Determination of the quark-gluon coupling constant*

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Presented here is a new method of applying perturbation theory to $e^+e^- \to$hadrons. Contact between theory and experiment is greatly enhanced by the introduction of a quantity that is both measurable and reliably calculable in perturbation theory. The method yields a value of approximately 700 MeV for Λ, the single parameter that characterizes the color-gluon force.

The chief problem in the application of quark gluon perturbation theory to $e^+e^- \to$ hadrons has been that of relating our knowledge from perturbation theory of the vacuum polarization $\Pi(s)$ in the deep Euclidean region (s, the c.m. energy squared, large and negative) to the behavior of its discontinuity $R(s) = \sigma(e^+e^- \to \text{hadrons})/\sigma(e^+e^- \to \mu^+\mu^-)$ measured in the timelike ($s > 4m_\pi^2$) region. In the deep Euclidean region one may calculate $\Pi(s)$ in a renormalization-group-improved series:

$$\Pi^{\text{theor}}(s) = \Pi^0(s) + \tfrac{4}{3}\alpha(s)\Pi^{(1)}(s) + \cdots, \tag{1}$$

where the running coupling constant for color SU(3) is[1]

$$\alpha(s) = \frac{\alpha(-M^2)}{1 + \dfrac{\alpha(-M^2)}{12\pi}\left[33\ln\left(\dfrac{s}{-M^2}\right) - 2\sum_i \ln\left(\dfrac{5m_i^2 - s}{5m_i^2 + M^2}\right)\right]} \tag{2}$$

where $\alpha(-M^2)$ is the coupling at a Euclidean point $s = -M^2$ and m_i is the quark mass for flavor i. The non-independence of M and $\alpha(-M^2)$ is made manifest by rewriting Eq. (2) as

$$\alpha(s) = \frac{12\pi}{33\ln\left(\dfrac{s}{-\Lambda^2}\right) - 2\sum_i \ln\left(\dfrac{5m_i^2 - s}{5m_i^2 + \Lambda^2}\right)} \tag{3}$$

in terms of a single parameter $\Lambda[M^2, \alpha(-M^2)]$ that defines the gluon force.

The perturbation expansion is expected to break down as we approach the physical region for several reasons, the absence of real quarks and gluons being one. However, Appelquist and Politzer[2] conjectured that if the approximate form of Eq. (1) was continued to the physical region its discontinuity would reproduce the essential features of R^{exp}, away from strong singularities and at energies where $\alpha(s)$ was small. I quote their result for future use:

$$R^{\text{theor}}(s) = \tfrac{3}{2}\sum_i Q_i^2 v_i(3 - v_i^2)[1 + \tfrac{4}{3}\alpha(s)f(v_i)] \tag{4}$$

$$= R^0 + \tfrac{4}{3}\alpha(s)R^1, \tag{5}$$

where Q_i is the quark change, v_i the quark velocity $= (s - 4m_i^2/s)^{1/2}$, and $f(v)$ the function calculated by Schwinger,[3]

$$f(v) = \frac{\pi}{2v} - \frac{3+v}{4}\left(\frac{\pi}{2} - \frac{3}{4\pi}\right). \tag{6}$$

If heavy leptons are not separated in experiment, they must enter Eq. (4) with $Q = 1$, $\alpha \equiv 0$, and the factor $\frac{3}{2}$ replaced by $\frac{1}{2}$.

Subsequently, Adler[4] and De Rújula and Georgi[1] went from the physical region to the Euclidean using as imput R^{exp} up to the highest energies, beyond which they made assumptions about its behavior consistent with the model being tested. Lastly, Poggio, Quinn, and Weinberg[5] compared the imaginary part of Π^{theor} 3 GeV^2 above the physical region with a suitably smeared form of R^{exp}.

