PHYSICAL REVIEW D VOLUME 15, NUMBER 3 1 FEBRUARY 1977

Pseudoscalar-meson mixing problem, spectral-function sum rules, and chiral symmetry

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(Received 24 March 1976)

The pseudoscalar-meson spectrum is first discussed in connection with SU(4) extended spectral-function sum rules. The spectrum can be fitted into the sum rule to within a well-known ambiguity in assigning the η' particle. In order to remove this ambiguity, the η' decay width as well as its mass are next investigated in the effective-chiral-Lagrangian approach. Experimental data seem to favor the assignment of $\eta' = X(958)$.

I. INTRODUCTION

Since the discovery of the new resonances ψ and ψ', an enormous number of theoretical proposals[1] as well as experimental searches[2] have been made in order to understand the possible mechanism. Among many theoretical attempts the simplest and probably the most successful one is a charm model based on SU(4) symmetry.[3] As a way of understanding the new resonances, Weinberg's first and second spectral-function sum rules[4] have recently been extended to SU(4) symmetry.[5,6,7] It has been found that the vector-meson mass spectrum can be fitted nicely into the sum rules constructed from the conserved vector currents.[7]

Although the vector-meson spectrum has been fitted into the Gell-Mann–Okubo mass formula by many people,[1,3] there are several advantages in using spectral-function sum rules. In the spectral-function approach once an SU(4)-breaking term is fixed, the mass spectrum comes out uniquely. This is not so in the group-theoretical approach. The question of whether one should use the quadratic or the linear mass formula is an old problem.[3] Our approach is independent of details of mixing models. The traditional way relies on the mass-mixing model, while there are other mixing models.[8]

Our approach has fewer ambiguities and parameters than a group-theoretical one. Matrix elements of vector currents, sandwiched between the vacuum state and a one-vector-meson state, are directly related to vector-meson masses, and hence SU(4)-breaking effects are automatically taken into account. Therefore, we do not have to assume a specific mass dependence of vector-meson–lepton–lepton coupling constants, for instance. This point has caused a serious theoretical ambiguity in other approaches.[9] It should also be observed that, since conserved vector currents can reproduce the vector-meson spectrum well, scalar mesons play a negligible role or none at all.

After the discovery of ψ, the pseudoscalar-meson system has been reinvestigated. The system needs a larger amount of mixing[3] which might critically depend upon the mixing scheme. For this reason we have decided to investigate the pseudoscalar-meson system using the spectral-function sum-rule approach.[7] Unfortunately, the pseudoscalar-meson system is more complicated than the vector-meson one, owing to the possible existence of axial-vector mesons. Since the experimental situation of axial-vector mesons is not clear at present, we ignore them entirely, thus avoiding the increase of unknown parameters.[10] In effect, we concentrate on the divergence of axial-vector currents rather than on the currents themselves.

In common with other approaches,[3] the following two assignments of the neutral member of the pseudoscalar meson are possible:

I. $\eta(549)$, $\eta' = X(958)$ and η_c

II. $\eta(549)$, $\eta' = E(1420)$ and η_c .

If we insist that the η_c, pseudoscalar counterpart of ψ, should be close to the pure $c\bar{c}$ state, the assignment II might be preferable.[3,11] Since these two assignments predict a different mass value for η_c, the matter will be settled experimentally.

In order to remove the theoretical ambiguity, we should investigate not only the mass but also the decay width. Contrary to $X(958)$ decay, the following process is energetically possible:

$$E(1420) \to K + \bar{K} + \pi \ . \tag{1.1}$$

In fact, through this decay mode the E meson was first observed.[12]

In this paper we are going to calculate the process (1.1) within the linear chiral-SU(3)$\times$SU(3) Lagrangian approach[13] assuming that E is a neutral member of 0^- mesons. In the limit of infinitely large scalar-meson masses, the obtained amplitude does not contain any free parameters and is found to be negligibly small. This certainly contradicts the present experiments. On the other hand, if we choose the assignment $\eta' = X$ within the same scheme[13] we can account for a process such as

$$X \to \eta + 2\pi \tag{1.2}$$

reasonably well. Therefore we conclude that the assignment of $\eta' = X(958)$ should be preferred.

II. THE SPECTRAL-FUNCTION SUM RULES

We start from the covariant propagator for axial-vector currents $A_\mu^{(i)}$ ($i=0,1,\ldots,15$; $\mu=1,2,3,4$),

$$\begin{aligned}\Delta_{\mu\nu}^{(ij)} &= -i\int d^4x e^{-iq\cdot x}\langle 0|T^*[A_\mu^{(i)}(x)A_\nu^{(j)}(0)]|0\rangle \\ &= -i\int dm^2\,\frac{1}{q^2+m^2}\Big[q_\mu q_\nu \rho_P^{(ij)}(m^2) \\ &\qquad +\Big(\delta_{\mu\nu}+\frac{q_\mu q_\nu}{m^2}\Big)\rho_A^{(ij)}(m^2)\Big],\end{aligned} \tag{2.1}$$

where $\rho_P^{(ij)}(m^2)$ [and $\rho_A^{(ij)}(m^2)$] are the spectral functions for pseudoscalar (and axial-vector) mesons, respectively. Ignoring possible contributions of axial-vector mesons, we restrict ourselves to the following object:

$$q_\mu q_\nu \Delta_{\mu\nu}^{(ij)}(q) = -i\int dm^2\,\frac{(q^2)^2\rho_P^{(ij)}(m^2)}{q^2+m^2}\ . \tag{2.2}$$

