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Quantum field theory about a Yang-Mills pseudoparticle*

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The quantum field theory built upon the SU(2) Yang-Mills pseudoparticle solution is studied in an O(5) covariant formalism. The gauge ghost and small oscillations of the Yang-Mills field problems are analyzed completely on the O(5) hypersphere, and it is shown that solutions of the various eigenvalue equations may be easily projected back into Euclidean space. Zero-frequency modes in the small oscillations are found, and their interpretation is given within the framework of the O(5,1) conformal group.

I. INTRODUCTION

Much interest has been generated recently by the existence of an exact solution to the SU(2) Yang-Mills theory in Euclidean four-space, called the pseudoparticle solution, found by Belavin, Polyakov, Schwartz, and Tyupkin.[1] It has been suggested[2] that this solution be used to dominate the Euclidean functional integral of the theory.

Jackiw and Rebbi[3] have shown that the pseudoparticle solution respects a certain O(5) subgroup of the O(5, 1) conformal group. 't Hooft,[4] working in four-space, and Jackiw and Rebbi,[3] using a five-dimensional hyperspherical formalism developed by Adler[5] for QED (quantum electrodynamics) and extended by these authors to non-Abelian gauge groups, have been studying the quantum theory built upon the pseudoparticle solution. In particular, they have studied the effects of adding fermions to the model.

Furthermore, Jackiw and Rebbi,[6] and independently Callan, Dashen, and Gross,[7] have developed an interesting description of the Yang-Mills vacuum in which the pseudoparticle solution plays a central role.

In the present paper, we present details of further calculations of interest in the quantum theory, specifically, we determine the lowest-order gauge ghost and small-oscillations eigenvalues and eigenfunctions for the Yang-Mills model. In this study, we use the O(5) hyperspherical formalism, which presents several calculational advantages over the O(4) formalism. Among these is the ability to obtain exact, analytic solutions to the differential equations which appear, while maintaining manifest O(5) invariance of the equations throughout the calculation. Furthermore, the O(5) formalism may be expected to play a major part in the future developments of the theory, when problems such as regularization and counting of states are approached: Since all differential operators in O(5) possess discrete spectra, these problems may be handled in a very well-defined, straightforward way. Finally, since the O(5) and O(4) formalisms are physically equivalent, one can trivially obtain the solutions to the O(4) equations by projecting back into O(4) the simple O(5) solutions.

The organization of the paper is as follows. In Sec. II we review the O(4) pseudoparticle solution and obtain the O(4) equations for the gauge ghosts and small oscillations of the Yang-Mills field. In Sec. III, we project these equations onto the surface of the O(5) unit hypersphere, and in Sec. IV, we solve the resulting equations in O(5). We find zero-frequency solutions of the small-oscillations problem, which are easily interpreted within the framework of the O(5, 1) conformal group. Finally, we present the details of our calculations in two appendixes, which treat, in particular, the group-theoretical aspects of the problem.

II. O(4) GAUGE THEORY

We study the O(4) gauge group[8] Yang-Mills theory in Euclidean four-space [hereafter O(4)] described by the action

$$I[A] = \tfrac{1}{8} \int d^4x \, \mathrm{Tr} F_{\mu\nu}(x) F_{\mu\nu}(x), \tag{1}$$

where

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) + [A_\mu(x), A_\nu(x)]$$

and the gauge fields $A_\mu(x)$ are anti-Hermitian matrices in the space of generators of the gauge group O(4): $A_\mu(x) = -iA_\mu^{\alpha\beta}(x)\Sigma_{\alpha\beta}$, where $A_\mu^{\alpha\beta}(x) = -A_\mu^{\beta\alpha}(x)$, and an explicit representation of the matrices $\Sigma_{\mu\nu}$ is

$$\Sigma_{\mu\nu} = \frac{1}{4i}[\alpha_\mu, \alpha_\nu],$$
$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \quad i = 1, 2, 3 \tag{2}$$
$$\alpha_4 = i \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}.$$

The $\Sigma_{\mu\nu}$ so chosen are block-diagonal, corresponding to the two SU(2)'s appearing in the decomposition of $O(4)=SU(2)\times SU(2)$.

