PHYSICAL REVIEW D VOLUME 15, NUMBER 4 15 FEBRUARY 1977

Line sources in general relativity*

Werner Israel†
W. K. Kellogg Radiation Laboratory, California Institute of Technology, Pasadena, California 91125
(Received 26 October 1976)

This paper is a preliminary study of how the field of a thin massive "wire" can be characterized in general relativity. For a class of "simple" line sources, a linear stress-energy-momentum tensor can be defined in terms of the extrinsic curvature of a tube of constant geodesic radius centered on the wire, in the limit when the radius shrinks to zero. A number of examples are considered, including the ring singularity of the Kerr metric. The Kerr ring is composed of dustlike material circulating about it with the speed of light. The mass distribution in a cross section is proportional to $\cos(\psi/2)$, where ψ is the angle of rotation about the ring in a plane normal to it. The "half-pole" structure is compatible with single-valuedness because of the two-sheeted character of the Kerr manifold.

I. INTRODUCTION

The nature of the singularities which, according to general relativity, arise inevitably[1] in gravitational collapse and cosmology is probably the most difficult unsolved problem facing the theory today. The simpler problem which will be our concern in this paper is to interpret physically certain types of singularity whose geometrical characteristics are assumed to be known. Such singularities may arise as convenient idealizations of realistic matter distributions (e.g., thin shells or wires), or as barriers to the analytic extension of known vacuum solutions (e.g., the ring singularity of the Kerr metric). To achieve a rudimentary understanding and control of such singularities, one seeks a way to infer the physical characteristics (e.g., mass, angular momentum, internal stresses) of the source from the nature of the geometry near the singularity.

In the case of thin shells or surface layers the singularity is so mild that a complete answer to this problem is easily given and is well known.[2] A timelike hypersurface Σ represents the history of a thin shell if its extrinsic curvature K_a^b suffers a discontinuity $[K_a^b]\neq 0$ when Σ is crossed (in the positive sense of the normal). The quantity

$$S_a^b = -(8\pi)^{-1}[K_a^b - \delta_a^b K_c^c] \qquad (1)$$

is called the surface energy-momentum-stress tensor of the layer and has the expected physical interpretation. For a surface layer in a vacuum one has the conservation law $S^b_{a;b}=0$, in which the covariant derivative refers to the intrinsic three-metric of Σ. The prescription [Eq. (1)] breaks down for null surface layers, which require special consideration.

Surface layers are easy to handle because the space-time metric remains nonsingular and in fact continuous at the singularity. This simplifying feature is already lost when we come to the next grade of singularity, line sources or thin wires, the subject of this paper. The general conclusion of the present work is that the classification of line singularities is a problem of some complexity. There exists no simple, general prescription, analogous to Eq. (1), for obtaining the physical characteristics of an arbitrary line source. Our work must be considered a preliminary reconnaissance rather than a systematic study, and no serious attempt is made at rigor or maximum generality.

In Sec. III we isolate a class of "simple" line sources for which a "line energy-momentum-stress tensor" is obtainable from the extrinsic curvature of a tube of constant geodesic radius enclosing the source in the limit when the radius shrinks to zero. Many of the line singularities that occur in the standard exact solutions (e.g., infinite rods, uniform circular rings, "Weyl struts") are "simple," and their internal structure can be inferred from our prescription (Sec. VI). However, the Kerr ring does not belong to this class. In Sec. VIII we plausibly infer its internal structure by considering it as the limit of a thin toroidal shell.

II. GENERAL CHARACTERIZATION OF LINE SOURCES

It seems reasonable to postulate, as a partial characterization, that a line source is a singularity which can be enclosed in a tube of arbitrarily small geodesic radius and circumference. In this section we attempt to formalize this idea.

A singular boundary L of space-time will be called a "line singularity" if the following three conditions hold:

(i) Each point p in a neighborhood of L can be connected to L by a spacelike curve of bounded arc length.

Remark. The spacelike curve through p which gives the arc length pL a stationary value $\rho(p)$ (a minimum under spacelike deformations and a maximum under timelike deformations) is necessarily a geodesic. We refer to this as a radial geodesic and to ρ as geodesic radius. A simple local argument shows that radial geodesics are orthogonal to the spaces $\rho=\text{const}$.

(ii) The spaces $\rho=\text{const}$ are timelike three-cylinders with topology $S^1\times M^2$, or (for closed ringlike sources) $S^1\times S^1\times M^1$, where M^n is Minkowski n-space.

(iii) Each three-cylinder $\rho=\text{const}$ is encircled by a congruence of simple, nonreducible closed spacelike curves C whose circumference tends to zero as they are Lie-transported inward to L along radial geodesics.