Assuming that $q_\mu q_\nu \Delta_{\mu\nu}^{(ij)}(q)/q^2$ becomes SU(4) symmetric as $q^2\to\infty$, we write

$$\int dm^2 \rho_P^{(ij)}(m^2) = A[\delta_{ij}+(X-1)\delta_{i0}\delta_{j0}] \quad (i,j=0,1,\ldots,15)\ , \tag{2.3}$$

which yields an extension of Weinberg's[4] first sum rule. Concerning an extension of Weinberg's second sum rule, we adopt the following form:

$$\int dm^2\rho_P^{(ij)}(m^2)m^2 = A[N\delta_{ij}+Y\delta_{i0}\delta_{j0} + C(d_{8ij}+\beta d_{15\,ij})]\ . \tag{2.4}$$

Here A, X, C, N, and β are some constants independent of SU(4) suffix i.[14]

Exactly the same form was investigated for the vector-meson case.[7] The explicit symmetry-breaking term (C term) is added in the spirit of Das, Mathur, and Okubo.[5]

We saturate only the low-lying pseudoscalar-meson states π, K, η, η', and η_c which are supposed to belong to the $\underline{15}+\underline{1}$ representation of SU(4). As discussed in the introduction, there are two possible candidates for η', and η_c has yet to be observed.

Let us introduce PCAC (partial conservation of axial-vector current) constants by

$$\langle 0|A_\mu^{(k)}(0)|\pi^{(j)}(q)\rangle = \frac{1}{(2q_0)^{1/2}}\frac{F_\pi}{\sqrt2}iq_\mu\delta_{jk} \quad (j,k=1,\ldots,3)\ , \tag{2.5}$$

$$\langle 0|A_\mu^{(k)}(0)|K^{(j)}(q)\rangle = \frac{1}{(2q_0)^{1/2}}\frac{F_K}{\sqrt2}iq_\mu\delta_{jk} \quad (j,k=4,\ldots,7)\ , \tag{2.6}$$

and

$$\langle 0|A_\mu^{(k)}(0)|P(q)\rangle = \frac{1}{(2q_0)^{1/2}}\frac{F_P^{(k)}}{\sqrt2}iq_\mu \quad (k=0,8,15;\ P=\eta,\eta',\eta_c)\ . \tag{2.7}$$

Substituting an expression like

$$\rho^{(ij)}(m^2) = \frac{F_K^{\,2}}{2}\delta(m^2-m_K^{\,2})\delta_{ij} \quad (i,j=4,\ldots,7) \tag{2.8}$$

in Eqs. (2.3) and (2.4), we obtain

$$2A = F_\pi^{\,2} = F_K^{\,2} = \sum_i (F_i^{(8)})^2 = \sum_i (F_i^{(15)})^2 = \frac{1}{X}\sum_i (F_i^{(0)})^2\ , \tag{2.9}$$

$$0 = \sum_i F_i^{(8)}F_i^{(15)} = \sum_i F_i^{(8)}F_i^{(0)} = \sum_i F_i^{(15)}F_i^{(0)} \tag{2.10}$$

(where $i=\eta,\eta'$, and η_c), and

$$\begin{aligned} &m_\pi^{\,2}F_\pi^{\,2} = 2A[N+C(1/\sqrt3+\beta/\sqrt6)]\ ,\\ &m_\pi^{\,2}F_K^{\,2} = 2A\{N+C[-1/(2\sqrt3)+\beta/\sqrt6]\}\ ,\\ &\sum_i m_i^{\,2}(F_i^{(8)})^2 = 2A[N+C(-1/\sqrt3+\beta/\sqrt6)]\ ,\\ &\sum_i m_i^{\,2}(F_i^{(0)})^2 = 2A[N+Y]\ ,\\ &\sum_i m_i^{\,2}(F_i^{(15)})^2 = 2A[N-(\tfrac23)^{1/2}\beta C]\ ,\end{aligned} \tag{2.11}$$

$$\begin{aligned} &\sum_i m_i^{\,2}F_i^{(8)}F_i^{(0)} = 2A\,\frac{C}{\sqrt2}\ ,\\ &\sum_i m_i^{\,2}F_i^{(8)}F_i^{(15)} = 2A\,\frac{C}{\sqrt6}\ ,\\ &\sum_i m_i^{\,2}F_i^{(15)}F_i^{(0)} = 2A\,\frac{C\beta}{\sqrt6}\ .\end{aligned} \tag{2.12}$$

Here we note that C and N are expressible in terms of β and masses only

$$C=-\frac{2}{\sqrt{3}}(m_K{}^2-m_\pi{}^2)\ , \tag{2.13}$$

$$N=m_\pi{}^2+\frac{2}{\sqrt{3}}(m_K{}^2-m_\pi{}^2)\left(\frac{1}{\sqrt{3}}+\frac{\beta}{\sqrt{6}}\right)\ . \tag{2.14}$$

