')c(p',\sigma')+L(p,\sigma;p',\sigma')c(p',\sigma')^*]. \tag{B10}$$

Carrying out the time differentiation and the spatial integration induced by Eq. (B9a) produces a momentum-conserving δ function in K and a $\delta^3(\vec p+\vec p')$ in L; however, the time differentiation in L produces a factor $E-E'$, which makes L vanish. So we find that

$$K(p,\sigma;p',\sigma')=i(2E)^{1/2}\epsilon^{\mu\nu}(p,\sigma)^*\bar\eta_a\gamma_{\mu ab}u_{b\nu}(p,\sigma')\delta^3(\vec p-\vec p')\,. \tag{B11}$$

The factorization of $\epsilon^{\mu\nu}$ and of $u_{b\nu}$ given by Eqs. (2) and (3), and the orthogonality condition given by Eq. (4b) allow us to simplify K further:

$$K(p,\sigma;p',\sigma') = i(2E)^{1/2}\bar{\eta}_a\epsilon^\mu(p,\sigma)^*\gamma_{\mu ab}u_b(p,\sigma)\delta_{\sigma\sigma'}\delta^3(\vec{p}-\vec{p}'), \quad \text{(B12)}$$

which leads immediately to the result quoted in Eq. (19).

A similar procedure, applied to $[Q(\eta), c(p,\sigma)]$, leads to the expression

$$[Q(\eta), c(p,\sigma)] = \sum_\sigma \int d^3p'[\tilde{K}(p,\sigma;p',\sigma')a(p',\sigma') + \tilde{L}(p,\sigma;p',\sigma')a(p',\sigma')^*]. \quad \text{(B13)}$$

Carrying out the spatial integration induced by Eq. (B9b) and the differentiation induced by Eq. (B5a) leads to the expressions

$$\tilde{K}(p,\sigma;p',\sigma') = \frac{-2u_a^\mu(p,\sigma)^*\sigma_{ab}^{\alpha\beta}p'_\beta\epsilon_{\mu\alpha}(p',\sigma')^*\eta_b}{(2E')^{1/2}}\delta^3(\vec{p}-\vec{p}'), \quad \text{(B14a)}$$

$$\tilde{L}(p,\sigma;p',\sigma') = \frac{2u_a^\mu(p,\sigma)^*\sigma_{ab}^{\alpha\beta}p'_\beta\epsilon_{\mu\alpha}(p',\sigma')^*\eta_b}{(2E')^{1/2}}e^{2iEt}\delta^3(\vec{p}+\vec{p}'). \quad \text{(B14b)}$$

The sum over μ in Eq. (B14a) produces a handedness-conserving $\delta_{\sigma\sigma'}$ from the factors $\epsilon^\mu(p,\sigma)^*$ and $\epsilon_\mu(p,\sigma')$ in the wave functions $u(\sigma)^*$ and $\epsilon(\sigma')$. Using the fact that $\sigma^{\alpha\beta}=\frac{1}{2}(g^{\beta\alpha}-\gamma^\beta\gamma^\alpha)$ and that $\epsilon(p,\sigma)\cdot p=0$ we obtain a simpler form for $\tilde{K}$:

$$\tilde{K} = \frac{1}{(2E)^{1/2}}\{u(p,\sigma)^*(\gamma\cdot p)[\gamma\cdot\epsilon(p,\sigma)]\eta\} \times \delta_{\sigma\sigma'}\delta^3(\vec{p}-\vec{p}'). \quad \text{(B15)}$$

From Eq. (4a) we deduce that

$$u(p,\sigma)^*\gamma\cdot p = 2iE\bar{u}(p,\sigma), \quad \text{(B16)}$$

from which we conclude that

$$\Delta(\eta,p,\sigma) = i(2E)^{1/2}\bar{u}(p,\sigma)\gamma\cdot\epsilon(p,\sigma)\eta. \quad \text{(B17)}$$

We have anticipated the demonstration that $\tilde{L}$ vanishes, which depends on showing that the factor

$$u_a(p,\sigma)^*\sigma_{ab}^{\alpha\beta}p'_\beta\epsilon_\alpha(p',\sigma')^*\eta_b \quad \text{(B18)}$$

within $\tilde{L}$ itself vanishes. The δ function in $\tilde{L}$ constrains the four-vector p' to be $(E,-\vec{p})$. Expression (B18) can be rewritten using the method applied to $\tilde{K}$, so it becomes

$$-\tfrac{1}{2}u(p,\sigma)^*(\gamma\cdot p')[\gamma\cdot\epsilon(p',\sigma')^*]\eta.$$

But Eq. (4a) shows that $u(p,\sigma)^*(\gamma\cdot p')=0$.

