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Angular momentum constraints on dimuon energy asymmetries*

T. L. Trueman
Brookhaven National Laboratory, Upton, New York 11973
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It is shown that if a dimuon system is produced with bounded angular momentum $j \le j_{\max}$, then $\xi = |\langle E_- - E_+\rangle/\langle E_- + E_+\rangle| \le \xi_0$, where ξ_0 is the maximum zero of $P_{j_{\max}+1}(\xi)$. No other assumptions regarding the source of the dimuons are made, and this bound can be reached. These bounds are considerably weaker than those of Pais and Treiman but are strong enough that they may be useful in future analyses. The results are applicable to a general two-body system and parity nonconservation is not required to reach the bound.

I. INTRODUCTION

Pais and Treiman have shown[1] that the energy asymmetry between positive and negative μ's from the decay of a neutral heavy lepton L of spin $\frac{1}{2}$,

$$L \to \mu^+ + \mu^- + \nu_L ,$$

is bounded by

$$\frac{9-4\sqrt{2}}{7} \le \frac{\langle E_-\rangle}{\langle E_+\rangle} \le \frac{9+4\sqrt{2}}{7} .$$

To derive this result they have assumed the most general four-fermion interaction, without derivative couplings or form factors. This result has important implications for the dimuon events produced in the Fermilab ν experiments.[2]

It is interesting to generalize this result. The analogous calculation, under the same general assumptions, has already been done for a spin-$\frac{3}{2}$ L (but spin-$\frac{1}{2}$ ν_L).[3] For reference the result is

$$\frac{2}{5} \le \frac{\langle E_-\rangle}{\langle E_+\rangle} \le \frac{5}{2} ,$$

which is a little less restrictive than the Pais-Treiman result.

The objective of this paper is to determine how strong a bound is obtained owing to angular momentum constraints alone. Suppose an L with spin J decays into a spin-$\frac{1}{2}$ ν_L and two muons. Suppose further that the decay mechanism limits the spin state j of the dimuon system, so that $j \le j_{\max}$, i.e., L decays into a coherent combination of dimuon spins $j \le j_{\max}$. (In the above-mentioned calculations, $j \le 1$.) We will make no assumptions about the dependence of the matrix element on the dimuon mass. Since the four-fermion couplings limit the rate of variation of the matrix element with the dimuon mass, the bound we obtain will necessarily be weaker. For $j \le 1$ it is weakened to

$$0.27 = \frac{\sqrt{3}-1}{\sqrt{3}+1} \le \frac{\langle E_-\rangle}{\langle E_+\rangle} \le \frac{\sqrt{3}+1}{\sqrt{3}-1} = 3.7 \quad (j \le 1) .$$

The bound becomes rapidly weaker with j. For $j \le 2$ it is

$$0.13 = \frac{\sqrt{5}-\sqrt{3}}{\sqrt{5}+\sqrt{3}} \le \frac{\langle E_-\rangle}{\langle E_+\rangle} \le \frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}} = 7.9 \quad (j \le 2) .$$

The experimental value[2] for all events in the ν experiment is

$$\frac{\langle E_-\rangle}{\langle E_+\rangle} = 3.7 \pm 0.65 ;$$

the value obtained by removing events with $E_+ > E_-$, which possibly come from $\bar{\nu}$ contamination, is

$$\frac{\langle E_-\rangle}{\langle E_+\rangle} = 6.1 \pm 0.8 .$$

Evidently, because of the errors and uncertainty about $\bar{\nu}$ contamination, we cannot draw any conclusions from these data for the general case.

Our results are the following: (a) *The extremum is independent* of J and M, spin and mass of L. This is a disappointing result, because it limits the information obtainable from this analysis. (b) If we define

$$\xi = \frac{E_- - E_+}{E_- + E_+} , \tag{1}$$

then

$$|\xi| \le \xi_0 ,$$

where ξ_0 is the largest zero of the Legendre polynomials,

$$P_{j_{\max}+1}(\xi_0) = 0 ; \tag{2}$$

$$\frac{1-\xi_0}{1+\xi_0} \le \frac{\langle E_-\rangle}{\langle E_+\rangle} \le \frac{1+\xi_0}{1-\xi_0} . \tag{3}$$

The bounds just cited are determined by the first zeros of $P_2(\xi)$ and $P_3(\xi)$, respectively.