First we observe that Eqs. (2.9) and (2.10) are trivially satisfied when $F_i^{(8)}/(2A)^{1/2}$, $F_i^{(15)}/(2A)^{1/2}$, and $F_i^{(0)}/(2AX)^{1/2}$ are regarded as three unit orthogonal vectors. On this basis, however, the mass matrix is not diagonal, as seen from Eqs. (2.12) and (2.11). In order to identify m_i to be physical, the mass matrix must be diagonalized. Thus, independent of PCAC constants, we must have the following mass relation:

$$\begin{vmatrix} E^{(8)}-m_i{}^2 & \dfrac{\sigma}{\sqrt{X}} & \epsilon \\ \dfrac{\sigma}{\sqrt{X}} & \dfrac{E^{(0)}}{X}-m_i{}^2 & \dfrac{\lambda}{\sqrt{X}} \\ \epsilon & \dfrac{\lambda}{\sqrt{X}} & E^{(15)}-m_i{}^2 \end{vmatrix}=0\ , \tag{2.15}$$

for $i=\eta,\eta',\eta_c$, where

$$\begin{aligned} E^{(8)}&=N+C\left(-\frac{1}{\sqrt{3}}+\frac{\beta}{\sqrt{6}}\right)\ ,\\ E^{(15)}&=N-\left(\frac{2}{3}\right)^{1/2}\beta C\ ,\\ E^{(0)}&=N+Y\ ,\\ \sigma&=\frac{C}{\sqrt{2}}\ ,\ \epsilon=\frac{C}{\sqrt{6}}\ ,\ \lambda=\frac{C\beta}{\sqrt{2}}\ . \end{aligned} \tag{2.16}$$

Equation (2.15) can be rewritten

$$\begin{aligned}\sigma^2(m_i{}^2-E^{(15)})+\epsilon^2(m_i{}^2X-E^{(0)})+\lambda^2(m_i{}^2-E^{(8)})+2\epsilon\sigma\lambda\\ =(m_i{}^2-E^{(15)})(m_i{}^2-E^{(8)})(m_i{}^2X-E^{(0)})\\ (i=\eta,\eta',\eta_c)\ .\end{aligned} \tag{2.17}$$

We note that the mass relation for pseudoscalar mesons has a form identical to the vector-meson one.[7] For given $m_\pi{}^2$ and $m_K{}^2$, Eq. (2.17) determines η, η', and η_c masses in terms of β, X, and Y or vice versa. Since the experimental η_c mass is not available at present, we cannot determine all the unknown parameters (β, X, and Y) completely. According to the quark model, however, β can be related to the quark mass by

$$\frac{2\sqrt{2}\ \beta+1}{3}=\frac{m_c-m_{\mathcal{P}}}{m_\lambda-m_{\mathcal{P}}}\ , \tag{2.18}$$

where m_α is a mass of the α quark. Thus, assuming universality, we may use the value of β determined from the analysis of vector mesons.

The 16-plet model gives

$$\beta=\frac{3m_\psi{}^2-2m_{K^*}{}^2-m_\rho{}^2}{4\sqrt{2}\,(m_{K^*}{}^2-m_\rho{}^2)}\ , \tag{2.19}$$

and a more refined model gives[7]

$$\beta=21.49\ . \tag{2.20}$$

Equation (2.17) can be written explicitly as

$$\begin{aligned} x_i{}^3+x_i{}^2\left[\tfrac{2}{3}\alpha'-\sqrt{2}\ \beta\alpha'+\mu^2-\frac{1}{X}\left(\mu^2-\tfrac{2}{3}\alpha'+\tfrac{1}{3}\sqrt{2}\ \beta\alpha'+Y\right)\right]\\ +x_i\alpha'\left\{(\tfrac{2}{3}-\sqrt{2}\ \beta)\left[\mu^2-\frac{1}{X}\left(\mu^2-\tfrac{2}{3}\alpha'+\tfrac{1}{3}\sqrt{2}\ \beta\alpha'+Y\right)\right]-\frac{2\alpha'}{9X}(3+X+3\beta^2)\right\}\\ -\frac{2\alpha'^2}{9X}\left(\tfrac{8}{3}\alpha'-\tfrac{16}{3}\sqrt{2}\ \beta\alpha'+\mu^2X-\mu^2-Y\right)=0\ , \end{aligned} \tag{2.21}$$

where $x_i\equiv m_i{}^2-\mu^2$ with $\mu^2=\frac{1}{3}(4m_K{}^2-m_\pi{}^2)$ and $\alpha'=m_K{}^2-m_\pi{}^2$ ($i=\eta,\eta'$, and η_c), or, equivalently,

$$x_\eta+x_{\eta'}+x_{\eta_c}=\frac{1}{X}\left(\mu^2-\tfrac{2}{3}\alpha'+\frac{\sqrt{2}}{3}\beta\alpha'+Y\right)-\mu^2-\tfrac{2}{3}\alpha'+\sqrt{2}\ \beta\alpha'\ , \tag{2.22a}$$

$$x_\eta x_{\eta'}+x_\eta x_{\eta_c}+x_{\eta'}x_{\eta_c}=(\tfrac{2}{3}-\sqrt{2}\ \beta)\alpha'\left[\mu^2-\frac{1}{X}\left(\mu^2-\tfrac{2}{3}\alpha'+\frac{\sqrt{2}}{3}\beta\alpha'+Y\right)\right]-\frac{2\alpha'^2}{9X}(3+X+3\beta^2)\ , \tag{2.22b}$$

$$x_\eta x_{\eta'}x_{\eta_c}=\frac{2\alpha'^2}{9X}\left(\tfrac{8}{3}\alpha'-\tfrac{16}{3}\sqrt{2}\ \beta\alpha'+\mu^2X-\mu^2-Y\right)\ . \tag{2.22c}$$

Substituting mass values[15] for π, K, η, and η', we have determined the unknown parameters X and Y and an η_c mass. Numerical results are listed in Table I. We note that parameter X is not close to 1 and that Y/α is not small at all. Thus the pseudoscalar-meson spectrum has quite a different structure from the vector-meson one where the nonet, or its SU(4)-extended 16-plet, model supplied us with such a good first approximation. In other words, our numerical analysis tells us that we should not apply nonet, or 16-plet, mass relations, which are the direct results of the assumptions $X=1$ and $Y=0$, to the pseudoscalar-meson spectrum. F-meson and D-meson[16] masses can be predicted similarly.