We introduce coordinates $x^a\equiv(z,\varphi,t)$, Lie-transported along radial geodesics, such that the closed curves C are parametric curves of φ ($0\le\varphi<2\pi$). In terms of $x^\alpha=(\rho,z,\varphi,t)$ the metric takes the Gaussian form

$$ds^2=d\rho^2+g_{ab}(\rho,x^c)dx^a dx^b\,. \tag{2}$$

(Greek and lower-case Latin indices have the range 1–4 and 2–4, respectively.) The extrinsic curvature K_{ab} of the cylinders $\rho=\text{const}$ and the corresponding density $\mathcal{K}_a^b$ are defined by

$$\partial g_{ab}/\partial\rho=2K_{ab},\quad \mathcal{K}_a^b=\sqrt{-g}K_a^b\,. \tag{3}$$

The field equations $G^{\alpha\beta}=-8\pi T^{\alpha\beta}$, projected onto and perpendicular to the cylinders, decompose into

$$\partial\mathcal{K}_a^b/\partial\rho+\sqrt{-g}\,{}^{(3)}R_a^b=-8\pi\sqrt{-g}\,(T_a^b-\tfrac{1}{2}\delta_a^bT_\alpha^\alpha)\,, \tag{4}$$

$$K_{,a}-K_{a;b}^b=-8\pi T_a^1\,, \tag{5}$$

$$K_{ab}K^{ab}-K^2-{}^{(3)}R=-16\pi T_1^1\,. \tag{6}$$

Here $K=K_a^a$, and ${}^{(3)}R_a^b$ and the semicolon denote the Ricci tensor and covariant derivative associated with the three-metric g_{ab}.

III. SIMPLE LINE SOURCES

The class of line singularities isolated by the criteria (i)–(iii) of Sec. II still includes many objects which cannot be considered as plausible analogs of Newtonian line sources or thin wires. It includes, for example, the (timelike) singularity at $r=0$ of the Schwarzschild solution with $m<0$, which one would normally consider as having a "pointlike" rather than "linelike" structure. The additional criteria to be imposed in this section pick out a subclass which in some respects is still too wide (the negative-mass Schwarzschild singularity is still included) and in other respects too narrow. One interesting and eligible source which is omitted is the Kerr ring singularity (treated separately in Sec. VIII).

This subclass of "simple" line sources is defined by the following additional conditions:

(iv) The intrinsic curvature density of the cylinders $\rho=\text{const}$ diverges less rapidly than ρ^{-1}:

$$\lim_{\rho\to 0}\rho^{1+\delta}\sqrt{-g}\,{}^{(3)}R_a^b=0\quad(\delta>0)\,. \tag{7}$$

According to Eq. (4) with vanishing or finite T_α^β (assuming the wire to be surrounded by vacuum or material of finite density), this implies existence of the limit

$$\lim_{\rho\to 0}\mathcal{K}_a^b\equiv\mathcal{C}_a^b(x^c)\,. \tag{8}$$

(v) Asymptotic axial symmetry and parity invariance: The closed curves of condition (iii) can be chosen so that

$$\partial\mathcal{C}_a^b/\partial\varphi=0\,,\quad \mathcal{C}_z^\varphi=\mathcal{C}_t^\varphi=0\,. \tag{9}$$

(vi) $\mathcal{C}_a^b$ has no null eigenvectors.

Condition (iv) characterizes a simple line source as a "normal-dominated" singularity,[3] a timelike analog of the "velocity-dominated" spacelike cosmological singularities studied by Eardley, Liang, and Sachs.[4]

Condition (v) expresses the assumption that the wire has uniform, circular cross sections and no angular momentum corresponding to spin about its axis (compare end of Sec. VI).

Condition (vi) excludes "lightlike" sources which (like null surface layers) require special treatment.

It follows from (v) and (vi) that $\mathcal{C}_a^b$ is diagonalizable by appropriate choice of the coordinates z,t. Let the diagonalized form be

$$\mathcal{C}_a^b(z,t)=\text{diag}(\alpha,\beta,\gamma)\,. \tag{10}$$

Equation (6) in the limit $\rho\to 0$ yields

$$\alpha\beta+\beta\gamma+\gamma\alpha=0\,. \tag{11}$$

Integration of Eq. (3) then gives the asymptotic form of the three-metric as $\rho\to 0$:

$$g_{ab}\approx\text{diag}(\gamma_{22}\rho^{2a},\gamma_{33}\rho^{2b},\gamma_{44}\rho^{2c})\,. \tag{12}$$

Here the diagonal "tensor" $\gamma_{ab}(z,t)$ is arbitrary up to constant scale transformations (which can be different along the three axes), and assigns a "metric structure" to the singularity L. (Note that this structure is three-dimensional, which belies the intuitive conception of the source history L as a two-dimensional object.) The exponents a,b,c are functions of z,t, defined by

$$a=\alpha/(\alpha+\beta+\gamma),\quad b=\beta/(\alpha+\beta+\gamma),$$
$$c=\gamma/(\alpha+\beta+\gamma)\,, \tag{13}$$

and satisfy

$$a+b+c=a^2+b^2+c^2=1 \tag{14}$$

by virtue of Eq. (11).

