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Can one dent a dyon?*

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We show that the exact monopole and dyon solutions found by Prasad and Sommerfield are stable by proving that they are absolute minima of the energy.

Two years ago, 't Hooft and Polyakov showed that certain classical gauge field theories admitted monopoles, nonsingular time-independent solutions of finite energy that carried magnetic charge.[1] Shortly afterwards, Julia and Zee showed that the same theories also admitted dyons, solutions that carried both electric and magnetic charge.[2]

Very little is known about the stability of these solutions under small perturbations. Of course, because of the conservation of magnetic charge, the solutions cannot radiate away all their energy and dissipate utterly, no matter how they are perturbed. However, all solutions obtained so far have been found under the assumption of spherical symmetry. This method of approach gives no information about the effects of perturbations that are not spherically symmetric; thus it remains a possibility that a monopole (or dyon), if tapped on its side, will radiate away some of its energy and settle down into an asymmetric configuration, a dyon with a dent.

This note reports some modest progress on the stability problem. We have been able to prove stability for the special limiting case of the 't Hooft-Polyakov theory in which Prasad and Sommerfield obtained exact analytic solutions for monopoles and dyons.[3] We emphasize that we define "stability" to include the possibility of neutral equilibrium. We had better define it this way, for it is trivial to construct a perturbation (a small Lorentz transformation) that sets the monopole as a whole in uniform motion with steady velocity. Also, of course, we prove stability only in an appropriately chosen gauge; without a gauge condition, we can trivially construct a perturbation that grows exponentially in time simply by applying an exponentially growing gauge transformation.

The theory considered by 't Hooft and Polyakov has for its dynamical variables an isotriplet of scalar fields $\vec{\phi}$ and an isotriplet of gauge fields $\vec{A}_\mu$ with dynamics determined by the Lagrangian density

$$\mathcal{L} = -\tfrac{1}{4}\vec{F}_{\mu\nu}\cdot\vec{F}^{\mu\nu} + \tfrac{1}{2}D_\mu\vec{\phi}\cdot D^\mu\vec{\phi} - \tfrac{1}{2}\lambda(\vec{\phi}^2 - 1)^2 .$$

Here

$$\vec{F}_{\mu\nu} = \partial_\mu\vec{A}_\nu - \partial_\nu\vec{A}_\mu + \vec{A}_\mu\times\vec{A}_\nu,$$

$$D_\mu\vec{\phi} = \partial_\mu\vec{\phi} + \vec{A}_\mu\times\vec{\phi} ,$$

λ is a positive number, and we have chosen our units of mass and length such that both the gauge coupling constant and the ground-state value of $\vec{\phi}^2$ are 1.

Three conserved quantities will be important to us. One is the energy

$$E = \frac{1}{2}\int d^3x[\vec{E}_i\cdot\vec{E}_i + \vec{B}_i\cdot\vec{B}_i + D_0\vec{\phi}\cdot D_0\vec{\phi} + D_i\vec{\phi}\cdot D_i\vec{\phi} + \lambda(\vec{\phi}^2 - 1)^2] ,$$

where

$$\vec{E}_i = \vec{F}_{0i}$$

and

$$\vec{B}_i = \tfrac{1}{2}\epsilon_{ijk}\vec{F}_{jk} .$$

Another is the electric charge

$$Q = \lim_{r\to\infty} r^2 \int d\Omega\,\vec{\phi}\cdot\vec{E}_r .$$

The third is the magnetic charge

$$\Phi = \lim_{r\to\infty} r^2 \int d\Omega\,\vec{\phi}\cdot\vec{B}_r .$$

For nonsingular solutions of finite energy, Φ must be an integral multiple of 4π.

Prasad and Sommerfield found analytic expressions for the monopole and dyon solutions in the limit $\lambda\to 0^+$. These solutions have $\Phi = \pm 4\pi$, arbitrary Q, and $E=(Q^2+\Phi^2)^{1/2}$. This is all the information that we will need for our proof.