Because of result (a), it will be seen that result (b) is generally applicable to lepton pairs produced with $j \le j_{\max}$ in any way whatever, and by simple extension the same results apply for any pairs of particles. These results should be useful

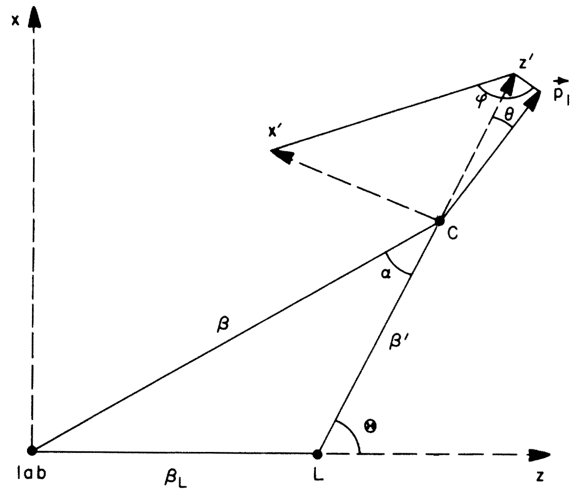

FIG. 1. The velocity diagram for the decay $L \to \nu_L + \mu^+ + \mu^-$.

in analyzing experiments where only energies are accurately measured. Of course, if momenta are accurately measured as well, the much more detailed conventional angular momentum analysis can be carried out.

II. KINEMATICS AND DEPENDENCE ON J

Suppose that L is moving with velocity β_L in the z direction in the laboratory frame and the c.m. of $\mu^+\mu^-$ (C) is moving with velocity $\vec{\beta}$ in the x-z plane with positive x component. (This defines the orientation of our coordinate system.) Let $\vec{\beta}'$ denote the velocity of C in the rest frame of L and take the z' direction in C so that L moves in the $-z'$ direction, and x' lies in the x-z plane. The velocity diagram and various angles defining a definite configuration is shown in Fig. 1.

In C, μ^- has energy E and momentum $\vec{p}_1 = \vec{p}$, and μ^+ has energy E and momentum $\vec{p}_2 = -\vec{p}$. The angles θ, φ specify the direction of $\vec{p}$ in C with respect to x', y', z'. In the lab frame the energies are given by

$$\begin{aligned} E_1^{\text{lab}} &= \gamma(E + \vec{\beta}\cdot\vec{p}) , \\ E_2^{\text{lab}} &= \gamma(E - \vec{\beta}\cdot\vec{p}) . \end{aligned} \tag{2.1}$$

By straightforward kinematics, this gives

$$\begin{aligned} E_1^{\text{lab}} = {} & \gamma'\gamma_L E(1 + \beta_L\beta' \cos\Theta) \\ & + p(\gamma'\beta'\gamma_L \cos\theta + \gamma'\beta_L\gamma_L \cos\theta \cos\Theta \\ & - \gamma_L\beta_L \sin\theta \cos\varphi \sin\Theta) , \end{aligned} \tag{2.2a}$$

$$\begin{aligned} E_2^{\text{lab}} = {} & \gamma'\gamma_L E(1 + \beta_L\beta' \cos\Theta) \\ & - p(\gamma'\beta' \cos\theta + \gamma'\beta_L\gamma_L \cos\theta \cos\Theta \\ & - \gamma_L\beta_L \sin\theta \cos\varphi \sin\Theta) . \end{aligned} \tag{2.2b}$$

The variables used here are useful because they are all expressed in terms of the decay angles of the L or the dimuon, the invariant mass ($2E$) of the dimuon, and the velocity of L. These expressions may then be readily used to calculate the average values $\langle E_1^{\text{lab}}\rangle$, $\langle E_2^{\text{lab}}\rangle$ in terms of β_L, the density matrix of L, and the decay amplitude.

Consider the case where L is produced unpolarized. Then all the terms in (2.2) proportional to β_L vanish on averaging and so

$$\xi \equiv \frac{\langle E_1^{\text{lab}} - E_2^{\text{lab}}\rangle}{\langle E_1^{\text{lab}} + E_2^{\text{lab}}\rangle} = \frac{\langle p\beta'\gamma' \cos\theta\rangle}{\langle E\gamma'\rangle} , \tag{2.3}$$

independent of the velocity of L. (The decay matrix is, of course, assumed to be independent of β_L.) We will now see that, in the absence of further dynamical assumptions, this case will give the maximum allowable asymmetry. This is in contrast to the Pais-Treiman situation where the maximum asymmetry is attained for a completely polarized L moving with $\beta_L = 1$.