The starting point for the present scheme is the observation that for the contour of Fig. 1,

$$\int_{C_1} \Pi\, ds = -\int_{C_2} \Pi\, ds. \tag{7}$$

Let us evaluate the left-hand side using R^{exp}. For the right-hand side we will use Π^{theor} of Eq. (1). If s is large enough for $\alpha(s)$ to be small, we can trust the series over most of the circle C_2 except for a small range $\pm\Delta\theta$ near the real axis (these ideas will be made more precise in a moment). Let us choose $M^2 = |s|$ in Eq. (2). We see that $\alpha(s)$ is essentially constant on C_2 and equal to $\alpha(-|s|)$, the small imaginary part it acquires away from the negative s axis being of the next order. The integrals of Π^0 and Π^1 may be expressed in terms of their discontinuities R^0 and R^1 of Eq. (5). We now have the prediction

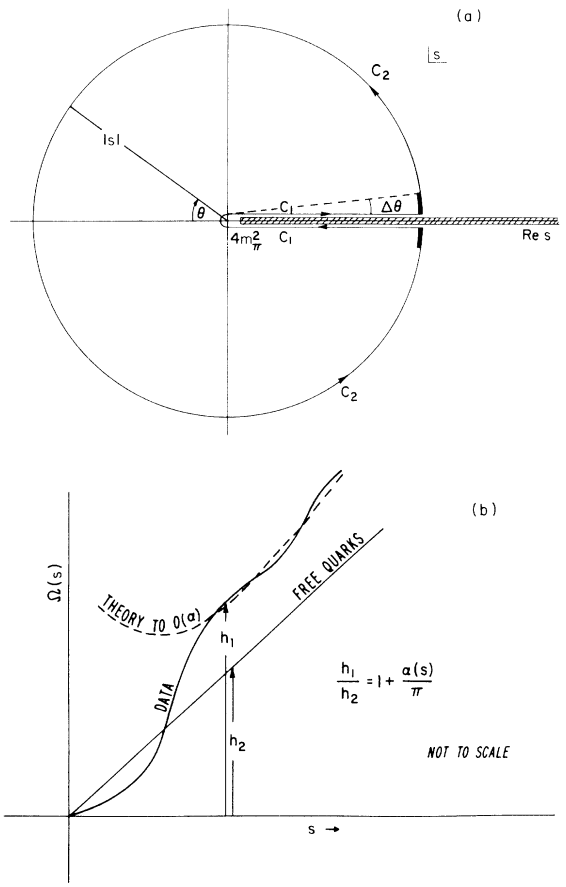

FIG. 1. (a) The integral of Π on C_1 (from experiment) must cancel that on C_2 (from theory). (b) Schematic plot of Ω^{theor} and Ω^{exp} for massless quarks. The ratio of heights to the free-quark model ($\simeq 1+\alpha/\pi$) may be in fair agrement even where the slopes (R) disagree substantially.

$$\int_{4m_\pi^2}^{s} R^{\text{exp}}\, ds = \Omega(s) = \int_0^s R^0\, ds + \tfrac{4}{3}\alpha(s)\int_0^s R^{(1)}\, ds. \tag{8}$$

This is the central equation and the following are its main features[6]:

(i) Ω is more reliably calculable than R. Whereas R is sensitive to the errors in the region $\Delta\theta$ [Fig. 1(a)], their impact on Ω is reduced by a factor of the order $\Delta\theta/\pi$. Conversely, in the calculation of R from Ω, we take the difference of two nearby contours, thereby discarding a large portion we trust and being left with a small piece that is suspect. It is evident from the above that in confronting theory with experiment or in determining its parameters the quantity to use is Ω and not R. There is the added advantage that in the determination of Ω^{exp}, the random errors in R^{exp} will tend to cancel. These ideas are depicted in Fig. 1(b).

(ii) If perturbation theory is reliable beyond some $s = s_{\min}$, Eq. (8) makes a prediction about the region of smaller s (down to threshold) hitherto considered unsusceptible to perturbative treatment.

(iii) The method calls for no assumptions about R^{exp} beyond the observed range.

