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Higher-order effects of asymptotically free field theories *†

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Several experimental consequences of non-Abelian asymptotically free gauge theories are derived, in particular, second-order corrections to the moments of the nonsinglet pieces of the structure functions of deep-inelastic lepton-hadron scattering are calculated. These results are then used to obtain corrections to the parton-model sum rules, as well as to derive some direct functional relationships among the structure functions. It is found that, if asymptotic behavior is assumed, the second-order correction terms involve calculable and uncalculable contributions; however, by considering suitable combinations of structure functions, the latter can be eliminated, thus obtaining some definite predictions. It is also found that some of these correction terms increase as one calculates higher moments of the structure functions. This remarkable fact, which is likely to persist in higher orders, suggests that all the functional relationships that are derived here and elsewhere have questionable validity near threshold; however, away from this region, where higher-order terms become negligible, they provide an important test of these theories. It is also shown that no similar predictions can be made for the singlet pieces of the structure functions since they always involve uncalculable constants. The question whether or not the present range of energies (25 GeV in electroproduction and 150 GeV in neutrino production) is sufficiently high to test these theoretical asymptotic predictions is discussed. By making some reasonable assumptions, it is found that the effective coupling constant in these ranges of energies is ≥ 1 so that according to these models, the asymptotic region has not yet been achieved. However, by regarding the expansion in terms of the effective coupling constant as an experimentally measurable parameter, it is possible (but not certain) that one obtains measurable theoretical predictions within the range of present energies.

I. INTRODUCTION

In the last years there has been considerable interest in the so-called asymptotically free theories of the strong interactions. These theories have the remarkable and unique feature of providing a possible explanation of the observed scaling properties of the structure functions of deep-inelastic scattering, within the framework of renormalizable field theories.

The typical feature of an asymptotically free gauge theory is manifested when studying the asymptotic behavior of renormalized Green's functions by means of the renormalization-group equation or their generalized version, the Callan-Symanzik equations. It is found there that for these theories, the renormalization-group equations have an ultraviolet-stable fixed point at zero coupling constant.

When the property of asymptotic freedom is combined with the assumptions of the operator-product expansion so as to study the short-distance behavior of products of local current operators, it is found that they provide a simple theoretical explanation of the scaling behavior of the moments of the structure functions. This property, which is *not* shared by any other renormalizable field theory, singles out the gauge theories as the only possible candidates for describing the strong interactions in the context of field theory. Whether these theories do in fact agree with the experiments is still an open question which will have to be settled by the outcome of forthcoming experiments.

In the remainder of this section we will briefly describe the basic features of non-Abelian gauge theories and the various techniques which are applied to study the asymptotic behavior of the structure functions within the framework of renormalization theory. Since there exist in the literature several excellent articles on the subject, in this discussion we will quote the most important results.[1-4]

A. Non-Abelian gauge theories

The non-Abelian gauge theory we will consider is a field theory of the Yang-Mills type in which the underlying gauge group G is noncommutative. (We will restrict ourselves to semisimple compact groups.) The Lagrangian density corresponding to these models is of the form

$$L = -\tfrac{1}{2}\mathrm{Tr} F_{\mu\nu}F^{\mu\nu}$$
$$= -\tfrac{1}{2}\mathrm{Tr}(\partial_\mu B_\nu - \partial_\nu B_\mu - g[B_\mu, B_\nu])^2\,, \qquad (1.1)$$

where

$$B_\mu = B_\mu(x) = B_\mu^a(x)\lambda_a \qquad (1.2)$$

is a matrix of vector fields (summation over repeated indices is understood) and the matrices λ_a are the generators of the Lie group G, satisfying the commutation relations

$$[\lambda_a, \lambda_b] = i f_{abc} \lambda_c \tag{1.3}$$

and being normalized according to

$$\mathrm{Tr}\lambda_a \lambda_b = \tfrac{1}{2}\delta_{ab} \,. \tag{1.4}$$

The incorporation of fermions into this Lagrangian can be carried out easily by adding to the free gauge field Lagrangian the free fermion Lagrangian with its derivatives replaced by gauge-covariant derivatives,

$$L = \bar{\Psi}\,(i\partial\!\!\!/ - g\sigma^a B\!\!\!/_a - M)\Psi\,, \tag{1.5}$$

where the σ^a are the matrices of the representation R of the group G, according to which the fermion field Ψ transforms.

For our future purposes we will consider a particular Lagrangian. We choose SU(3) as the gauge symmetry group and assume that the fermions transform according to the triplet representation; furthermore, let us also require our model to have the approximate chiral SU(3)×SU(3) symmetry, which is broken by mass terms and which we assume commutes with the gauge group G. The fermion field will then belong to a representation of SU(3)×SU(3)×SU(3). If we take the fermions to be the ordinary quark triplet then the spinor field will be represented by the following 3×3 matrix:

$$\Psi = \begin{bmatrix} \mathcal{P}_1 & \mathfrak{N}_1 & \lambda_1 \\ \mathcal{P}_2 & \mathfrak{N}_2 & \lambda_2 \\ \mathcal{P}_3 & \mathfrak{N}_3 & \lambda_3 \end{bmatrix}. \tag{1.6}$$

The generators of chiral SU(3)×SU(3) transform the columns of this matrix, whereas the generators of G [SU(3)] transform its rows.

B. The renormalization group and the Wilson coefficients[5]

In deep-inelastic scattering processes one is always interested in the Fourier transform of the commutator of weak or electromagnetic currents sandwiched between hadron states,

$$W_{\mu\nu}^{a,b}(p,q) = \int d^4x\, e^{iq\cdot x}\,\langle p|\,[J_\mu^a(x), J_\nu^b(0)]\,|p\rangle_{\text{spin average}}\,, \tag{1.7}$$

where q is the momentum carried by the currents, p is the hadron momentum (usually taken to be a nucleon), and a, b are the SU(3)×SU(3) labels of the currents. This expression can also be written as

$$W_{\mu\nu}^{a,b}(p,q) = \frac{p_\mu p_\nu}{M^2} F_2^{a,b}(\nu, q^2) - g_{\mu\nu} F_1^{a,b}(\nu, q^2) + \frac{i\epsilon_{\mu\nu\rho\sigma}}{2M} p^\rho q^\sigma F_3^{a,b}(\nu, q^2) + \cdots\,, \tag{1.8}$$

where the deleted terms are proportional to q_μ or q_ν.

If one considers this expression in the limit

$$q^2 \to -\infty, \quad \nu \to \infty, \quad \text{and } -q^2/2\nu = x \text{ fixed},$$

one can show that it is precisely the light-cone singularity of the current commutator which determines the functions $F_i^{a,b}(x, q^2)$ $(i = 1, 2, 3)$ in this limit. According to the operator-product-expansion ideas, we can write

$$\begin{aligned}
[J_\mu^a(x), J_\nu^b(0)] = {} & \tfrac{1}{2} g_{\mu\nu} \Box^2 \frac{1}{x^2 - i\epsilon} \sum_{n=0}^{\infty} \sum_i C_{i,1}^n(a,b;x^2 - i\epsilon) x_{\mu_1} \cdots x_{\mu_n} \hat{O}_i^{\mu_1 \cdots \mu_n}(0) \\
& + \frac{1}{x^2 - i\epsilon} \sum_{n=0}^{\infty} \sum_i C_{i,2}^n(a,b;x^2 - i\epsilon) x_{\mu_1} \cdots x_{\mu_n} \hat{O}_i^{\mu\nu\mu_1 \cdots \mu_n}(0) \\
& + \tfrac{1}{2} i\epsilon_{\mu\nu\sigma\rho} \frac{\partial}{\partial x_\rho} \frac{1}{x^2 - i\epsilon} \sum_{n=0}^{\infty} \sum_i C_{i,3}^n(a,b;x^2 - i\epsilon) x_{\mu_1} \cdots x_{\mu_n} \hat{O}^{\sigma\mu_1 \cdots \mu_n}(0) + \cdots\,,
\end{aligned} \tag{1.9}$$

where the deleted terms are proportional to $\partial/\partial x^\mu$ or $\partial/\partial x^\nu$. The operator $\hat{O}^i_{\mu_1 \cdots \mu_n}$ has spin n and dimensions $n+2$, and the index i labels the various operators of the same character [i.e., spin, SU(3), and parity] which may appear in the expansion. In our model these operators are

$${}^n\hat{O}^V_{\mu_1 \cdots \mu_n} = i^{n-2} S(\mathrm{Tr} F_{\mu_1 \alpha}(\nabla_{\mu_2} \cdots \nabla_{\mu_{n-1}}) F^\alpha{}_{\mu_n}) - \text{trace terms}\,, \tag{1.10a}$$

$${}^n\hat{O}^{F\pm;s}_{\mu_1 \cdots \mu_n} = \tfrac{1}{2} i^{n-1} S(\bar{\Psi}_{\mu_1} \nabla_{\mu_2} \cdots \nabla_{\mu_n} (1 \pm \gamma_5)\Psi) - \text{trace terms}\,, \tag{1.10b}$$

and

$${}^n\hat{O}^{F\pm;a}_{\mu_1 \cdots \mu_n} = \tfrac{1}{2} i^{n-1} S(\bar{\Psi}\gamma_{\mu_1} \nabla_{\mu_2} \cdots \nabla_{\mu_n} (1 \pm \gamma_5)\tfrac{1}{2}\lambda^a \Psi) - \text{trace terms}\,, \tag{1.10c}$$

with (1.10a) and (1.10b) being SU(3) singlets while

(1.10c) is an SU(3) nonsinglet. Furthermore, ∇_μ is the covariant derivative, which is $\partial_\mu + i\sigma^a B^a_\mu$ acting on fermions and $\partial_\mu + i\lambda^a B^a_\mu$ acting on bosons. The letter S denotes symmetrization of tensor indices.

Substituting experession (1.9) into Eq. (1.7) one can easily derive the relations

$$\int_0^1 dx\, x^n F_1^{a,b}(x,q^2) = \sum_i \tilde{C}^{n+1}_{i,1}(a,b;q^2) M_i^{n+1}\,,$$
$$\int_0^1 dx\, x^n F_2^{a,b}(x,q^2) = \sum_i \tilde{C}^{n+2}_{i,2}(a,b;q^2) M_i^{n+2}\,, \qquad (1.11)$$
$$\int_0^1 dx\, x^n F_3^{a,b}(x,q^2) = \sum_i \tilde{C}^{n+1}_{i,3}(a,b;q^2) M_i^{n+1}\,,$$

where the constants M_i^n are defined by

$$\langle p\,|\,\hat{O}^i_{\mu_1\cdots\mu_n}(0)|p\rangle_{\text{spin average}} = i^n \frac{1}{M} p_{\mu_1}\cdots p_{\mu_n} M_n^i + \cdots \qquad (1.12)$$

and the coefficients $\tilde{C}^n_{i,k}$ are given by

$$\tilde{C}^n_{i,k}(a,b;q^2) = \tfrac{1}{2} i (q^2)^{n+1} \left(-\frac{\partial}{\partial q^2}\right)^n \times \int d^4y\, e^{iq\cdot y} \frac{C^n_{i,k}(a,b;y^2 - i\epsilon)}{y^2 - i\epsilon}\,. \qquad (1.13)$$

In a free-field theory these coefficients are constants independent of q^2, but for interacting theories they become nontrivial functions of q^2 whose form can be determined by the renormalization-group equations. In fact it can be easily shown that they satisfy[2]

$$\left[\mu \frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} + \gamma^n(g)\right] \tilde{C}^n(q^2/\mu^2, g) = 0 \qquad (1.14)$$

for the nonsinglet case, where γ_n, is the anomalous dimension of the operator (1.10c). The general solution of Eq. (1.14) can be expressed in terms of the effective coupling constant $\bar{g}(g,t)$ as follows:

$$\tilde{C}^n(q^2/\mu^2, g) = \tilde{C}^n(1,\bar{g}) \exp\left[-\int_0^t dt'\, \gamma^n(\bar{g}(g,t'))\right]. \qquad (1.15)$$

In the case of the singlet operators, which are not multiplicatively renormalizable, the corresponding renormalization-group equations are

$$\left[\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g}\right] \tilde{C}^n_{i,k} = \sum_j [\tilde{\gamma}^n(g)]_{i,j} \tilde{C}^n_{j,k}\,. \qquad (1.16)$$

The solutions of Eq. (1.16) are

$$\tilde{C}^n_{i,k}(q^2/\mu^2, g) = \sum_j \left\{ T \exp\left[-\int_0^t \tilde{\gamma}^n(\bar{g}(g,t'))dt'\right]\right\}_{i,j} \times \tilde{C}^n_{j,k}(1,\bar{g})\,, \qquad (1.17)$$

where T is the t-ordering operation.

For an asymptotically free theory $\bar{g}(g,t)$ tends to zero in the limit $q^2 \to -\infty$. For small g the functions $\beta(g)$, $\gamma^n(g)$, and $\tilde{\gamma}^{(n)}(g)$ are given by the expressions

$$\beta(g) = -\tfrac{1}{2} b_0 g^3 + \tfrac{1}{2} b_1 g^5 + O(g^7)\,,$$
$$\gamma^n(g) = \gamma_0^n g^2 + \gamma_1^n g^4 + O(g^6)\,, \qquad (1.18)$$
$$\tilde{\gamma}^n(g) = \tilde{\gamma}_0^n g^2 + \tilde{\gamma}_1^n g^4 + O(g^6)\,,$$

with

$$b_0 = \frac{1}{8\pi^2}\left[\tfrac{11}{3} C_2(G) - \tfrac{4}{3} T(R)\right],$$
$$\gamma_0^n = \frac{C_2(R)}{(4\pi)^2}\left[1 - \frac{2}{n(n+1)} + 4\sum_{j=2}^n \frac{1}{j}\right], \qquad (1.19)$$
$$\tilde{\gamma}^n = \begin{pmatrix} {}^n\gamma^F_{FF} & {}^n\gamma^V_{FF} \\ {}^n\gamma^F_{VV} & {}^n\gamma^V_{VV} \end{pmatrix},$$

where ${}^n\gamma^F_{VV}$ is given by $\mu(\partial/\partial\mu)(Z)^F_{VV}$, and so forth.

The entries of the above matrix are (for n = even, $n \geq 2$)[1]

$${}^n\gamma^V_{VV} = \frac{g^2}{8\pi^2}\left\{C_2(G)\left[\frac{1}{3} - \frac{4}{n(n-1)} - \frac{4}{(n+1)(n+2)} + 4\sum_2^n \frac{1}{j}\right] + \tfrac{4}{3} T(R)\right\},$$
$${}^n\gamma^F_{FF} = \frac{g^2}{8\pi^2}\left\{C_2(R)\left[1 - \frac{2}{n(n+1)} + 4\sum_2^n \frac{1}{j}\right]\right\}, \qquad (1.20)$$
$${}^n\gamma^F_{VV} = \frac{-g^2}{8\pi^2}\,\frac{4(n^2+n+2)}{n(n+1)(n+2)}\, T(R)\,,$$
$${}^n\gamma^V_{FF} = \frac{-g^2}{8\pi^2}\,\frac{2(n^2+n+2)}{n(n^2-1)}\, C_2(R)\,,$$

where $C_2(G)$ is the quadratic Casimir operator of G evaluated in the adjoint representation, $C_2(R)$ is the evaluation of the quadratic Casimir operator of G in the irreducible representation R to which the fermions belong, and $T(R)$ is the trace of the square of a matrix in the Lie algebra of the representation R. In our case $C_2(R) = \frac{3}{4}$, $C_2(G) = 3$, and $T = \frac{1}{2}$.

One can derive that for large $t = \frac{1}{2}\ln(Q^2/\mu^2)$ $(Q^2 = -q^2)$

$$\bar{g}^2(t,g) \to b_0^{-1} t^{-1} + O\left(\frac{1}{t^2}\right). \qquad (1.21)$$

Therefore in this limit we can expand Eqs. (1.15) and (1.16) in powers of $\bar{g}^2$, and calculate each term as in ordinary perturbation theory, thereby obtaining an asymptotic expression for the moments of the structure functions. For the lowest-order terms it was found[1,6] that

$$\int_0^1 dx\, x^n F_i(x, Q^2) \sim \text{const.} \times \left(\ln \frac{Q^2}{\mu^2}\right)^{-\gamma_0^{n+1}/b_0} \tag{1.22}$$

Hence, according to these theories, scaling is violated by powers of $\ln Q^2$. In the next sections we will discuss the effect of considering the higher-order terms of this "perturbation" expansion.