IV. ENERGY-MOMENTUM TENSOR OF SIMPLE LINE SOURCE

It is straightforward to assign a "line energy-momentum tensor" density $\mathcal{L}_a^b(z,t)$ to a simple line source. We have to consider two cases.

(*a*) *Generic case*: $\mathfrak{C}^a_{[c}\mathfrak{C}^b_{d]}\neq 0$. We define

$$-4\mathcal{L}_a^b(z,t)\equiv \mathfrak{C}_a^b-\delta_a^b\mathfrak{C}_c^c\,. \tag{15}$$

Then Eq. (5) gives, for a line source in a vacuum, the conservation law

$$\nabla_b\mathcal{L}_a^b=0\,, \tag{16}$$

where the covariant derivative ∇ refers to the metric γ_{ab}. Equation (16) is invariant under constant scale transformations of γ_{ab}.

(*b*) *Conical case*: $\mathfrak{C}^a_{[c}\mathfrak{C}^b_{d]}=0$. In this case two of the functions a,b,c must vanish. Since our assumption (iii) of "thinness" of the source forbids b to vanish, the only nongeneric case is $a=c=0$, $b=1$. The asymptotic form of the metric as $\rho\to 0$ is

$$ds^2\approx d\rho^2+A^2(z,t)dz^2+B^2\rho^2d\varphi^2-C^2(z,t)\,dt^2\,. \tag{17}$$

Asymptotically, the extrinsic curvature of the cylinders $\rho=$ const has only one nonvanishing component: $K_\varphi^\varphi\approx\rho^{-1}$. From Eqs. (6) and (5) we deduce, for a line source in a vacuum,

$$\lim_{\rho\to 0}{}^3R=0\,,\quad \partial B/\partial z=\partial B/\partial t=0\,. \tag{18}$$

Thus, B is a constant, and $\rho=0$ is a quasiregular conelike singularity in the classification scheme of Ellis and Schmidt.[5] The tensor density $\mathcal{L}_a^b(z,t)$ is now appropriately defined by

$$\mathcal{L}_a^3=0\,,\quad 4\mathcal{L}_m^n=AC(B-1)\delta_m^n\ \ (m,n=2,4)\,, \tag{19}$$

and again satisfies the conservation law, Eq. (16), in a vacuum by virtue of Eq. (5), with ∇ referring to the metric $\gamma_{ab}=\mathrm{diag}(A,B,C)$.

Under certain conditions it is possible to interpret $\mathcal{L}_a^b$, defined by Eqs. (15) or (19), as the integral

$$\mathcal{L}_a^b=\int_0^\epsilon d\rho'\int_0^{2\pi}d\varphi\sqrt{-g}\,T_a^b \tag{20}$$

for a thin tube of material of geodesic radius $\rho'=\epsilon$ in the limit $\epsilon\to 0$, when T_a^b becomes a distribution with support at $\rho'=0$. The following argument makes this plausible; with some effort it can be made more rigorous.

We consider a cylindrical tube of material with geodesic radius ϵ. The history of the axis is a regular two-space L. Let ρ',z,φ,t be Gaussian coordinates with L represented by $\rho'=0$. As $\epsilon\to 0$ and the concentration of material increases, the geometry inside and near the tube is determined more and more exclusively by the local matter distribution. We choose a material filling T_α^β such that $\partial T_\alpha^\beta/\partial\varphi=0$. Then the metric inside and near the tube will be "locally axisymmetric" ($\partial g_{\alpha\beta}/\partial\varphi=0$ for $0<\rho'\lesssim\epsilon$) to an increasingly good approximation as $\epsilon\to 0$, even though the tube is not in general straight and the geometry not axisymmetric at a finite distance from the source.

At the tube surface Σ: $\rho'=\epsilon$, the interior geometry joins (for sufficiently small ϵ) to the asymptotic exterior geometry (12) with continuity of metric and extrinsic curvature. The interior Gaussian coordinates z,φ,t are most naturally fixed by Lie-transporting the exterior coordinates inward along radial geodesics. Thus, z,φ,t are continuous. However, the exterior and interior radial coordinates ρ,ρ' do not agree on Σ; the relation between their values $\rho=\rho_0(\epsilon)$, $\rho'=\epsilon$ is found from continuity of the induced three-metric.

