Comment on hard-photon theorems*

G. Grammer, Jr.
Institute for Theoretical Physics, State University of New York, Stony Brook, New York 11794
(Received 24 March 1975)

We examine the possibility of generalizing the Low soft-photon theorem in a model-independent way and conclude that meaningful hard-photon theorems do not exist.

I. INTRODUCTION

Soft-photon techniques have been used extensively in the study of radiative processes.[1] In such an approach, the radiative amplitude is assumed to be smooth enough to expand in powers of the photon momentum, k

$$A = a/k + b + ck + \cdots \tag{1.1}$$

for k small. Then, according to the theorem of Low,[2,3] the coefficients a and b can be calculated exactly from the on-shell nonradiative amplitude and its derivatives and the static electromagnetic properties of the external particles. Model dependence appears only in the term ck and higher orders. Thus on the one hand, Low's theorem provides a model-independent method for calculating the dominant contributions to a soft-photon radiative amplitude. On the other hand, it assures us that the radiative process provides no new information about the underlying dynamics, at least in the soft-photon regime.

Attempts to generalize Low's theorem to avoid the soft-photon limitation have resulted in so-called hard-photon theorems.[4,5] The intention is to forego expansion of the radiative amplitude by employing finite different techniques; the resulting form is then

$$A = a(k)/k + b(k) \ , \tag{1.2}$$

which is valid to all orders in k. According to the hard-photon theorem, $a(k)$ and $b(k)$ can again be computed exactly from the on-shell nonradiative amplitude.[6] Only separately gauge-invariant contributions involving internal radiation are supposedly unaccounted for in (1.2). Consequently, disagreement between experiment and the predictions of (1.2) could be interpreted as evidence for internal structure radiation,[5] for example, as being due to the magnetic moment of the $\Delta^{++}(1236)$ in radiative π^+p scattering.[8-11]

Our intention is to show that such hard-photon theorems do not exist, a conclusion reached more than ten years ago by Feshbach and Yennie.[12] What follows is, in fact, little more than a translation of the Feshbach-Yennie arguments to the present-day language.[4,5,7,9-11] Basically, once it is clear how soft-photon theorems can be derived, it should be obvious why hard-photon theorems cannot be. Accordingly, in Sec. II a Low-type theorem is constructed; in Sec. III the corresponding hard-photon theorem is obtained by the conventional arguments; and finally the fallacy involved in obtaining this generalized Low theorem is explicated.

II. SOFT-PHOTON THEOREM

The essential features of the Low theorem can be illustrated by considering the scattering of a charged pion from a neutral target as shown in Fig. 1. The nonradiative amplitude, $\pi^+N^0 \to \pi^+N^0$, is a function of several variables: the square of the c.m. scattering energy $s = (\pi + P)^2$, the momentum transfer $t = (P - P')^2$, which we take to be fixed, and the masses of the external pion legs which are off-shell by the amounts

$$\delta = \pi^2 - \mu^2 \tag{2.1}$$

and

$$\delta' = \pi'^2 - \mu^2 \ . \tag{2.2}$$

This amplitude will be referred to as $\mathcal{T}(s, \delta, \delta')$, while its on-shell value will be abbreviated $\mathcal{T}(s) \equiv \mathcal{T}(s, 0, 0)$.

The structure of graphs contributing to the radiative amplitude $\pi^+N^0 \to \pi^+N^0\gamma$ is indicated in Fig. 2, where the photon has momentum k_μ and polarization ϵ_μ with $\epsilon \cdot k = k^2 = 0$. First we will discuss radiation by the final external pion, Fig. 2(a). Even though one pion leg of the electromagnetic vertex is off-shell by an amount

$$\delta' = (q' + k)^2 - \mu^2 = 2k \cdot q' \ , \tag{2.3}$$

that vertex is unchanged from one with $\delta' = 0$ owing to the Ward-Takahashi identity.[13-15] Then this part of the amplitude can be written in terms of the off-shell nonradiative amplitude as

$$\frac{1}{e_\pi} \epsilon \cdot E_f = \frac{\epsilon \cdot q'}{k \cdot q'} \mathcal{T}(s_i, 0, \delta') \ , \tag{2.4}$$

where e_π is defined to be a product of the pion

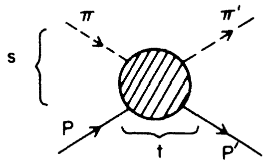

FIG. 1. Kinematic labeling of graphs for scattering of a charged pion from a neutral target, $\pi^+ N^0 \to \pi^+ N^0$.