II. CALCULATION OF THE CORRECTIONS

We have seen that the moments of the structure functions are proportional to the Fourier transform of the Wilson coefficients Eq. (1.11), and that these coefficients satisfy the renormalization-group equations, Eq. (1.14). Let us first consider the nonsinglet piece of these moments. The general solution of Eq. (1.14) is of the form

$$\tilde{C}_i^n(q^2/\mu^2, g) = \tilde{C}_i^n(1, \bar{g}(t, g_p)) \times \exp\left[-\int_0^t dx\, \gamma^n(\bar{g}(x, g_p))\right], \tag{2.1}$$

where $\bar{g}(t, g_p) \to 0$ in the limit $q^2 \to -\infty$. The leading asymptotic form of this solution is

$$\tilde{C}_i^n(q^2/\mu^2, g) \simeq \text{const.} \times \left(\ln \frac{-q^2}{\mu^2}\right)^{-\gamma_0^n/b_0} \times [\tilde{C}_i^n(1,0) + O(\bar{g}^2)], \tag{2.2}$$

as can be seen by expanding Eq. (2.1) in powers of $\bar{g}$ and keeping only the leading term.

In this section our main purpose will be to determine the next-leading corrections to Eq. (2.1) and therefore to Eq. (1.23). The motivation for this is that, presumably, careful measurements of the structure functions will yield and experimental test for these corrections, provided of course scaling is violated by powers of logarithms as predicted by Eq. (2.2). Correction terms to Eq. (1.11) may arise from different sources; however, for the moment we will restrict ourselves to the evaluation of those corrections which arise from expanding the Wilson coefficients to second order in the effective coupling constant $\bar{g}$, that is

$$\tilde{C}_1^n(1,0) + \frac{\partial^2 \tilde{C}^n(1,0)}{\partial \bar{g}^2}\, \bar{g}^2 . \tag{2.3}$$

In order to evaluate Eq. (2.3) one proceeds as follows: First calculate the moments of the structure functions to second order in perturbation theory; then make use of Eq. (1.11) to obtain the desired coefficient to the same order.

Let us start by considering the amplitude[7]

$$T_{\mu\nu}^{a\bar{a}}(p, q) = i \int e^{iq\cdot x} \langle p | T(J_\mu^a(x) J_\nu^{\bar{a}}(0)) | p \rangle , \tag{2.4}$$

where spin average is understood, a stands for the SU(3) label of the current, and $J_\nu^{\bar{a}} = (J_\nu^a)^\dagger$. The tensor $T_{\mu\nu}$ can be written as

$$\begin{aligned} T_{\mu\nu}^{a\bar{a}}(p, q) &= \left(\frac{q_\mu q_\nu}{q^2} - g_{\mu\nu}\right) T_1^{a\bar{a}}(q^2, \nu) \\ &+ \frac{1}{M^2}[p_\mu - (\nu/q^2) q_\mu][p_\nu - (\nu/q^2) q_\nu] T_2^{a\bar{a}}(q^2, \nu) \\ &- \frac{i\epsilon_{\mu\nu\rho\sigma}}{2M^2} p^\rho q^\sigma T_3^{a\bar{a}}(\nu, q^2) + \cdots , \end{aligned} \tag{2.5}$$

where the deleted terms are proportional to q_μ. In the case of the electromagnetic currents only the first two terms are present. The functions $T_1(q^2, \nu)$ and $T_3(q^2, \nu)$ satisfy unsubtracted dispersion relations in $\nu = p\cdot q$ for q^2 fixed, while $T_2(q^2, \nu)$ requires one subtraction; therefore one can write

$$T_1^{a\bar{a}}(q^2, \nu) = T_1^{a\bar{a}}(q^2, \infty) + \int_{-1}^{+1} dx' \frac{F_1^{a\bar{a}}(x', q^2)}{x - x'} , \tag{2.6a}$$

$$T_2^{a\bar{a}}(q^2, x) = \frac{M^2}{\nu} \int_{-1}^{+1} dx' \frac{F_2^{a\bar{a}}(x', q^2)}{x - x'} , \tag{2.6b}$$

and

$$T_3^{a\bar{a}}(q^2, x) = \frac{M^2}{\nu} \int_{-1}^{+1} dx' \frac{F_3^{a\bar{a}}(x', q^2)}{x - x'} , \tag{2.6c}$$

where

$$\begin{aligned} F_1^{a\bar{a}}(x', q^2) &= \frac{1}{\pi} \operatorname{Im} T_1^{a\bar{a}}(x', q^2) , \\ F_{2,3}^{a\bar{a}}(x', q^2) &= \frac{\nu}{M^2} \frac{1}{\pi} \operatorname{Im} T_{2,3}^{a\bar{a}}(x', q^2) \end{aligned} \tag{2.7}$$

are the structure functions of interest and $x = -q^2/2\nu = Q^2/2\nu$.

Let us consider two projections of the tensor $T_{\mu\nu}^{a\bar{a}}$, namely

$$T^{a\bar{a}}{}_\mu{}^\mu = \frac{1}{2x} \frac{\nu T_2^{a\bar{a}}}{M^2} - 3T_1^{a\bar{a}} + \cdots \tag{2.8}$$

and

$$p^\mu p^\nu T_{\mu\nu}^{a\bar{a}} = \frac{\nu}{2x}\left(\frac{\nu T_2^{a\bar{a}}}{2xM^2} - T_1^{a\bar{a}}\right) + \cdots , \tag{2.9}$$

where the deleted terms are down by a factor of $1/q^2$, which is assumed to be very small. If one substitutes Eqs. (2.6a) and (2.6b) into the above

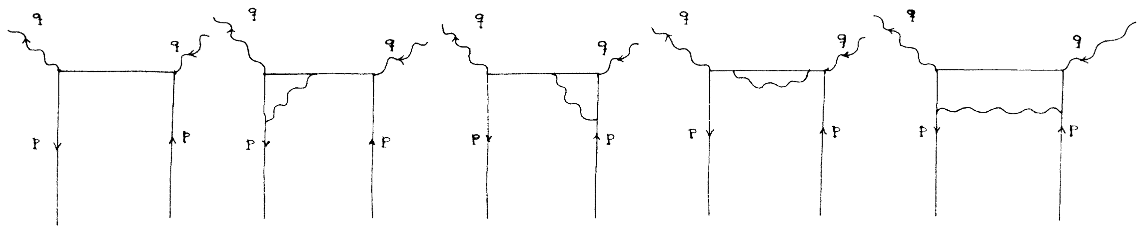

FIG. 1. Diagrams contributing to the vector-vector current amplitude $T_{\mu}^{a\bar{a}\mu}$.

expressions and lets x lie outside the physical region, i.e., $x>1$, one can easily derive that

$$T^{a\bar{a}}{}_{\mu}{}^{\mu}-\frac{6x}{\nu}T^{a\bar{a}}_{\mu\nu}p^{\mu}p^{\nu}-6T_1^{a\bar{a}}(q^2,\infty) = -\sum_{n=2}^{\infty}\frac{1}{x^n}\int_{-1}^{+1}dx'\,F_2^{a\bar{a}}(x',q^2)x'^{\,n-2} \quad (2.10)$$

and

$$\frac{1}{2}\left(\frac{2x}{\nu}T^{a\bar{a}}_{\mu\nu}p^{\mu}p^{\nu}-T^{a\bar{a}}{}_{\mu}{}^{\mu}\right) = \sum_{n=1}^{\infty}\frac{1}{x^n}\int_{-1}^{+1}dx'\,x'^{\,n-1}F_1^{a\bar{a}}(x',q^2)\,. \quad (2.11)$$

In order to derive the corresponding relation for the moments of $F_3(x,Q^2)$, it is convenient to introduce two independent four-vectors y and y^* such that

$$y\cdot q=y^*\cdot q=y\cdot p=y^*\cdot p=y\cdot y^*=0\,, \qquad y^{*2}=y^2=-1\,. \quad (2.12)$$

Then, contracting them with the amplitude $T^{a\bar{a}}_{\mu\nu}$, we project out $T_3^{a\bar{a}}(x,q^2)$ as follows:

$$y^{\nu}y^{*\mu}T^{a\bar{a}}_{\nu\mu}=\frac{-i\epsilon_{\nu\mu\sigma\gamma}y^{\nu}y^{*\mu}p^{\sigma}q^{\gamma}}{2M^2}\,T_3^{a\bar{a}}(x,Q^2)\,. \quad (2.13)$$

We now use Eq. (2.6c) to write (2.13) as

$$y^{\nu}y^{*\mu}T^{a\bar{a}}_{\nu\mu}=\frac{-i\epsilon(y,y^*,p,q)}{2M^2}\frac{M^2}{\nu}\times\sum_{n=1}^{\infty}\frac{1}{x^n}\int_{-1}^{+1}F_3^{a\bar{a}}(x',Q^2)x'^{\,n-1}dx'\,, \quad (2.14)$$

where

$$\epsilon(y,y^*,p,q)\equiv\epsilon_{\nu\mu\sigma\lambda}y^{\nu}y^{*\mu}p^{\sigma}q^{\lambda}\,.$$

We shall proceed to calculate these amplitudes to second order in perturbation theory. Our model Lagrangian was described in the preceding section.[1] To this order in the coupling constant g, the diagrams of Fig. 1 and the crossed graphs will contribute to $T^{a\bar{a}}{}_{\mu}{}^{\mu}$. All radiative corrections involve gauge bosons (gluons) and are therefore of second order.

Incidentally, one may well proceed to calculate the absorptive part of this amplitude by cutting the above diagrams. This would automatically give us the structure functions from which all their moments could be computed. However simple it looks, this procedure becomes troublesome because of the infrared singular behavior of the structure functions at $x=1$. By introducing an infrared energy cutoff the singularity can be avoided, and one can show that this cutoff is absent in the expression for the moments; however, the exact evaluation of some of the integrals turns out to be involved. Fortunately there exists an alternative method for obtaining these moments, which consists of evaluating the amplitudes for $x>1$ then expanding it in powers of $1/x$ and finally using Eqs. (2.6), (2.11), and (2.14) to read off the desired moments. Here we shall follow this last method.[8] (See Appendix A for details.)

For the electromagnetic currents the final expression one finds is

$$T_{\mu}{}^{\mu}(p,q)=\sum_{\substack{n\ \mathrm{even}\\ n>0}}\left[\frac{1}{x^n}\left(4+\frac{g^2C_2(R)}{\pi^2}\left\{\left[\frac{1}{4}\left(1-\frac{2}{n(n+1)}+4\sum_{j=2}^{n}\frac{1}{j}\right)\ln\frac{Q^2}{M^2}\right]+\frac{1}{2}-\frac{5}{2n}+\frac{n^2+2n-1}{2n^2(n+1)^2} -\left(\frac{7}{4}-\frac{1}{2n(n+1)}\right)\sum_{j=1}^{n}\frac{1}{j}+\sum_{j=1}^{n}\frac{1}{j^2}+\sum_{s=1}^{n}\frac{1}{s}\sum_{j=1}^{s}\frac{1}{j}\right\}\right)\right] \quad (2.15)$$

[with $C_2(R)=\sum_{c,d}f_{abc}f_{bcd}\,\delta_{ab}$], whereas for the vector part of the isospin-raising currents the corresponding expression

$$T_V^{\pm\mp}{}_\mu{}^\mu(p,q) = \frac{\pm 1}{x}\left(2+\frac{3g^2C_2(R)}{8\pi^2}\right)$$
$$+\sum_{n>1}\frac{(\pm 1)^n}{x^n}\left(2+\frac{g^2}{2\pi^2}C_2(R)\left\{\frac{1}{4}\left(1-\frac{2}{n(n+1)}+4\sum_2^n\frac{1}{j}\right)\ln\frac{Q^2}{M^2}\right]+\frac{1}{2}-\frac{5}{4n}+\frac{n^2+2n-1}{2n^2(n+1)^2}\right.$$
$$\left.+\sum_{j=1}^n\frac{1}{j^2}-\left(\frac{7}{4}-\frac{1}{2n(n+1)}\right)\sum_{j=1}^n\frac{1}{j}+\sum_{s=1}^s\frac{1}{s}\sum_{j=1}^s\frac{1}{j}\right\}\right) . \tag{2.16}$$

We notice that the infrared terms are not present and also that the $\ln(Q^2/M^2)$ term appears multiplied by the anomalous dimension of the nonsinglet operator, namely

$$^n\gamma_{FF}^F=\frac{g^2}{8\pi^2}C_2(R)\left[1-\frac{2}{n(n+1)}+4\sum_{j=2}^n\frac{1}{j}\right]. \tag{2.17}$$

The later was an expected result. We mentioned before that the functions $\tilde{C}^n(Q^2/M^2,g)$ are proportional to the coefficient of $1/x^n$ in the expansions (2.15) and (2.16). Moreover, they must also be solutions of the renormalization-group equations Eq. (1.14) to any finite order in perturbation theory. It is then an easy exercise to show that the coefficient function $\tilde{C}^n(Q^2/M^2,g)$ must contain a term proportional to $^n\gamma_{FF}^F\ln(Q^2/M^2)$ if it satisfies Eq. (1.14) to the second order.

In the case of neutrino scattering most of the analysis proceeds in complete analogy with the previous electroproduction case except for the slight modifications due to the presence of axial-vector currents. For the amplitude $T^{\pm\mp}{}_\mu{}^\mu$, we must also add the additional contributions from the diagrams in Fig. 2 and the crossed graphs (the slashed lines denote pure axial-vector currents). For large Q^2, the contribution of these graphs is equal to that of the vector currents; therefore the expression for the weak amplitude is

$$T^{\pm\mp}{}_\mu{}^\mu(p,q) = \pm\frac{1}{x}\left(4+\frac{3g^2C_2(R)}{4\pi^2}\right)$$
$$+\sum_{n>1}\frac{(\pm 1)^n}{x^n}\left(4+\frac{g^2}{\pi^2}C_2(R)\left\{\left[\frac{1}{4}\left(1-\frac{2}{n(n+1)}+4\sum_2^n\frac{1}{j}\right)\ln\frac{Q^2}{M^2}\right]+\frac{1}{2}-\frac{5}{2n}+\frac{n^2+2n-1}{2n^2(n+1)^2}\right.\right.$$
$$\left.\left.+\sum_{j=1}^n\frac{1}{j^2}-\left(\frac{7}{4}-\frac{1}{2n(n+1)}\right)\sum_{j=1}^n\frac{1}{j}+\sum_{s=1}^n\frac{1}{s}\sum_{j=1}^s\frac{1}{j}\right\}\right) .$$

In a similar fashion one computes the moments of the structure function F_3 by means of Eq. (2.14). The relevant diagrams are those given in Fig. 3 plus the crossed graphs. The contribution of these graphs to the amplitude T^{+-} is

$$y^\nu y^{*\mu}T_{\nu\mu}^{+-}=\frac{-i\epsilon(y,y^*,p,q)}{2M^2}\left(\frac{M^2}{\nu}\right)\left[\frac{1}{x}\left(-4+\frac{3g^2C_2(R)}{2\pi^2}\right)\right.$$
$$+\sum_{n=2}^\infty\frac{(-1)^n}{x^n}\left(-4+\frac{g^2}{\pi^2}C_2(R)\left\{\left[\frac{1}{4}\left(1-\frac{2}{n(n+1)}+4\sum_{j=2}^n\frac{1}{j}\right)\ln\frac{Q^2}{M^2}\right]+\frac{13}{4}-\frac{3}{4n}\right.\right.$$
$$+\frac{3n^2-1}{2n^2(n+1)}+\sum_{j=1}^n\frac{1}{j^2}$$
$$\left.\left.\left.-\left(\frac{7}{4}+\frac{1}{2n(n+1)}\right)\sum_{j=1}^n\frac{1}{j}-\sum_{s=1}^n\frac{1}{s}\sum_{j=1}^s\frac{1}{j}\right\}\right)\right]. \tag{2.19}$$

It is clear that, for the amplitude T^{+-} and to this order in g, the crossed diagrams do not contribute,

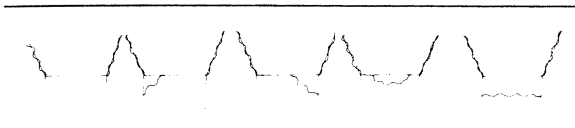

FIG. 2. Diagrams contributing to the axial-vector–axial-vector current amplitude $T^{\pm\mp}{}_\mu{}^\mu$.