Let the interior metric be

$$ds^2=d\rho'^2+g_{mn}dx^m dx^n+g_{33}d\varphi^2\quad (0\le\rho'\le\epsilon)\,, \tag{21}$$

where g_{ab} are functions of ρ', x^m, and m,n take the values 2 and 4. Near the axis, regularity requires $g_{33}\approx\rho'^2$.

The radial stresses and momentum T_α^1 must vanish at the surface of a tube in a vacuum. Sources for which the definitions (15) or (19) are appropriate have the property that the ratio $T_\alpha^1/(\max|T_\alpha^\beta|)\to 0$ uniformly throughout the interior as $\epsilon\to 0$, so that we may neglect $T_\alpha^\alpha-T_a^a=T_1^1$. This means, for example, that a static tube of this type is supported primarily by hoop stresses rather than a steep gradient of radial stress.

Integration of Eq. (4) then yields

$$\mathcal{K}_a^b\Big|_{\rho'=0}^{\rho'=\epsilon}=-4(\mathcal{L}_a^b-\delta_a^b\mathcal{L}_c^c)-\int_0^\epsilon\sqrt{-g}\,{}^{(3)}R_a^b d\rho'\,, \tag{22}$$

with $\mathcal{L}_a^b$ defined by Eq. (20). The last integral can be neglected in the limit $\epsilon\to 0$ if a condition of normal dominance analogous to Eq. (7) is satisfied throughout the interior. We shall assume this to be so for a reasonable material filling, since it is certainly satisfied (by hypothesis) near the surface and also near the axis, where we have ${}^{(3)}R_a^b\sim(\rho' l)^{-1}$, $\sqrt{-g}\,{}^{(3)}R_a^b\sim l^{-1}$ (l is the radius of curvature of the tube axis).

Some care is required in the evaluation of $\mathcal{K}_a^b$ at $\rho'=0$. We have

$$\mathcal{K}_m^n\big|_{\rho'=0}=0\quad (m,n=2,4) \tag{23}$$

since $K^n_m \sim l^{-1}$ is finite. On Σ, we obtain from Eq. (12)

$$K^m_m = (1-b)/\rho_0 > 0 \quad (b \neq 1)\,. \tag{24}$$

Positivity of K^m_m must be maintained in the interior, because Eq. (6) yields

$$K^3_3 = \tfrac{1}{2}[-(\det K^n_m) - {}^{(3)}R + 16\pi T^1_1]/K^m_m\,. \tag{25}$$

This shows that in general K^m_m cannot vanish for $0<\rho'<\epsilon$ if the cylinders $\rho'=\text{const}$ are regular. It then follows from Eq. (3) that $(-\det g_{mn})^{1/2}$ is an increasing function of ρ'. Hence

$$\mathcal{K}^3_3\big|_{\rho'=0} = (-\det g_{mn})^{1/2}\big|_{\rho'=0} < (-\det g_{mn})^{1/2}\big|_{\rho'=\epsilon} = \rho_0{}^{1-b}(-\det\gamma_{mn})^{1/2}\,. \tag{26}$$

For $b<1$, this vanishes in the limit $\epsilon\to 0$, since $\rho_0(\epsilon)$ tends to zero with ϵ. Thus, in the generic case $b\neq 1$, Eq. (22) leads to the definition (15).

We now treat the conical case $a=c=0$, $b=1$. The exterior metric is given by Eqs. (17) and (18). The simplest interior metric which joins smoothly to this, and which is regular at $\rho'=0$, is

$$ds^2 = d\rho'^2 + A^2(z,t)dz^2 + \rho'^2 d\varphi^2 - C^2(z,t)dt^2\,. \tag{27}$$

Then

$$\mathcal{K}^3_3\big|^{\rho'=\epsilon}_{\rho'=0} = B-1\,,\quad \lim_{\epsilon\to 0}\mathcal{K}^n_m\big|^{\rho'=\epsilon}_{\rho'=0} = 0\,, \tag{28}$$

and Eq. (22) yields the definition (19).