charge and whatever normalization factors are appropriate. For sufficiently small k we can expand (2.3) about $\delta' = 0$ to get

$$\frac{1}{e_\pi} \epsilon \cdot E_f = \frac{\epsilon \cdot q'}{k \cdot q'} \mathcal{T}(s_i) + 2\epsilon \cdot q' \frac{\partial}{\partial \delta'} \mathcal{T}(s_i, 0, \delta')\Big|_{\delta'=0} + O(k) \ , \tag{2.5}$$

where we have only kept terms to order k. The dominant contribution to (2.5) in this limit is from the infrared-divergent $O(k^{-1})$ term; that part involving the virtual-mass derivative is entirely independent of k.

The amplitude for radiation from the initial-pion line, Fig. 2(b), can be treated similarly except that the πN scattering energy is now shifted to a value $s_f = (p' + q')^2 = s_i - 2k \cdot (p + q)$, so that

$$\frac{1}{e_\pi} \epsilon \cdot E_i = -\frac{\epsilon \cdot q}{k \cdot q} \mathcal{T}(s_f, \delta, 0) \ . \tag{2.6}$$

The small-k expansion can be taken about $\delta = 0$ and $s_f = s_i$,

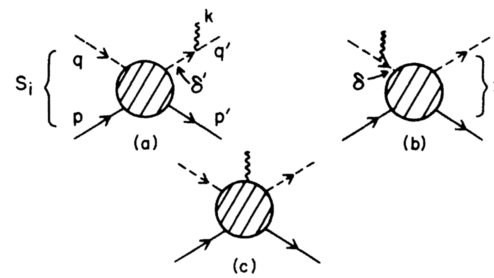

FIG. 2. Structure of graphs contributing to the radiative amplitude $\pi^+ N^0 \to \pi^+ N^0 \gamma$: (a) radiation by final external pion, (b) radiation by initial external pion, (c) internal radiation. The extent to which the pion legs are off-shell is indicated by the quantities δ and δ'.

$$\frac{1}{e_\pi} \epsilon \cdot E_i = -\frac{\epsilon \cdot q}{k \cdot q} \mathcal{T}(s_i) + 2\epsilon \cdot q \frac{\partial}{\partial \delta} \mathcal{T}(s_i, \delta, 0)\Big|_{\delta=0} + \epsilon \cdot q \frac{2k \cdot (p+q)}{k \cdot q} \frac{\partial \mathcal{T}(s_f)}{\partial s_f}\Big|_{s_f = s_i} + O(k) \ , \tag{2.7}$$

where the k dependence of the first two terms is that of the corresponding terms in Eq. (2.5). The energy-variation term is of order $(k)^0$ as distinquished from k-independent. For later convenience we will rewrite Eq. (2.7) by adding and subtracting a term independent of k,

$$\frac{1}{e_\pi} \epsilon \cdot E_i = -\frac{\epsilon \cdot q}{k \cdot q} \mathcal{T}(s_i) + \frac{(\epsilon^\mu k^\nu - \epsilon^\nu k^\mu)}{k \cdot q} \left(2q_\mu (p+q)_\nu \frac{\partial \mathcal{T}(s_f)}{\partial s_f}\Big|_{s_f = s_i} \right) + 2\epsilon \cdot (p+q) \frac{\partial \mathcal{T}(s_f)}{\partial s_f}\Big|_{s_f = s_i} + 2\epsilon \cdot q \frac{\partial}{\partial \delta} \mathcal{T}(s_i, \delta, 0)\Big|_{\delta=0} + O(k) \ ; \tag{2.8}$$

the term in question is the one proportional to $\epsilon^\nu k^\mu$.