In order to complete the computation of

$$\Gamma(\eta,p,\sigma) = i(2E)^{1/2}\bar{\eta}\gamma\cdot\epsilon(p,\sigma)^*u(p,\sigma) \quad \text{(B19)}$$

we use the following facts: For our purpose $\vec{p}$ may be taken in the x-z plane, so that p has the form given in Eq. (23),

$$p=(E, E\sin\tilde{\theta}, 0, E\cos\tilde{\theta}).$$

Then

$$\epsilon(p,\pm) = \mp(\tfrac{1}{2})^{1/2}(0, \cos\tilde{\theta}, \pm i, -\sin\tilde{\theta}), \quad \text{(B20a)}$$

$$u(p,+) = (\tfrac{1}{2})^{1/2}(\cos\tfrac{1}{2}\tilde{\theta}, \sin\tfrac{1}{2}\tilde{\theta}, \cos\tfrac{1}{2}\tilde{\theta}, \sin\tfrac{1}{2}\tilde{\theta}), \quad \text{(B20b)}$$

$$u(p,-) = (\tfrac{1}{2})^{1/2}(-\sin\tfrac{1}{2}\tilde{\theta}, \cos\tfrac{1}{2}\tilde{\theta}, \sin\tfrac{1}{2}\tilde{\theta}, -\cos\tfrac{1}{2}\tilde{\theta}). \quad \text{(B20c)}$$

Our three spatial γ's have the form

$$\gamma_k = \begin{pmatrix} 0 & -i\sigma_k \\ i\sigma_k & 0 \end{pmatrix}. \quad \text{(B21)}$$

The Majorana spinor $\bar{\eta}=\eta^*\gamma_4$ is computed using Eq. (B8) for γ_4 and Eq. (25), which imposes the Majorana constraint $\eta_a = D_{ab}\eta_b^*$.

Then straightforward computation reveals that

$$\Gamma(\eta,p,+) = (2E)^{1/2}(\eta_1\cos\tfrac{1}{2}\tilde{\theta}+\eta_2\sin\tfrac{1}{2}\tilde{\theta}), \quad \text{(B22a)}$$

$$\Gamma(\eta,p,-) = (2E)^{1/2}(-\eta_1^*\cos\tfrac{1}{2}\tilde{\theta}-\eta_2^*\sin\tfrac{1}{2}\tilde{\theta}). \quad \text{(B22b)}$$

We now discuss briefly the supersymmetry operations within the Wess-Zumino supermultiplet consisting of the scalar and pseudoscalar fields $A(x), B(x)$, the auxiliary fields $F(x), G(x)$, and the Majorana spin-$\frac{1}{2}$ field $\psi_a(x)$. They satisfy

$$[Q(\eta), A] = i\bar{\eta}\psi, \quad \text{(B23a)}$$

$$[Q(\eta), B] = i\bar{\eta}\gamma_5\psi, \quad \text{(B23b)}$$

$$[Q(\eta), \psi] = i\gamma\cdot\partial(A+\gamma_5 B)\eta + i(F+\gamma_5 G)\eta, \quad \text{(B23c)}$$

$$[Q(\eta), F] = i\bar{\eta}\gamma\cdot\partial\psi, \quad \text{(B23d)}$$

$$[Q(\eta), G] = i\bar{\eta}\gamma_5\gamma\cdot\partial\psi. \quad \text{(B23e)}$$

The constraint equations

$$F = -mA - \beta(A^2-B^2), \quad G = -mB + 2\beta AB \quad \text{(B24)}$$

imply that $F(\text{in}) = -mA(\text{in})$ and $G(\text{in}) = -mB(\text{in})$ so that the "in" fields satisfy

$$[Q(\eta), A] = i\bar{\eta}\psi, \quad \text{(B25a)}$$

$$[Q(\eta), B] = i\bar{\eta}\gamma_5\psi, \quad \text{(B25b)}$$

$$[Q(\eta), \psi] = i(\gamma\cdot\partial - m)(A+\gamma_5 B)\eta. \quad \text{(B25c)}$$

Defining the chiral fields $\mathcal{A}_\pm$ via Eq. (30), we find that

$$[Q(\eta), b(p,\pm)] = i\sqrt{E}\sum_{\sigma}\bar{\eta}(1\pm i\gamma_5)u(p,\sigma)d(p,\sigma), \tag{B26a}$$

$$[Q(\eta), d(p,\sigma)] = -i\sqrt{E}\{[\bar{u}(p,\sigma)(1-i\gamma_5)\eta]b(p,+) + [\bar{u}(p,\sigma)(1+i\gamma_5)\eta]b(p,-)\}, \tag{B26b}$$

where $b(p,+)$ and $b(p,-)$ are the destruction operators associated with the chiral fields $\mathfrak{a}_+$ and $\mathfrak{a}_-$, and $d(p,\sigma)$ is associated with ψ_a.

In the massless case the $u(p,\sigma)$ are eigenstates of $1\pm i\gamma_5$. Our γ_5 has the form

$$\gamma_5 = -i\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}. \tag{B27}$$

We find that

$$[Q(\eta), b(p,\sigma)] = [2i\sqrt{E}\,\bar{\eta}u(p,\sigma)]d(p,\sigma), \tag{B28a}$$

$$[Q(\eta), d(p,\sigma)] = [-2i\sqrt{E}\,\bar{u}(p,\sigma)\eta]b(p,\sigma). \tag{B28b}$$

Using Eqs. (B20b), (B20c) we find that

$$2i\sqrt{E}\,\bar{\eta}u(p,\sigma) = i\Gamma(\eta,p,\sigma)^*. \tag{B29}$$

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