Let $\rho^{jj'}_{mm'}(E, \lambda_1, \lambda_2)$ denote the density matrix for the dimuon system. λ_1, λ_2 denote the μ^-, μ^+ helicities and will be suppressed frequently. Without any assumptions about the E dependence of the decay, such as centrifugal barrier effects, the only remnant of the original J in ρ is the condition $|m|, |m'| \le J \pm \frac{1}{2}$ depending on the handedness of ν_L. This condition will turn out to be of no consequence. That is, angular momentum considerations have no bearing on the j, j' dependence of ρ. Now (with $\mu = \lambda_1 - \lambda_2$),

$$\xi = \frac{\displaystyle\sum_{\substack{j,j',m\\ \lambda_1,\lambda_2}} \int dE\, d\Omega\, \rho^{jj'}_{mm'}(E) \cos\theta\, \beta'\gamma' p D^j_{m\mu}(\varphi,\theta,-\varphi) D^{j'*}_{m'\mu}(\varphi,\theta,-\varphi)}{\displaystyle\sum_{\substack{j,j',m\\ \lambda_1,\lambda_2}} \int dE\, d\Omega\, \rho^{jj'}_{mm'}(E) \gamma' E D^j_{m\mu}(\varphi,\theta,-\varphi) D^{j'}_{m'\mu}(\varphi,\theta,-\varphi)} , \tag{2.4}$$

If we neglect the mass of the ν_L, then

$$\gamma' = \frac{M^2 + 4E^2}{4EM} ,$$

$$\beta'\gamma' = \frac{M^2 - 4E^2}{4EM} ,$$

where M is the mass of L. If $\rho(E)$ peaks in the region where

$$\mu \ll E \ll M \,, \tag{2.5}$$

for which there is presumably ample room, then the energy factors in the numerator and denominator are the same. Outside this region, the numerator factors are always less than the demoninator.

Define

$$\rho^{jj'}_{mm'} = \int dE\, \rho^{jj'}_{mm'}(E)\,; \tag{2.6}$$

then

$$\xi \le \frac{\sum \rho^{jj'}_{mm'} \int d\Omega \cos\theta\, D^{j}_{m\mu}(\varphi,\theta,-\varphi) D^{j'*}_{m'\mu}(\varphi,\theta,-\varphi)}{\sum \rho^{jj'}_{mm'} \int d\Omega\, D^{j}_{m\mu}(\varphi,\theta,-\varphi) D^{j'*}_{m'\mu}(\varphi,\theta,-\varphi)}\,, \tag{2.7}$$

with approximate equality attained if $\rho(E)$ peaks sharply as in (2.5). But this is precisely the equation one obtains for the asymmetry, assuming only a distribution of j's for the dimuon but with no assumptions at all about the source. This is result (a) cited in Sec. I.

It may seem *a priori* obvious that the asymmetry will be independent of J. We have chosen to present the result in this way to emphasize the fact that it is strictly related to maximizing over all possible energy dependences of the amplitude. One can envision cases in which ρ is peaked for $E \approx M/2$. In such cases, β' is small and, from Eq. (2.2), we see that the asymmetry will be proportional to a combination of $\langle\cos\Theta\rangle$ and $\langle\sin\Theta\rangle$ and hence be proportional to the polarization vector of L. It will be maximum for $|\beta_L| \to 1$, as in the Pais-Treiman case, but obviously will be smaller than the bound we obtain in the next section.

III. THE MAXIMIZATION PROBLEM

The first step is to evaluate the angular integrals in (2.7), the result of which is

$$\xi \le \frac{\displaystyle\sum_{\substack{j=1\\ m,\lambda_1,\lambda_2}}^{j_{\max}} \frac{4m\mu}{(2j+2)(2j+1)(2j)}\rho^{jj}_{mm}(\lambda_1,\lambda_2) + \sum_{\substack{j=0\\ m,\lambda_1,\lambda_2}}^{j_{\max}-1} \frac{4[(j+m+1)(j-m+1)(j+\mu+1)(j-\mu+1)]^{1/2}}{(2j+3)(2j+2)(2j+1)} \mathrm{Re}\rho^{jj+1}_{mm}(\lambda_1,\lambda_2)}{\displaystyle\sum_{\substack{j=0\\ m,\lambda_1,\lambda_2}}^{j_{\max}} \frac{\rho^{jj}_{mm}(\lambda_1,\lambda_2)}{(2j+1)}}\,. \tag{3.1}$$

Positivity requires that

$$(\mathrm{Re}\rho^{jj+1}_{mm})^2 \le \rho^{jj}_{mm}\,\rho^{j+1,j+1}_{mm}\,. \tag{3.2}$$

Since we can vary ρ^{jj+1}_{mm} independently of ρ^{jj}_{mm}, the maximum will clearly be attained if the equality in (3.2) holds. Assume that it does.

