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Gauge-invariant extension of the σ model with electromagnetic interactions

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The SU_2 symmetry of an extension of the σ model containing massive vector and axial-vector bosons is broken by adding electromagnetic interactions. This is done in a way formally similar to the way in which the intrinsic symmetry—but not the "gauge" invariance—of space-time is broken by gravitational interactions.

I. INTRODUCTION

The unified gauge theories of Weinberg[1] and Salam[2] have been recently extended to include strong interactions (Bardakci,[3] de Wit,[4,5] and Bars, Halpern, and Yoshimura[6]). Using the Higgs-Kibble mechanism[7] locally chiral-invariant extensions of the σ model can be constructed which include strongly as well as weakly interacting Yang-Mills fields. From the general results of 't Hooft,[8] one can show that the resulting Lagrangian is renormalizable. Furthermore, de Wit[5] has shown that although the current-algebra hypothesis does not seem to be satisfied in these models, the results of chiral-Lagrangian field theory (see, for example, Gasiorowicz and Geffen[9]) which follow directly from partial conservation of axial-vector current (PCAC) remain valid.

In this paper we shall present a locally chiral-invariant extension of the σ model which includes electromagnetic interactions. Our approach differs from that, for example, of de Wit[5] in two respects. First of all, the Yang-Mills fields which we introduce acquire a mass in the same way that the nucleon acquires a mass in the σ model. In the σ model the nucleon has no mass and there is a term in the Lagrangian linear in σ which explicitly breaks the gauge symmetry and makes the vacuum unstable. The fields are then shifted to give a stable vacuum and a nucleon mass. We introduce a second scalar field ϕ which gives a mass to the Yang-Mills fields in a similar way. However, we pursue the general logic of the σ model in the inverse direction. Instead of breaking gauge symmetry and shifting the fields, we shift the fields in the classical Lagrangian by fiat [see formulas (2.19), (2.25)]. This means that we keep the gauge invariance but the quantum Lagrangian will have an unstable vacuum. This approach has a drawback in that it cannot be directly shown to be renormalizable, although we have explicitly constructed the Lagrangian to be polynomial and to have a minimum number of derivative couplings as well as to be locally gauge-invariant. However, it allows the classical Lagrangian to have massive gauge fields and these masses are fixed as they are in the unified gauge theories which use the Higgs-Kibble mechanism (for example, see de Wit[5]). Also, it leads naturally to a second, and more important, difference.

We have attempted to interpret the photon not as a gauge field arising from an additional U_1 gauge group to be juxtaposed to the strong-interaction gauge group $SU_2 \times SU_2$, but as a gauge field associated with the breaking of $SU_2 \times SU_2$. Therefore, the total gauge group remains $SU_2 \times SU_2$ even in the presence of electromagnetic interactions; the intrinsic symmetry of this gauge group is broken, however, by these interactions. We have here been inspired by an analogous situation in general relativity.

If one has a Lorentz-invariant theory which one wishes to write in a form invariant under the general covariant (pseudo-) group (or under the local Lorentz group), one introduces a general (flat) metric and a general (flat) connection or covariant derivative. One has then a gauge theory which is Lorentz-invariant. If one wishes to introduce gravity, one requires that the metric and the connection acquire curvature. One has then a gauge theory which is no longer Lorentz invariant; the gauge invariance remains as it was, but the intrinsic symmetry of the gauge group is broken.

In order to lend support to the approach we use, we present two mass formulas, (3.13), and (6.11). However, neither of these formulas can be considered to be exclusively a consequence of our Lagrangian. Formulas like (3.22) connecting the ratio of the axial-vector mass to the vector mass have appeared often in chiral-Lagrangian theory (see, for example, Barnes and Isham[10]) and they also follow from spontaneous symmetry breaking (for example, de Wit[5]). Also, formula (6.11) is identical to that connecting m_Z, m_W, and θ_W in the Weinberg-Salam model of weak interactions.

The free Lagrangian for the nucleon system $\psi = \binom{p}{n}$ is

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi . \tag{1.1}$$

This Lagrangian is invariant under constant transformations of the isospin group. It is not, however, invariant under transformations which are functions of the point in space-time because of the derivation in the kinematical part. If a is an element of SU_2 (or of $SU_2 \times SU_2$) then, in general,

$$\partial_\mu(a\psi) \neq a\,\partial_\mu \psi .$$

Yang and Mills[11] solved this problem by replacing the partial derivative ∂_μ by a covariant derivative D_μ defined such that

$$D_\mu(a\psi') = aD'_\mu \psi', \quad \psi = a\psi' .$$

If we write

$$D_\mu = \partial_\mu + \Gamma_\mu ,$$

then the Γ_μ are the Yang-Mills fields; they also constitute what is called a connection (see, for example, Kobayashi and Nomizu[12]). We shall be led to introduce three different types of connections—the general connection or Yang-Mills fields, and two special connections. We shall discuss in Sec. II the theory of connections in sufficient detail to be able to describe the three types of connections which interest us.

The Lagrangian (1.1) is not invariant under even constant transformations of the chiral group $SU_2 \times SU_2$ because of the γ^0 in the mass term. If a is an element of $SU_2 \times SU_2$ then, in general,

$$a\gamma^0 \neq \gamma^0 a .$$

This problem was solved by Gürsey[13] by replacing the matrix γ^0 by a general metric g defined such that

$$ag' = ga .$$

We shall discuss in Sec. II the theory of metrics within the context of the chiral group and we shall in particular define a metric connection, one of the special types of connections mentioned above.

Define $\tilde{\psi} = \psi^* g$. We now have a Lagrangian,

$$\mathcal{L} = \tilde{\psi} i\gamma^\mu D_\mu \psi - m\tilde{\psi}\psi , \tag{1.2}$$

which describes the nucleons in interaction with the metric fields g and the Yang-Mills fields Γ_μ and which is invariant under local transformations of the chiral group $SU_2 \times SU_2$. These first two steps have also been described by Mainland and O'Raifeartaigh.[14] They refer to them as a generalized minimal principle. The principal remaining problem is the mass of the Yang-Mills fields. No invariant mass term can be constructed from the connection Γ_μ alone.

We shall solve this problem by using a generalization of an idea due to Stueckelberg[15] which has also been used by Schwinger[16] and by Wess and Zumino.[17] We shall introduce a third connection Γ^0_μ which is flat and we shall use it with the Yang-Mills connection to construct an invariant mass term. We shall discuss this connection in Sec. II. In particular we shall show that it was necessary to introduce it; the metric connection mentioned above could not be used because it would lead to a nonpolynomial Lagrangian.

The introduction of the mass term for the Yang-Mills field in a gauge-invariant manner fixes the value of the mass. To see this, we recall the Stueckelberg argument. Let p be the proton field. Then

$$\mathcal{L} = \bar{p} i\gamma^\mu(\partial_\mu + ieA_\mu)\, p - \tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} \tag{1.3}$$

is invariant under space-time-dependent phase transformations $e^{i\alpha}$. The connection in this case is ieA_μ. $ieF_{\mu\nu}$ is what is called the curvature of the connection. A connection is flat if its curvature vanishes. In this case, this means that it is of the form $f\,\partial_\mu \chi$ where f is a constant. An invariant mass term may be constructed using eA_μ and $f\partial_\mu\chi$. The resulting Lagrangian is

$$\mathcal{L} = \bar{p} i\gamma^\mu(\partial_\mu + ieA_\mu)p - \tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2f^2}(f\partial\chi - eA)^2 . \tag{1.4}$$

The mass of the field A_μ is equal to e/f. We shall find similar relations for the masses of the vector and axial-vector mesons in the more complicated case of $SU_2 \times SU_2$.