V. SIMPLE LINE SOURCES IN STATIONARY SPACE-TIMES

We consider any stationary asymptotically flat space-time containing a stationary simple line source L. Introduce "stationary coordinates" x^1, x^2, x^3, t in which the timelike Killing vector has components δ^α_4, so that $\partial_4 g_{\alpha\beta}=0$. Let Σ_2 denote an instantaneous section $t=\text{const}$ of the geodesic cylinder Σ: $\rho=\delta$ with axis on L, and V_3 the three-space $t=\text{const}$ exterior to Σ_2. We may assume that the unit normal n^α to Σ is also the three-normal to Σ_2 in V_3, i.e., that V_3 intersects Σ orthogonally:

$$n^\alpha\partial_\alpha t\big|_{\rho=\delta} = 0\,. \tag{29}$$

This can always be achieved by the (admissible) transformation $t\to t+f(x^1,x^2,x^3;\delta)$ with a suitable choice of f. According to Eq. (29), t is constant along radial geodesics in an infinitesimal neighborhood of Σ, and hence can be made to agree locally with the Gaussian coordinate t introduced in Sec. II.

If X^α are any functions such that $\partial_4 X^\alpha = X^4 = 0$, Gauss's divergence theorem applied to V_3 yields

$$\iiint_{V_3}\partial_\alpha([{}^{(3)}g]^{1/2}X^\alpha)dx^1dx^2dx^3 = \int_{r=\infty} X^\alpha n_\alpha dS - \int_{\Sigma_2} X^\alpha n_\alpha d\Sigma_2\,. \tag{30}$$

If Σ is parametrized by the Gaussian coordinates $x^a = z, \varphi, t$ of Sec. II,

$$d\Sigma_2 = \sqrt{-g}\,|\nabla t|\,dz\,d\varphi\,, \tag{31}$$

where $|\nabla t| = (-g^{44})^{1/2}$ is the gradient of t, and $\sqrt{-g}$ refers to the Gaussian metric.

We choose $X^\alpha = -|\nabla t|^{-1}\xi^{\alpha|4}$, where ξ^α is a Killing vector. Then

$$\partial_\alpha([{}^{(3)}g]^{1/2}X^\alpha) = \partial_\alpha(-\sqrt{-g}\,\xi^{\alpha|4}) = -\sqrt{-g}\,\xi^{\alpha|4}{}_{|\alpha} = \sqrt{-g}\,\xi^\alpha R^4_\alpha\,, \tag{32}$$

with the aid of the Ricci commutation relations, where $\sqrt{-g}$ now refers to the metric associated with the stationary coordinates. Also, $\xi^\alpha n_\alpha = 0$ on Σ_2, since a Killing vector of the four-space is necessarily a generator of Σ. Hence

$$X^\alpha n_\alpha\big|_{\Sigma_2} = |\nabla t|^{-1}\xi^\alpha n^{|4}_\alpha = |\nabla t|^{-1}\xi^a K^4_a\,, \tag{33}$$

where K^b_a is the extrinsic curvature of Σ. Substituting Eqs. (31)–(33) into Eq. (30), and using the Einstein field equations and the definition (8) in the limit $\delta\to 0$, we obtain the expression

$$C = -8\pi\int_{V_3}(T^4_\alpha - \tfrac{1}{2}\delta^4_\alpha T^\beta_\beta)\xi^\alpha\sqrt{-g}\,d^3x + \iint_L \mathcal{C}^4_a\xi^a dz\,d\varphi\,, \tag{34}$$

for the value C of the surface integral at infinity.

For the timelike Killing vector $\xi^\alpha = \delta^\alpha_4$ the surface integral is $C=4\pi m$, assuming asymptotic flatness with $g_{44}\approx -(1-2m/r)$. For a field with axial symmetry there is a second Killing vector ($\xi^\alpha_{(\psi)} = \delta^\alpha_3$ with the obvious choice of axial coordinate $x^3=\psi$) and $C=-8\pi L$ if $g_{34}\approx -(2L/r)\sin^2\theta$ in quasispherical coordinates.

Using now the definitions (15) and (19), we find for any stationary field containing a simple line source L, the gravitational mass

$$m = -\int_{V_3}(T^4_4 - T^1_1 - T^2_2 - T^3_3)\sqrt{-g}\,dx^1dx^2dx^3 - \int_L(\mathcal{L}^4_4 - \mathcal{L}^2_2 - \mathcal{L}^3_3)dz\,, \tag{35}$$

and for a stationary field with axial symmetry, the angular momentum

$$L = \int_{V_3} T^4_3\sqrt{-g}\,d^3x + \int_L \mathcal{L}^4_a\xi^a_{(\psi)}dz\,. \tag{36}$$

These formulas, which generalize the well-known integrals of Tolman,[6] confirm the interpretation (20) by an argument which is independent of the one in Sec. IV. The integrals in Eqs. (35) and (36) may thus be identified as densities of "effective" gravitational mass and angular momentum for a stationary simple line source. For a source of conical type, Eq. (17) or Eq. (19) shows that both effective densities vanish.