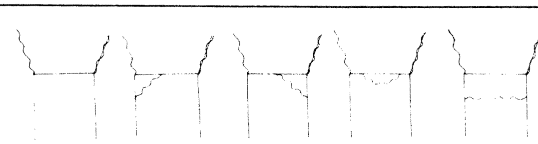

FIG. 3. Diagrams contributing to the vector–axial-vector current amplitude $y_\mu y_\nu^* T^{\pm\mp\mu\nu}$.

whereas for the amplitude T^{-+} the situation is just the opposite. (Note also that if $\mathcal{P}$, the charged quark, is replaced by $\mathfrak{N}$, the neutral quark, then the direct diagram gives T^{-+} and the crossed one T^{+-}).

Therefore,

$$y^{\nu}y^{\mu*}T^{-+}_{\nu\mu}=\frac{-i\epsilon(y,y^*,p,q)}{2M^2}\left(\frac{M^2}{\nu}\right)\left[\frac{1}{x}\left(4-\frac{3g^2C_2(R)}{2\pi^2}\right)+\sum_{n=2}\frac{(-1)^n}{x^n}\left(4-\frac{g^2}{\pi^2}C^2(R)\left\{\left[\frac{1}{4}\left(1-\frac{2}{n(n+1)}+4\sum_2^n\frac{1}{j}\right)\ln\frac{Q^2}{M^2}\right]+\frac{13}{4}-\frac{3}{4n}+\frac{3n^2-1}{2n^2(n+1)^2}+\sum_{j=1}^{n}\frac{1}{j^2}-\left(\frac{7}{4}+\frac{1}{2n(n+1)}\right)\sum_{j=1}^{n}\frac{1}{j}-\sum_{s=1}^{n}\frac{1}{s}\sum_{j=1}^{s}\frac{1}{j}\right\}\right)\right]. \tag{2.20}$$

Next we consider the amplitude

$$p^{\mu}p^{\nu}T^{a\bar{a}}_{\mu\nu}(p,q); \tag{2.21}$$

the evaluation of it is much simpler than in the previous two cases. Here only one graph is involved, namely that in Fig. 4. This is a result of the fact that all the others yield nonleading contributions, which are smaller by a factor of M^2/Q^2. In the electromagnetic case the corresponding expression is

$$p^{\mu}p^{\nu}T_{\mu\nu}(p,q)=\frac{g^2}{2\pi^2}C_2(R)\frac{\nu}{2}\sum_{\substack{n=2\\n\text{ even}}}^{\infty}\frac{1}{x^n}\frac{1}{n}. \tag{2.22}$$

Similarly, for the case of weak currents

$$p^{\mu}p^{\nu}T^{\mp\pm}_{\mu\nu}=\frac{g^2}{2\pi^2}C_2(R)\frac{\nu}{2}\sum_{n=2}^{\infty}\frac{(\pm)^n}{x^n}\frac{1}{n} \tag{2.23}$$

(with an analogous expression for the neutral quark). We observe that the lowest-order diagram does not contribute, so that the leading logarithmic deviation from scaling for this particular combination of structure functions arises precisely from the term

$$\frac{\bar{g}^2}{2\pi^2}C_2(R)\frac{1}{n+2}\left(\ln\frac{Q^2}{M^2}\right)^{-\gamma_0^{n+2}/b_0}$$

for the nth moment.

Finally, we compute the expression for the coefficients of the singlet operators. Restricting ourselves to the parity-conserving case, it is obvious that for the singlet operators of the form

$${}^{n}\hat{O}^{F,0}_{\mu_1\cdots\mu_n}=\tfrac{1}{2}i^{n-1}(S\bar{\psi}\gamma_{\mu_1}\nabla_{\mu_2}\cdots\nabla_{\mu_n}\psi-\text{trace terms}), \tag{2.24}$$

the analysis follows the same pattern as in the previous case; consequently, the corresponding numerical results are the same except for the trivial isospin factor. All the novelties will come from the remaining singlet operators

$${}^{n}\hat{O}^{V}_{\mu_1\cdots\mu_n}=i^{n-2}S\,\mathrm{Tr}_{\mu_1\alpha}(\nabla_{\mu_2}\cdots\nabla_{\mu_{n-1}})F^{\alpha}_{\mu_n}-\text{trace terms}. \tag{2.25}$$

In order to calculate the Wilson coefficients associated with these operators, we consider the amplitude for the scattering process

gluon + current → gluon + current ,

which is given by

$$R_{\mu\nu}(k,q)=i\int d^4x e^{iq\cdot x}\langle G(k)|T(J^a_{\mu}(x)J^{\bar{a}}_{\nu}(0))|G(k)\rangle, \tag{2.26}$$

where $|G(k)\rangle$ denotes a gluon state of momentum k and where a sum over gluon polarizations is understood. As a consequence of requiring gauge invariance this amplitude must be of the form

$$R_{\mu\nu}=\left(\frac{q_{\mu}q_{\nu}}{q^2}-g_{\mu\nu}\right)A_1(q^2,k\cdot q)+\left(k_{\mu}-\frac{k\cdot q}{q^2}q_{\mu}\right)\left(k_{\nu}-\frac{k\cdot q}{q^2}q_{\nu}\right)A_2(q^2,k\cdot q). \tag{2.27}$$

If one assumes that $A_1(q^2,k\cdot p)$ and $A_2(q^2,k\cdot p)$ satisfy, respectively, once-subtracted and unsubtracted dispersion relations in k, q with q^2 fixed,[9]

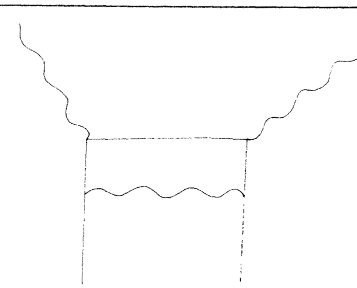

FIG. 4. Diagram contributing to the vector-vector current amplitude $p^{\mu}p^{\nu}T^{a\bar{a}}_{\mu\nu}$.

one can then derive in analogy with the previous case the relations

$$\int_0^1 d\omega\, \omega^n \tilde{a}_1(\omega, q^2) = C_{SV,1}^{n+1}(q^2) B_{SV}^{n+1}\,,$$
$$\int_0^1 d\omega\, \omega^n \tilde{a}_2(\omega, q^2) = C_{SV,2}^{(n+2)}(q^2) B_{SV}^{n+2}\,, \tag{2.28}$$

where

$$\tilde{a}_1(q^2, \omega) = \frac{1}{\pi}\, \mathrm{Im} A_1(q^2, \omega)\,,$$
$$\tilde{a}_2(q^2, \omega) = \frac{1}{\pi} \mathrm{Im} A_2(q^2, \omega)\,, \tag{2.29}$$

where $\omega = -q^2/2k\cdot q$ and B_{SV}^n are the unknown constants which appear in the matrix element of the operators ${}^n\hat{O}^V$ between gluon states.

However, this time, in order to evaluate the above moments, we will directly calculate the "structure functions" $\tilde{a}_1(q^2, k\cdot q)$ and $\tilde{a}_2(q^2, k\cdot q)$. It is clear that no infrared delicacy will arise in this case since there are no massless intermediate states. Taking the projections R_μ^μ and $k_\mu k_\nu R^{\mu\nu}$, one finds, respectively

$$\int_0^1 \omega^n\left(\frac{\tilde{a}_2}{2\omega} - 3\tilde{a}_1\right) d\omega = \frac{4g^2}{\pi^2}\, C_2(G)\left\{\frac{n^2+3n+4}{(n+1)(n+2)(n+3)}\left[\left(\ln\frac{Q^2}{M^2}\right) - \sum_{j=1}^{n+1}\frac{1}{j}\right] - \frac{n}{(n+1)^2} - \frac{2}{(n+2)(n+3)}\right\} \tag{2.30}$$

and

$$\int_0^1 \omega^n\left(\frac{\tilde{a}_2}{2\omega} - \tilde{a}_1\right) d\omega = \frac{4g^2}{\pi^2}\,\frac{C_2(G)}{(n+1)(n+2)} \tag{2.31}$$

for n even, and zero for n odd. Again, taking suitable combinations of these expressions, one obtains the moments of the functions $\tilde{a}_1$ and $\tilde{a}_2$ themselves.

An immediate consequence of these results is the fact that in the large-n limit the contribution from this last singlet Wilson coefficient is smaller by at least a factor $1/n$ compared to the previous one.

Thus, we have obtained the Wilson coefficients in the limit $Q^2 \gg M^2$, to second order in perturbation theory. However, these quantities have no direct interest to us, since, according to Eq. (2.3), the general solutions of the renormalization-group equations involve these coefficients as obtained from perturbation theory but for $Q^2 = \mu^2$, where μ is some arbitrary mass parameter at which one performs the subtractions. We will now relate these two quantities (see also Appendix B). First we choose $\mu^2 = M^2$ and consider a Taylor expansion of the general solution of the renormalization-group equation (focusing on the nonsinglet case first):

$$C^n\left(\frac{Q^2}{M^2}, g\right) = C^n(1, \bar{g}) \exp\left[-\int_0^t \gamma^n(\bar{g}(g, t'))\, dt'\right]. \tag{2.32}$$

Let us then expand both sides of Eq. (2.32) in powers of g and retain g^2 terms only. We obtain for the left-hand side

$$C^n\left(\frac{Q^2}{M^2}, g\right) = C_0^n + D_2^n\left(\frac{Q^2}{M^2}\right) g^2\,, \tag{2.33}$$

where

$$D_2^n\left(\frac{Q^2}{M^2}\right) = \frac{\partial^2}{\partial g^2}\, C^n\left(\frac{Q^2}{M^2}, g\right)\Big|_{g=0}.$$

This is precisely the quantity we obtained previously from perturbation theory. For the right-hand side we get

$$C^n(1,0) + \frac{\partial}{\partial g^2}\left\{C^n(1, \bar{g}) \exp\left[-\int_0^t \gamma^n(\bar{g}(g, t'))dt'\right]\right\}\Big|_{g=0} g^2. \tag{2.34}$$

Explicit evaluation of this derivative at $g=0$ gives

$$C^n(1,0) + \left\{\frac{\partial^2 C^n}{\partial \bar{g}^2}\left(\frac{\partial \bar{g}}{\partial g}\right)^2\Big|_{g=0} - C^n(1,0)\int_0^t dt'\, \frac{\partial^2\gamma^n}{\partial \bar{g}^2}[\bar{g}(g, t')]\left(\frac{\partial \bar{g}}{\partial g}\right)^2\right\} g^2$$
$$= C^n(1,0) + g^2\, \frac{\partial^2 C^n(1, \bar{g})}{\partial \bar{g}^2}\Big|_{\bar{g}=0} - g^2\, \tfrac{1}{2}\,\gamma_0^n\left(\ln\frac{Q^2}{M^2}\right) C^n(1,0)\,, \tag{2.35}$$

where we have used the following conditions:

$$\left.\frac{\partial\gamma^n(g)}{\partial g}\right|_{g=0}=0,\quad \left.\frac{\partial^2\gamma^n(g)}{\partial g^2}\right|_{g=0}=\gamma_0^n,\quad \left.\frac{\partial\bar g}{\partial g}\right|_{g=0}=1,\quad \left.\frac{\partial^2\bar g}{\partial g^2}\right|_{g=0}=0\,. \tag{2.36}$$

Then, comparing both sides of Eq. (2.32), one can read off the desired quantities. For the mixed combination, corresponding to $F_2/2x-3F_1=F_L-2F_T$, we find [we denote

$(1/2x)F_2(x,Q^2)-F_1(x,Q^2)=F_L(x,Q^2)\sim\sigma_{\text{longitudinal}}$,

$F_1(x,Q^2)=F_T(x,Q^2)\sim\sigma_{\text{transverse}}$]

$$\left.\frac{\partial^2 C_M^n(1,\bar g)}{\partial\bar g^2}\right|_{\bar g=0}=\frac{C_2(R)}{\pi^2}\left[\frac12-\frac{5}{2n}+\frac{n^2+2n-1}{2n^2(n+1)^2}+\sum_{j=1}^{n}\frac{1}{j^2}-\left(\frac74-\frac{1}{2n(n+1)}\right)\sum_{j=1}^{n}\frac1j+\sum_{s=1}^{n}\frac1s\sum_{j=1}^{s}\frac1j\right]\quad(n\text{ even}). \tag{2.37}$$

For the longitudinal combination, corresponding to $F_2/2x-F_1=F_L$,

$$\left.\frac{\partial^2 C_L^n(1,\bar g)}{\partial\bar g^2}\right|_{\bar g=0}=\frac{1}{2\pi^2}\,\frac{C_2(R)}{n}\quad(n\text{ even}). \tag{2.38}$$

Finally for the parity-mixing coefficient C_3^n,

$$\left.\frac{\partial^2 C_3^n(1,\bar g)}{\partial\bar g^2}\right|_{\bar g=0}=\frac{1}{\pi^2}C_2(R)\left[\frac{13}{4}-\frac{3}{4n}+\frac{3n^2-1}{2n^2(n+1)}+\sum_{j=1}^{n}\frac{1}{j^2}-\left(\frac74+\frac{1}{2n(n+1)}\right)\sum_{j=1}^{n}\frac1j-\sum_{s=1}^{n}\frac1s\sum_{j=1}^{s}\frac1j\right]\quad(n\text{ even}). \tag{2.39}$$

For the lowest-order terms we find

$$C_M^n(1,0)=4,\quad C_L^n(1,0)=0,\quad C_3^n(1,0)=-4\,. \tag{2.40}$$

Let us now consider the singlet case; the general solution to the renormalization-group equation is in this case

$$C_i^n\left(\frac{Q^2}{M^2},g\right)=\sum_{j=V,F}\left\{T\exp\left[-\int_0^t\tilde\gamma^n(\bar g(g,t'))\,dt'\right]\right\}_{i,j}C_j^n(1,\bar g)\,, \tag{2.41}$$

where the indices i,j label the Wilson coefficients corresponding to fermion or gluon field operators.

If again we expand both sides of Eq. (2.41) and retain second-order terms we find for the left-hand side

$$C_j^n(1,0)+\left(\left.\frac{\partial}{\partial g^2}C_j^n(1,g)\right|_{g=0}\right)g^2\,, \tag{2.42}$$

whereas for the right-hand side we get

$$C_i^n(1,0)+\left.\frac{\partial^2}{\partial g^2}\left(\sum_j\left\{T\exp\left[-\int_0^t\tilde\gamma^n(\bar g(g,t'))dt'\right]\right\}_{i,j}C_j^n(1,\bar g)\right)\right|_{g=0}g^2\,. \tag{2.43}$$

Explicit evaluation of the second-derivative term yields

$$\left.\frac{\partial^2}{\partial g^2}\left(\sum_j\left\{T\exp\left[-\int_0^t\tilde\gamma^n(\bar g(g,t'))dt'\right]\right\}_{i,j}C_j^n(1,\bar g)\right)\right|_{g=0}g^2=g^2\sum_j\left[\delta_{ij}\left.\frac{\partial C_j^n(1,g)}{\partial g^2}\right|_{g=0}-\tfrac12\tilde\gamma^n_{0ij}C_j^n(1,0)\ln\frac{Q^2}{M^2}\right], \tag{2.44}$$

where we have made use of the following facts:

$$\left.\frac{\partial\tilde\gamma^n}{\partial g}\right|_{g=0}=0,\quad \left.\frac{\partial^2\bar g}{\partial g^2}\right|_{g=0}=0,\quad \left.\frac{\partial\bar g}{\partial g}\right|_{g=0}=1,\quad \left.\frac{\partial^2\tilde\gamma^n(g)}{\partial g^2}\right|_{g=0}=\tilde\gamma_0^{(n)}\,, \tag{2.45}$$

with the matrix $\tilde\gamma_0^n$ given by Eq. (1.32). Thus one finally obtains the results

$$\left.\frac{\partial}{\partial g^2}C_V^n\left(\frac{Q^2}{M^2},g\right)\right|_{g=0}g^2=\left.\frac{\partial^2 C_V^n(1,g)}{\partial g^2}\right|_{g=0}g^2-\tfrac12({}^n\gamma^F_{VV})\left(\ln\frac{Q^2}{M^2}\right)C_F^n(1,0)g^2 \tag{2.46}$$

and

$$\left.\frac{\partial}{\partial g^2}C_F^n\left(\frac{Q^2}{M^2},g\right)\right|_{g=0}g^2=\left.\frac{\partial^2 C_F^n(1,g)}{\partial g^2}\right|_{g=0}g^2-\tfrac12({}^n\gamma^F_{FF})\left(\ln\frac{Q^2}{M^2}\right)C_F^n(1,0)g^2\,.$$

For the lowest-order terms we find

$$C_j^n(1,0)=\begin{cases}0 & \text{if } j=V\\ \pm 4 & \text{if } j=F.\end{cases} \tag{2.47}$$

The expressions for the fermion singlet pieces are identical to those for the fermion nonsinglets. For the gluon singlet pieces we have

$$\left.\frac{\partial^2 C^{n,s}_{\mathrm{mix},V}(1,g)}{\partial g^2}\right|_{g=0}=\frac{4}{\pi^2}C_2(G)\left[\frac{n^2+n+2}{n(n+1)(n+2)}\sum_1^n\frac{1}{j}+\frac{n-1}{n^2}+\frac{3}{(n+1)(n+2)}\right]\ (n\text{ even}) \tag{2.48}$$

and

$$\left.\frac{\partial^2 C^{n,s}_{\mathrm{long},V}(1,g)}{\partial g^2}\right|_{g=0}=\frac{4}{\pi^2}\frac{C_2(G)}{n(n+1)}\ \ (n\text{ even}). \tag{2.49}$$

Now combining these results, we finally obtain the following relations for the longitudinal coefficients:

$$C^n_{\mathrm{NS,long}}(1,\bar g)=\frac{1}{2\pi^2}\frac{C_2(R)}{n}\bar g^2\ \ (n\text{ even}), \tag{2.50a}$$

$$C^n_{SF,\mathrm{long}}(1,\bar g)=\frac{1}{2\pi^2}\frac{C_2(R)}{n}\bar g^2\ \ (n\text{ even}), \tag{2.50b}$$

$$C^n_{SV,\mathrm{long}}(1,\bar g)=\frac{4}{2\pi^2}\frac{C_2(G)}{n(n+1)}\bar g^2\ \ (n\text{ even}) \tag{2.50c}$$

where NS, SF, and SV indicate nonsinglet fermion, singlet fermion, and singlet gluon, respectively.