We see then that by three successive applications of the general idea of a compensating field we shall be led from the free-nucleon Lagrangian to one which is invariant under local transformations of the chiral group $SU_2 \times SU_2$. The three additional conditions mentioned at the beginning of this section place severe restrictions on the final Lagrangian.

The additional fields which we shall be led to introduce are the Yang-Mills fields ρ_μ and a_μ for the chiral group $SU_2 \times SU_2$; the σ-model fields σ and π come from the metric g and four isospin-$\frac{1}{2}$ fields come from the flat connection. The notation is that of Bjorken and Drell.[18]

In Sec. II we shall introduce some general mathematical formulas and write down the invariants which can be formed. The only essential conclusion to be drawn from this section is that the metric connection of g cannot be used in the Lagrangian since it would lead to nonpolynomial terms; this serves as motivation for introducing the flat connection. We use some of the elementary concepts (but none of the results) of differential geometry in this section since in Sec. VI we wish to stress a vague analogy between the way we break the $SU_2 \times SU_2$ symmetry by electromagnetic interactions and the way the Lorentz symmetry of space-time is broken by gravitational interactions.

In Sec. III the Lagrangian is written down and expanded in terms of the fields. We also discuss here the constraints on the masses of the axial-vector and vector mesons. In Sec. IV the equations of motion are discussed. In Sec. V the currents are introduced; the fields are formally quantized and the charge algebra is shown to be formally equal to $SU_2 \times SU_2$. In Sec. VI, the symmetry is broken by introducing electromagnetic interactions. This is done in a way which conserves local gauge invariance and an analogy is drawn with general relativity. The extra invariants are discussed which the electromagnetic interaction introduces and various interaction terms are explicitly written down.

II. INVARIANTS

The nucleon field ψ, defined by

$$\psi = \begin{pmatrix} p \\ n \end{pmatrix}, \tag{2.1}$$

takes its values in the eight-dimensional complex space $\mathbf{C}^2 \otimes \mathbf{C}^4$. The chiral group $SU_2 \times SU_2$ operates on this space, the Pauli matrices operate on $\mathbf{C}^2$, and the Dirac matrices γ^5 and I operate on $\mathbf{C}^4$. We are interested in defining a derivative of the nucleon field ψ which is covariant under transformations of ψ by elements a of the chiral group. That is, if

$$\psi = a\psi' \tag{2.2}$$

is a transformation of ψ, then we wish to define D_μ of ψ such that

$$aD_\mu\psi' = D_\mu\psi . \tag{2.3}$$

Define Γ_μ by

$$D_\mu = \partial_\mu + \Gamma_\mu . \tag{2.4}$$

Then

$$a(\partial_\mu + \Gamma'_\mu)\psi' = (\partial_\mu + \Gamma_\mu)a\psi' , \tag{2.5}$$

and we see that Γ_μ must transform as

$$\Gamma'_\mu = a^*\Gamma_\mu a + a^*\partial_\mu a . \tag{2.6}$$

The additive term $a^*\partial_\mu a$ is an element of the Lie algebra of $SU_2 \times SU_2$, so it is sufficient to suppose that Γ_μ takes its values also in this Lie algebra. We shall call Γ_μ a connection.[12] We may write Γ_μ as

$$\Gamma_\mu = \rho_\mu + \gamma^5 a_\mu , \tag{2.7}$$

where ρ_μ and a_μ are functions with values in the Lie algebra of SU_2. They are the classical Yang-Mills fields. Let τ_i be the Pauli matrices. For convenience we shall expand ρ_μ and a_μ in terms of the matrices $\sigma_j = -\frac{1}{2} i \tau_j \otimes I$.

It is interesting to note that if C is a curve in Minkowski space with tangent vector dx^α/dt, then the condition

$$\frac{dx^\alpha}{dt} D_\alpha \psi = 0 \tag{2.8}$$

uniquely defines a nucleon field along C in terms of its value at one point of C. Fields which satisfy such equations are the closest things we have in a gauge theory to constant fields. That is, the equation $D_\alpha\psi = 0$ in general has no solutions. The reason for this is a quantity known as curvature.

The curvature $R_{\mu\nu}$ of the connection Γ_μ is defined by

$$R_{\mu\nu} = \partial_\mu\Gamma_\nu - \partial_\nu\Gamma_\mu + [\Gamma_\mu, \Gamma_\nu] . \tag{2.9}$$

It is an antisymmetric (Lorentz) tensor with values in the Lie algebra of $SU_2 \times SU_2$. One sees easily that

$$[D_\mu, D_\nu]\psi = R_{\mu\nu}\psi . \tag{2.10}$$

So it is evident that the equation $D_\alpha\psi = 0$ has no solutions unless $R_{\mu\nu} = 0$. One can in fact show[12] that if C is a curve which starts and finishes at the same point then a solution of Eq. (2.8) will not have the initial and final values of ψ equal in general unless $R_{\mu\nu}$ vanishes.

A connection for which $R_{\mu\nu}$ vanishes is called a flat connection. One can show that $R_{\mu\nu}$ vanishes if and only if Γ_μ is of the form

$$\Gamma_\mu = \Gamma^0_\mu = b^{-1}\partial_\mu b , \tag{2.11}$$

where b is a function with values in $SU_2 \times SU_2$. From (2.6) it is evident that such connections can be made to vanish by an appropriate choice of a. (We have written b^{-1} instead of b^* since we shall be lead to consider a slightly more general form of flat connection later.)

A metric in the space of ψ's is a matrix-valued function whose determinant is not zero, which to ψ_1 and ψ_2 associates a real number $\psi_1^* g \psi_2$. It follows that g must be Hermitian:

$$g = g^* . \tag{2.12}$$

Under a gauge transformation $\psi_i = a\psi'_i$ $(i = 1, 2)$, g must transform as

$$g' = a^* g a \tag{2.13}$$

if we wish to have $\psi_1^* g \psi_2 = \psi_1'^* g' \psi_2'$.

Consider a curve C and two solutions ψ_1 and ψ_2 to Eq. (2.8); then we have

$$\frac{d}{dt}(\psi_1^* g \psi_2) = \frac{dx^\alpha}{dt}(\partial_\alpha\psi_1^* g\psi_2 + \psi_1^*\partial_\alpha g\psi_2 + \psi_1^* g\,\partial_\alpha\psi_2)$$

$$= \frac{dx^\alpha}{dt}\psi_1^*(\partial_\alpha g - \Gamma_\alpha^* g - g\Gamma_\alpha)\psi_2 . \tag{2.14}$$

Suppose that this vanishes for any curve C; then

$$\partial_\alpha g = \Gamma^*_\alpha g + g\Gamma_\alpha . \tag{2.15}$$

A connection which satisfies this equation is called a metric connection. In particular, with such a connection a field ψ has constant length $\psi^* g\psi$ along a curve C if it satisfies Eq. (2.8). Since $\Gamma^*_\alpha = -\Gamma_\alpha$ we have

$$D_\alpha g \equiv \partial_\alpha g + [\Gamma_\alpha , g] = 0 \tag{2.16}$$

if Γ_α is a metric connection for g.