VI. SOME EXAMPLES

The Kasner vacuum metric

$$ds^2 = d\rho^2 + \rho^{2a}dz^2 + \rho^{2b}d\varphi^2 - \rho^{2c}dt^2 , \tag{37}$$

in which the constants a, b, c satisfy

$$a + b + c = a^2 + b^2 + c^2 = 1 , \tag{38}$$

is static, axially symmetric, and cylindrically symmetric. It represents the field of an *infinite rod*. The line energy-momentum tensor density is obtained from Eq. (15) for $b \neq 1$:

$$\mathfrak{L}_a^b = \tfrac{1}{4}\,\mathrm{diag}(1-a, 1-b, 1-c) . \tag{39}$$

Thus, all components are non-negative, indicating that the source has positive pressures and a negative energy density. However, the effective gravitational mass per unit (coordinate) length is, according to Eq. (35),

$$\mathfrak{L}_2^2 + \mathfrak{L}_3^3 - \mathfrak{L}_4^4 = \tfrac{1}{2}c , \tag{40}$$

and will be positive if $c > 0$, in which case Eq. (37) shows that the source is infinitely red-shifted.

Asymptotically flat fields which have simple line sources with a structure similar to Eq. (39) include the Weyl axisymmetric field of a uniform finite rod and the Weyl field of a uniform circular ring recently given by Thorne.[7] In both cases the metric close to the source approaches the form (37) asymptotically. On the other hand, the "uniform ring" solution of Bach and Weyl[8] appears highly anisotropic close to the source and the source is not simple.

Our assumption that the source be "thin" requires that $b > 0$. In that case $ac < 0$, so there is a correlation between infinite or zero proper length and infinite red-shift or blue-shift of the source. For sources with $b < 0$, so that the circumference of a tube $\rho = \text{const}$ diverges as $\rho \to 0$, the limit (8) still exists in many cases, and Eqs. (15), (16), (35), and (36) remain valid. However, we shall not pursue further the question of what physical interpretation is to be given to $\mathfrak{L}_a^b$ in this case.

Familiar examples of conical-type simple line sources are the "Weyl struts" which maintain equilibrium in static, axisymmetric two-body solutions of the field equations. We recall that all static axisymmetric vacuum metrics can be reduced to the Weyl canonical form[9]

$$ds^2 = e^{2(\nu - V)}(d\rho^2 + dz^2) + \rho^2 e^{-2V} d\varphi^2 - e^{2V}dt^2 , \tag{41}$$

in which the functions $V(\rho, z), \nu(\rho, z)$ satisfy

$$\nabla^2 V \equiv V_{\rho\rho} + \rho^{-1}V_\rho + V_{zz} = 0 , \tag{42}$$

$$\nu_\rho = \rho(V_\rho^{\,2} - V_z^{\,2}) , \quad \nu_z = 2\rho V_\rho V_z . \tag{43}$$

The geometry is regular on the axis if the condition of "elementary flatness," $\nu(0, z) = 0$, is satisfied. From Eq. (43) it follows that the difference of the values of ν at two points of the axis connected by a curve C lying entirely in a vacuum is

$$\nu_2 - \nu_1 = -4\int \left(\frac{1}{4\pi}\nabla^2 V\right)\left(-\frac{\partial V}{\partial z}\right)(2\pi\rho\, d\rho\, dz) , \tag{44}$$

where the integral is to be taken over the volume enclosed by the surface of revolution S generated by C. Formally, $\nu_2 - \nu_1$ is (-4) times the gravitational force acting on the material enclosed by S in the analogous Newtonian problem with potential V. Hence, in a static two-body solution which is asymptotically flat and regular, the segment of the axis between the two bodies will have a constant but nonvanishing value of $\nu = \nu_0$, equal to (-4) times the Newtonian force between them. This singular segment of the axis was interpreted by Bach and Weyl[8] as a strut which holds the two bodies apart.

A Weyl strut is a simple line source of conical type. Equations (19) give

$$\mathfrak{L}_2^2 = \mathfrak{L}_4^4 = \tfrac{1}{4}(1 - e^{\nu_0}) \tag{45}$$

as the only nonvanishing components of $\mathfrak{L}_a^b$. The strut therefore contains a pressure $\mathfrak{L}_2^2$ (which reduces in the weak-field limit to the Newtonian value $-\tfrac{1}{4}\nu_0$) and a negative energy density numerically equal to it. The effective gravitational mass of the strut vanishes; this "explains" why the strut makes no contribution to the gravitational potential V.