(iv) Equation (8) has the following relation to Eq. (5) of Appelquist and Politzer.[7] If their formula for R^{theor} is integrated from 0 to s, one obtains Eq. (8), the equality between the two being obvious in the complex plane. [Although $\alpha(s)$ increases without bound in the range of integration, cancellation of large quantities near the pole yields a finite result.] In the present formulation the finiteness of the result is manifest on C_2 and more importantly, *the result is shown to rest on the use of perturbation theory at the upper limit* s, *and not, as may seem, down to threshold.*

(v) Equation (8) provides us with possibly the most reliable way to determine $\alpha(s)$ and Λ to date. Not only is the prediction for Ω more likely to be accurate, it is also expected to become reliable at lower energies [see Fig. 1(b)] where the coupling is fairly strong and the determination of Λ not too sensitive to experimental errors.

We now turn to experiment to find Λ. The data fall into three regions: the resonance region ($s \lesssim 1$ GeV^2), the middle region (1–6 GeV^2), and the SPEAR region (6–49 GeV^2). I first considered the region 0–1 GeV^2. The light quarks were assumed massless and strangeness was excluded from theory and experiment: the ϕ, which contributes 0.73 GeV^2 to Ω, according to the parameters of Ref. 4 was omitted as was the strange quark assumed to have mass $m_s = 500$ MeV. The ρ was described by the Gournaris-Sakurai fit to the Orsay data,[8] and the ω was treated as a narrow spike contributing 0.32 GeV^2 to Ω.[4] A comparison of $\Omega^{\text{theor}} = \frac{5}{3}(1+\alpha/\pi)$ yields a value of $\Lambda \simeq 700$ MeV [Fig. 2(a)]. A ±10% change in the data leads to $\Lambda = 700^{+50}_{-100}$ MeV. [It is worth pointing out that at $s = 25$ (50) GeV^2, the error in Ω due to $\Delta\Lambda = 100$ MeV is 2% (1%).] What if the quarks are assigned a mass? There are two nearly compensating factors: the threshold increases, reducing Ω, while the perturbative correction increases, as an examination of Eq. (4) near threshold will indicate. The net result is that as the quark mass goes from 0 to 300 MeV, Λ drops by an average of 75 MeV in this range. I therefore ignore this effect, since it lies within our quoted uncertainty.

A surprising feature of the analysis is that even

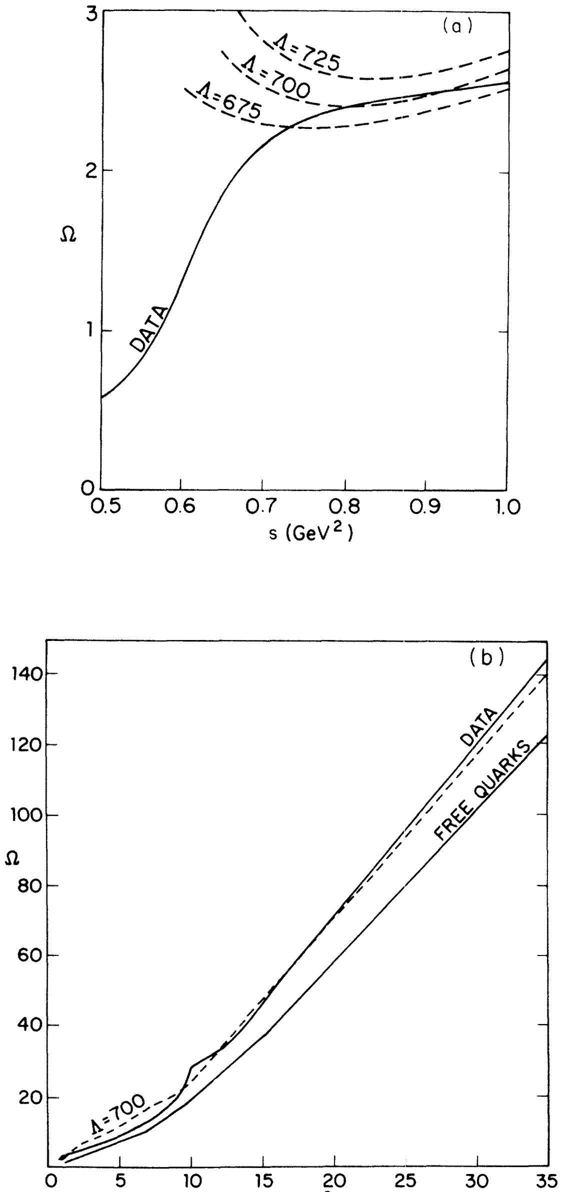

FIG. 2. (a) The determinations of Λ (in MeV) from the ρ and ω contributions. (b) Comparison between experiment and theory for $\Lambda = 700$ MeV and quark parameters given in the text.