From Eqs. (2.3) and (2.4) we obtain

$$2A=F_\pi^2=F_K^2=F_D^2=F_F^2 , \tag{2.23}$$

$$m_D^2A=N+C\left(\frac{1}{2\sqrt{3}}-\frac{\beta}{\sqrt{6}}\right), \tag{2.24}$$

$$m_F^2A=N+C\left(-\frac{1}{\sqrt{3}}-\frac{\beta}{\sqrt{6}}\right), \tag{2.25}$$

where

$$\langle 0|A_\mu^{(j)}(0)|D(q)\rangle=\frac{1}{(2q_0)^{1/2}}\frac{F_D}{\sqrt{2}}iq_\mu \quad (j=9,10,11,12) , \tag{2.26}$$

$$\langle 0|A_\mu^{(j)}(0)|F(q)\rangle=\frac{1}{(2q_0)^{1/2}}\frac{F_F}{\sqrt{2}}iq_\mu \quad (j=13,14) . \tag{2.27}$$

Substituting Eqs. (2.13) and (2.14) we obtain

$$m_D^2=\mu^2-\alpha'+\frac{2\sqrt{2}}{3}\beta\alpha' , \tag{2.28}$$

$$m_F^2=\mu^2+\frac{2\sqrt{2}}{3}\beta\alpha' . \tag{2.29}$$

With a choice $\beta=21.49$, masses are predicted to be

$$m_D=2230.2 \text{ MeV} \tag{2.30}$$

$$m_F=2178.2 \text{ MeV} . \tag{2.31}$$

Earlier, essentially the same values were predicted by the group-theoretical approach provided one used the quadratic mass formula.[3] As we stressed in the Introduction, the spectral-function sum rules supply us with information not only on masses but also on PCAC constants. We find

$$|F_D|=|F_F|=|F_K|=|F_\pi| , \tag{2.32}$$

which is, apart from uninteresting phases, just an SU(4) relation.[1,17] After some calculations, other PCAC constants F_η, $F_{\eta'}$, and F_{η_c} can be determined. The ratios of F_i are, for instance,

$$\frac{F_i^{(0)}}{F_i^{(8)}}=a_i \text{ and } \frac{F_i^{(15)}}{F_i^{(8)}}=b_i \quad (i=\eta,\eta',\eta_c) , \tag{2.33}$$

TABLE I. Numerical results for Y, X, and η_c with a choice of $\beta=21.49$ and with known pseudoscalar-meson masses as an input.

	Y/α	X	m_{η_c}
$\eta'=X$	18.371	4.6021	2776.6 MeV
$\eta'=E$	14.038	1.3004	3092.0 MeV

where

$$a_i=\frac{\sqrt{3}\,X[\sqrt{2}\,\alpha'+3\beta(\mu^2-m_i^2)]}{4\sqrt{2}\,\alpha'\beta+3(\mu^2-\frac{2}{3}\alpha'+Y-Xm_i^2)} , \tag{2.34}$$

$$b_i=\frac{\sqrt{2}\,\alpha'+3\beta(\mu^2-m_i^2)}{4\sqrt{2}\,\alpha'\beta+3(\mu^2-\frac{2}{3}\alpha'-m_i^2)} . \tag{2.35}$$

In terms of a_i and b_i, PCAC constants are given by

$$F_i^{(8)}=\pm F_\pi\left[(\vec{a}\times\vec{b})_i\Big/\sum_j(\vec{a}\times\vec{b})_j\right]^{1/2} , \tag{2.36}$$

$$F_i^{(0)}=\pm F_\pi a_i\left[(\vec{a}\times\vec{b})_i\Big/\sum_j(\vec{a}\times\vec{b})_j\right]^{1/2} , \tag{2.37}$$

$$F_i^{(15)}=\pm F_\pi b_i\left[(\vec{a}\times\vec{b})_i\Big/\sum_j(\vec{a}\times\vec{b})_j\right]^{1/2} . \tag{2.38}$$

Here a comment is in order. Suppose η_c is a pure $c\bar{c}$ state, then we should have

$$F_{\eta_c}^{(8)}=0 ,$$

or equivalently from Eqs. (2.33) and (2.35)

$$4\sqrt{2}\,\beta\alpha'=3(m_{\eta_c}^2-\mu^2+\tfrac{2}{3}\alpha') . \tag{2.39}$$

With a choice of $\beta=21.49$, Eq. (2.39) gives

$$m_{\eta_c}=3077 \text{ MeV} ,$$

which is close to the previously predicted value if $\eta'=E$ is assigned (see Table I). This is not so for the assignment of $\eta'=X(958)$. Thus if we accept the assignment of $\eta'=X(958)$ either the impure $c\bar{c}$ state should be allowed or the nonuniversal value of β must be entertained, agreeing with earlier observations.[3] Since the numerical values of F_{η_c}, etc. are not of immediate experimental interest, we shall not discuss them any further.