Another well-known metric sometimes associated with an axial singularity is Newman-Unti-Tamburino (NUT) space:

$$\begin{aligned} ds^2 &= U^{-1}dr^2 + (r^2 + a^2)(d\theta^2 + \sin^2\theta\, d\varphi^2) \\ &\quad - U[dt + 4a \sin^2(\tfrac{1}{2}\theta)\, d\varphi]^2 , \\ U &= 1 - 2(mr + a^2)/(r^2 + a^2) . \end{aligned} \tag{46}$$

If the space is considered to be asymptotically Minkowskian, there is a singularity along the half-axis $\theta = \pi$. Misner[10] showed how this singularity could be removed by reidentifying t as a periodic coordinate. Bonnor[11] has considered the alternative of retaining the singularity and a casual structure for t, and has suggested that $\theta = \pi$ be interpreted as a "massless source of angular momentum." The only point to be made here is that the Bonnor singularity is not a line source in our sense of the term, since its circumference $4\pi a U^{1/2}$ is different from zero, so that condition (iii) of Sec. II is not satisfied. Presumably it is this which makes a nonvanishing angular momentum possible for this source.

VII. A NONSIMPLE LINE SOURCE

As a preliminary to the discussion of the Kerr ring source, it is useful to consider the metric

$$ds^2 = d\rho^2 + \rho^2 d\varphi^2 + V(\rho, \varphi)dz^2 + 2dzdt\,, \tag{47}$$

which is stationary and cylindrically symmetric, though not axially symmetric. The substitution

$$z = 2^{-1/2}(Z - T)\,, \quad t = 2^{-1/2}(T + Z) \tag{48}$$

converts Eq. (47) to a manifestly Kerr-Schild form[12]

$$ds^2 = d\rho^2 + \rho^2 d\varphi^2 + dZ^2 - dT^2 + \tfrac{1}{2}V(\rho, \varphi)(dZ - dT)^2\,, \tag{49}$$

showing that the curvature is of Petrov type D. The condition for Eq. (47) to satisfy the vacuum equations is

$$\nabla^2 V \equiv V_{\rho\rho} + \rho^{-1}V_\rho + \rho^{-2}V_{\varphi\varphi} = 0\,. \tag{50}$$

The extrinsic curvature density of the cylinders $\rho = \text{const}$ is

$$\mathcal{K}^\varphi_\varphi = 1\,, \quad \mathcal{K}^t_z = \tfrac{1}{2}\rho V_\rho\,, \quad \text{all other } \mathcal{K}^b_a = 0\,. \tag{51}$$

Solutions of Eq. (50) which are singular at $\rho = 0$ do not have simple line sources, since $|\mathcal{K}^t_z| \to \infty$ as $\rho \to 0$; also the condition of asymptotic axial symmetry is not in general satisfied.

To elicit the source structure of Eq. (47) we first construct a thin-shell source. We consider an exterior ($\rho > \rho_0$) and an interior ($\rho < \rho_0$) vacuum metric, both of the form (47), which join continuously at $\rho = \rho_0$. Let us choose specifically

$$V(\rho, \varphi) = C\cos n\varphi \times \begin{cases} \rho^{-n} & (\rho > \rho_0)\,, \\ (\rho/\rho_0{}^2)^n & (\rho < \rho_0)\,, \end{cases} \tag{52}$$

where C and $n > 0$ are constants. The boundary $\rho = \rho_0$ is the site of a thin shell whose surface energy tensor can be found from Eq. (51) and the jump conditions (1):

$$2\pi S^{ab} = C\rho_0{}^{-2}(\cos n\varphi)l^a l^b\,. \tag{53}$$

We have introduced the null vector

$$l^a \equiv (l^z, l^\varphi, l^t) = (\tfrac{1}{2}n)^{1/2}\rho_0{}^{-n/2}(0, 0, 1)\,, \tag{54}$$

which is a principal null vector of the type D Riemann tensor.

For all finite ρ_0, and hence also in the limit $\rho_0 \to 0$, the source is composed of dustlike material with a 2^n-pole mass distribution, streaming along the axis with the speed of light. In the limit $\rho_0 \to 0$ we may identify

$$\mathcal{L}^b_a = 2\pi\rho_0\sqrt{-g}S^b_a = C(\cos n\varphi)l_a l^b \tag{55}$$

as the line energy-momentum tensor density of the source.

VIII. THE KERR RING

Finally, we examine the remarkable source structure of the ring singularity in the Kerr metric. In terms of Boyer-Lindquist coordinates, the Kerr metric is

$$ds^2 = \Sigma(\Delta^{-1}dr^2 + d\theta^2) + (r^2 + a^2 + 2ma^2 r\Sigma^{-1}\sin^2\theta)d\varphi^2 - 4mar\Sigma^{-1}\sin^2\theta\, d\varphi dt - (1 - 2mr\Sigma^{-1})dt^2\,, \tag{56}$$

where

$$\Sigma = r^2 + a^2\cos^2\theta\,, \quad \Delta = r^2 - 2mr + a^2\,. \tag{57}$$

The singularity is located on the ring $r = 0$, $\theta = \pi/2$.