A very simple solution $\phi_\mu(x)$ to the classical theory described by the action I has been found by Belavin, Polyakov, Schwartz, and Tyupkin[1]:

$$\phi_\mu(x)=-\frac{2i\Sigma_{\mu\nu}x_\nu}{1+x^2}. \tag{3}$$

Recalling the block-diagonality of $\Sigma_{\mu\nu}$, we note that

$$\phi_\mu(x)=\begin{pmatrix}\varphi_\mu(x) & 0\\ 0 & \bar\varphi_\mu(x)\end{pmatrix},$$

where $\varphi_\mu(x)$ and $\bar\varphi_\mu(x)$ are 2×2 anti-Hermitian matrices which represent, respectively, in the language of Jackiw and Rebbi,[3] the "pseudoparticle" and the "antipseudoparticle."

Jackiw and Rebbi[3] have shown that the solution $\phi_\mu(x)$ is invariant[9] under a certain O(5) subgroup of the O(5, 1) conformal group of space-time transformations. The ten generators of this O(5) group are $M_{\mu\nu}$ and $\frac{1}{2}(P_\mu+K_\mu)$, where $M_{\mu\nu}$ generates O(4) rotations, P_μ generates translations, and K_μ generates conformal transformations.[10] To make this invariance of $\phi_\mu(x)$ manifest, these authors find it convenient to project the O(4) space onto the four-dimensional surface of the unit hypersphere imbedded in a five-dimensional Euclidean space [hereafter O(5)] and to further extend the gauge group from O(4) to O(5). We shall find it useful to make these modifications in the theory, since they render all equations trivial to solve. For the moment, however, we remain in the O(4) space with O(4) gauge group for purposes of reviewing the theory.

In order to use the pseudoparticle solution for physical application,[2] one writes the quantum Yang-Mills field as

$$A_\mu(x)=\phi_\mu(x)+a_\mu(x), \tag{4}$$

where $a_\mu(x)$ is taken as a small quantum correction to $\phi_\mu(x)$. The complete, quantum effective action [i.e., including ghost fields and gauge-fixing terms] is expanded in a power series in $a_\mu(x)$, and one obtains differential equations depending upon $\phi_\mu(x)$, which must be satisfied by the propagators of the theory continued to Euclidean space. One may solve these equations, in principle, by determining complete sets of eigenfunctions of the differential operators involved. The eigenfunctions so obtained may be thought of as the linearized fields themselves, and we shall refer to them as such in the sequel. We proceed, therefore, to the derivation of the equations satisfied by these fields. The actual determination of the eigenvalues and eigenfunctions will be taken up in Sec. IV, in the context of the O(5) formalism.

A. The ghost equation

The O(4) ghost field is an anti-Hermitian matrix in the space of generators of the gauge group O(4): $\theta(x)=-i\Sigma_{\mu\nu}\theta_{\mu\nu}(x)$, where $\theta_{\mu\nu}(x)=-\theta_{\nu\mu}(x)$. We obtain a differential equation for $\theta(x)$ by the usual Faddeev-Popov prescription.

First, we specify a gauge condition upon the Yang-Mills fields $A_\mu(x)$:

$$D_\mu(\phi)A_\mu(x)=\frac{4A(x)\cdot x}{1+x^2}, \tag{5}$$

where $D_\mu(\phi)$ is the gauge-covariant derivative in the field of the pseudoparticle solution: $D_\mu(\phi)G=\partial_\mu G+[\phi_\mu(x),G]$ for an arbitrary field G. The gauge condition (5), though unconventional in appearance in O(4), will take on a very simple form in the O(5) formalism. Next, we perform an infinitesimal gauge transformation $e^{\theta(x)}\simeq 1+\theta(x)$ upon the fields $A_\mu(x)$:

$$A_\mu(x)\to A'_\mu(x)=A_\mu(x)+\partial_\mu\theta(x)+[A_\mu(x),\theta(x)]. \tag{6}$$