III. $E\to K+\bar{K}+\pi$

In this section we are going to discuss the question of which assignment of η' can be consistent with the pseudoscalar-meson system.

Since the spectral-function-sum-rules approach is awkward for treating problems like 3-body decay processes, we rely instead on the supposedly equivalent phenomenological-Lagrangian approach from now on. For simplicity, we limit ourselves to the SU(3) level and concentrate on the 0^--meson nonet and discuss the possible assignment of the η' meson. [A complete discussion on the pseudoscalar-meson 16-plet system in $SU(4)\times SU(4)$ symmetry will be given elsewhere.]

Using the linear chiral-$SU(3)\times SU(3)$ approach developed earlier,[13] we are going to compute the decay mode

$$E(p)\to K^+(q_+)+K^-(q_-)+\pi^0(q_\pi)\ , \tag{3.1}$$

where the momentum of each particle is indicated in parentheses. Corresponding to the Feynman diagrams of Fig. 1, the T-matrix element in the tree approximation is given by

$$T=f-\frac{1}{\sqrt{2}}\left[\frac{g_{\epsilon K\bar{K}}g_{\epsilon\eta'\pi}}{m_\epsilon^2+(p-q_\pi)^2}+\frac{g_{\kappa K\pi}g_{\kappa\eta' K}}{m_\kappa^2+(p-q_-)^2}+\frac{g_{\kappa K\pi}g_{\kappa\eta' K}}{m_\kappa^2+(p-q_+)^2}\right], \tag{3.2}$$

where three-point vertices and a four-point vertex are defined by

$$\begin{aligned}-\mathcal{L}=&\,g_{\epsilon\eta'\pi}(\vec{\epsilon}\cdot\vec{\pi})\eta'+\frac{1}{\sqrt{2}}g_{\epsilon K\bar{K}}(K(\vec{\tau}\cdot\vec{\epsilon})\bar{K})\\&+[g_{\kappa\eta' K}\eta'(\kappa\bar{K})+\text{H.c.}]\\&+\left[\frac{1}{\sqrt{2}}g_{\kappa K\pi}(\kappa(\vec{\tau}\cdot\vec{\pi})\bar{K})+\text{H.c.}\right]+f\eta'\pi^0K^+K^-\ .\end{aligned} \tag{3.3}$$

Here ϵ (or κ) is a scalar meson partner of the pseudoscalar meson π (or K), respectively.

In the linear-chiral-Lagrangian approach these three- and four-point vertices are all calculated and can be related to two-point vertices, i.e., essentially to masses. Three-point vertices were previously evaluated as[13]

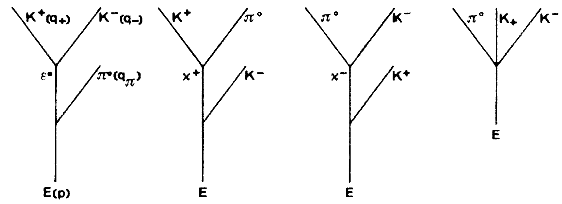

FIG. 1. Feynman diagrams for $E\to K^++K^-+\pi^0$ decay.

$$g_{\kappa\eta' K}=\frac{m_\kappa^2-m_{\eta'}^2}{\alpha(1+W)}(a+\sqrt{2}\,b)\ , \tag{3.4}$$

$$g_{\kappa K\pi}=\frac{m_\kappa^2-m_\pi^2}{\alpha(1+W)}=\frac{m_\kappa^2-m_K^2}{2\alpha}=\frac{m_K^2-m_\pi^2}{\alpha(W-1)}\ , \tag{3.5}$$

$$g_{\epsilon K\bar{K}}=\frac{m_\epsilon^2-m_K^2}{\alpha(1+W)}\ , \tag{3.6}$$

$$g_{\epsilon\eta'\pi}=\frac{a}{\alpha}(m_\epsilon^2-m_{\eta'}^2)\ , \tag{3.7}$$

where parameters α and W were introduced in Ref. 13 and can be related to F_K and F_π as

$$\alpha=\tfrac{1}{2}F_\pi\ , \tag{3.8}$$

$$W=2F_K/F_\pi-1\ . \tag{3.9}$$

Parameters a and b depend upon a mixing angle of η and η'. Its explicit form will be given later [see Eq. (3.12)]. We must evaluate only a four-point vertex f. In the notation of Ref. 13, let us write the chiral-invariant four-point and also three-point vertices as

$$-\mathcal{L}=\frac{1}{4!}\sum_{a,b,c,d,e,f,g,h}\left\langle\frac{\partial^4V_0}{\partial\varphi_a^b\partial\varphi_c^d\partial\varphi_e^f\partial\varphi_g^h}\right\rangle_0\varphi_a^b\varphi_c^d\varphi_e^f\varphi_g^h+\frac{1}{2}\sum_{a,b,c,d,e,f}\left\langle\frac{\partial^3V_0}{\partial s_a^b\partial\varphi_c^d\partial\varphi_e^f}\right\rangle_0 s_a^b\varphi_c^d\varphi_e^f\ , \tag{3.10}$$

where V_0 is the most general chiral-$SU(3)\times SU(3)$-invariant potential made of nine pseudoscalar-meson fields and nine scalar-meson fields, and the notation $\langle\ \rangle_0$ means that the enclosed object is evaluated at the equilibrium point of the system. (For further discussion on the choice of an explicit symmetry-breaking term, etc. see Ref. 13.) In Eq. (3.10) $\varphi_a^b(s_a^b)$ represent nine physical pseudoscalar (scalar) mesons, respectively. The previously defined parameter f is now rewritten as

$$f\equiv\left\langle\frac{\partial^4V_0}{\partial\eta'\partial\pi^0\partial\varphi_1^3\partial\varphi_3^1}\right\rangle_0=\sum_{g,m}\left\langle\frac{\partial^4V_0}{\partial\varphi_m^m\partial\varphi_g^g\partial\varphi_1^3\varphi_3^1}\right\rangle_0\frac{\partial\varphi_g^g}{\partial\pi_0}\frac{\partial\varphi_m^m}{\partial\eta'}\ .$$