If b is a function with values in $SU_2 \times SU_2$, then

$$g^0 = b^* \gamma^0 b \tag{2.17}$$

is a metric. We shall call it a flat metric since it has as metric connection the flat connection Γ^0_μ. In a particular gauge we have $g^0 = \gamma^0$ (from 2.13), so that in this gauge

$$\psi^* g^0 \psi = \psi^* \gamma^0 \psi = \bar\psi\psi . \tag{2.18}$$

The gauge in which $g^0 = \gamma^0$ is the gauge in which $\Gamma^0_\mu = 0$.

The nucleon metric which we shall introduce is the metric

$$g = \gamma^0 (1 + \sigma + 2\gamma^5 \pi) \tag{2.19}$$

of the σ model (Schwinger,[19] Gell-Mann and Lévy[20]). See also Gürsey[21] and Lee.[22] We have chosen units so that the pion decay constant f_π is equal to one,

$$f_\pi = 1 \text{ (unit of length)} . \tag{2.20}$$

This fixes the factor 2 in front of π in (2.19). The factor 1 in front of σ is fixed by the free kinematical Lagrangian. As with the Yang-Mills fields we expand π in terms of the σ_i matrices: $\pi = \pi^i \sigma_i$.

Let $a = e^{\alpha + \gamma^5 \beta}$ be an arbitrary element of $SU_2 \times SU_2$. The requirement that g transform according to (2.13) induces the following transformations of the σ and π fields:

$$\begin{aligned} \sigma' &= \sigma + 2\beta^2 + 2\{\beta, \pi\} + 2\beta^2 \sigma + \{\beta, [\pi, \alpha]\} + \cdots , \\ \pi' &= \pi + \beta + [\pi, \alpha] + \beta\sigma + \tfrac{1}{2}[\beta, \alpha] + \tfrac{1}{2}[[\pi, \alpha], \alpha] \\ &\quad + \tfrac{1}{2}[\beta, \alpha]\sigma + \{\beta, \pi\}\beta + \tfrac{1}{6}[[\beta, \alpha], \alpha] + \tfrac{2}{3}\beta^2 \beta + \cdots . \end{aligned} \tag{2.21}$$

The metric g is Hermitian and it defines one invariant j:

$$j = g^2 = (1+\sigma)^2 - 4\pi^2 . \tag{2.22}$$

We shall identify invariants with the corresponding multiple of the unit matrix in the 8-dimensional space we are using, so that $\mathrm{Tr}(j) = 8j$.

In order to give a mass to the Yang-Mills fields using the Stueckelberg mechanism we need a second connection in order to form a gauge-invariant mass term for the Lagrangian. We see from Eq. (2.6) that the difference of two connections transforms without the additional term $a^* \partial_\mu a$. Once we have introduced a metric it is natural to inquire whether an associated metric connection could play this rôle.

Let B_μ be such a connection; therefore, B_μ satisfies Eq. (2.15) or (2.16). By choosing the gauge $\pi = 0$ one can see immediately that g has in fact no associated metric connections, but that the conformally related metric $g/\sqrt{j}$ has such connections. If we expand B_μ as follows:

$$B_\mu = X_\mu + \gamma^5 Y_\mu ,$$

then we find from (2.16) that X_μ , Y_μ satisfy the equations

$$\begin{aligned} Y_\mu &= \frac{\partial_\mu \pi}{1+\sigma} + \frac{[X_\mu, \pi]}{1+\sigma} - \frac{\partial_\mu j\pi}{2j(1+\sigma)} , \\ \{Y_\mu , \pi\} &= \frac{\partial_\mu \sigma}{2} - (1+\sigma)\frac{\partial_\mu j}{4j} . \end{aligned} \tag{2.23}$$

The general solution to these equations is given by

$$\begin{aligned} X_\mu &= j^{-1}[\partial_\mu \pi, \pi] + V_\mu , \\ Y_\mu &= j^{-1}(\partial_\mu \pi + \sigma\partial_\mu \pi - \pi\partial_\mu \sigma) + \frac{[\pi, V_\mu]}{1+\sigma} . \end{aligned} \tag{2.24}$$

The solution depends on an arbitrary vector V_μ which takes its values in the Lie algebra of SU_2.

We cannot set V_μ equal to zero, since it must transform so that B_μ is a connection. One solution would be to identify V_μ with the ρ-meson fields and to use B_μ as the Yang-Mills connection. This would do away with the necessity of introducing the A_1 mesons as fundamental fields. It would, however, lead to derivative couplings between the pions and the nucleons and it would be impossible to introduce either a ρ-meson mass term or a pion kinetic term in a gauge-invariant manner.

If we wish to use B_μ to form a mass term for the Yang-Mills fields then V_μ must be constructed from π and from a scalar field τ with isospin equal to one. From the known transformation properties of σ, π, and B_μ one can in fact easily see that B_μ must contain a term $\partial_\mu \tau$ and that it cannot be independent of π. We have not succeed in constructing such a V_μ which would lead to a B_μ polynomial in the field variables, or even one which would yield a B_μ such that $j^n B_\mu$ is polynomial, for any integer n.

We shall use this negative result as motivation for introducing yet a third connection—a flat connection Γ^0_μ given by (2.11). Define the matrix function b by

$$b = 1 + \frac{f\phi}{2} + \gamma^5 \frac{f\phi_5}{2} + f\chi + \gamma^5 f\kappa . \tag{2.25}$$

χ and κ are functions with values in the Lie algebra of SU_2. We expand them in terms of the σ_i matrices:

$$\chi = \chi^i \sigma_i\,, \quad \kappa = \kappa^i \sigma_i\,.$$

(ϕ, χ^i) are scalar fields; (ϕ_5, κ^i) are pseudoscalar fields. We shall see that each of these sets of fields constitutes two isospin-$\frac{1}{2}$ multiplets. The parameter f is the ratio of the (ϕ_5, κ) "decay constant" f_κ to f_π (which we have set equal to one by a choice of units). This fixes the factor 1 in front of κ; the factor $\frac{1}{2}$ in front of ϕ_5 comes from the free kinematical Lagrangian.