Similarly, for the "mixed" coefficients we find

$$C^n_{\mathrm{NS,mix}}(1,\bar g)=\left\{4+\frac{\bar g^2}{\pi^2}C_2(R)\left[\frac{1}{2}-\frac{5}{4n}+\frac{n^2+2n-1}{2n^2(n+1)^2}+\sum_{j=1}^n\frac{1}{j^2}-\left(\frac{7}{4}-\frac{1}{2n(n+1)}\right)\sum_{j=1}^n\frac{1}{j}+\sum_{s=1}^n\frac{1}{s}\sum_{j=1}^s\frac{1}{j}\right]\right\}\ (n\text{ even}), \tag{2.51a}$$

$$C^n_{SF,\mathrm{mix}}(1,\bar g)=\left\{4+\frac{\bar g^2}{\pi^2}C_2(R)\left[\frac{1}{2}-\frac{5}{4n}+\frac{n^2+2n-1}{2n^2(n+1)^2}+\sum_{j=1}^n\frac{1}{j^2}-\left(\frac{7}{4}-\frac{1}{2n(n+1)}\right)\sum_{j=1}^n\frac{1}{j}+\sum_{s=1}^n\frac{1}{s}\sum_{j=1}^s\frac{1}{j}\right]\right\}\ (n\text{ even}), \tag{2.51b}$$

$$C^n_{SV,\mathrm{mix}}(1,\bar g)=\frac{-4\bar g^2}{\pi^2}C_2(G)\left[\frac{n^2+n+2}{n(n+1)(n+2)}\sum_{j=1}^n\frac{1}{j}+\frac{n-1}{n^2}+\frac{2}{(n+1)(n+2)}\right]\ (n\text{ even}). \tag{2.51c}$$

Finally, for the parity-mixing fermion singlet and nonsinglet coefficients, we find these relations:

$$C^n_{\mathrm{NS},3}(1,\bar g)=-\left\{4-\frac{\bar g^2}{\pi^2}C_2(R)\left[\frac{13}{4}-\frac{3}{4n}+\frac{3n^2-1}{2n^2(n+1)^2}+\sum_{j=1}^n\frac{1}{j^2}-\left(\frac{7}{4}+\frac{1}{2n(n+1)}\right)\sum_{j=1}^n\frac{1}{j}-\sum_{s=1}^n\frac{1}{s}\sum_{j=1}^s\frac{1}{j}\right]\right\}, \tag{2.52a}$$

$$C^n_{SF,3}(1,\bar g)=-\left\{4-\frac{\bar g^2}{\pi^2}C_2(R)\left[\frac{13}{4}-\frac{3}{4n}+\frac{3n^2-1}{2n^2(n+1)^2}+\sum_1^n\frac{1}{j^2}-\left(\frac{7}{4}+\frac{1}{2n(n+1)}\right)\sum_{j=1}^n\frac{1}{j}-\sum_{s=1}^n\frac{1}{s}\sum_{j=1}^s\frac{1}{j}\right]\right\}. \tag{2.52b}$$

These results are crucial for most of the forthcoming applications.

III. EXPERIMENTAL PREDICTIONS

The first and most direct consequence of these results is obtained by putting them into the equations for the moments, Eqs. (1.11), and thereby determining corrections to their leading asymptotic form. However, unless the anomalous dimensions of the operators relevant to the moment in question are zero, which is true only in some exceptional cases, the correction terms arising from the exponential factor of the nonsinglet Wilson coefficient,

$$\exp\left[-\int_0^t \gamma^n(\bar{g}(g,t'))\,dt'\right], \tag{3.1}$$

are, in most cases, much larger than those which arise from the expansion of $C^n(1,\bar{g})$. Although a detailed analysis of these corrections was not our main purpose, we include them here for completeness. Later on we shall consider the expansion in terms of the effective coupling constant, which can be determined experimentally, rather than its asymptotic form.

For small g, $\gamma^n(g)$ has the expansion

$$\gamma^n(g)=\gamma_0^n g^2+\gamma_1^n g^4+O(g^6), \tag{3.2}$$

where γ_0^n is a well-known quantity but γ_1^n has not so far been calculated. Furthermore, one also has that the function $\beta(g)$ can be written in the form

$$\beta(g)=-\tfrac{1}{2}b_0g^3+\tfrac{1}{2}b_1g^5+\tfrac{1}{2}b_2g^7+O(g^9), \tag{3.3}$$

where

$$b_0=\frac{1}{8\pi^2}\left[\tfrac{11}{3}C_2(G)-\tfrac{4}{3}T(R)\right]$$
$$=9/8\pi^2 \tag{3.4}$$

and

$$b_1=\frac{2}{16\pi^2}\left[-\tfrac{34}{3}C_2{}^2(G)+4C_2(R)T(R)+\tfrac{20}{3}C_2(G)T(R)\right]$$
$$=-\tfrac{128}{16}. \tag{3.5}$$

Using these results,[10] one can then easily show that in the large-t limit $\bar{g}^2$ behaves asymptotically as

$$\bar{g}^2(t,g)\rightarrow\frac{1}{b_0t}+\frac{b_1}{b_0^3}\frac{\ln b_0t}{t^2}+\frac{c}{(b_0t)^2}+O\left(\frac{1}{t^3}\right), \tag{3.6}$$

where the constant c is uncalculable since it involves the unknown physical coupling constant g_p. [Incidentally one can automatically conclude that it is of no use trying to calculate the coefficient b_2 so as to determine the large-t behavior of $\bar{g}^2$, since it would only provide a smaller correction to the already unknown term $c/(b_0t)^2$.]

Inserting all these terms into the exponential of Eq. (3.1) we obtain in the large-t limit

$$\exp\left(-\int_0^t \gamma(\bar{g}(t',g))\,dt'\right)\rightarrow(t)^{-\gamma_0^n/b_0}\left[1+\frac{\gamma_0^n b_1\ln b_0t}{b_0^3t}+\frac{\gamma_0^n(b_1+cb_0^3)+\gamma_1^nb_0}{b_0^3t}\right]. \tag{3.7}$$

From this result one learns that the second term inside the square brackets gives the largest correction to the leading term. It is in fact $\ln t$ times larger than the next term, which is in turn of the same magnitude as the correction terms that one obtains by expanding $C^n(1,\bar{g})$ and retaining up to terms proportional to $\bar{g}^2\sim 1/b_0t$. Moreover, since γ_1^n has not so far been calculated one would expect that the corrections arising from the Wislon coefficients have little practical importance.

The total ignorance of c implies that corrections proportional to $1/t$ cannot be fully calculated, at least by present techniques. Nevertheless, as we will see later, it is possible in some cases to get rid of c and thus obtain perfectly calculable predictions. Another viewpoint is to regard this expansion as an expansion in $\bar{g}$ and not its asymptotic form Eq. (3.6); then the moments of the structure functions will have the form

$$M_n=\text{const}\times(\bar{g}^2)^{-\gamma_0^n/b_0}\left[1-\bar{g}^2\left(\frac{\gamma_1^n}{b_0}+\frac{2b_1\gamma_0^n}{b_0^2}\right)\right]\times[C_n(1,0)+D_n(1,0)\bar{g}^2].$$

One can experimentally determine $\bar{g}^2$ from one of the moments and use it to test the others. This procedure would require the knowledge of the structure function for all values of x, the explicit calculation of γ_1^n, and that $\bar{g}^2\ll 1$, so that higher-order terms can be neglected. We shall return to this point at the end.

Before entering into a detailed analysis of all these terms we shall first concentrate on the largest correction term in Eq. (3.7). Substituting Eq. (3.7) into Eqs. (1.11) we find, for the nonsinglet piece, the following equations for the moment integrals:

$$\int_0^1 dx\,x^nF_1^{\rm NS}(x,Q^2)=M_{\rm NS}^{n+1}C_{1,\rm NS}^{n+1}(1,0)(t^{-\gamma_0^{n+1}/b_0})\times\left(1+\frac{\gamma_0^{n+1}b_1\ln t}{b_0^3t}\right),$$

$$\int_0^1 dx\,x^nF_2^{\rm NS}(x,Q^2)=M_{\rm NS}^{n+2}C_{2,\rm NS}^{n+2}(1,0)(t^{-\gamma_0^{n+2}/b_0})\times\left(1+\frac{\gamma_0^{(n+2)}b_1\ln t}{b_0^3t}\right), \tag{3.8}$$

$$\int_0^1 dx\,x^nF_3^{\rm NS}(x,Q^2)=M_{\rm NS}^{n+1}C_{3,\rm NS}^{n+1}(1,0)(t^{-\gamma_0^{n+1}/b_0})\times\left(1+\frac{\gamma_0^{n+2}b_1\ln t}{b_0^3t}\right),$$

where the $M_{\rm NS}^n$ are unknown constants. There are similar relations for the singlet pieces. We will now proceed to derive a functional relationship between the structure functions themselves following a technique discussed by Gross.[11] Let us consider the ratio between the nth moment of $F_2^{NS}(Q^2,x)$ for two distinct, but large values of Q^2:

$$\frac{\int_0^1 dx\, x^n F_2(x,t)}{\int_0^1 dx\, x^n F_2(x,t')} = \left(\frac{t}{t'}\right)^{-\gamma_0^{n+2}/b_0} \left(\frac{1+\dfrac{\gamma_0^{n+2} b_1 \ln b_0 t}{b_0^3 t}}{1+\dfrac{\gamma_0^{n+2} b_1 \ln b_0 t'}{b_0^3 t'}}\right). \quad (3.9)$$

Assuming that $\gamma_0^n b_1 \ln b_0 t'/b_0^3 t' \ll 1$, we obtain[12]

$$\left(\frac{t'}{t}\right)^{\gamma_0^{n+2}/b_0}\left[1+\frac{\gamma_0^{n+2} b_1}{b_0^2}\left(\frac{\ln b_0 t}{b_0 t}-\frac{\ln b_0 t'}{b_0 t'}\right)\right] = \left(\frac{t'}{t}\right)^{A_n}[1+A_n H(t,t')]\,, \quad (3.10)$$

where

$$H(t,t') = \frac{b_1}{b_0}\left(\frac{\ln b_0 t}{b_0 t} - \frac{\ln b_0 t'}{b_0 t'}\right)$$

and

$$A_n = \frac{\gamma_0^{n+2}}{b_0} \simeq G[4\ln(N+2) - 0.69]\,,$$

where G is totally determined by the gauge-group parameters and the representation of the quarks, and in our case is $G = \frac{4}{27}$.

Let us now construct a function $R(t,t';x)$ such that its Mellin transform gives the function[13]

$$\left(\frac{t'}{t}\right)^{A(s)}[1+A(s)H(t,t')]\,, \quad (3.11)$$

where $A(s)$ is the analytic continuation of A_n to $\mathrm{Re}\, s > -1$. This function can be constructed easily in terms of the already known function $T(t/t',x)$ whose Mellin transform is[11]

$$\left(\frac{t'}{t}\right)^{A(s)},$$

by noting that

$$\left(\frac{t'}{t}\right)^{A(s)}[1+A(s)H(t,t')] = \left(\frac{t'}{t}\right)^{A(s)} + H(t,t')t'\frac{\partial}{\partial t'}\left(\frac{t'}{t}\right)^{A(s)} \quad (3.12)$$

$$T\left(\frac{t}{t'},\,\omega\right) = \left(\frac{t}{t'}\right)^{0.69G}\frac{(\ln\omega)^{P-1}}{\omega\Gamma(P)}\,, \quad (3.13)$$

with

$$P = 4G\ln t/t' \text{ and } x = 1/\omega\,.$$

Thus the Mellin transform of $F_2(x,Q^2)$ is equal to the product of the Mellin transforms of $F_2(x,Q'^2)$ and $R(x;t,t')$. One can then use the convolution theorem of Mellin transforms to derive

$$F_2(\omega,t) = \int_1^\omega \frac{d\omega'}{\omega'}\,F_2\left(\frac{\omega}{\omega'},\,t'\right)T\left(\frac{t}{t'},\omega'\right) + \int_1^\omega\frac{d\omega'}{\omega'}F_2\left(\frac{\omega}{\omega'},t'\right)H(t,t')t'\frac{\partial}{\partial t'}T\left(\frac{t}{t'},\omega'\right). \quad (3.14)$$

This relation can be converted into a more useful one by taking the ratio of $F_2(\omega,t)$ and $F_2(\omega,t')$ with $\omega \simeq 1$ and assuming that $F_2(\omega,t) \sim (1-\omega)^d$ (experimentally $d=3$ for $Q^2 \sim 5$ GeV^2). One can thereby derive for Eq. (3.14) that

$$\frac{F_2(\omega,t)}{F_2(\omega,t')} = R(\omega;t,t') = R_0(\omega;t,t') - tH(t,t')\frac{\partial}{\partial t}R_0(\omega;t,t')\,. \quad (3.15)$$

Inserting the approximation

$$R_0(\omega;t,t') \simeq \left(\frac{t}{t'}\right)^{0.69G}\frac{\Gamma(d+1)}{\Gamma(d+1+P)}(\ln\omega)^P \quad (3.16)$$

in Eq. (3.15), we finally obtain

$$R(\omega;t,t') = R_0(\omega;t,t') - H(t,t')[0.69G + \ln\ln\omega - \psi(P+d+1)] \times R_0(\omega;t,t')\,, \quad (3.17)$$

where $\psi(P) = \Gamma'(P)/\Gamma(P)$ is the digamma function. One can now make a numerical estimate of the size of this correction. Choosing $Q^2 = 50$ GeV^2, $Q'^2 = 5$ GeV, and $\mu^2 \sim 1$ GeV^2, for $\omega = 1.1$, we have

$$R(\omega;t,t')\big|_{\omega=1.1} \simeq R_0(\omega=1.1)\times(1-\tfrac{1}{5}) = R_0(\omega=1.1)\times 0.8\,, \quad (3.18)$$

so the correction term is 5 times smaller than the leading term. For $\omega = 2.0$,

$$R(\omega;t,t')\big|_{\omega=2.} = R_0(\omega=2.0)\,(1-\tfrac{5}{130})\,; \quad (3.19)$$

the correction term is 25 times smaller than the leading term.