Our proof will rest on a general theorem of Lagrangian mechanics: Given a system with generalized coordinates q^a, and Lagrangian of the form

$$L=\tfrac{1}{2}\dot{q}^a\dot{q}^a+B^a(q)\dot{q}^a-V(q)\,,$$

Then the minima of the energy are points of stable equilibrium.[4] If the theory admits other conserved quantities, then the same is true for the minima of the energy with these other quantities held fixed. The Lagrangian of our theory is in this form if we choose the gauge

$$\vec{A}_0=\vec{\bar{A}}_0\,,$$

where $\vec{\bar{A}}_0$ is the value of $\vec{A}_0$ in the known solution.[5]

We shall now show that the solutions of Ref. 3 are minima of the energy with fixed Φ and Q.

From the definition of $\vec{B}_i$,

$$D_i\vec{B}_i=0\,.$$

Thus, by integration by parts,

$$\int d^3x\, D_i\vec{\phi}\cdot\vec{B}_i=\Phi\,.$$

Likewise, one of the field equations is

$$D_i\vec{E}_i=D_0\vec{\phi}\times\vec{\phi}\,.$$

Thus, by integration by parts,

$$\int d^3x\, D_i\vec{\phi}\cdot\vec{E}_i=Q\,.$$

Hence, for arbitrary angle α,

$$\begin{aligned}E&\geq\frac{1}{2}\int d^3x\,[(\vec{E}_i)^2+(\vec{B}_i)^2+(D_i\vec{\phi})^2]\\&=\frac{1}{2}\int d^3x\,[(\vec{E}_i-\sin\alpha\, D_i\vec{\phi})^2+(\vec{B}_i-\cos\alpha\, D_i\vec{\phi})^2]\\&\quad+Q\sin\alpha+\Phi\cos\alpha\\&\geq Q\sin\alpha+\Phi\cos\alpha\,.\end{aligned}$$

If we choose

$$\sin\alpha=Q/(Q^2+\Phi^2)^{1/2},\quad \cos\alpha=\Phi/(Q^2+\Phi^2)^{1/2}\,,$$

we obtain

$$E\geq(Q^2+\Phi^2)^{1/2}\,.$$

This bound is saturated by the solutions of Ref. 3. Q.E.D.

As a by-product of this proof, we observe that saturation of the inequality implies that the solutions must obey

$$\vec{E}_i-\sin\alpha\, D_i\vec{\phi}=\vec{B}_i-\cos\alpha\, D_i\vec{\phi}=0\,.$$

These are first-order differential equations, and they can be solved trivially once one makes the assumption of rotational invariance. This is a method of finding the analytic form of the solutions alternative to that of the original paper, to wit, guesswork.

Note added. After this manuscript was submitted we discovered that substantially identical results had been obtained independently by L. Faddeev (private communication) and E. B. Bogomolny [ITP Chernogolovka report, 1975 (unpublished)].

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[1]G. 't Hooft, Nucl. Phys. B79, 276 (1974); A. M. Polyakov, Zh. Eksp. Teor. Fiz. Pis'ma Red. 20, 430 (1974) [JETP Lett. 20, 194 (1974)]. There has been considerable subsequent activity; for a review, see S. Coleman, in Proceedings of the 1975 International School of Subnuclear Physics "Ettore Majorana" (unpublished).

[2]B. Julia and A. Zee, Phys. Rev. D 11, 2227 (1975).

[3]M. K. Prasad and C. M. Sommerfield, Phys. Rev. Lett. 35, 760 (1975).

[4]We sketch the proof here:

$$E=\dot{q}^a(\partial L/\partial\dot{q}^a)-L=\tfrac{1}{2}\dot{q}^a\dot{q}^a+V\,.$$

Let us add a constant to V such that the minimum of E is zero, and let us denote by S_0 the surface of zeros of V. Because E is the sum of two positive terms, the two statements that the configuration of the system is close to S_0 and that the velocity of the system is small are equivalent to the single statement that E is small. By conservation of energy, this is preserved by time evolution. This is stability, including the possibility of neutral equilibrium for motion along S_0. We emphasize that this proof does not depend on a linear approximation to motion near a point of equilibrium; such an approximation may be deceptive, notoriously so in the case of neutral equilibrium.

[5]For any initial field configuration, we can always make a gauge transformation such that $\vec{A}_0$ takes on any desired value. For a proof, see S. Coleman, Ref. 1. We don't gauge $\vec{A}_0$ away altogether because this would make the dyon a time-dependent solution.