The Hermitian adjoint of b is given by

$$b^* = 1 + \frac{f\phi}{2} + \gamma^5 \frac{f\phi_5}{2} - f\chi - \gamma^5 f\kappa\,. \tag{2.26}$$

This yields the invariants

$$k = b^* b = b b^* = \xi + \gamma^5 \zeta\,, \tag{2.27}$$

where ξ, ζ are given by

$$\begin{aligned} \xi &= \left(1 + \frac{f\phi}{2}\right)^2 + \frac{f^2 \phi_5{}^2}{4} - f^2\chi^2 - f^2\kappa^2\,, \\ \zeta &= f\phi_5\left(1 + \frac{f\phi}{2}\right) - f^2\{\chi, \kappa\}\,. \end{aligned} \tag{2.28}$$

The inverse of k is given by

$$k^{-1} = \frac{\xi - \gamma^5\zeta}{\xi^2 - \zeta^2}\,, \tag{2.29}$$

and the inverse of b is given by

$$b^{-1} = k^{-1} b^*\,. \tag{2.30}$$

We require that under a gauge transformation a and b transform as

$$b \to b' = ba\,. \tag{2.31}$$

This yields the following transformations on the field variables:

$$\begin{aligned} \phi' &= \phi + \frac{1}{f}(\alpha^2 + \beta^2) + \{\chi, \alpha\} + \{\kappa, \beta\} \\ &\quad + \frac{\phi}{2}(\alpha^2 + \beta^2) + \frac{\phi_5}{2}\{\alpha, \beta\} + \cdots\,, \\ \phi_5' &= \phi_5 + \{\chi, \beta\} + \{\kappa, \alpha\} + \frac{\phi_5}{2}(\alpha^2 + \beta^2) \\ &\quad + \left(\frac{1}{f} + \frac{\phi}{2}\right)\{\alpha, \beta\} + \cdots\,, \\ \chi' &= \chi + \frac{\alpha}{f} + \tfrac{1}{2}[\chi, \alpha] + \tfrac{1}{2}[\kappa, \beta] + \frac{\phi\alpha}{2} + \frac{\phi_5\beta}{2} \\ &\quad + \frac{\chi}{2}(\alpha^2 + \beta^2) + \frac{\kappa}{2}\{\alpha, \beta\} + \cdots\,, \\ \kappa' &= \kappa + \frac{\beta}{f} + \tfrac{1}{2}[\chi, \beta] + \tfrac{1}{2}[\kappa, \alpha] + \frac{\phi\beta}{2} + \frac{\phi_5\alpha}{2} \\ &\quad + \frac{\kappa}{2}(\alpha^2 + \beta^2) + \frac{\chi}{2}\{\alpha, \beta\} + \cdots\,. \end{aligned} \tag{2.32}$$

Define Γ^0_μ by (2.11). If one calculates the adjoint of Γ^0_μ, one finds

$$\Gamma^{0\,*}_\mu = -\Gamma^0_\mu + \partial_\mu(\ln k)\,; \tag{2.33}$$

so that Γ^0_μ is not strictly a connection as we have defined them above. In fact the anti-Hermitian part of Γ^0_μ is a connection and the Hermitian part,

$$\tfrac{1}{2}(\Gamma^0_\mu + \Gamma^{0\,*}_\mu) = \tfrac{1}{2}\partial_\mu(\ln k)\,, \tag{2.34}$$

is an invariant. We shall ignore this rather trivial difference. Written out in terms of the field variables, Γ^0_μ is given by

$$\begin{aligned} \frac{k}{f^2}\Gamma^0_\mu &= \left(\frac{1}{f} + \frac{\phi}{2} + \gamma^5\frac{\phi_5}{2}\right)\partial_\mu(\chi + \gamma^5\kappa) \\ &\quad - \tfrac{1}{2}(\chi + \gamma^5\kappa)\partial_\mu(\phi + \gamma^5\phi_5) \\ &\quad + \tfrac{1}{2}[\partial_\mu(\chi + \gamma^5\kappa), (\chi + \gamma^5\kappa)] + \frac{1}{2f^2}\partial_\mu k\,. \end{aligned} \tag{2.35}$$

The curvature $R^0_{\mu\nu}$ of Γ^0_μ, given by (2.9), vanishes:

$$R^0_{\mu\nu} = 0\,. \tag{2.36}$$

Since we have introduced a flat connection it is of interest to enquire whether we could have used the flat metric g^0 (2.17) instead of the σ-model metric (2.19). In fact, we could have done so but the fields we introduce using b are necessarily of isospin $\frac{1}{2}$ and could not be used as pion fields. If we use g^0 as metric, then the pions would disappear from the Lagrangian.

We have introduced two connections Γ_μ and Γ^0_μ. One easily sees from (2.6) that

$$\Gamma'_\mu = p\Gamma_\mu + q\Gamma^0_\mu\,, \quad p + q = 1 \tag{2.37}$$

is also a connection. The most general covariant derivative we can introduce is therefore

$$D'_\mu = \partial_\mu + \Gamma'_\mu\,.$$

However, if we do not wish to have derivative couplings with the nucleons we must set $q = 0$ and $p = 1$. This yields the covariant derivative (2.4). It also implies ρ universality.

From Γ_μ and Γ^0_μ we can construct the covariant quantity

$$C_\mu = \Gamma_\mu - \Gamma^0_\mu\,. \tag{2.38}$$

From (2.6) we see that C_μ transforms as

$$C_\mu \to C'_\mu = a^* C_\mu a\,. \tag{2.39}$$

The Hermitian adjoint of C_μ is given by

$$C^*_\mu = -C_\mu - \partial_\mu(\ln k)\,. \tag{2.40}$$

We are now in a position to discuss the invariants which can be formed from the covariant quantities which we have introduced. Since we wish to form a Lagrangian from the invariants we shall restrict our attention to those which have at most first-order derivatives in the field variables and which yield diagonal second-order terms.

Consider first of all those invariants formed using the nucleon field. We have

$$\tilde{\psi}\psi, \quad \tilde{\psi}=\psi^* g \tag{2.41a}$$

$$\tilde{\psi}j^n\psi, \tilde{\psi}k^n\psi, \quad n \text{ a positive integer} \tag{2.14b}$$

$$\psi^* g^0\psi, \tag{2.14c}$$

$$\psi^* g^0 j^n\psi, \psi^* g^0 k^n\psi, \quad n \text{ a positive integer} \tag{2.41d}$$

$$\overline{\psi}i\gamma^\mu D_\mu\psi, \tag{2.41e}$$

$$\overline{\psi}i\gamma^\mu j^n D_\mu\psi, \overline{\psi}i\gamma^\mu k^n D_\mu\psi, \quad n \text{ a positive integer.} \tag{2.41f}$$

We could choose as mass term any linear combination of (2.41a), (2.41b), (2.41c), and (2.41d). For example, (2.41c) would lead to direct nucleon-kaon couplings. We shall choose (2.41a) for the mass term since it corresponds to the σ-model choice. Also it is the only one which contributes to the Lagrangian a term of degree not greater than 4 (see, for example Bessis and Turchetti[23]). The only term which we can choose as the nucleon kinematical term is (2.41e); the invariants (2.41f) lead to derivative couplings.

Consider now those invariants which can be formed using only the pseudoscalar and scalar fields. The most general such invariant is a general function,

$$F(j,\partial_\mu j,k,\partial_\nu k), \tag{2.42a}$$

which is a Lorentz scalar. However, if we do not wish to have derivative couplings between these fields then the most general function we can choose is

$$F(j,k). \tag{2.42b}$$

We shall restrict ourselves to using

$$\mathrm{Tr}(j), \ \mathrm{Tr}(k) \tag{2.42c}$$

as mass terms. The only other invariant of degree not greater than 4 which we could have used is

$$\mathrm{Tr}(j^2) \tag{2.42d}$$

as in the σ model.