The equation simplifies in appearance if we define

$$\rho^{jj}_{mm} = (2j+1)^2 (a^j_m)^2\,. \tag{3.3}$$

Then

$$\mathrm{Re}\rho^{jj+1}_{mm} = (2j+1)(2j+3) a^j_m a^{j+1}_m \tag{3.4}$$

and Eq. (3.1) becomes

$$\xi \le \left[\sum \frac{4m\mu(2j+1)}{(2j+2)(2j)}(a^j_m)^2 + \sum \frac{2[(j+1)^2-\mu^2]^{1/2}[(j+1)^2-m^2]^{1/2}}{(j+1)} a^j_m a^{j+1}_m\right]\left[\sum (a^j_m)^2(2j+1)\right]^{-1}. \tag{3.1$'$}$$

We determine the maximum possible value ξ_0 for the right-hand side of this inequality by differentiating with respect to each a^j_m and setting the result equal to zero. We can clearly choose $m \ge 0$, $\mu \ge 0$ without loss of generality. The result is the set of equations

$$\frac{4m\mu(2j+1)}{(2j+2)(2j)} a^j_m + \frac{[(j+1)^2-\mu^2]^{1/2}[(j+1)^2-m^2]^{1/2}}{j+1} a^{j+1}_m + \frac{(j^2-\mu^2)^{1/2}(j^2-m^2)^{1/2}}{j} a^{j-1}_m = \xi_0(2j+1) a^j_m\,, \tag{3.5}$$

with

$$a^{j_{\max}+1}_m = 0\,.$$

The recursion relation for Jacobi polynomials is given by[4]

$$m\mu(2j+1)P^{(m-\mu,\,m+\mu)}_{j-m}(\xi_0) + j(j-m+1)(j+m+1)P^{m-\mu,\,m+\mu}_{j-m+1}(\xi_0) + (j+1)(j-\mu)(j+\mu)P^{(m-\mu,\,m+\mu)}_{j-m-1}(\xi_0) = j(j+1)(2j+1)\xi_0 P^{(m-\mu,\,m+\mu)}_{j-m}(\xi_0)\,. \tag{3.6}$$

Hence, the set of equations (3.5) is solved by

$$a^j_m(\lambda_1,\lambda_2)=\left[\frac{(j+m)!\,(j-m)!}{(j+\mu)!\,(j-\mu)!}\right]^{1/2}P^{(m-\mu,\,m+\mu)}_{j-m}(\xi_0)\,, \quad (3.7)$$

provided

$$a^{j_{\max}+1}_m(\lambda_1,\lambda_2)=0\,,$$

i.e., (3.8)

$$P^{(m-\mu,\,m+\mu)}_{j_{\max}-m+1}(\xi_0)=0\,.$$

We must now determine which value of m and μ gives the maximum value of ξ_0. It is possible to prove that the maximum value occurs for $m=\mu=0$. This seems very reasonable from Eq. (3.1′), but we will now indicate a proof.

The proof relies on certain theorems given by Szegö[5] and a recurrence relation. Denote the various first zeros of Eq. (3.8) by $\xi_0(m-\mu, m+\mu, j_{\max}-m+1)$. By simple application of the quoted theorems (Szegö's numbers), we have for $m\geq 1$

$$\xi_0(m-1,m+1,j_{\max}-m+1)>\xi_0(m,m,j_{\max}-m+1)$$
(Theorem 6.21.1),

$$\xi_0(m-1,m+1,j_{\max}-m+1)\leq\xi_0(0,2,j_{\max}-m+1)$$
(Theorem 6.12.1),

$$\xi_0(0,2,j_{\max}-m+1)<\xi_0(0,2,j_{\max}+1)$$
(Theorem 3.3.2).

Thus, the maximum zero is either $\xi(0,2,j_{\max})$ or $\xi(0,0,j_{\max}+1)$. From the Rodrigues formula[4] for $P^{(\alpha,\beta)}_n$ we derive the recursion relation

$$(1+x)^2P^{(0,2)}_{n-1}(x)-(1-x)^2P^{(2,0)}_{n-1}(x)=4P^{(0,0)}_n(x)\,.$$

When

$$x=\xi_0(0,2,n-1)\,,$$

$$P^{(2,0)}_{n-1}>0 \quad \text{(Theorem 6.21.1)}\,.$$

Hence $P^{(0,0)}_n<0$, but since $P^{(2,0)}_n(1)=1$ we have

$$\xi(0,0,j_{\max}+1)>\xi(0,2,j_{\max})\,. \quad \text{Q.E.D.}$$

Thus, we reach result (b) cited in Sec. I.

Note that because the maximum is attained for $m=\mu=0$, the bound $J\geq m$ is of no consequence, as stated; furthermore, the bound can be attained in a *parity-conserving* production and decay. Hence, these results may also be useful for pairs of hadrons. Evidently, similar results could be obtained for higher energy correlations, i.e., higher moments of $\cos\theta$.

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