Demanding that both $A_\mu(x)$ and $A'_\mu(x)$ satisfy Eq. (5), and evaluating the resulting equation for $\theta(x)$ at $A_\mu(x)=\phi_\mu(x)$, we obtain

$$\left\{[D_\lambda(\phi)]^2-\frac{4x^\lambda}{1+x^2}D_\lambda(\phi)\right\}\theta(x)=0\ . \tag{7a}$$

Equation (7a) can now be converted to an eigenvalue equation by writing

$$\left\{[D_\lambda(\phi)]^2+\frac{\mu}{[\frac{1}{2}(1+x^2)]^2}-\frac{4x^\lambda}{1+x^2}D_\lambda(\phi)\right\}\theta^{(\mu)}(x)=0\ , \tag{7b}$$

where μ is a dimensionless numerical constant. The choice of eigenvalue in Eq. (7b) is not arbitrary; we shall see in Sec. III that this equation, when projected into O(5), takes a very simple and natural form.

B. The small-oscillations equation

The field equation for $A_\mu(x)$ which follows from the action $I[A]$ is

$$\partial_\mu F^{\mu\nu}(x)+[A_\mu(x),F^{\mu\nu}(x)]=0. \tag{8}$$

If Eqs. (4) and (5) and the fact that $\phi_\mu(x)$ satisfies $\delta I/\delta A_\mu(x)|_{A_\mu=\phi_\mu}=0$ are used, one obtains from Eq. (8) a nonlinear differential equation for $a_\mu(x)$ which, when linearized in $a_\mu(x)$, takes the form

$$[D_\lambda(\phi)]^2a_\nu(x)-D_\nu(\phi)\left[\frac{4x^\lambda a^\lambda(x)}{1+x^2}\right]+2[a_\lambda(x),f_{\lambda\nu}(x)]=0\ , \tag{9a}$$

where $f_{\lambda\nu}(x)$ is the field tensor formed of $\phi_\mu(x)$. We convert Eq. (9a) to an eigenvalue equation exactly as we did in the case of the ghost equation, obtaining

$$\left\{ [D_\lambda(\phi)]^2 + \frac{\mu}{[\frac{1}{2}(1+x^2)]^2} \right\} a_\nu^{(\mu)}(x) - D_\nu(\phi)\left[\frac{4x^\lambda a^{(\mu)\lambda}(x)}{1+x^2}\right] + 2[a_\lambda^{(\mu)}(x), f_{\lambda\nu}(x)] = 0. \quad (9b)$$

As with the ghost equation, Eq. (9b) will acquire especial simplicity when it is written in the O(5) formalism.

III. THE PROJECTION INTO O(5)

In this section we shall indicate how the equations of Sec. II may be written in the O(5)-covariant language developed by Adler[5] for QED and extended by Jackiw and Rebbi[3] to non-Abelian gauge groups. Our notation will be that of the latter authors.

We first review the details of the projection of the O(4) space onto the four-dimensional surface of the unit hypersphere imbedded in an O(5) space. The O(4) coordinates x_μ (in what follows, Greek indices $\mu, \nu, \lambda, \ldots$ take the values 1, 2, 3, 4 and Latin indices $a, b, c, \ldots$ take the values 1, 2, 3, 4, 5) are projected into the O(5) coordinates r_a, with $r_a r_a = 1$, as follows:

$$r_\mu = \frac{2x_\mu}{1+x^2}, \qquad r_5 = \frac{1-x^2}{1+x^2}, \quad (10)$$

where $x^2 = x_\mu x_\mu$.