Finally, consider those invariants containing the Yang-Mills fields. For simplicity we do not write down terms with g^0 factors; they can be eliminated in the same way as the terms with g factors. We have

$$\mathrm{Tr}(k^p j^q C_\mu g^r C^{*\mu} g^s), \quad p,q,r,s \text{ integers} \tag{2.43a}$$

$$\mathrm{Tr}(k^p j^q R_{\mu\nu} g^r R^{\mu\nu} g^s), \quad p,q,r,s \text{ integers} \tag{2.43b}$$

$$\mathrm{Tr}(k^p j^q D_\mu g g^r D^\mu g g^s), \quad p,q,r,s \text{ integers}. \tag{2.43c}$$

We shall choose as mass term for the Yang-Mills fields (2.43a) with $p=1$ and $q=r=s=0$. We shall see when we write this invariant out in detail in the next section that it is the only one which has no derivative couplings. As the pion kinematical term we shall choose (2.43c) with $p=q=r=s=0$. We shall see also when we write this invariant out in detail that it is the only possible choice. It contributes a term to the axial-vector mass. The standard Yang-Mills kinematical Lagrangian is given by (2.43b) with $p=q=r=s=0$. We shall also make this choice. Any other values of p,q,r,s would lead to derivative couplings between the vector, axial-vector, scalar, and pseudoscalar fields containing second-order derivatives. Similar restrictions on chiral Lagrangians due to the absence of derivative couplings have been considered by Ogievetsky and Zupnik.[24]

III. THE LAGRANGIAN

From the discussion of the preceding sections, we have as Lagrangian the following invariant:

$$\begin{aligned}\mathcal{L} = \overline{\psi}i\gamma^\mu D_\mu\psi - m\tilde{\psi}\psi + \frac{1}{4f^2}\,\mathrm{Tr}(kC_\mu C^{*\mu}) \\ + \tfrac{1}{8}\mathrm{Tr}(R_{\mu\nu}R^{\mu\nu}) + \tfrac{1}{16}\mathrm{Tr}(D_\mu g D^\mu g) \\ - \frac{\mu_K^2}{4}\mathrm{Tr}(k) - \frac{\mu_\pi^2}{16}\mathrm{Tr}(j).\end{aligned} \tag{3.1}$$

This Lagrangian depends on five parameters $f_\pi, f_K, \mu_\pi, \mu_K, m$. We have set the (bare) ρ coupling constant g_ρ equal to one.

A short calculation yields the following expansion for the third term in $\mathcal{L}$:

$$\frac{1}{4f^2}\mathrm{Tr}(kC_\mu C^{*\mu}) = \frac{1}{4f^2}\mathrm{Tr}(\partial_\mu b\,\partial^\mu b^* + 2\Gamma^\mu b^*\partial_\mu b - bb^*\Gamma_\mu\Gamma^\mu). \tag{3.2}$$

We can see now why this was the only invariant which we could choose from the family (2.43a). Any other values of p,q,r,s would not have yielded the simple $\partial_\mu b\,\partial^\mu b^*$ term on the right-hand side and this is the only one which does not give derivative couplings between the scalar and pseudoscalar fields.

Similarly for the fifth term in $\mathcal{L}$. We find the following expansion:

$$\begin{aligned}\tfrac{1}{16}\mathrm{Tr}(D_\mu g D^\mu g) = \tfrac{1}{16}\mathrm{Tr}(\partial_\mu g\,\partial^\mu g) + \tfrac{1}{8}\mathrm{Tr}(\Gamma_\mu(g\partial^\mu g - \partial^\mu g g)) \\ + \tfrac{1}{8}\mathrm{Tr}(\Gamma_\mu g\Gamma^\mu g - j\Gamma_\mu\Gamma^\mu).\end{aligned} \tag{3.3}$$

Any other values of p,q,r,s in (2.43c) would not have yielded the simple $\partial_\mu g\,\partial^\mu g$ term and this is the only one which does not give derivative pion-

pion couplings.

We write $\mathcal{L}$ as the sum of three terms:

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3 , \tag{3.4}$$

where

$$\mathcal{L}_1 = \bar{\psi} i\gamma^\mu \partial_\mu \psi - m\bar{\psi}\psi + \tfrac{1}{8}\operatorname{Tr}(R_{\mu\nu} R_{\mu\nu}) + \frac{1}{4f^2}\operatorname{Tr}(\partial_\mu b\, \partial^\mu b^*) + \tfrac{1}{16}\operatorname{Tr}(\partial_\mu g\, \partial^\mu g) - \frac{\mu_\kappa^2}{4}\operatorname{Tr}(k) - \frac{\mu_\pi^2}{16}\operatorname{Tr}(j) , \tag{3.5}$$

$$\mathcal{L}_2 = \bar{\psi} i\gamma^\mu \Gamma_\mu \psi - m(\bar{\psi}\psi - \bar{\psi}\psi) , \tag{3.6}$$

$$\mathcal{L}_3 = \frac{1}{2f^2}\operatorname{Tr}(\Gamma^\mu b^* \partial_\mu b)\, \tfrac{1}{8}\operatorname{Tr}(\Gamma_\mu (g\, \partial^\mu g - \partial^\mu g g)) - \frac{1}{4f^2}\operatorname{Tr}(bb^* \Gamma_\mu \Gamma^\mu) + \tfrac{1}{8}\operatorname{Tr}(\Gamma_\mu g \Gamma^\mu g - j\Gamma_\mu \Gamma^\mu) . \tag{3.7}$$

It is straightforward to expand $\mathcal{L}_i$ in terms of the field variables:

$$\mathcal{L}_1 = \bar{\psi} i\gamma^\mu \partial_\mu \psi - m\bar{\psi}\psi + \tfrac{1}{8}\operatorname{Tr}(R_{\mu\nu} R^{\mu\nu}) + \tfrac{1}{2}(\partial_\mu \phi\, \partial^\mu \phi - \mu_\kappa^2 \phi^2) + \tfrac{1}{2}(\partial_\mu \phi_5 \partial^\mu \phi_5 - \mu_\kappa^2 \phi_5^2)$$
$$- \tfrac{1}{4}\operatorname{Tr}(\partial_\mu \chi\, \partial^\mu \chi - \mu_\kappa^2 \chi^2 + \partial_\mu \kappa\, \partial^\mu \kappa - \mu_\kappa^2 \kappa^2) + \tfrac{1}{2}(\partial_\mu \sigma\, \partial^\mu \sigma - \mu_\pi^2 \sigma^2) - \tfrac{1}{4}\operatorname{Tr}(\partial_\mu \pi\, \partial^\mu \pi - \mu_\pi^2 \pi^2) - 2\mu_\kappa^2 \phi - \mu_\pi^2 \sigma , \tag{3.8}$$

$$\mathcal{L}_2 = \bar{\psi} i\gamma^\mu (\rho_\mu + \gamma^5 a_\mu)\psi - \sigma m\bar{\psi}\psi - 2m\bar{\psi}\gamma^5 \pi \psi , \tag{3.9}$$

$$\mathcal{L}_3 = \tfrac{1}{2}\operatorname{Tr}[(j^{\mu(\pi)} + j^{\mu(\kappa)})\rho_\mu + (j^{\mu 5(\pi)} + j^{\mu 5(\kappa)})a_\mu] - \frac{1}{4f^2}\operatorname{Tr}(\rho^2) - \tfrac{1}{4}(1 + 1/f^2)\operatorname{Tr}(a^2)$$
$$- \frac{1}{4f^2}\operatorname{Tr}[(\tfrac{1}{4}\phi^2 + \phi + \tfrac{1}{4}\phi_5^2 - \chi^2 - \kappa^2)(\rho^2 + a^2)] - \tfrac{1}{4}\sigma(\sigma + 2)\operatorname{Tr}(a^2) - \frac{1}{2f^2}\operatorname{Tr}[(\phi_5(1 + \tfrac{1}{2}\phi) - \{\chi, \kappa\})a_\mu \rho^\mu]$$
$$+ \tfrac{1}{2}(1 + \sigma)\operatorname{Tr}(\rho_\mu[\pi, a^\mu]) - \tfrac{1}{4}\operatorname{Tr}([\rho^\mu, \pi][\rho_\mu, \pi] + [a^\mu, \pi][a_\mu, \pi]) . \tag{3.10}$$