That part of the radiative amplitude describing internal emisssion of a photon, $\epsilon \cdot I(k)$, is as yet undetermined. We will not consider the case where $I(k)$ has pathological contributions of order k^{-1} or order k^0, but rather assume that in the soft-photon limit, $I(k)$ can be decomposed into a part independent of k, $I^{(0)}(0)$, and a part of order k and higher, $I^{(1)}(k)$.[12,16,17] The full radiative amplitude

$$\epsilon \cdot T = \epsilon \cdot E_i + \epsilon \cdot E_f + \epsilon \cdot I(k) \tag{2.9}$$

can now be written in expanded form, using Eqs. (2.5) and (2.8), as

$$\frac{1}{e_\pi} \epsilon \cdot T = \epsilon^\mu \left(\frac{q'_\mu}{k \cdot q'} - \frac{q_\mu}{k \cdot q} \right) \mathcal{T}(s_i) + \frac{(\epsilon^\mu k^\nu - \epsilon^\nu k^\mu)}{k \cdot q} \left(2q_\mu (p+q)_\nu \frac{\partial \mathcal{T}(s_f)}{\partial s_f}\Big|_{s_f = s_i} \right) + 2\epsilon \cdot V(0) + O(k) \ . \tag{2.10}$$

The four-vector

$$V_\mu(0) = q'_\mu \frac{\partial \mathcal{T}}{\partial \delta'}(s_i, 0, \delta')\Big|_{\delta'=0} + q_\mu \frac{\partial \mathcal{T}}{\partial \delta}(s_i, \delta, 0)\Big|_{\delta=0} + (p+q)_\mu \frac{\partial \mathcal{T}(s_f)}{\partial s_f}\Big|_{s_f = s_i} + I^{(0)}_\mu(0) \tag{2.11}$$

is just a collection of all the k-independent terms which occur in (2.9).

The requirement of electromagnetic-current conservation, $k \circ T = 0$, is adequate to completely determine the internal radiation, $\epsilon \circ I^{(0)}$. The first two terms in (2.10) vanish identically for $\epsilon_\mu \to k_\mu$, leaving

$$k \cdot T = 0 = k \cdot V(0) \quad . \tag{2.12}$$

As a constant vector cannot be orthogonal to all k_μ, $V_\mu(0)$ must be null to satisfy (2.12).

The radiative amplitude is thus simply

$$\frac{1}{e_\pi}\, \epsilon \cdot T = \epsilon^{\mu}\left(\frac{q'_\mu}{k\cdot q'} - \frac{q_\mu}{k\cdot q}\right)\mathcal{T}(s_i) + \frac{(\epsilon^{\mu}k^{\nu} - \epsilon^{\nu}k^{\mu})}{k\cdot q}\left(2q_\mu(p+q)_\nu \left.\frac{\partial \mathcal{T}(s_f)}{\partial s_f}\right|_{s_f=s_i}\right) + O(k) \quad , \tag{2.13}$$

which depends only on the *nonradiative* amplitude $\mathcal{T}(s)$ and its derivative. No off-mass-shell dependence appears to order k. This is the Low soft-photon theorem for the particular process we have considered.

III. HARD-PHOTON THEOREM

As mentioned earlier, only hard-photon emission provides new dynamical information; soft-photon amplitudes are model independent. The following generalization of the Low theorem to account for high-energy photon emission has been suggested as a model-independent procedure for extracting the interesting part of the internal radiation from experimental data, that is, for subtracting the contribution of external radiation, etc.[5] The reader should note that the derivation given here is designed to elucidate the fallacious aspect of the argument, rather than obscure it as in the original work.[4,5]

The essential new ingredient is the finite-difference ratio; those we will need are defined as

$$D_\delta \mathcal{T}(s_i, \delta, 0) = [\mathcal{T}(s_i, \delta, 0) - \mathcal{T}(s_i)]/\delta \quad , \tag{3.1}$$

$$D_{\delta'} \mathcal{T}(s_i, 0, \delta') = [\mathcal{T}(s_i, 0, \delta') - \mathcal{T}(s_i)]/\delta' \quad , \tag{3.2}$$

and

$$D_{s_f} \mathcal{T}(s_f) = [\mathcal{T}(s_f) - \mathcal{T}(s_i)]/(s_f - s_i) \quad . \tag{3.3}$$

Each difference ratio becomes the corresponding derivative in the limit $k_\mu \to 0$ [e.g., $D_\delta \to (d/d\delta)_{\delta=0}$]. The procedure is then to mimic the steps of Sec. II, replacing all derivatives by finite-difference ratios. The power-series expansion is thereby avoided, so that the result is no longer restricted to soft photons.