One can easily show that this term slowly increase as ω approaches 1, attaining its maximum value for $\omega \simeq 1.02$, where it gives a 28% modification to the leading term. The most important conclusion one draws from Eq. (3.17) is that the $(\ln\omega)^P$ dependence of R that was predicted by Gross is unchanged by this term.[11]

Let us now turn to the analysis of the corrections which arise from the Wilson coefficients. The first and most direct application of our results will be a calculation of the leading correction terms for some of the known sum rules (for them the anomalous dimension is zero, consequently the exponential term is identically unity). Consider first the Bjorken backward sum rule;[7] by combining Eq. (1.11) and Eqs. (2.51) we derive

$$\int_0^1 dx[F_1^{\nu p}(x,Q^2)-F_1^{\bar\nu p}(x,Q^2)]=-2-\frac{\bar g^2}{4\pi^2}\,C_2(R). \tag{3.20}$$

In a similar way, we obtain for the quark baryon-number sum rule[7]

$$\int_0^1 dx[F_3^{\nu p}(x,Q^2)+F_3^{\nu n}(x,Q^2)]=-2+\frac{5\bar g^2}{16\pi^2}\,C_2(R). \tag{3.21}$$

Another interesting application of our results is the functional relationship that can be derived between $F_L(x,Q^2)=F_2(x,Q^2)-2xF_1(x,Q^2)$ and $F_1(x,Q^2)$ (for nonsinglet pieces). Taking the ratio between the moments of these two functions for equal values of Q^2, one has that

$$\frac{\int_0^1 dx\, x^n[F_2(x,Q^2)-2xF_1(x,Q^2)]}{\int_0^1 x^{n+1}F_1(x,Q^2)dx}=\frac{\bar g^2 C_2(R)}{2\pi^2(n+3)}=\frac{\bar g^2}{2\pi^2}\,C_2(R)\int_0^1 x^{n+2}\,dx. \tag{3.22}$$

Then, using the convolution theorem, one obtains

$$F_L(x,Q^2)=\int_1^\omega\frac{d\omega'}{\omega'^3}\,F_1\left(\frac{\omega}{\omega'},Q^2\right)\frac{\bar g^2}{2\pi^2}\,C_2(R). \tag{3.23}$$

For ω near threshold, and assuming that $F_1\sim(\omega-1)^d$ in this region, we calculate that

$$\frac{F_L(\omega,Q^2)}{F_1(\omega,Q^2)}\underset{\omega\simeq 1}{\simeq}\frac{\bar g^2}{2\pi^2}\,C_2(R)\,\frac{(\omega-1)}{d+1}. \tag{3.24}$$

Similarly, for large ω and assuming $F_1\sim c\omega^{\alpha-1}$ as $\omega\to\infty$ ($\alpha=\frac{1}{2}$ for nonsinglet), one finds that

$$\frac{F_L(\omega,Q^2)}{F_1(\omega,Q^2)}\underset{\omega\to\infty}{\simeq}\frac{\bar g^2C_2(R)}{4\pi^2}\,\frac{1}{1+\alpha}. \tag{3.25}$$

Our next application will be the analysis of the corrections to the ratio of the moments of any of the strucutre functions and the functional relation that we can extract from it. Let us take, for example, the ratio of the structure function F_2 (nonsinglet) for different values of Q^2:

$$\frac{\int_0^1 F_2(x,Q^2)x^n\,dx}{\int_0^1 F_2(x,Q'^2)x^n dx}=\frac{C_2^{n+2}(1,\bar g)\exp[-\int_0^t\gamma^{n+2}(\bar g(\tau,g))d\tau]}{C_2^{n+2}(1,\bar g')\exp[-\int_0^{t'}\gamma^{n+2}(\bar g(\tau,g))d\tau]}=\frac{C_2^{n+2}(1,\bar g)}{C_2^{n+2}(1,\bar g')}\,\exp\left[\int_t^{t'}\gamma^{n+2}(\bar g(\tau,g))d\tau\right] \tag{3.26}$$

with ($Q^2>Q'^2\gg\mu^2$). Expanding both factors in powers of $\bar g^2$, one finds, after some manipulations, that Eq. (3.26) can be rewritten as

$$\left(\frac{t'}{t}\right)^{A_n}\left\{1-A_n\,\frac{b_1}{b_0}\left(\frac{\ln b_0t}{b_0t}-\frac{\ln b_0t'}{b_0t'}\right)+\left[\frac{1}{b_0}\,\frac{D_2^{n+2}(1,0)}{C_2^{n+2}(1,0)}-\frac{A_nb_1+c\gamma_0^{n+2}-\gamma_1^{n+2}}{b_0^{\,2}}\right]\left(\frac{1}{t}-\frac{1}{t'}\right)\right\}+O\left(\frac{1}{t^2}\right). \tag{3.27}$$

The effect of the first two terms inside the above curly brackets was already analyzed. The contribution from the term

$$\frac{A_nb_1}{b_0^{\,2}}\left(\frac{1}{t}-\frac{1}{t'}\right) \tag{3.28}$$

is very similar to the previous one and gives rise to a correction of the order of magnitude $\ln b_0t$ times that of the term

$$A_N\,\frac{b_1}{b_0}\left(\frac{\ln b_0t}{b_0t}-\frac{\ln b_0t'}{b_0t'}\right).$$

For $Q^2/\mu^2\sim 50$ and $Q'^2/\mu^2\sim 5$ one can easily see that these two terms are of comparable magnitude.

For the term

$$\frac{\gamma_1^{n+2}+c}{b_0^{\,2}}\left(\frac{1}{t}-\frac{1}{t'}\right) \tag{3.29}$$

nothing can be said since γ_1^{n+2} has not yet been calculated and c is unknown. However, there are some indications that its effect is small.[14]

Finally, the contribution of the term

$$\frac{1}{b_0}\frac{D_2^{n+2}(1,0)}{C_2^{n+2}(1,0)}\left(\frac{1}{t}-\frac{1}{t'}\right) \tag{3.30}$$

can be easily analyzed since all the relevant numbers are known. Before drawing any conclusion concerning the effect of this term on the functional relation between $F_2(x,Q^2)$ and $F_2(x,Q'^2)$, let us consider a particular situation for which only this last term contributes. We focus on the following combinations:

$$\frac{\int_0^1 F_2(x,Q^2)x^n dx}{\int_0^1 F_2(x,Q'^2)x^n dx} - \frac{\int_0^1 F_3(x,Q^2)n^{n+1}dx}{\int_0^1 F_3(x,Q'^2)x^{n+1}dx} = \left(\frac{t'}{t}\right)^{A_n}\frac{1}{b_0}\left[\frac{D_2^{n+2}(1,0)}{C_2^{n+2}(1,0)} - \frac{D_3^{n+2}(1,0)}{C_3^{n+2}(1,0)}\right]\left(\frac{1}{t}-\frac{1}{t'}\right) + O\left(\frac{1}{t^2}\right). \tag{3.31}$$

Using Eqs. (2.37) and (2.51a) we find that for large n we can approximate:

$$\frac{D_2^{n+1}(1,0)}{C_2^{n+2}(1,0)} - \frac{D_3^{n+2}(1,0)}{C_3^{n+2}(1,0)} \simeq \frac{-C_2(R)}{4\pi^2}\ln^2(n+2). \tag{3.32}$$

Let $S(t,t';x)$ be the inverse Mellin transform

$$\left(\frac{t'}{t}\right)^{A(\tau)}\left(\frac{1}{b_0 t}-\frac{1}{b_0 t'}\right)\left[\frac{-C_2(R)}{4\pi^2}\ln^2(n+2)\right]. \tag{3.33}$$

As before, this expression is defined for Re $\tau > -1$. The function $S(t,t',x)$ can be easily constructed from the function $T(t/t',x)$ of Eq. (3.13); the result is

$$\frac{-C_2(R)}{4\pi^2}\left(\frac{1}{b_0 t}-\frac{1}{b_0 t'}\right)\left(\frac{t'}{t}\right)^{0.69G}\frac{\partial^2}{\partial p^2}\left[\frac{(\ln\omega)^{P-1}}{\omega\Gamma(P)}\right]. \tag{3.34}$$

Finally, invoking the convolution theorem of Mellin transforms, we obtain

$$\int_0^\omega\left[F_3\left(\frac{\omega}{u},t\right)F_2(u,t') - F_3\left(\frac{\omega}{u},t'\right)F_2(u,t)\right]\frac{du}{u} = \int_1^\omega\frac{du}{u}\int_1^u\frac{d\nu}{\nu}S\left(t,t';\frac{\omega}{u}\right)F_3\left(\frac{u}{\nu},t\right)F_2(\nu,t). \tag{3.35}$$

This result can be tested easily since it only requires the knowledge of $F_3(\omega',t)$, $F_3(\omega',t')$, $F_2(\omega',t)$, and $F_2(\omega',t')$ for ω' between 1 and ω. Clearly, similar relationships can be established between the functions F_1 and F_3 or F_2 and F_3.

Let us now return to study the effect of term (3.30) in Eq. (3.27). We assume that this term provides the dominant correction so that the terms in Eqs. (3.28) and (3.29) can be neglected. It is possible to estimate expression (3.30) by using Eqs. (2.37) and (2.38). We find that for large n

$$\frac{D_2^{n+2}(1,0)}{C_2^{n+2}(1,0)} \simeq -\frac{C_2(R)}{4\pi^2}\ln^2(n+2) = -\frac{\ln^2(n+2)}{3\pi^2}, \tag{3.36}$$

whence

$$\frac{1}{b_0}\frac{D_2^{n+2}(1,0)}{C_2^{n+2}(1,0)} \simeq -0.3\ln^2(n+2) + O(\ln n). \tag{3.37}$$

Choosing $Q^2/\mu^2=50$ and $Q'^2/\mu^2=5$ we get

$$\frac{1}{b_0}\frac{D_2^{n+2}(1,0)}{C_2^{n+2}(1,0)}\left(\frac{1}{t}-\frac{1}{t'}\right) \simeq \tfrac{1}{2}\ln^2(n+2). \tag{3.38}$$

Clearly this term is not negligible for small values of n since it is roughly equal to the leading term; indeed, for large n it is much larger than the leading term and the other corrections.[15] Hence, there exists a region in Q^2 for which this term will be by far the most important correction, so it is perhaps not a bad approximation to set

$$\frac{\int_0^1 F_2(x,Q^2)x^n dx}{\int_0^1 F_2(x,Q'^2)x^n dx} \simeq \left(\frac{t'}{t}\right)^{A_n}\left[1+\frac{D_2^{n+2}(1,0)}{b_0 C_2^{n+2}(1,0)}\left(\frac{1}{t}-\frac{1}{t'}\right)\right] = \left(\frac{t'}{t}\right)^{A_n}\left[1-0.3[\ln^2(n+2)]\left(\frac{1}{t}-\frac{1}{t'}\right)\right]. \tag{3.39}$$

Let us now find some function $R(x;t,t')$ whose nth moment gives the above expression. Using our previous results, Eq. (3.12), we see that the problem actually reduces to finding a function $R_1(x;t,t')$ whose nth moment gives

$$\left(\frac{t'}{t}\right)^{A_n}\ln^2(n+2), \tag{3.40}$$

then clearly

$$R(\omega;t,t')=T\left(\frac{t}{t'},\omega\right)-0.3R_1(\omega;t,t')\left(\frac{1}{t}-\frac{1}{t'}\right). \tag{3.41}$$

Again, using the convolution theorem for Mellin transforms, one obtains

$$F_2(\omega,t)=\int_1^\omega \frac{d\omega'}{\omega'}\,F_2\left(\frac{\omega}{\omega'},t'\right)\,R(\omega';t,t'). \tag{3.42}$$

From Eq. (3.42) we observe that

$$R_1(\omega;t,t')=\left(\frac{t'}{t}\right)^{0.69G}\frac{\partial^2}{\partial p^2}\left[\frac{(\ln\omega)^{P-1}}{\omega\Gamma(P)}\right], \tag{3.43}$$

so that $R(\omega;t,t')$ becomes

$$R(\omega;t,t')=\left(\frac{t'}{t}\right)^{0.69G}\left\{\frac{(\ln\omega)^{P-1}}{\omega\Gamma(P)}-0.3\frac{\partial^2}{\partial p^2}\left[\frac{(\ln\omega)^{P-1}}{\omega\Gamma(P)}\right]\left(\frac{1}{t}-\frac{1}{t'}\right)\right\}. \tag{3.44}$$

Inserting this in Eq. (3.37) one finds that

$$\begin{aligned}F_2(\omega,t)=&\left[1+0.3(\psi'(P)-\psi^2(P))\left(\frac{1}{t}-\frac{1}{t'}\right)\right]\int_1^\omega\frac{d\omega'}{\omega'}\,F_2\left(\frac{\omega}{\omega'},t'\right)T\left(\omega,\frac{t}{t'}\right)\\&-0.3\left(\frac{1}{t}-\frac{1}{t'}\right)\int_1^\omega\frac{d\omega'}{\omega'}\,F_2\left(\frac{\omega}{\omega'},t'\right)T\left(\omega',\frac{t}{t'}\right)\left[(\ln\ln\omega')^2-2\psi(P)\ln\ln\omega'\right].\end{aligned} \tag{3.45}$$

The $\ln\ln\omega'$ terms in this equation provide important corrections when ω is near threshold. These corrections are particularly important in calculating the hadronic form factors from the structure functions.[16]

The final application we shall consider is the corrections to the Llewellyn Smith relations.[17] These relations are

$$\begin{aligned}\frac{\int_0^1 dx\,x^n\,[F_2^{ep}(Q^2,x)-F_2^{en}(Q^2,x)]}{\int_0^1 dx\,x^{n+1}[F_3^{\nu p}(Q^2,x)-F_3^{\nu n}(Q^2,x)]}&=\frac{4+(\bar g^2/2\pi^2)\,C_2(R)\ln^2(n+2)}{-4+(\bar g^2/2\pi^2)\,C_2(R)\ln^2(n+2)}\\&=-\left[1+\frac{\bar g^2}{4\pi^2}\,C_2(R)\ln^2(n+2)\right],\end{aligned} \tag{3.46}$$

valid for $n\gg1$, but such that $(\bar g^2/4\pi^2)\ln^2(n+2)\ll1$. However, here we cannot invoke the convolution theorem to relate these two combinations of structure functions since the relation (3.42) holds only for large n.

So far we have been analyzing the effects of correction terms upon the assumption that we are in the large-Q^2 region, so that they are always smaller than the next-order term by a factor of $\bar g^2\sim1/(\ln Q^2/\mu^2)$. However, for the accessible values of Q^2 (50 GeV^2 in electroproduction and about 250 GeV^2 in neutrino production) and the "natural" choice of the mass scale parameter $\mu^2\sim1$ GeV^2, one finds that $\bar g^2$ is a number close to unity. Therefore in this range of values of Q^2/μ^2, the correction terms which we have considered and the higher-order terms which we have dropped will probably produce substantial deviations from the results given by the lowest-order calculations.

Let us finally consider the nonsinglet pieces of the structure functions. As was mentioned before, their analysis is complicated by the mixing of the operators, and as a consequence, only weak predictions can be made. To illustrate this let us consider the nth moment of the single part of the structure function F_2. According to Eq. (1.11) we have that

$$\int_0^1 F_2^s(x,Q^2)x^n dx=M_{n+2}^F C_{F,2}^{n+2}\left(\frac{Q^2}{\mu^2},g\right)+M_{n+2}^V C_{V,2}^{n+2}\left(\frac{Q^2}{\mu^2},g\right), \tag{3.47}$$

where $C_{F,2}^{n+2}$ and $C_{V,2}^{n+2}$ are the Wilson coefficients corresponding, respectively, to

$${}^{n+2}\hat O^F_{\mu_1\cdots\mu_{n+2}}=\tfrac12 i^{n+1}S\bar\psi(x)(\gamma_{\mu_1}\partial_{\mu_2}\cdots\partial_{\mu_{n+2}})\psi(x)$$

and

$${}^{n+2}\hat O^V_{\mu_1\cdots\mu_{n+2}}=i^nS\,\mathrm{Tr}F_{\mu,\alpha}(\nabla_{\mu_2}\cdots\nabla_{\mu_{n+1}})F^\alpha_{\mu_{n+2}}-\text{trace terms},$$

and the M^F_{n+2},M^V_{n+2} are unknown constants coming from the matrix element of the operators. Furthermore, the coefficients $C_{i,2}^{n+2}(Q^2/\mu^2,g)$ have the form

$$C_{i,2}^{n+2}(Q^2/\mu^2,g)\xrightarrow[t\to\infty]{}\left[\exp\left(\frac{\gamma_0^{n+2}}{b_0t}\right)\tilde M^{n+2}\right]_{i,j}C_{j,2}^{n+2}(1,\bar g), \tag{3.48}$$

with $i,j=F$ (fermion), V (vector meson).