The currents (Sec. V) are defined by the Eqs. (5.15), (5.16). In terms of the field variables, they are given by

$$j^{\mu(\pi)} = [\partial^\mu \pi, \pi] ,$$
$$j^{\mu 5(\pi)} = \partial^\mu \pi + \sigma\, \partial^\mu \pi - \pi\, \partial^\mu \sigma ,$$
$$j^{\mu(\kappa)} = \left(\frac{1}{f} + \frac{\phi}{2}\right)\partial^\mu \chi + \frac{\phi_5}{2}\partial^\mu \kappa - \frac{\chi}{2}\partial^\mu \phi - \frac{\kappa}{2}\partial^\mu \phi_5 + \tfrac{1}{2}[\partial^\mu \kappa, \kappa] + \tfrac{1}{2}[\partial^\mu \chi, \chi] , \tag{3.11}$$
$$j^{\mu 5(\kappa)} = \left(\frac{1}{f} + \frac{\phi}{2}\right)\partial^\mu \kappa + \frac{\phi_5}{2}\partial^\mu \chi - \frac{\chi}{2}\partial^\mu \phi_5 - \frac{\kappa}{2}\partial^\mu \phi + \tfrac{1}{2}[\partial^\mu \chi, \kappa] + \tfrac{1}{2}[\partial^\mu \kappa, \chi] .$$

$\mathcal{L}_1$ contains the free Lagrangian plus the Yang-Mills fields' self-interactions terms; $\mathcal{L}_2$ contains the nucleon-boson interaction terms; $\mathcal{L}_3$ contains the mass terms for the Yang-Mills fields plus the boson-boson interaction terms.

$\mathcal{L}$ contains 6 fictitious degrees of freedom corresponding to the 6 gauge degrees of freedom. We shall discuss how best to eliminate them in the next section. Here we would like to point out that $\mathcal{L}_1$ has two linear terms $-2(\mu_\kappa^2/f)\phi$ and $-\mu_\pi^2\sigma$; and therefore the vacuum expectation value of the ϕ and σ fields do not vanish. In order to obtain a stable vacuum we would have to add to $\mathcal{L}$ two linear terms $+2(\mu_\kappa^2/f)\phi, + \mu_\pi^2\sigma$ as in the σ model. These terms break the chiral symmetry and give a non-vanishing value to the divergence of the axial-vector currents. It is important that this be so since we shall see later that if we formally quantize the Yang-Mills fields, the axial-vector charge can be used to transform a ρ field into an A_1 field. Since these particles have different masses we arrive at a contradiction with conserved axial-vector currents.

We see from (3.10) that the (bare) masses of the Yang-Mills fields are fixed. They are given by

$$m_\rho^2 = \frac{1}{f^2}, \quad m_{A_1}^2 = 1 + \frac{1}{f^2} . \tag{3.12}$$

This yields the ratio

$$\frac{m_{A_1}^2}{m_\rho^2} = 1 + f^2 \tag{3.13}$$

for the masses in terms of the "decay constant" of the fields (ϕ_5, κ) [see (3.11)].

The splitting of the vector and axial-vector masses arises because of the extra contribution to the latter coming from the fifth term in (3.1). This is identical to the origin of the splitting in Barnes and Isham[10] (formulas 20 and 21).

IV. EQUATIONS OF MOTION

By varying the Lagrangian with respect to the field variables we obtain the classical equations of motion. The only ones which we shall consider here are those for the Yang-Mills fields since they imply constraints. We shall show that the constraint equations are identical to the equations for the conservation of the currents.

The equation of motion reads

$$\partial_\mu R^{\mu\nu} = \mathcal{J}^\nu + \gamma^5 \mathcal{J}^{\nu 5} \tag{4.1}$$

where the right-hand side is the total current. The constraint equation is therefore

$$\partial_\nu(\mathcal{J}^\nu + \gamma^5 \mathcal{J}^{\nu 5}) = 0 . \tag{4.2}$$

Equation (4.1) may be written in a manifestly gauge-invariant manner by introducing the covariant derivative of $R_{\mu\nu}$:

$$D_\lambda R_{\mu\nu} = \partial_\lambda R_{\mu\nu} + [\Gamma_\lambda, R_{\mu\nu}] . \tag{4.3}$$

We have therefore, for the equation of motion,

$$D_\mu R^{\mu\nu} = \tfrac{1}{4}[g, D^\nu g] - \frac{k}{f^2} C^\nu - \frac{\partial^\nu k}{2f^2} . \tag{4.4}$$

From the identity $D_\nu D_\mu R^{\mu\nu} = 0$ we obtain

$$D_\lambda(kC^\lambda) + \tfrac{1}{2} \Box k = \tfrac{1}{4} f^2 [g, D_\lambda D^\lambda g] . \tag{4.5}$$

This is the same equation as (4.2).

If we were to calculate the equations of motion for the spin-0 fields we would find also that (4.2) [or (4.5)] followed from them.

In fact, it is of interest to consider these equations in the lowest approximation since they indicate the most natural gauge condition to choose:

$$(\Box + \mu_\kappa^2)\chi = \frac{1}{f} \partial_\mu \rho^\mu ,$$

$$(\Box + \mu_\kappa^2)\kappa = \frac{1}{f} \partial_\mu a^\mu , \tag{4.6}$$

$$(\Box + \mu_\pi^2)\pi = \partial_\mu a^\mu - 2m \bar{\psi} \gamma^5 \sigma_i \psi \sigma_i .$$

The most natural gauge condition is

$$\chi = \kappa = 0 . \tag{4.7}$$

In this gauge, the longitudinal components of the Yang-Mills fields decouple.