Neutral components of φ_a^b are related to π^0, η, and η' by

$$\begin{pmatrix}\varphi_1^1\\ \varphi_2^2\\ \varphi_3^3\end{pmatrix}=\begin{pmatrix}\sqrt{\frac{1}{2}} & b & a\\ -\sqrt{\frac{1}{2}} & b & a\\ 0 & -\sqrt{2}\,a & \sqrt{2}\,b\end{pmatrix}\begin{pmatrix}\pi^0\\ \eta\\ \eta'\end{pmatrix}, \tag{3.11}$$

where

$$a=\frac{1}{\sqrt{6}}(\sin\theta+\sqrt{2}\cos\theta),\quad b=\frac{1}{\sqrt{6}}(-\sqrt{2}\sin\theta+\cos\theta)\ . \tag{3.12}$$

Using the basic formula

$$\alpha(1+W)\left\langle\frac{\partial^4 V_0}{\partial\varphi_m^m\,\partial\varphi_g^g\,\partial\varphi_1^3\,\partial\varphi_3^1}\right\rangle_0 \quad \text{(no sum)}$$
$$=\left\langle\frac{\partial^3 V_0}{\partial\varphi_m^m\,\partial\varphi_g^g\,\partial s_1^1}\right\rangle_0+\delta_g^3\left\langle\frac{\partial^3 V_0}{\partial\varphi_m^m\,\partial\varphi_1^3\,\partial s_3^1}\right\rangle_0+\delta_m^3\left\langle\frac{\partial^3 V_0}{\partial\varphi_g^g\,\partial\varphi_1^3\partial s_3^1}\right\rangle_0+\left\langle\frac{\partial^3 V_0}{\partial\varphi_m^m\partial\varphi_g^g\partial s_3^3}\right\rangle_0+\delta_1^g\left\langle\frac{\partial^3 V_0}{\partial\varphi_m^m\partial\varphi_1^3\partial s_3^1}\right\rangle_0+\delta_1^m\left\langle\frac{\partial^3 V_0}{\partial\varphi_g^g\,\partial\varphi_1^3\partial s_3^1}\right\rangle_0, \tag{3.13}$$

we derive

$$f=\frac{1}{\alpha(1+W)}\left[\left\langle\frac{\partial^3 V_0}{\partial\eta'\partial\pi^0\partial s_1^1}\right\rangle_0+\left\langle\frac{\partial^3 V_0}{\partial\eta'\partial\pi^0\partial s_3^3}\right\rangle_0+\left\langle\frac{\partial^3 V_0}{\partial\eta'\partial\varphi_1^3\partial s_3^1}\right\rangle\frac{1}{\sqrt{2}}+\left\langle\frac{\partial^3 V_0}{\partial\pi^0\partial\varphi_1^3\partial s_3^1}\right\rangle(a+\sqrt{2}\,b)\right]$$
$$=\frac{1}{\alpha(1+W)}\left[\frac{a(m_\epsilon^2-m_{\eta'}^2)}{\sqrt{2}\,\alpha}+\frac{g_{\kappa\eta' K}}{\sqrt{2}}+\frac{g_{\kappa K\pi}}{\sqrt{2}}(a+\sqrt{2}\,b)\right]. \tag{3.14}$$

After combining everything, we find

$$T=\frac{(a+\sqrt{2}\,b)}{\sqrt{2}\,\alpha^2(1+W)^2}\left\{2m_\kappa^2-m_{\eta'}^2-m_\pi^2-(m_\kappa^2-m_\pi^2)(m_\kappa^2-m_{\eta'}^2)\left[\frac{1}{m_\kappa^2+(p-q_-)^2}+\frac{1}{m_\kappa^2+(p-q_+)^2}\right]\right\}$$
$$+\frac{a}{\sqrt{2}\,\alpha^2(1+W)}(m_\epsilon^2-m_{\eta'}^2)\left[1-\frac{m_\epsilon^2-m_K^2}{m_\epsilon^2+(p-q_\pi)^2}\right]\ . \tag{3.15}$$

Consistent with Adler, the above amplitude vanishes in the soft-kaon limit $q_+=0$ (keeping other particles on the mass shell). The final result contains scalar-masses on which we do not have much experimental information at present.