The amplitude for radiation from the final-pion line, Eq. (2.5), becomes

$$\frac{1}{e_\pi}\, \epsilon \cdot E_f = \frac{\epsilon \cdot q'}{k \cdot q'}\, \mathcal{T}(s_i) + 2\epsilon \cdot q' D_{\delta'} \mathcal{T}(s_i, 0, \delta') \quad , \tag{3.4}$$

while that for initial radiation, Eq. (2.7), is

$$\frac{1}{e_\pi}\, \epsilon \cdot E_i = -\frac{\epsilon \cdot q}{k \cdot q}\, \mathcal{T}(s_i) + 2\epsilon \cdot q D_\delta \mathcal{T}(s_f, \delta, 0) + \epsilon \cdot q\, \frac{2k\cdot(p+q)}{k\cdot q} D_{s_f}\mathcal{T}(s_f) \quad . \tag{3.5}$$

We combine these with the internal-radiation contribution to obtain the full amplitude

$$\frac{1}{e_\pi}\, \epsilon \cdot T = \epsilon^{\mu}\left(\frac{q'_\mu}{k\cdot q'} - \frac{q_\mu}{k\cdot q}\right)\mathcal{T}(s_i) + \frac{(\epsilon^{\mu}k^{\nu} - \epsilon^{\nu}k^{\mu})}{k\cdot q}\,[2q_\mu(p+k)_\nu D_{s_f}\mathcal{T}(s_f)] + 2\epsilon \cdot V(k) \quad , \tag{3.6}$$

which is exact to all orders in k. The four-vector

$$V_\mu(k) = q'_\mu D_{\delta'} \mathcal{T}(s_f, 0, \delta') + q_\mu D_\delta \mathcal{T}(s_i, \delta, 0) + (p+q)_\mu D_{s_f}\mathcal{T}(s_f) + I_\mu(k) \tag{3.7}$$

contains, among other things, the internal radiation contribution.

Imposition of current conservation yields

$$k \cdot T = 0 = k \cdot V(k) \quad , \tag{3.8}$$

whereby it is concluded [*sic*] that

$$V_\mu(k) = 0 \tag{3.9}$$

in analogy with Sec. II.[18] The resulting hard-photon theorem

$$\epsilon \cdot T = \epsilon^{\mu}\left(\frac{q'_\mu}{k\cdot q'} - \frac{q_\mu}{k\cdot q}\right)\mathcal{T}(s_i) + \frac{(\epsilon^{\mu}k^{\nu} - \epsilon^{\nu}k^{\mu})}{k\cdot q}\,[2q_\mu\,(p+q)_\nu D_{s_f}\mathcal{T}(s_f)] \tag{3.10}$$

would appear to have the following properties:

(a) It is exact to all orders in k and is thus valid for hard photons.

(b) The off-mass-shell dependence cancels to all orders.[6]

(c) Only separately-gauge-invariant contributions from the *internal* structure, such as magnetic-moment radiation, are not accounted for in Eq. (3.10).

Any disagreement between the predictions of Eq. (3.10) and experimental data could then be interpreted as evidence for internal structure.

We note here that if the finite-difference ratios are applied in reverse order in arriving at Eq. (3.5), the final result (3.10) will acquire off-shell dependence. This ambiguity was pointed out in Ref. 7; its origin will be clear momentarily.

IV. THE CRITICISM

The very crux of the Low soft-photon theorem lies in our ability to determine the internal radiation by imposing current conservation, that is, to conclude that $V_\mu(0)=0$ from Eq. (2.12). It is at precisely the corresponding point, Eq. (3.9), that the derivation of the generalized Low theorem fails. Of course, the constant part $V_\mu(0)$ will still vanish as in the derivation of the soft-photon theorem; however, this is simply not true of those parts of $V_\mu(k)$ which are of order k and higher. Thus it is clear that there is an *approximation* involved in arriving at a hard-photon theorem. Whether or not it is a reasonable one we will now discuss.