V. CURRENTS

Since the Lagrangian is an invariant the equations of motion yield the following identity:

$$\begin{aligned} \delta\mathcal{L} = \partial_\mu \Bigg\{ & \frac{\delta\mathcal{L}}{\delta \partial_\mu \psi} \delta\psi + \frac{\delta\mathcal{L}}{\delta \partial_\mu \rho_\nu^i} \delta\rho_\nu^i + \frac{\delta\mathcal{L}}{\delta \partial_\mu a_\nu^i} \delta a_\nu^i + \frac{\delta\mathcal{L}}{\delta \partial_\mu \chi^i} \delta\chi^i + \frac{\delta\mathcal{L}}{\delta \partial_\mu \kappa^i} \delta\kappa^i \\ & + \frac{\delta\mathcal{L}}{\delta \partial_\mu \phi} \delta\phi + \frac{\delta\mathcal{L}}{\delta \partial_\mu \phi_5} \delta\phi_5 + \frac{\delta\mathcal{L}}{\delta \partial_\mu \sigma} \delta\sigma + \frac{\delta\mathcal{L}}{\delta \partial_\mu \pi^i} \delta\pi^i \Bigg\} \\ = 0 . \end{aligned} \tag{5.1}$$

This yields the equation for the conservation of the vector and axial-vector currents, defined by the equations

$$J^\mu = -i \bar{\psi} \gamma^\mu \sigma_i \psi \sigma_i , \tag{5.2}$$

$$J^{\mu 5} = -i \bar{\psi} \gamma^\mu \gamma^5 \sigma_i \psi \sigma_i , \tag{5.3}$$

$$j^{\mu(\rho)} + \gamma^5 j^{\mu 5(\rho)} = -[R^{\mu\lambda}, \Gamma_\lambda] - \frac{k}{f^2} \Gamma^\mu - \tfrac{1}{4}[[\Gamma^\mu, g], g] , \tag{5.4}$$

$$j^{\mu(\kappa)} + \gamma^5 j^{\mu 5(\kappa)} = \frac{1}{2f^2}(b^* \partial^\mu b - \partial^\mu b^* b) , \tag{5.5}$$

$$j^{\mu(\pi)} + \gamma^5 j^{\mu 5(\pi)} = \tfrac{1}{4}[g, \partial^\mu g] , \tag{5.6}$$

$$\mathcal{J}^\mu = J^\mu + j^{\mu(\rho)} + j^{\mu(\pi)} + j^{\mu(\kappa)} , \tag{5.7}$$

$$\mathcal{J}^{\mu 5} = J^{\mu 5} + j^{\mu 5(\rho)} + j^{\mu 5(\pi)} + j^{\mu 5(\kappa)} . \tag{5.8}$$

We remark here that the usual definition of the currents (see, for example, Gasiorowicz and Geffen[9]) is such that a gauge-invariant term would yield no contribution to them. Since we have a gauge-invariant Lagrangian this would yield a total current which is identically zero. The reason is that this definition includes in the current (5.4), minus the left-hand side of (4.1), so that (4.1) becomes simply the expression of the fact that the total current vanishes.

The total vector and axial-vector charges are

$$Q = \int \mathcal{J}^0 d^3x , \quad Q^5 = \int \mathcal{J}^{05} d^3x . \tag{5.9}$$

It is convenient to introduce the notation

$$Q_\gamma = \tfrac{1}{2} i\,\mathrm{Tr}(Q\alpha + Q^5\beta), \quad \gamma = \alpha + \gamma^5\beta\,. \tag{5.10}$$

We now formally quantize the fields. Using the standard (anti-) commutation relations between the fields and their conjugate momenta, we obtain the charge-field commutation relations:

$$[Q_\gamma, \psi] = \gamma\psi\,, \tag{5.11}$$

$$[Q_\gamma, \chi + \gamma^5\kappa] = -\left(\gamma + \tfrac{1}{2}[\chi + \gamma^5\kappa, \gamma] + \tfrac{1}{2}(\phi + \gamma^5\phi_5)\gamma\right), \tag{5.12}$$

$$[Q_\gamma, \phi + \gamma^5\phi_5] = -\{\chi + \gamma^5\kappa, \gamma\}\,, \tag{5.13}$$

$$[Q_\gamma, \rho_\lambda + \gamma^5 a_\lambda] = -[\rho_\lambda + \gamma^5 a_\lambda, \gamma]\,, \tag{5.14}$$

$$[Q_\gamma, \pi] = -\beta - \beta\sigma - [\pi, \alpha]\,, \tag{5.15}$$

$$[Q_\gamma, \sigma] = -2\{\beta, \pi\}\,. \tag{5.16}$$

We also obtain the charge-algebra commutation relations

$$[Q_\gamma, Q_{\gamma'}] = Q_{[\gamma, \gamma']}\,. \tag{5.17}$$

As we remarked earlier, (5.14) is inconsistent with the existence of vector and axial-vector mesons of different masses.

From (5.12) and (5.13) we can deduce the isospin content of the (ϕ_5, κ) fields. To do this it is convenient to expand the charge and the fields in terms of isospin components. We have

$$Q = -Q^i\sigma_i\,, \quad Q^5 = -Q^{5i}\sigma_i\;; \tag{5.18}$$

so we have

$$\begin{aligned} &[Q^i, \kappa^j] = \tfrac{1}{2} i\phi_5\delta^{ij} + \tfrac{1}{2} i\epsilon^i{}_{jk}\kappa^k\,, \\ &[Q^i, \phi_5] = -\tfrac{1}{2} i\,\kappa^i\,. \end{aligned} \tag{5.19}$$

We see then that we have two isospin-$\frac{1}{2}$ multiplets.

VI. ELECTROMAGNETIC INTERACTIONS

In the general theory of relativity, interaction with the gravitational field is introduced by supposing that a flat connection develops curvature. We introduced in Sec. II a flat connection Γ_μ^0 which we expanded in terms of the (ϕ_5, κ) fields and their scalar parity partners. We shall suppose that electromagnetic interactions cause this connection to acquire nonzero curvature. That is, we suppose that it is replaced by the connection Δ_μ given by

$$\Gamma_\mu^0 \to \Delta_\mu = \Gamma_\mu^0 - \tan\theta A_\mu b^{-1}\sigma_3 b\,. \tag{6.1}$$

θ is a mixing angle between the photon and the ρ^0 meson which we shall determine later in terms of the charge. A_μ is a vector field which is invariant under $SU_2 \times SU_2$ transformations. A straightforward calculation yields the following value for the curvature of Δ_μ:

$$R_{\mu\nu}^0 \to S_{\mu\nu} = -\tan\theta F_{\mu\nu} b^{-1}\sigma_3 b\,. \tag{6.2}$$

$F_{\mu\nu}$ is the curl of A_μ. We define C_μ now as

$$C_\mu = \Gamma_\mu - \Delta_\mu\,. \tag{6.3}$$

If we were to maintain a closer analogy with general relativity we would set C_μ equal to zero and use Δ_μ as the Yang-Mills connection. We did not do this since it would introduce derivative couplings with the nucleons. Such couplings exist between the nucleons and the gravitational field. An attempt has been made[25] to push the analogy we are here considering in the other direction and to introduce a nonvanishing C_μ into gravitational interactions.