Let $\{\Gamma_i^{n+2}\}$ be the eigenvalues of the matrix $\tilde{\gamma}_0^{n+2}$; then introducing a set of projection operators $\{\tilde{P}_i^{n+2}\}$ such that

$$\tilde{\gamma}_0^{n+2}=\sum_i \Gamma_i \tilde{P}_i^{n+2}\,,$$
$$\sum_i \tilde{P}_i^n=\tilde{I}\,,\quad \tilde{P}_i{}^2=\tilde{P}_i\,,\ \text{and}\ \tilde{P}^i\cdot\tilde{P}^j=\delta_{ij}\tilde{P}_i\,, \tag{3.49}$$

one can rewrite Eq. (3.56),

$$C_{i,2}^{n+2}\left(\frac{Q^2}{\mu^2},g\right)\sum_j\left(\sum_k \tilde{P}_k e^{-\Gamma_k^{n+2}/b_0 t}\tilde{M}\right)_{i,j}C_{j,2}^{n+2}(1,\bar{g})$$
$$=\sum_j\left(\sum_k \tilde{P}_k t^{-a_k}\tilde{M}^{n+2}\right)_{i,j}C_{j,2}^{n+2}(1,\bar{g})\,, \tag{3.50}$$

with $a_k=\Gamma_k^{n+2}/b_0$. Since neither $\tilde{M}^{n+2}$ nor M_F^{n+2} and M_V^{n+2} are known, the ratio of the moments of $F_2^s(x,Q^2)$ and $F_2^s(x,Q'^2)$ is not known. However, we have seen that in the large-n limit

$$\frac{C_{V,2}^{n+2}}{C_{F,2}^{n+2}}\simeq\frac{1}{(n+2)} \tag{3.51}$$

[see. Eqs. (2.23) and (2.31)]. Therefore in this limit C_V^n is negligible compared to C_F^n;[18] moreover, it is also known that the eigenvalues of $\tilde{\gamma}_0^n$ approaches the same value; thus we conclude that for large n

$$\int_0^1 F_2^s(x,Q^2)x^n dx\to\text{const}\times\left(\ln\frac{Q^2}{M^2}\right)^{-A_n}C_{F,2}^{n+2}(1,\bar{g})\,. \tag{3.52}$$

Consequently, if the criterion given below holds then we will have that for large n

$$\frac{\int_0^1 F_2^s(x,Q^2)x^n dx}{\int_0^1 F_2^s(x,Q'^2)x^n dx}\to\left(\frac{t'}{t}\right)^{A_n}[1+O(\bar{g}^2)]\,. \tag{3.53}$$

However, here we cannot apply the Mellin-transform theorem since this relation holds only for large n.[19]

IV. FINAL REMARKS

Throughout the last section we have derived several results which will be valid in the asymptotic region, i.e., for sufficiently large values of Q^2, such that the effective coupling constant $\bar{g}$ is a small parameter and therefore the perturbation expansion in powers of $\bar{g}^2$ is valid. Then the different moments of the structure functions, as well as their ratios, will exhibit deviations from exact scaling in the fashion we have described above. This is an unquestionable prediction. We have also derived direct relations between the structure functions; however, the validity of these relations depends critically upon the behavior of higher-order terms. To elucidate this statement let us again consider the ratio of the nth moment of F_2 for two different values of Q^2:

$$\frac{\int_0^1 F_2(x,Q^2)x^n dx}{\int_0^1 F_2(x,Q'^2)x^n dx}=\frac{C_2^{n+2}(1,\bar{g})}{C_2^{n+2}(1,\bar{g}')}\times\exp\left[\int_t^{t'}\gamma^n(g(\tau,g))d\tau\right]. \tag{4.1}$$

If we imagine expanding the right-hand side of Eq. (4.1) in powers of $\bar{g}^2$ up to some arbitrary order, we will find that the coefficients of this series are increasing functions of n, which will grow faster with n for higher orders of $\bar{g}$. This fact is motivated in part by our $\bar{g}^2$ calculation of $C_2^n(1,\bar{g})$ for which we found a $\ln^2 n$ behavior, and also by the fact that if one calculates the structure function $F_2(x,Q^2)$ to some order in perturbation theory, so as to construct from its moments the coefficients $C^n(1,\bar{g})$, one will find that the higher the order in g one considers the more severe the infrared singularity of $F_2(x,Q^2)$ becomes as $x\to 1$, thus producing the n growing behavior that we anticipated above. Therefore, for sufficiently large n the correction terms will eventually become dominant and the expansion meaningless. The important point is that the large-n behavior of (4.1) will determine the behavior of its inverse-Mellin-transformed function

$$T(\omega;t,t')=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}ds\left[\frac{\int_0^1 F_2(x,Q^2)x^s dx}{\int_0^1 F_2(x,Q'^2)x^s dx}\right]\omega^{s+1} \tag{4.2}$$

for values of ω near 1. This in turn will affect the convolution integral,

$$F_2(\omega,t)=\int_1^\omega\frac{d\omega'}{\omega'}F_2\left(\frac{\omega}{\omega'},t\right)T(\omega';t,t'), \tag{4.3}$$

particularly if ω is close to threshold. For values ω away from threshold the relation (4.3) may or may not be sensitive to how $T(\omega;t,t')$ behaves when ω is close to unity depending, of course, on the particular function $T(\omega;t,t')$. However, if we accept as a reasonable guess that the expansion of (4.1) will give an expression of the form

$$\left(\frac{t'}{t}\right)^{A_n}(1+\bar{g}^2\ln^2 n+\bar{g}^4\ln^4 n+\cdots+\bar{g}^{2r}\ln^{2r}n),$$

then (4.2) will be as follows:

$$T(\omega;t,t')\sim T_0\left(\omega;\frac{t}{t'}\right)[1+\bar{g}^2\ln\ln\omega+\cdots+\bar{g}^{2r}(\ln\ln\omega)^r],$$

with $T_0(\omega, t/t')$ given by Eq. (3.13). Thus the correction terms will be important only for

$$\ln(\omega - 1) \geq \bar{g}^2$$

but will be negligible for ω away from threshold. If this assumption on the behavior of (4.1) is correct, then all the results derived here are true for ω away from 1 and will thus provide a test to these theories.

The final and more serious problem is whether or not we are really in the asymptotic region. Pharased another way: Is $\bar{g}^2$ a sufficiently small parameter so that the lowest-order terms in the expansions we have calculated yield the true asymptotic behavior of the moments of the structure functions, for the present range of energies?

The present available energies yield a maximum of $Q^2_{\max} \sim 50$ GeV^2 in electroproduction and 250 GeV^2 in neutrinoproduction. On the other hand, the value of the mass scale parameter μ has to be determined by experiment, but it seems reasonable that its value will be in the domain of the hadron masses, namely $\mu \sim 1$ GeV. If this is so, we can make a numerical estimate of the limiting value of the effective coupling constant $\bar{g}^2$:

$$\bar{g}_{\lim}{}^2 \simeq \frac{1}{b_0 t} = \frac{16\pi^2}{9\ln 50} = 4.5$$

for electroproduction and

$$\bar{g}_{\lim}{}^2 \simeq \frac{1}{b_0 t} = \frac{16\pi^2}{9\ln 250} = 3.2$$

for neutrinoproduction.

In both cases we observe that $\bar{g}^2 > 1$, therefore the leading asymptotic form, at these energies, is not a good approximation; the contribution of nonleading terms will probably produce substantial deviations from this value. Furthermore, since $g_{\lim}{}^2$ will not be a small parameter the series may converge to something quite different from the lowest-order terms we have calculated. (In particular, the logarithmic deviations from scaling will be unreliable.)

Some improvement is obtained if one observes that the natural expansion parameter is not $\bar{g}^2$ but rather $\bar{g}^2/(2\pi)^2$, the factor $(2\pi)^2$ arising from the momentum-space integrations; if this number is small, then we may consider the expansions in powers of $\bar{g}^2/(2\pi)^2$ with the hope that the lowest-order terms will provide an adequate description of the behavior of the structure functions. (Note that at present energies we have no way to estimate $\bar{g}^2$; it *must* be determined from experiment by assuming that the lowest-order terms of the series give approximately the exact behavior of the structure functions. This assumption must of course be experimentally checked by comparison of the data at different values of Q^2. Any deviation from experiment can be either an indication that the range of Q^2 is not sufficiently high so as to justify the assumption that higher-order terms in $\bar{g}^2$ can be neglected or that these theories are wrong.)

If one incorrectly uses the asymptotic expression for $\bar{g}^2$ in the range of experimental values of Q^2 so as to estimate $\bar{g}^2/(2\pi)^2$, one obtains the values 0.11 and 0.08 for $Q^2 = 50$ GeV^2 and $Q^2 = 250$ GeV^2, respectively. On the other hand, since nothing can be said about the corresponding sizes of the coefficients in the expansion, it is impossible to estimate the error made by ignoring terms of order higher than the second.[20] Moreover, the increasing singular infrared behavior of the structure functions with the order of perturbation theory suggests that these coefficients will also increase, particularly for large values to n.

ACKNOWLEDGMENT

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APPENDIX A

This appendix contains the details of the evaluation of the diagrams that appeared in the discussions of Sec. II. We first list the relevant variables that will enter into the calculations.

We will be mainly concerned with the amplitude for the "process"

quark + current → quark + current,

in which:

q = momentum of incoming current = momentum of outgoing current,

p = momentum of incoming quark = momentum of outgoing quark,

$p' = p + q$, M = quark mass, $p \cdot q = \nu$, and $x = -q^2/2\nu = Q^2/2\nu = y + 1$.

We shall consider the above amplitude in the limit $Q^2 \gg M^2$ and $x > 1$; therefore, unless otherwise specified, throughout the calculations we will deliberately ignore terms of order M^2/Q^2.

The currents satisfy the standard SU(3) × SU(3) algebra,[21] but for simplicity we will omit their labeling. The proper SU(3) indices will be inserted at the end. We shall adopt the conventions of Bjorken and Drell[22] for Feynman rules, representation of γ matrices, etc. However, our states will be normalized as follows:

$$\langle ps|p's'\rangle=(2\pi)^3 2E\delta^3(\vec{p}-\vec{p}\,')\delta_{s,s'}. \qquad (A1)$$

First, let us calculate the expression corresponding to Fig. 5:

$$\left(\tfrac{1}{2}i\sum_s \bar{u}(p,s)\gamma_\mu\frac{i}{\not{p}'-M}[-i\Sigma(p')]\frac{i}{\not{p}'-M}\gamma^\mu u(p,s)\right)C_2(R), \qquad (A2)$$

where

$$\Sigma(p')=\frac{-ig^2}{(2\pi)^4}\int d^4k\,\frac{\gamma_\nu(\not{p}'-\not{k}+M)\gamma^\mu}{[(p'-k)^2-M^2](k^2-\lambda^2)}\,. \qquad (A3)$$

This last integral is divergent and therefore must be regularized; the resulting finite expression is

$$\bar{\Sigma}(p')=-\frac{g^2}{(4\pi)^2}\left(\ln\frac{\Lambda^2}{M^2}\right)+\frac{g^2}{(4\pi)^2}\left[\left(\ln\frac{Q^2}{M^2}\right)+\ln\left(1-\frac{1}{x}\right)\right]. \qquad (A4)$$

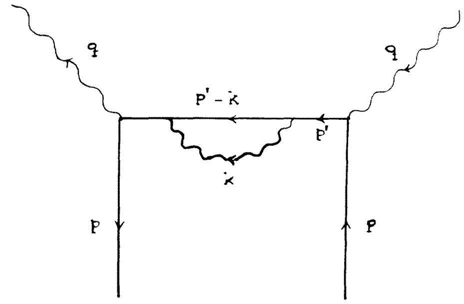

FIG. 5. Amplitude for the "process" quark + current → quark + current. Second-order diagram contributing to the vector-vector current amplitude $T^{a\bar{a}}{}_\mu{}^\mu$.

Then, performing a subtraction on-shell, so that the renormalization constant Z_2 is given by[23]

$$\frac{1}{Z_2}-1=\frac{g^2}{(4\pi)^2}\left[\left(\ln\frac{\Lambda^2}{M^2}\right)-2\left(\ln\frac{M^2}{\lambda^2}\right)+\frac{9}{2}\right], \qquad (A5)$$

the renormalized amplitude expanded in powers of $1/x$ becomes

$$\frac{-g^2}{2\pi^2}C_2(R)\tau^{P}_{a\bar{a}}\left\{\frac{1}{x}\left[\left(\ln\frac{\lambda}{M}\right)+\frac{3}{4}+\frac{1}{4}\ln\frac{Q^2}{M^2}\right]+\sum_{n=2}^{\infty}\frac{1}{x^n}\left[\left(\ln\frac{\lambda}{M}\right)+\frac{1}{4}\left(\ln\frac{Q^2}{M^2}\right)+\frac{3}{4}-\frac{1}{4}\sum_{1}^{n-1}\frac{1}{j}\right]\right\}, \qquad (A6)$$

where we have introduced the proper SU(3) factor of the currents

$$\tau^{P}_{a\bar{a}}=X^\dagger(P)(\lambda^a\lambda^{a\dagger})X(P), \qquad (A7)$$

in which $X(P)$ is the standard SU(3) spinor wave function corresponding to the (+) charged quark P.

Next we calculate the expression corresponding to the diagram in Fig. 6. It can be easily shown that, for $Q^2 \gg M^2$, both graphs contribute by equal amounts. Thus the resulting expression is

$$2C_2(R)\tfrac{1}{2}i\sum_s \bar{u}(p,s)\gamma_\mu\frac{i}{\not{p}'-M}\tilde{\Lambda}^\mu(p',p)u(p,s)=\frac{C_2(R)}{p'^2}\,\mathrm{Tr}[\not{p}\gamma_\mu\not{p}'\Lambda^\mu(p',p)]. \qquad (A8)$$

where

$$\Lambda^\mu(p',p)=ig^2\int\frac{d^4k}{(2\pi)^4}\,\frac{\gamma_\nu(\not{p}'-\not{k})\gamma_\mu(\not{p}-\not{k})\gamma^\nu}{(k^2-\lambda^2)[(p'-k)^2-M^2][(p-k)^2-M^2]}\,, \qquad (A9)$$

in which mass terms have been ignored wherever they do not give rise to leading terms. Introducing

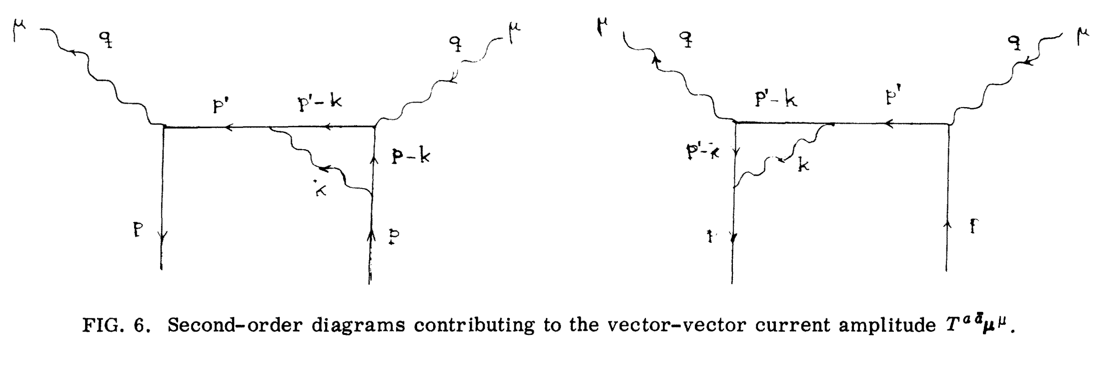

FIG. 6. Second-order diagrams contributing to the vector-vector current amplitude $T^{a\bar{a}}{}_\mu{}^\mu$.