According to the analysis of vector-meson masses[7] based on the spectral-function sum rules, however, scalar mesons are not needed at all. Therefore, motivated by this fact and for the sake of simplicity, we let scalar-meson masses m_ϵ^2 and m_κ^2 be infinitely heavy. Then

$$T(E\to K^++K^-+\pi^0)$$
$$=\frac{2}{\sqrt{3}\,F_\pi^2(1+W)}\left\{\frac{(2\sqrt{2}\,\cos\theta-\sin\theta)}{(1+W)}[(m_E-m_\pi)^2-2m_K^2-2m_ET_\pi]+(\sqrt{2}\,\cos\theta+\sin\theta)[m_K^2-(m_E-m_\pi)^2+2m_ET_\pi]\right\}, \tag{3.16}$$

where T_π is a kinetic energy of the π^0 meson in the rest frame of the E meson. This expression does not contain any free parameter. As was mentioned earlier, W can be determined from

$$W=2F_K/F_\pi-1\ .$$

From semileptonic decays, $|F_K/F_\pi|$ is conventionally taken to be 1.28, but there is some uncertainty associated with this determination. Alternatively, $F_K/F_\pi=1$ [SU(3) value] might be used. The former gives $W=1.56$, while the latter gives $W=1$. Regarding W as a free parameter, we can independently determine W from the following mass formula[18,13]:

$$m_{\eta'}^2=\frac{2W^2m_K^2(m_\eta^2-m_K^2)+2W(m_K^2-m_\pi^2)(m_\eta^2-2m_K^2)+m_\pi^2(m_\eta^2-m_\pi^2)-2(m_K^2-m_\pi^2)^2}{2W^2(m_\eta^2-m_K^2)-2W(m_K^2-m_\pi^2)+m_\eta^2-m_\pi^2}\ . \tag{3.17}$$

With an assignment $\eta'=E$, W is determined to be 1.055, which is close to the SU(3) value. Earlier we let scalar mesons be infinitely heavy. Theoretically speaking, this procedure is valid only if W is close to 1. With an assignment of $\eta'=E$, the above criterion is satisfied. With a choice of $\eta'=E$ (hence with $W=1.055$) the mixing angle θ turns out to be[19]

$$\theta=-0.0977\ .$$

Substituting numbers, we obtain

$$T=-6.212(m_{\pi^0}/F_\pi)^2[1+0.2041(T_\pi/m_\pi)]\ . \tag{3.18}$$

This in turn gives the following decay rate[20]:

$$\Gamma(E\to K^+K^-\pi^0)=0.26\text{ MeV}\ .$$

The reason why we obtain a vanishingly small amplitude is due to the fact that the two terms originating from κ and ϵ interfere destructively. When we rewrite Eq. (3.16) in the form

$$T(E\to K^+ + K^- + \pi^0)$$
$$=\frac{2}{\sqrt{3}\,F_\pi^{\,2}(1+W)}\left\{-m_K^{\,2}(\sqrt{2}\cos\theta+\sin\theta)+\left[(\sqrt{2}\cos\theta+\sin\theta)-\frac{(2\sqrt{2}\cos\theta-\sin\theta)}{(1+W)}\right][2m_K^{\,2}-(m_E-m_\pi)^2+2m_ET_\pi]\right\}, \quad (3.19)$$

we note that in the limit of $W=1$ and $\theta=0$ the energy-dependent term disappears completely, leaving a small constant term

$$T(E\to K^+K^-\pi^0)=-\sqrt{\tfrac{2}{3}}\;\frac{m_K^{\,2}}{F_\pi^{\,2}} \quad (3.20)$$
$$(W=1,\ \theta=0)\ .$$

Numerical results are not sensitive to the choice of W or θ as long as W stays close to 1 and θ stays close to 0.

Present experimental data are scanty. Earlier Baillon *et al.*[12] gave, within large error bars,

$$\Gamma(E\to K\bar{K}\pi)\simeq 40\ \text{MeV}.$$

The latest particle data[2] give, as an educated guess, the following total decay rate of E:

$$\Gamma_E(\text{total})=60\pm 20\ \text{MeV}\ ,$$

of which roughly $\frac{1}{5}$ might be the $K\bar{K}\pi$ decay mode.[2] Hence, the experimental width is roughly

$$\Gamma(E\to K\bar{K}\pi)\sim 12\ \text{MeV}\ . \quad (3.21)$$

This should be compared with the theoretical value

$$\Gamma(E\to KK\pi)=6\Gamma(E\to K^+K^-\pi^0)=1.56\ \text{MeV}\ , \quad (3.22)$$

where isospin invariance is assumed.[21]

In spite of large experimental uncertainties, it is clear that the theory cannot account for the broad width which present experiments seem to indicate.

IV. $\eta'\to\eta+2\pi$

We can also calculate

$$E(p)\to\eta(q_\eta)+\pi^+(q_+)+\pi^-(q_-)$$

in the same $SU(3)\times SU(3)$ model.

In the infinitely-heavy-scalar-meson limit the T matrix element for $\eta'\to\eta+\pi^++\pi^-$ was obtained as[13,22]

$$T(\eta'\to\eta+\pi^++\pi^-)=\frac{1}{3F_\pi^{\,2}}\,(2\sqrt{2}\cos 2\theta-\sin 2\theta)$$
$$\times\left[(m_{\eta'}-m_\eta)^2-2\pi^2-2m_{\eta'}T_\eta\right]\ , \quad (4.1)$$

where T_η is a kinetic energy of η in the rest frame of η'

$$(T_\eta\equiv q_\eta^0-m_\eta)\ .$$

The above formula holds for $\eta'=X$ as well as $\eta'=E$. We note that, contrary to the $\eta'\to K\bar{K}\pi$ decay, no delicate cancellation occurs here. If we assign $\eta'=E$, we then have

$$T(E\to\eta+\pi^++\pi^-)=38.88\left(1-0.5341\frac{T_\eta}{m_{\pi^0}}\right). \quad (4.2)$$

Since the kinetic-energy term is not necessarily small, we evaluate the phase space exactly. Then the width for $E\to\eta+2\pi$ turns out to be

$$\Gamma(E\to\eta+2\pi)=\tfrac{3}{2}\,\Gamma(E\to\eta+\pi^++\pi^-)$$
$$=15.41\ \text{MeV}\ . \quad (4.3)$$

Unfortunately the experimental width is not accurately known yet. The following width may be guessed from the data[2]:

$$\Gamma(E\to\eta+2\pi)\sim\tfrac{3}{5}\,\Gamma_E(\text{total})\sim 36\ \text{MeV}\ . \quad (4.4)$$

Thus we cannot draw any conclusion from the analysis of this decay at present.