The hard-photon theorem becomes approximate in order k, the order in which it is intended to improve on the Low theorem. Can one argue that the $O(k)$ terms being dropped are smaller than those retained? One might hope that if $V(k)$ is smooth in k it would remain relatively small even for hard photons. For example, in Ref. 5 the finite-difference technique is alleged to improve the expansion parameter from k/Γ in the vicinity of an s-channel resonance of mass M_R and width Γ to k/M_R by taking into account some of the rapid energy variation of the amplitude. However, one can easily see from the structure of $V_\mu(k)$ in Eq. (3.7) that it has contributions of the same origin as each of the higher-order terms retained in the hard-photon theorem, and can thus be expected to be just as large. It has off-shell contributions, energy-variation terms, as well as internal-radiation effects.

To the extent that order-k contributions are important in a radiative amplitude, there appears to be no way to justify neglect of $V_\mu(k)$. Accordingly, hard-photon theorems of this type do not exist. That one should expect different results in order k for different parametrizations of the strong amplitude should now be clear.[7,10]

ACKNOWLEDGMENT

I would like to thank Professor J. Smith and Professor D. R. Yennie for useful discussions and for reading the manuscript.

---

*Work supported in part by the National Science Foundation under Grant No. GP-32998X.

[1]Reference to much of this work is contained in the review by H. W. Fearing, E. Fischbach, and J. Smith, Phys. Rev. D 2, 542 (1970).

[2]F. E. Low, Phys. Rev. 110, 974 (1958).

[3]S. L. Adler and Y. Dothan, Phys. Rev. 151, 1267 (1966).

[4]W. E. Fischer and P. Minkowski, Nucl. Phys. B 36, 519 (1972).

[5]R. P. Haddock and K. C. Leung, Phys. Rev. D 9, 2151 (1974).

[6]Although this result is not obtained in Ref. 5, it can be a consequence of the "theorem." This ambiguity is symptomatic of difficulties with hard-photon theorems as pointed out in Ref. 7.

[7]C. Picciotto, Nucl. Phys. B89, 357 (1975).

[8]There is a considerable literature on this subject. Most earlier references are given in Refs. 9 and 10.

[9]R. H. Thompson, Nuovo Cimento 16A, 290 (1973).

[10]D. S. Beder, Nucl. Phys. B84, 362 (1975).

[11]A. H. Huffman, Bull. Am. Phys. Soc. 20, 88 (1975).

[12]H. Feshbach and D. R. Yennie, Nucl. Phys. 37, 150 (1962).

[13]J. C. Ward, Phys. Rev. 77, 293 (1950); 78, 182 (1950); Proc. Phys. Soc. 64, 54 (1951).

[14]Y. Takahashi, Nuovo Cimento 6, 371 (1957).

[15]See also Refs. 2, 4, and 5.

[16]S. Barshay and T. Yao, Phys. Rev. 171, 1708 (1968).

[17]V. I. Zakharov, L. A. Kondratyuk, and L. A. Ponomarev, Yad. Fiz. 8, 783 (1968) [Sov. J. Nucl. Phys. 8, 456 (1969).].

[18]It is not clear in other papers that the equivalent of Eq. (3.9) is being used; an additional step obscures this fact. [See, in particular, Eqs. (38) and (39) of Ref. 4 and Eq. (8) of Ref. 5.] For example, suppose

$$V_\mu(k)=Q_\mu B(k)+I_\mu(k),$$

so that current conservation gives

$$k\cdot V=0=k\cdot QB+k\cdot I,$$

or, in other words,

$$k\cdot I(k)=-k\cdot QB(k).$$

This, of course, is fine; it is when one concludes that "therefore"

$$\epsilon\cdot I(k)=-\epsilon\cdot QB(k)$$

that the (erroneous) statement $V_\mu(k)=0$ is made. We note, and comment later, that the knowledge a four-vector $V_\mu(k)$ is orthogonal to a variable four-vector k_μ does not specify $V_\mu(k)$. However, we can say that the only *constant* (i.e., k-independent) four-vector which satisfies $k\cdot V(0)=0$ is $V_\mu=(0,\vec{0})$.