Feynman parameters, the above expression becomes

$$\Lambda^{\mu}(p', p) = \frac{2ig^2}{(2\pi)^4} \int_0^1 d\alpha_1 \cdots d\alpha_3 \delta\left(1 - \sum_{i=1}^{3} \alpha_i\right) \int d^4k \frac{A(k)}{[(k-\bar{p})^2 - C(\alpha)]^3}, \tag{A10}$$

where

$$C(\alpha) = \alpha_3\lambda^2 + \alpha_2(M^2 - p'^2) + \bar{P}^2,$$
$$\bar{P} = \alpha_1 p + \alpha_2 p', \text{ and } A(k) = \gamma_\nu(\not{p}' - \not{k})\gamma_\mu(\not{p} - \not{k})\gamma^\nu. \tag{A11}$$

This expression gives rise to a divergent quantity; the resulting regularized expression is then

$$\bar{\Lambda}_\mu(p', p) = \frac{2g^2}{(4\pi)^2} \int d\alpha_1 \cdots d\alpha_3 \delta\left(1 - \sum_{i=1}^{3} \alpha_i\right)\left[i\pi^2\gamma_\mu \ln\frac{C(\Lambda^2)}{C(\lambda^2)} + \frac{i\pi^2(\not{p} - \bar{\not{p}})\gamma_\mu(\not{p}' - \not{p})}{C(\lambda)}\right]. \tag{A12}$$

Finally integrating, collecting all the terms, expanding in powers of $1/x$, and inserting the SU(3) factor of the currents, one obtains the following expression:

$$\frac{g^2}{2\pi^2} T^{p}_{a\bar{a}} C_2(R)\left\{ \frac{1}{x}\left[2\left(\ln\frac{\lambda}{M}\right) + \frac{1}{2}\left(\ln\frac{Q^2}{M^2}\right) + 3\right] + \sum_{n=2}^{\infty} \frac{1}{x^n}\left[2\left(\ln\frac{\lambda}{M}\right) + 2 - \sum_{j=1}^{n}\frac{1}{j} + \sum_{j=1}^{n}\frac{1}{j^2} + \sum_{s=1}^{n}\frac{1}{s}\sum_{j=1}^{s}\frac{1}{j} + \left(\frac{1}{2} + \sum_{j=2}^{n}\frac{1}{j}\right)\ln\frac{Q^2}{M^2}\right]\right\}. \tag{A13}$$

Finally we calculate the expression corresponding to Fig. 7:

$$\frac{i}{2}\frac{g^2}{(2\pi)^4} C_2(R) \int d^4k \frac{\mathrm{Tr}[(\not{p}+M)\gamma^\rho(\not{p}-\not{k}-M)\gamma_\mu(\not{p}'-\not{k}+M)\gamma^\mu(\not{p}-\not{k}+M)\gamma_\rho]}{[(p-k)^2 - M^2]^2[(p'-k)^2 - M^2](k^2 - \lambda^2)} \tag{A14}$$

Let

$$A(k) = \mathrm{Tr}[(\not{p}+M)\gamma^\rho(\not{p}-\not{k}+M)\gamma_\mu(\not{p}'-\not{k}+M)\gamma^\mu(\not{p}'-\not{k}+M)\gamma_\rho]. \tag{A15}$$

Then, by introducing Feynman parameters into the integral we obtain

$$\frac{i}{2}\frac{g^2}{2(\pi)^4} C_2(R) 6 \int_0^1 d\alpha_1 d\alpha_2 d\alpha_3 \delta\left(1 - \sum_{i=1}^{3}\alpha_i\right)\alpha_1 \int d^4k \frac{A(k)}{[(k-\bar{P})^2 - C(\lambda)]^4}, \tag{A16}$$

where

$$\bar{P} = \alpha_1 p + \alpha_2 p' \text{ and } C(\lambda) = \bar{P}^2 - \alpha_2 p'^2 + \alpha_3\lambda^2. \tag{A17}$$

In evaluating the trace $A(k)$ one must be careful when dropping mass terms, since some of them give rise to leading contributions. The resulting expression is

$$A(k) = 16\{2p\cdot k(p'\cdot k) - k^2(p\cdot p' + p\cdot k) - 32M^2[\nu - k\cdot(p'-p)]\}. \tag{A18}$$

Performing the integrations, expanding the result in powers of $1/x$, and inserting the SU(3) factor, one

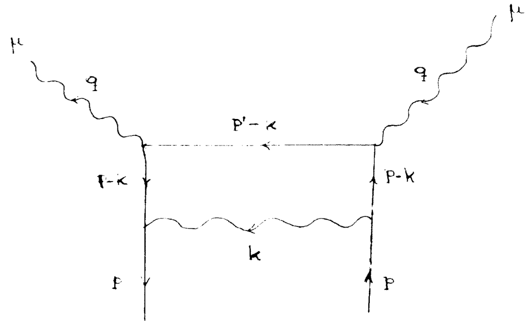

FIG. 7. Second-order diagrams contributing to the vector-vector current amplitude $T^{a\bar{a}}{}_{\mu}{}^{\mu}$.

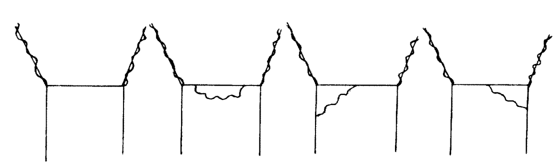

FIG. 8. Diagrams contributing to the axial-vector–axial-vector current amplitude $T^{\pm\mp}{}_{\mu}{}^{\mu}$.

finally obtains the following expression:

$$\frac{-g^2}{2\pi^2}C_2(R)\tau^P_{a\bar{a}}\left\{\frac{1}{x}\left[\frac{3}{2}+\left(\ln\frac{\lambda}{M}\right)+\tfrac{1}{4}\ln\frac{Q^2}{M^2}\right]+\sum_{n=2}^{\infty}\frac{1}{x^n}\left[\left(\ln\frac{\lambda}{M}\right)+\frac{\ln Q^2/M^2}{2n(n+1)}+\left(1+\frac{1}{2n(n+1)}\right)\sum_{j=1}^{n}\frac{1}{j}-\frac{n^2+2n-1}{2n^2(n+1)^2}\right]\right\}. \tag{A19}$$

Clearly the evaluation of the axial-vector–axial-vector current amplitude is identical to the previous case except that each current vertex will now contain an additional γ_5 matrix. However, for the graphs (see Fig. 8; the slashed line stands for axial-vector current) in which M can be neglected, the result is the same as before; by anticommuting the γ_5 through, one obviously obtains the same expression as for the previous vector-vector current graphs. For the diagram in Fig. 9, for which some M^2 terms must be kept, a little algebra again shows that the resulting expression is equal to that of the corresponding vector-currents graph.

Next we consider the vector–axial-vector current diagrams. These graphs will contribute to the pseudoscalar part of the weak current scattering amplitude.

Let us evaluate first the expression corresponding to Fig. 10:

$$\tfrac{1}{2}iC_2(R)\sum_s\bar{u}(p,s)\not{y}^*\frac{i}{\not{p}'-M}\left[-i\Sigma(p')\right]\frac{i}{\not{p}'-M}\not{y}\gamma_5 u(p,s), \tag{A20}$$

where we can write

$$\Sigma(p')=\not{p}'\,\Gamma(p'^2), \tag{A21}$$

whose finite part is

$$\tilde{\Gamma}(p'^2)=\frac{g^2}{(4\pi)^2}\left[3+2\left(\ln\frac{\lambda^2}{M^2}\right)+\left(\ln\frac{Q^2}{M^2}\right)+\ln(1-1/x)\right]. \tag{A22}$$

Expanding the renormalized amplitude in powers of $1/x$ and inserting the SU(3) factor, we obtain

$$\frac{-i\epsilon(y,y^*,p,q)}{2M^2}\left[\frac{-g^2C_2(R)}{2\pi^2}\,\tau^{p}_{a\bar{a}}\,\frac{M^2}{\nu}\right]\left\{\frac{1}{x}\left[\left(\frac{1}{4}\ln\frac{Q^2}{M^2}\right)+\left(\ln\frac{\lambda}{M}\right)+\frac{3}{4}\right]+\sum_{n=2}^{\infty}\frac{1}{x^n}\left[\frac{3}{4}+\left(\ln\frac{\lambda}{M}\right)+\frac{1}{4}\left(\ln\frac{Q^2}{M^2}\right)-\frac{1}{4}\sum_{j=1}^{n-1}\frac{1}{j}\right]\right\}. \tag{A23}$$

Next we evaluate the expression corresponding to Fig. 11. As in the previous case, Eq. (A8), both graphs contribute by equal amounts. Hence, the total expression is

$$2(i/2)C_2(R)\sum_s\bar{u}(p,s)\not{y}^*_\mu\Lambda^\mu(p,p')\frac{i}{\not{p}'-M}\not{y}\gamma_5u(p,s), \tag{A24}$$

where

$$\Lambda_\mu(p,p')=ig^2\int\frac{d^4k}{(2\pi)^4}\,\frac{\gamma_\rho(\not{p}-\not{k})\gamma_\mu(\not{p}'-\not{k})\gamma^\rho}{(k^2-\lambda^2)[(p'-k)^2-M^2][(p-k)^2-M^2]}\,. \tag{A25}$$

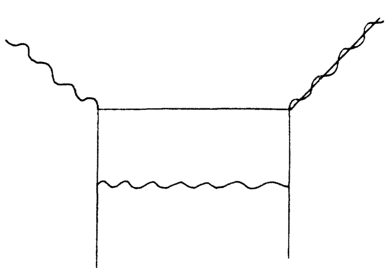

FIG. 9. Diagram contributing to the axial-vector–axial vector current amplitude $T^{\pm\mp}{}_{\mu}{}^{\mu}$ for which some mass terms must be kept.

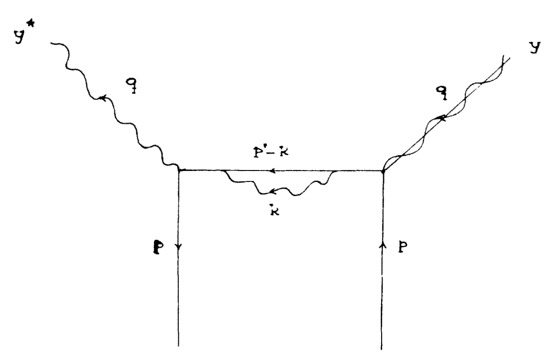

FIG. 10. Second-order diagram contributing to the vector–axial-vector current amplitude $y^*_\mu y_\nu T^{\pm\mp\mu\nu}$.

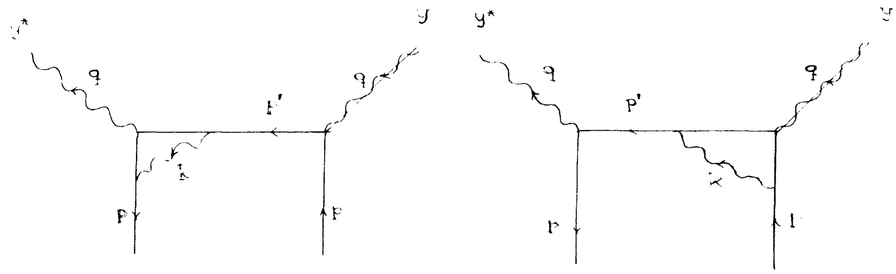

FIG. 11. Second-order diagram contributing to the vector–axial-vector current amplitude $y_\mu^* y_\nu T_{-+}^{(+-)\,\mu\nu}$.

A similar procedure as for the vector-vector currents case yields

$$\frac{-i\epsilon(y,y^*,p,q)}{2M^2}\left(\frac{g^2}{\pi^2}C_2(R)\tau_{aa}^{P}\left(\frac{M^2}{\nu}\right)\left\{\frac{1}{x}\left[\left(\ln\frac{\lambda}{M}\right)+\frac{1}{4}\left(\ln\frac{Q^2}{M^2}\right)+\frac{5}{4}\right]\right.\right.$$
$$+\sum_{n=2}^{\infty}\frac{1}{x^n}\left[\left(\ln\frac{\lambda}{M}\right)+\frac{1}{2}\left(\frac{1}{2}+\sum_{j=2}^{n}\frac{1}{j}\right)\left(\ln\frac{Q^2}{M^2}\right)+\frac{1}{4n}+2\right.$$
$$\left.\left.\left.+\frac{1}{2}\sum_{j=1}^{n}\frac{1}{j^2}-\frac{1}{2}\sum_{j=1}^{n}\frac{1}{j}-\frac{1}{2}\sum_{s=1}^{n}\frac{1}{s}\sum_{j=1}^{s}\frac{1}{j}\right]\right\}\right). \qquad (A26)$$

Finally, we compute the expression corresponding to Fig. 12:

$$\tfrac{1}{2}iC_2(R)\frac{g^2}{(2\pi)^4}\int d^4k\,\frac{\mathrm{Tr}(\not{p}+M)\gamma^\rho(\not{p}-\not{k}+M)\not{y}^*(\not{p}'-\not{k}+M)\not{y}\gamma_5(\not{p}-k+M)\gamma^\rho}{[(p-k)^2-M^2]^2[(p'-k)^2-M^2](k^2-\lambda^2)}. \qquad (A27)$$

Introducing Feynman parameters in Eq. (A27) we obtain

$$\frac{i}{2}\frac{g^2}{(2\pi)^4}C_2(R)6\int d\alpha_1 d\alpha_2 d\alpha_3\,\alpha_1\,\delta\left(1-\sum_1^3\alpha_i\right)\int d^4k\,\frac{A(k)}{[(k-P)^2-C(\alpha)]^4}, \qquad (A28)$$

where

$$C(\alpha)=p'^2\left(\alpha_2-\left(1+\frac{1}{y}\right)\right)\left(\alpha_2-\frac{M^2}{p'^2}\,\frac{\alpha_1^2+\lambda^2/M^2}{1+\alpha_1/y}\right)$$

and

$$A(k)=\mathrm{Tr}(\not{p}+M)\gamma^\rho(\not{p}-\not{k}+M)\not{y}^*(\not{p}'-\not{k}+M)\not{y}\gamma_5(\not{p}-\not{k}+M)\gamma^\rho.$$

Then evaluating the integrals, expanding the result in powers of $1/x$, and inserting the SU(3) factor, one finally obtains

$$\frac{-i\epsilon(y,y^*,p,q)}{2M^2}\left[\frac{g^2}{2\pi^2}C_2(R)\tau_{aa}^{P}\frac{M^2}{\nu}\right]\left\{\frac{1}{x}\left[-\frac{1}{4}-\left(\ln\frac{\lambda}{M}\right)-\frac{1}{4}\left(\ln\frac{Q^2}{M^2}\right)\right]\right.$$
$$\left.-\sum_{n=2}^{\infty}\frac{1}{x^n}\left[\left(\ln\frac{\lambda}{M}\right)+\frac{\ln Q^2/M^2}{2n(n+1)}+\left(1+\frac{1}{2n(n+1)}\right)\sum_{j=1}^{n}\frac{1}{j}-\frac{3n^2-1}{2n^2(n+1)^2}\right]\right\}. \qquad (A29)$$

Let us now compute the amplitude $p_\mu p_\nu T^{\mu\nu}(p,q)$. To second order in g, one can easily show that the only diagram which contributes to the leading term is that shown in Fig. 13, while all the others are down by a factor of M^2/Q^2. This is essentially due to the fact that when the current vertex γ_μ is contracted with p^μ and acts on the spinor wave function, it automatically gives a factor M.

The expression corresponding to this leading diagram is

$$\frac{i}{2}\frac{g^2}{(2\pi)^4}C_2(R)\int d^4k\,\frac{\mathrm{Tr}\not{p}\gamma_\nu\not{k}\not{p}(\not{p}'-\not{k})\not{k}\gamma^\nu}{[(p-k)^2]^2(p'-k)^2k^2}. \qquad (A30)$$

By introducing Feynman parametrization and performing the corresponding k and α integrations one obtains

$$-\frac{g^2}{(2\pi)^2}C_2(R)\frac{\nu}{y}\left\{\frac{y^2}{2}[\ln(1-1/x)]+\frac{y^2}{2x}\right\}. \qquad (A31)$$

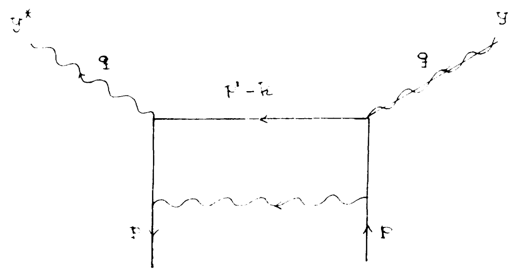

FIG. 12. Second-order diagram contributing to the vector–axial-vector current amplitude $y^*_\mu y_\nu T^{\pm\mp\mu\nu}$.