Let us next discuss the $X\to\eta+\pi^++\pi^-$ decay with an assignment of $\eta'=X$. From Eq. (3.17), etc. we find $W=1.7$ and $\theta=0.005$. Since W is not close to 1 the idea of infinitely heavy scalar mesons may be more questionable here.[23] The above value of W is, however, rather close to that determined from $F_K/F_\pi=1.28$.[24]

Because the phase space available is small, the decay width turns out to be small,

$$\Gamma(X\to\eta+\pi^++\pi^-)=4.08\times 10^{-2}\ \text{MeV}\ ,$$

or equivalently

$$\Gamma(X\to\eta+2\pi)=\tfrac{3}{2}\,\Gamma(X\to\eta+\pi^++\pi^-)=0.0611\ \text{MeV}\ . \quad (4.5)$$

Here again the phase space was evaluated exactly. The above value is smaller than the nonrelativistic one.[13] The theoretical result is compatible with the present experimental upper limit

$$\Gamma(X \to \eta + 2\pi) < 1 \text{ MeV} . \quad (4.6)$$

In conclusion, if we assign $\eta' = E$, then we have difficulty in explaining the broad width for $E \to K + \bar{K} + \pi$ decay. On the other hand, if we assign $\eta' = X(958)$, such a problem does not exist. In fact, the theoretical width for $X \to \eta + 2\pi$ decay is compatible with the small experimental upper limit. Although the assignment of $\eta' = X(958)$ should be favored at present, more accurate experimental data on the energy spectrum of an emitted particle or the widths are highly desirable in order to test the theory more precisely.

ACKNOWLEDGMENTS

The author is grateful to Professor J. Strathdee, Professor A. Salam, and Professor A. Zimmerman for discussions. He is also grateful to Professor Abdus Salam, the International Atomic Energy Agency, and UNESCO for hospitality at the International Centre for Theoretical Physics, Trieste.

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[15]For numerical calculations we choose $m_{\pi^0} = 134.96$ MeV, $m_\eta = 548.8$ MeV, $m_{\eta'}(958) = 957.6$ MeV, $m_E(1420) = 1416$ MeV, $m_{K^0} = 497.7$ MeV as listed in Ref. 2. Our F_π is numerically close to the π^0 mass, $F_\pi = m_{\pi^0}$.

[16]We follow the notation of M. Gaillard *et al.* of Ref. 1. Quark contents of F and D are $F_+ = c\bar{\lambda}$ and $D_+ = c\bar{n}$.

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[19]See the first article of Ref. 18. Explicitly, the mixing angle can be determined from

$$\frac{3(m_{\eta'}^2+m_\eta^2-2m_\pi^2)}{m_{\eta'}^2-m_\eta^2}=(4W-1)\cos 2\theta-\sqrt{2}\,(2+W)\sin 2\theta$$

and

$$\frac{3\sqrt{2}}{m_{\eta'}^2-m_\eta^2}\,[W(m_\eta^2+m_{\eta'}^2)+2m_\pi^2-2m_K^2(1+W)]=\sqrt{2}\,(2+W)\cos 2\theta+(4W-1)\sin 2\theta\,.$$

These are not independent. If we eliminate the mixing angle θ from the above, we reproduce Eq. (3.17).

[20]The decay rate is evaluated from

$$\Gamma(E\to K^+ + K^- + \pi^0)=\frac{1}{64\pi^3 m_E}\int dq_+{}^0 dq_-{}^0|T|^2=\frac{1}{64\pi^3 m_E}|T|^2\times 1.0388 m_{\pi_0}{}^2\,,$$

where phase space is evaluated nonrelativistically, omitting a small kinetic-energy term.

[21]Since isospin must be conserved, the final state is in the isospin-0 state. This means that

$$\Gamma(E\to\bar{K}^0K^+\pi^-)=\Gamma(E\to K^0K^-\pi^+)=2\Gamma(E\to K^0\bar{K}^0\pi^0)=2\Gamma(E\to K^+K^-\pi^0)\,.$$

Therefore we have

$$\Gamma(E\to K\bar{K}\pi)=6\Gamma(E\to K^+K^-\pi^0)\,.$$

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[24]There is a slight discrepancy between the value determined here and the earlier one in Eq. (2.32), although they are still roughly within the experimental ambiguities. Since $F_K/F_\pi-1$ happens to depend upon the SU(3) breaking only, its small discrepancy merely reflects the fact that two methods treat symmetry-breaking effects in a different manner. On the one hand, the linear-chiral-Lagrangian model respects the spontaneous-symmetry-breaking mechanism, while the sum-rule approach does not. Of course in the symmetric limit ($W=1$), both methods yield the same result and our main conclusions on width as well as masses are not sensitive to such symmetry-breaking effects. (For instance, the width for $E\to K+\bar{K}+\pi$ does not depend upon W much for the range of W of interest.)