Introduce new coordinates ρ, ψ defined by

$$a + \rho\cos\psi = (r^2 + a^2)^{1/2}\sin\theta\,, \quad \rho\sin\psi = r\cos\theta\,. \tag{58}$$

Close to the ring ($\rho \to 0$), ρ and ψ become plane polar coordinates in a plane perpendicular to the ring. For $\rho \to 0$, we have

$$r \approx (2a\rho)^{1/2}\cos\tfrac{1}{2}\psi\,, \qquad 2mr\Sigma^{-1} \approx m[2/(a\rho)]^{1/2}\cos\tfrac{1}{2}\psi \equiv V(\rho, \psi)\,, \tag{59}$$

and the asymptotic form of the metric is

$$ds^2 \approx d\rho^2 + \rho^2 d\psi^2 + a^2 d\varphi^2 - dt^2 + V(\rho, \psi)(a\,d\varphi - dt)^2\,. \tag{60}$$

This agrees with the metric (49) we have already studied, with the change of notation $\varphi \to \psi$, $Z \to a\varphi$, $T \to t$.

We infer that the Kerr ring has a line energy-momentum tensor of the form

$$\mathcal{L}^b_a = C(\cos\tfrac{1}{2}\psi)l_a l^b\,, \tag{61}$$

where l_a is a principal null vector of the Kerr geometry and is tangential to the history of the ring. Dustlike material circulates about the ring with the speed of light and with angular speed $d\varphi/dt = a^{-1}$. The mass distribution in the ring has a "demipole" structure ($\propto \cos\frac{1}{2}\psi$). Such a structure is, of course, not possible in Newtonian theory since it is not single-valued in a Euclidean topology. However, it fits in with the two-sheeted structure of the Kerr geometry (sheets of positive and negative r, linked at the disk $r=0$, which serves as a branch cut). A complete circuit $-\pi \leq \psi < \pi$ of the ring in the $r>0$ sheet reveals only positive-mass densities; in the subsequent circuit $\pi \leq \psi < 3\pi$ in the $r<0$ sheet the mass is negative. These properties of the source provide an intuitive explanation for some of the peculiarities of test-particle behavior in the Kerr geometry, such as the repulsive gravitational force in the $r<0$ sheet, and the fact that test particles in the plane of the disk $r=0$ are unaccelerated. (Compare also Ref. 13.)

ACKNOWLEDGMENTS

It is a pleasure to acknowledge the kind hospitality afforded by K. S. Thorne and his group at the California Institute of Technology. I am indebted to K. S. Thorne, S. W. Hawking, G. F. R. Ellis, C. J. S. Clarke, and A. H. Taub for valuable discussions and suggestions.

---

*Work supported in part by the National Science Foundation under Grant No. AST75-01398 A01.

†Sherman Fairchild Distinguished Scholar at Caltech 1974–75; on leave of absence from the Theoretical Physics Institute, Department of Physics, University of Alberta, Edmonton, Canada.

[1]S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space-Time* (Cambridge U.P., Cambridge, 1973), Chap. 8.

[2]C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973), p. 551.

[3]E. P. T. Liang, Commun. Math. Phys. 32, 51 (1973); Lett. Nuovo Cimento 6, 459 (1973).

[4]D. Eardley, E. P. T. Liang, and R. K. Sachs, J. Math. Phys. 13, 99 (1972).

[5]G. F. R. Ellis and B. G. Schmidt, Gen. Relativ. Gravit. (to be published).

[6]R. C. Tolman, *Relativity, Thermodynamics and Cosmology* (Clarendon Press, Oxford, 1934), p. 234.

[7]K. S. Thorne, J. Math. Phys. 16, 345 (1975).

[8]R. Bach and H. Weyl, Mat. Zeits. 13, 134 (1922).

[9]J. L. Synge, *Relativity: The General Theory* (North-Holland, Amsterdam, 1960), p. 309.

[10]C. W. Misner, in *Relativity Theory and Astrophysics*, edited by J. Ehlers (Amer. Math. Soc., Providence, Rhode Island, 1967), Vol. 1, p. 160.

[11]W. B. Bonnor, Proc. Camb. Philos. Soc. 66, 145 (1969).

[12]R. P. Kerr and A. Schild, in *Proceedings of the Galileo Galilei Centenary Meeting on General Relativity* (G. Barbara, Florence, Italy, 1965), p. 222.

[13]W. Israel, Phys. Rev. D 2, 641 (1970).