though $\alpha/\pi \simeq 0.5$ near $s \simeq 1$ GeV^2, the first-order curve for Ω^{theor} follows the data down to 0.7 GeV^2. It seems, however, that we must redetermine Λ in the range 1–6 GeV^2, where the coupling is small enough to trust first-order perturbation theory but large enough to fix Λ within reasonable limits, given the inevitable experimental errors. Unfortunately, the data are very poor here with a lack of agreement among various groups. Although I have used the weighted mean given by Silvistrini[8] for later use, I have not relied on it to find Λ. Instead, I have chosen to predict Ω in this range and it will be interesting to compare it with better data that will, it is hoped, be available in the near future. The prediction, in the range 1–6 GeV^2 in steps of 0.5 GeV^2, is 2.66, 3.97, 5.13, 6.28, 7.41, 8.53, 9.65, 10.77, 11.89, 13.00, 14.12 GeV^2.

The SPEAR region[9] is once again not well suited for a determination of Λ but for different reasons: to minimize the role of the 1–6 GeV^2 region, we must go to high energies, but when we do, the coupling is too small to allow a determination of Λ with reasonable uncertainty and on top of that there is uncertainty regarding the number of quarks and heavy leptons involved. I have therefore opted to compare the theory, with $\Lambda = 700$ MeV, assuming a charmed quark of mass 1.5 GeV and one heavy lepton of mass 1.7 GeV (Ref. 10) [see Fig. 2(b)]. The theory continues to droop all the way to $s = 49$ GeV^2 (not shown). The introduction of a fifth quark of charge $\frac{1}{3}$ boosts it up by 5% or 8% depending on whether it is introduced at $s = 25$ GeV^2 (Ref. 5) or 9 GeV^2 (Ref. 11). Our analysis can be taken as evidence for a fifth quark were it not for the systematic errors in the data estimated to be $\pm 10\%$.[12] Should these errors be reduced to a few percent, it will be possible to rule out some models and deduce the number of quarks and their masses.

Let us conclude by going over the key points. The heart of the paper is the finite-energy sum rule, Eq. (8). Unlike its counterpart in S-matrix theory, which relates high-energy Regge parameters to low-energy resonance parameters, neither of which are really calculable, the present sum rule relates low-energy data to a high-energy quantity calculable in perturbation theory. In this paper we have exploited this relation and deduced Λ from the low-energy data. The value of 700 MeV is perhaps the most reliable estimate to date. *Our analysis also provides strong evidence supporting the asymptotically free nature of the dynamics: The agreement between theory and experiment, once established, persists for higher energies [Figs. 2(a) and 2(b)].*

The notion of using such finite-energy sum rules to relate low-energy data to high-energy quantities calculable in perturbation theory is general and may be used to study other currents besides the electromagnetic. Unlike the usual spectral-function sum rules that allow us to consider only those combinations of currents with the right convergence properties at infinity (in s), the finiteness of the contour in the present case allows us to consider any pair of currents. This idea is currently being studied.

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[6]The contribution of a quark of mass m and charge Q to Ω^{theor} is

$$3Q^2 s\left\{v^3+\frac{\alpha(s)}{3}\,[2\pi-\delta-3\delta v^3+(2m^2/s)(2\pi+\delta)\ln(s/4m^2)+(6\delta-8\pi)m^2/s-8m^4\delta/s^2]\right\}\theta(s-4m^2),$$

where $v=(1-4m^2/s^{1/2}$ and $\delta=\pi/2-3/4\pi$.

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