Then expanding in powers of $1/x$ and inserting the proper SU(3) factor we get

$$\nu \frac{g^2}{2\pi^2} C_2(R)\tau^P_{aa} \sum_{n=2}^{\infty} \frac{1}{x^n}\frac{1}{4n} . \qquad \text{(A32)}$$

Finally, let us consider the lowest-order diagrams. It is clear that for the amplitude $p^\mu p^\nu T_{\mu\nu}(\nu, q^2)$, the lowest-order diagram does not contribute to leading order, while those graphs corresponding to the amplitudes

T^μ_μ and $y^\mu y^{*\nu} T_{\mu\nu}$

yield nonvanishing contributions. For $T_\mu{}^\mu$ the corresponding diagram is given in Fig. 14, and it gives the following result:

$$\tau^P_{aa}\tfrac{1}{2} i\sum_s \bar{u}(p,s)\gamma_\mu \frac{1}{\not{p}' - M}\gamma^\mu u(p,s) = \sum_{n=0}^{\infty} \frac{2\tau^P_{aa}}{x^n}. \qquad \text{(A33)}$$

The lowest-order diagram contributing to $y^\mu y^\nu T_{\mu\nu}(p, q)$ is given in Fig. 15, and the resulting expression is

$$\frac{-i\epsilon(y, y^*, p, q)}{2M^2}\frac{M^2}{\nu}\tau^P_{aa}\sum_{n=0}\frac{-2}{x^n} . \qquad \text{(A34)}$$

Next we proceed to calculate the Wilson coefficients associated with the singlet gluon operators

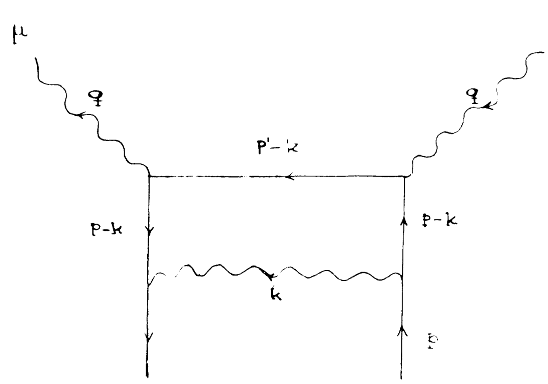

FIG. 13. Lowest-order diagram contributing to the amplitude $p_\mu p_\nu T^{\mu\nu}$.

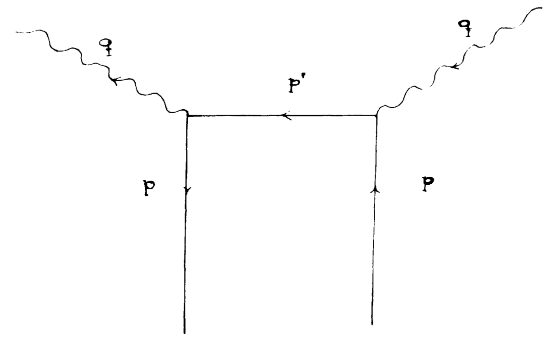

FIG. 14. Lowest-order diagram contributing to the vector-vector current amplitude $T_\mu{}^\mu$.

[see Eq. (1.10a)]. As mentioned in Sec. II, this calculation involves the absorptive part of the amplitude for the following process:

current + gluon → current + gluon,

which can be diagrammatically represented by Fig. 16.

In order to obtain this absorptive part, we first relate the above amplitude (call it M_{fi}) to the corresponding S-matrix element:

$$S_{fi} = \delta_{fi} - i(2\pi)^4\delta^4(p_f - p_i)\frac{1}{2k_0}\frac{1}{(2\pi)^3}M_{fi}; \qquad \text{(A35)}$$

f and i are the initial and final gluon-current "states." Then by invoking unitarity of S, we derive

$$\mathrm{Im}M_{ii} = -\tfrac{1}{2}\sum_n (2\pi)^4\delta^4(P_n - k - q)\left(\frac{1}{\sqrt{N_n}}\right)^2 M^*_{ni}M_{ni}, \qquad \text{(A36)}$$

where $1/\sqrt{N_n}$ is the normalization factor corresponding to the intermediate states n. To second

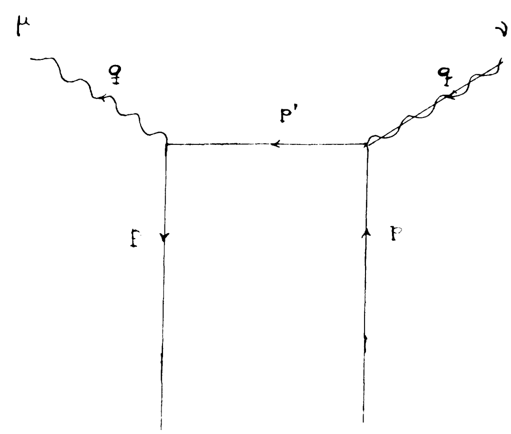

FIG. 15. Lowest-order diagram contributing to the vector–axial-vector current amplitude $y^*_\mu y_\nu T^{\mu\nu}$.

order in g, the only possible intermediate state which can contribute to (A36) is clearly a quark-antiquark pair. Let us denote by p and p' their respective momenta and work in the center-of-mass frame. Then we have that

$$\mathrm{Im}M_{ii} = -\tfrac{1}{2}(2\pi)^4 \frac{C_2(R)}{(2\pi)^3} \times \int\int \frac{d^3p}{2E}\frac{d^3p'}{2E'}\,\delta^4(p+p'-k-q)\,|M_{ni}|^2, \tag{A37}$$

with

$$M_{ni} = i\bar{v}(p', s')\left[\gamma_\mu \frac{i}{\not p - \not k - M}(-ig\gamma^\rho) + (-ig\gamma^\rho)\frac{i}{\not p' - \not k - M}\gamma_\mu\right]u(p, s). \tag{A38}$$

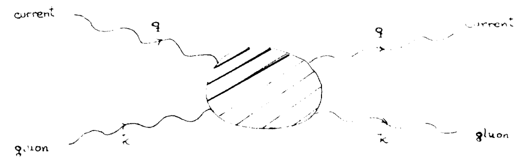

FIG. 16. Amplitude for the "process" gluon + current → gluon + current.

Introducing spherical coordinates and orienting the 2 axis along $\vec{k}$, so that $\vec{p}\cdot\vec{k} = |\vec{p}\,||\vec{k}|\cos\theta$, Eq. (A37) becomes

$$-\frac{C_2(R)}{16\pi}\frac{|\vec{p}|}{E}\int_{-1}^{+1} d(\cos\theta)\,|M_{ni}|^2. \tag{A39}$$

Thus, for the contracted currents amplitude we obtain the following result:

$$|M_{ni}|^2 = \frac{g^2}{4}\,\mathrm{tr}\,\not p\left(\frac{\gamma^\rho(\not p - \not k)\gamma^\nu}{p\cdot k} + \frac{\gamma^\nu(\not p' - \not k)\gamma^\rho}{-p\cdot k}\right)\not p'\left(\frac{\gamma_\nu(\not p - \not k)\gamma_\rho}{p\cdot k} + \frac{\gamma_\rho(\not p' - \not k)\gamma_\nu}{-p'\cdot k}\right)$$
$$= g^2\left\{8\left[\frac{1+(|\vec{p}|/E)\cos\theta}{1-(|\vec{p}|/E)\cos\theta} + \frac{1-(|\vec{p}|/E)\cos\theta}{1+(|\vec{p}|/E)\cos\theta}\right] + 1.6p\cdot p'\left(\frac{p\cdot p'}{(p\cdot k)(p\cdot k)} - \frac{1}{p\cdot k} - \frac{1}{p'\cdot k}\right)\right\}. \tag{A40}$$

Substituting it into the integral (A39) and noting that

$$p\cdot p' = s/2,\quad k_0 = k\cdot q/\sqrt{s},\quad E = \sqrt{s}/2, \tag{A41}$$

where $s = (k+q)^2 = 2k\cdot q(1-\omega)$ with $\omega = -q^2/2k\cdot q = Q^2/2k\cdot q$,

we derive

$$\mathrm{Im}M_{ii} = \frac{-2g^2C_2(R)}{\pi}\left\{[2\omega(1-\omega)-1]\left[\ln\left(\frac{1-\omega}{\omega}\right)\right] + 1\right\}. \tag{A42}$$

Let us now evaluate the moments

$$\frac{-2g^2}{\pi}C_2(R)\int_0^1 d\omega\left\{[2\omega - 2\omega^2 - 1]\left[\left(\ln\frac{Q^2}{M^2}\right) + \ln(1-\omega)\right] - (\ln\omega) + 1\right\}\omega^n$$
$$= \frac{2g^2C_2(R)}{\pi}\left[\frac{n^2+3n+4}{(n+1)(n+2)(n+3)}\left(\ln\frac{Q^2}{M^2}\right) - \frac{n^2+3n+4}{(n+1)(n+2)(n+3)}\sum_{j=1}^{n+1}\frac{1}{j} - \frac{n}{(n+1)^2} - \frac{2}{(n+2)(n+3)}\right\}. \tag{A43}$$

Finally, adding the contribution from the crossed graph and inserting the SU(3) factors we obtain

$$\frac{4g^2}{\pi}C_2(R)\left[\frac{n^2+3n+4}{(n+1)(n+2)(n+3)}\left(\ln\frac{Q^2}{M^2}\right) - \frac{n^2+3n+4}{(n+1)(n+2)(n+3)}\sum_{j=1}^{n+1}\frac{i}{j} - \frac{n}{(n+1)^2} - \frac{2}{(n+2)(n+3)}\right] \tag{A44}$$

for n even, and zero for n odd.

In a similar fashion we can compute the absorptive part of the longitudinally projected amplitude

$$k_\mu k_\nu T^{\mu\nu}(k, q)\,.$$

The corresponding expression is

$$\frac{-4g^2C_2(R)}{\pi}\frac{kq}{(n+1)(n+2)}$$

for n even, and zero for n odd.

As a final remark we observe that in Eq. (A44) the coefficient of $\ln(Q^2/M^2)$ is precisely the mixing term of the operator.

APPENDIX B

Throughout the calculations, we have been dealing with two masses, namely, the quark mass and the subtractions point; the latter is being determined only by the onset of scaling. It is true that in the large-Q^2 region one should be able to set the quark mass equal to zero and only retain the subtraction mass μ. However, in our calculations there have been good reasons for not doing so; in fact, if one does all the calculations of Sec. II with $M_{\rm quark}=0$ and performs all subtractions off-shell, then one will find that *only* the graph in Fig. 17 is infrared divergent. Therefore, the amplutude itself will also be divergent. Eventually, this infrared term will cancel against the similar term that then appears in the matrix element of the operators $\hat{O}^n$, thereby giving a finite expression for the Wilson coefficients. But to see this happen, one would have to renormalize the operators $\hat{O}^n$ off shell, which is considerably more difficult. It is therefore convenient to keep the quark mass nonzero and subtract on shell; then, by performing an intermediate renormalization at some point $\mu \gg M_{\rm quark}$, we can see that $T_{\mu\nu}$ is unaltered (at least to this order in g), except that now it be-

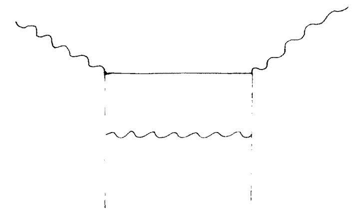

FIG. 17. Infrared-divergent diagram.

comes a function of μ, the new subtraction point, and of $M_{\rm quark}$. However, the Wilson coefficients that only depend on the subtraction point μ are then multiplied by some finite renormalization constants whose value is irrelevent for the applications of Sec. III. Thus we have

$$\tilde{C}^n\left(\frac{Q^2}{M^2}, g\right) = \frac{1}{z_n^{(i)}}\, \tilde{C}^{n\prime}\left(\frac{Q^2}{M^2}, g\right),$$

where $z_n^{(i)}$ is the intermediate renormalization constant of the operator $\hat{O}^n_{(i)}$.

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[1]D. Gross and F. Wilczek, Phys. Rev. D 8, 3633 (1973); 9, 980 (1974); Phys. Rev. Lett. 30, 1343 (1973).

[2]C. Callan, in *Developments in High-Energy Physics*, proceedings of the International School of Physics "Enrico Fermi," Course LIV, edited by R. R. Gatto (Academic, New York, 1972).

[3]L. D. Faddeev and V. N. Popov, Fermilab Report No. NAL-THY-57 (unpublished).

[4]B. Lee and J. Zinn-Justin, Phys. Rev. D 5, 3121 (1972); 5, 3137 (1972); 5, 3155 (1972).

[5]W. Zimmermann, in *Lectures on Elementary Particles and Quantum Field Theory*, proceedings of the 1970 Brandeis University Summer Institute, edited by S. Deser *et al.* (M.I.T., Cambridge, Mass., 1971), Vol. 1.

[6]H. D. Politzer, Phys. Rev. Lett. 26, 1346 (1973); H. Georgi and H. D. Politzer, Phys. Rev. D 9, 416 (1974).

[7]For a complete discussion see article by D. Gross in S. Treiman, R. Jackiw, and D. Gross, *Lectures on Current Algebra and Applications*, (Princeton Univ. Press, Princeton, New Jersey, 1972).

[8]The other method is useful in checking the numerical results obtained by this procedure.

[9]That this is a plausible assumption can be seen by noting that the structure of this amplitude is similar to that of the pseudoscalar pion-nucleon theory for which the assumption is consistent with Regge theory.

[10]W. Casell, Princeton Univ. thesis, 1974 (unpublished).

[11]D. Gross, Phys. Rev. Lett. 32, 1145 (1974).

[12]The fact that for any given values of t and t' there always exists some N_0 such that if $n > N_0$ the condition $g^{-2}\ln^2(n+1) \ll 1$ is violated implies that the approximation we have made in Eq (3.10) breaks down for $n > N_0$. This in turn makes the results that follow Eq. (3.10) questionable since, as we will see later, they depend critically on the large-n behavior of Eq. (3.9).

[13]Had we set $H(t,t')=0$ in Eq. (3.11) we would have obtained the results of Gross (Ref. 11); however, here we shall retain both terms in order to see the effect of the next-leading corrections.

[14]The smallness of $F_{\rm long}(x, Q^2)$ for all accessible values of Q^2 is an indication that this term should be small.

[15]It is very likely that the term proportional to γ_1^{n+2} also grows as $\ln^2 n$, but, owing to phase-space factors, its numerical coefficient is small compared to unity.

[16]D. Gross and S. Treiman, Phys. Rev. Lett. 32, 1145 (1974).

[17]C. H. Llewellyn Smith, Nucl. Phys. B17, 277 (1970).

[18]This is an assumption which will depend on how the entries of the matrix $(M)_{ij} = M_{ij}^{n+2}$, as well as the constants M_F^{n+2} and M_V^{n+2}, behave for large n. The criterion is the following: If

$$\frac{M_F^{n+2}\, M_{11}^{n+2} + M_V^{n+2}\, M_{21}^{n+2}}{M_F^{n+2}\, M_{12}^{n+2} + M_V^{n+2}\, M_{22}^{n+2}} \xrightarrow[\text{large n}]{} F(n) < \text{const} \times n$$

Eq. (3.49) holds. Otherwise nothing can be said for the singlet piece of F_2.

[19]One would be tempted to apply it so as to derive relations between $F_2^s(\omega,t)$ and $F_2^s(\omega,t')$ for ω near threshold, since in this region it is the large n behavior of (3.70) that dominates. However, higher-order terms will also become important for such values of ω.

[20]The fact that for $n \ge 10$ the size of the coefficients is of the order of $[g^{-2}/(2\pi)^2]^{-1}$ indicates that the second-order correction terms are of the order of the leading term and consequently the expansion ceases to be

valid.
[21]S. Adler and R. Dashen, *Current Algebra and Applications to Particle Physics* (Benjamin, New York, 1967), section on conventions and notation.
[22]J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill, New York, 1964).
[23]R. Feynman, *Quantum Electrodynamics* (Benjamin, New York, 1962), p. 144.