Using A_μ we can write down two additional invariants,

$$A_\mu \bar\psi\gamma^\mu\psi, \quad \mathrm{Tr}(S_{\mu\nu}S^{\mu\nu})\,, \tag{6.4}$$

and our final Lagrangian, including electromagnetic interactions, is given by

$$\begin{aligned} \mathfrak{L} = {}& \bar\psi i\gamma^\mu D_\mu\psi - m\bar\psi\psi - \tfrac{1}{2}\tan\theta A_\mu\bar\psi\gamma^\mu\psi \\ & + \frac{1}{4f^2}\mathrm{Tr}(kC_\mu C^{*\mu}) + \tfrac{1}{16}\mathrm{Tr}(D_\mu g D^\mu g) \\ & + \tfrac{1}{8}\mathrm{Tr}(R_{\mu\nu}R^{\mu\nu}) + \frac{1}{8\tan^2\theta}\mathrm{Tr}(S_{\mu\nu}S^{\mu\nu}) \\ & - \frac{\mu_\kappa^2}{4}\mathrm{Tr}(k) - \frac{\mu_\pi^2}{16}\mathrm{Tr}(j)\,. \end{aligned} \tag{6.5}$$

The mass term for the ρ^0 meson is contained in the invariant $\mathrm{Tr}(kC_\mu C^{*\mu})$. With the new definition of C_μ (6.3), this term is no longer diagonal and we have to introduce a mixing between the ρ_μ^3 and A_μ fields to make it so. The physical ρ^0 and photon fields are given by

$$\begin{aligned} &\tilde\rho_\mu^3 = \cos\theta\,\rho_\mu^3 + \sin\theta A_\mu\,, \\ &\tilde A_\mu = -\sin\theta\,\rho_\mu^3 + \cos\theta A_\mu\,. \end{aligned} \tag{6.6}$$

To first order, C_μ is given then by

$$\begin{aligned} C_\mu = {}& \sum_1^2 (\rho_\mu^i + \gamma^5 a_\mu^i)\sigma_i + (\tilde\rho_\mu^3\sec\theta + \gamma^5 a_\mu^3)\sigma_3 \\ & - \partial_\mu\chi - \gamma^5\partial_\mu\kappa - \partial_\mu\phi - \gamma^5\partial_\mu\phi_5\,. \end{aligned} \tag{6.7}$$

With this new definition of the ρ^0 field, the invariant $\mathrm{Tr}(kC_\mu C^{*\mu})$ is diagonal to second order.

The Lagrangian (6.5) includes the following interaction term between the nucleons and the vector bosons ρ^0 and γ:

$$\rho_\mu^3\bar\psi i\sigma_3\gamma^\mu\psi - \tfrac{1}{2}\tan\theta A_\mu\bar\psi i\gamma^\mu\psi\,.$$

Written out in terms of the physical fields this becomes

$$\rho_\mu^3 \bar{\psi} i\sigma_3\gamma^\mu\psi - \tfrac{1}{2}\tan\theta A_\mu \bar{\psi} i\gamma^\mu\psi$$
$$= i\cos\theta\,\tilde{\rho}_\mu^3\bar{\psi}\sigma_3\gamma^\mu\psi - \frac{\sin^2\theta}{2\cos\theta}\tilde{\rho}_\mu^3\bar{\psi}\gamma^\mu\psi$$
$$- \sin\theta\tilde{A}_\mu\bar{\psi}(i\sigma_3 + \tfrac{1}{2})\gamma^\mu\psi\,. \qquad (6.8)$$

So we see that the charge e of the proton is related to the mixing angle θ by

$$\sin\theta = e\,. \qquad (6.9)$$

In the Lagrangian (6.5), the mass of the ρ^0 meson is no longer equal to that of the $\rho^\pm$ mesons. To see this we write out explicitly the appropriate terms:

$$\mathfrak{L} = \frac{1}{4f^2}\mathrm{Tr}(kC_\mu C^{*\mu}) + \cdots$$
$$= -\frac{1}{4f^2}\mathrm{Tr}(k\tilde{\Gamma}_\mu\tilde{\Gamma}^\mu) + \cdots$$
$$= -\frac{1}{4f^2}\mathrm{Tr}[a^2 + (\tilde{\rho}^1\sigma_1 + \tilde{\rho}^2\sigma_2 + \sec\theta\tilde{\rho}^3\sigma_3)^2] + \cdots\,. \qquad (6.10)$$

We have therefore $m_{\rho^0} = (1/f)\sec\theta$ and the following ratio[26]:

$$\frac{m_{\rho^0} - m_{\rho^\pm}}{m_\rho^\pm} = \frac{1}{(1-e^2)^{1/2}} - 1\,. \qquad (6.11)$$

The terms describing the interactions between the (ϕ_5, κ) fields and the electromagnetic field are contained in the same invariant. We have

$$\mathfrak{L} = \frac{1}{4f^2}\mathrm{Tr}(kC_\mu C^{*\mu}) + \cdots$$
$$= e^2\tilde{A}^2\kappa^-\kappa^+ - ie\tilde{A}_\mu(\partial^\mu\kappa^+\kappa^- - \partial^\mu\kappa^-\kappa^+) + \cdots\,, \qquad (6.12)$$

where we have set $\kappa^\pm = (1/\sqrt{2})(\kappa^1 \pm i\kappa^2)$.

We must expand another invariant to obtain the terms describing the interactions between the pions and the electromagnetic field.

$$\mathfrak{L} = \tfrac{1}{16}\mathrm{Tr}(D_\mu g D^\mu g) + \cdots$$
$$= \tfrac{1}{8}\mathrm{Tr}\big(\Gamma_\mu(g\,\partial^\mu g - \partial^\mu g g)\big)$$
$$+ \tfrac{1}{8}\mathrm{Tr}(\Gamma_\mu g\Gamma^\mu g - j\Gamma_\mu\Gamma^\mu) + \cdots$$
$$= e^2\tilde{A}^2\pi^-\pi^+ - ie\tilde{A}_\mu(\partial^\mu\pi^+\pi^- - \partial^\mu\pi^-\pi^+) + \cdots\,. \qquad (6.13)$$

We have broken the intrinsic symmetry of the Lagrangian (3.1) so as to maintain the local gauge invariance, so we still have conserved currents. The currents for the vector and axial-vector mesons are the only ones which are modified. They become

$$j^{\mu(\rho)} + \gamma^5 j^{\mu 5(\rho)} = -[R^{\mu\lambda}, \Gamma_\lambda] - k\Gamma^\mu - \tan\theta A^\mu b^*\sigma_3 b$$
$$- \tfrac{1}{4}[[\Gamma^\mu, g], g]\,. \qquad (6.14)$$

The charge algebra (5.17) remains unaltered, but the charge-field commutation relations involving the $\tilde{\rho}^3$ field are modified. Instead of (5.14) we have (for $\beta = 0$)

$$[Q_\alpha, \tilde{\rho}_\lambda] = -[\tilde{\rho}_\lambda, \alpha] - \tilde{\rho}_\lambda^3[\sigma_3, \alpha](\cos\theta - 1)$$
$$+ \tilde{A}_\lambda[\sigma_3, \alpha]\sin\theta\,,$$
$$[Q_\alpha, \tilde{A}_\lambda] = \tfrac{1}{2}\mathrm{Tr}([\alpha, \tilde{\rho}_\lambda]\sigma_3)\sin\theta\,. \qquad (6.15)$$

We see then that although we have conserved vector currents we can have different masses for the ρ^0 and the $\rho^\pm$ because the vector charge Q_α is not realized as an operator on the space $(\tilde{\rho}_\mu^1, \tilde{\rho}_\mu^2, \tilde{\rho}_\mu^3)$, but on the extended space $(\tilde{\rho}_\mu^1, \tilde{\rho}_\mu^2, \tilde{\rho}_\mu^3, \tilde{A}_\mu)$.

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