In order that one not venture off the surface of the unit sphere, it is necessary to require that only angular derivatives appear in the O(5) theory. These are given by the angular momentum operators

$$L_{ab} = -i\left(r_a \frac{\partial}{\partial r_b} - r_b \frac{\partial}{\partial r_a}\right),$$

with

$$L_{\mu\nu} = -i\left(x_\mu \frac{\partial}{\partial x_\nu} - x_\nu \frac{\partial}{\partial x_\mu}\right),$$
$$L_{5\mu} = -i\left[x_\mu x_\nu \frac{\partial}{\partial x_\nu} + \left(\frac{1-x^2}{2}\right)\frac{\partial}{\partial x_\mu}\right]. \quad (11)$$

The operators $L_{\mu\nu}$ and $L_{5\mu}$ furnish a differential realization of the operators $M_{\mu\nu}$ and $\frac{1}{2}(P_\mu + K_\mu)$, respectively. It should be noted that the relevance of the O(5) formalism to the pseudoparticle solution lies in this fact: It is precisely these operators that generate symmetries under which $\phi_\mu(x)$ is invariant, as pointed out by Jackiw and Rebbi.[3]

One further defines O(5)-covariant fields in terms of their O(4) counterparts. We shall, by way of notation, place a caret above O(5) fields whenever any confusion with O(4) fields may arise. The O(4) fields are characterized by a number d (their scale dimension in mass units) and two sets of indices, one set for ordinary spin (i.e., tensor indices) and another set for internal symmetry (in our case, gauge group indices). The projection to O(5) of these fields only involves the scale dimension d and the tensor indices; internal symmetry indices are not affected. O(5) fields are chosen to be dimensionless, which is achieved by multiplying the O(4) fields by factors of $\frac{1}{2}(x^2+1)$. This factor[11] has dimension −1. Also, tensor indices undergo a certain transformation upon projection into O(5), which we shall indicate only for the gauge fields.

An O(4) scalar field $\phi(x)$ of dimension d becomes, in O(5), the scalar field

$$\hat{\phi}(r) = \left(\frac{1+x^2}{2}\right)^d \phi(x). \quad (12)$$

The transformation law for an O(4) vector field $A_\mu(x)$ of dimension d is as follows. The O(5) field $\hat{A}_a(r)$ is chosen to satisfy the constraint equation $r_a \hat{A}_a(r) = 0$; then, one defines

$$\left(\frac{1+x^2}{2}\right)^d A_\mu(x) = \hat{A}_\mu(r) - x_\mu \hat{A}_5(r),$$
$$\hat{A}_5(r) = -\left(\frac{1+x^2}{2}\right)^{d-1} x_\mu A_\mu(x). \quad (13)$$

Using Eq. (13), with $d = 1$ for Yang-Mills fields, one can exhibit the pseudoparticle solution in O(5), $\hat{\phi}_a(r)$:

$$\hat{\phi}_\mu(r) = -i\Sigma_{\mu\nu} x_\nu,$$
$$\hat{\phi}_5(r) = 0. \quad (14)$$

Following Jackiw and Rebbi,[3] we now extend the gauge group from O(4) to O(5) by adding to the set of generators the four matrices $\Sigma_{\mu 5} = \frac{1}{2}\alpha_\mu$. This extension is merely a mathematical device; it allows one to write the pseudoparticle solution in a form in which its spatial O(5) symmetry is manifest, but the physics of the situation remains unchanged. Indeed, we shall demand that all solutions we obtain using the O(5) gauge group be reducible by a specific gauge transformation to the O(4) group spanned by $\Sigma_{\mu\nu}$.

The pseudoparticle solution $\hat{\phi}_a$ may be transformed into a manifestly O(5)-covariant form by a gauge transformation $U(r)$ as follows. We let

$$U(r) = \frac{1 - 2i\Sigma_{\mu 5} x_\mu}{(1+x^2)^{1/2}}. \quad (15)$$

Then, using the general form for a gauge transformation U upon a gauge field A_a,

$$A_a \xrightarrow[U]{} A_a' = U^{-1} A_a U + U^{-1} l_a U, \tag{16}$$

where $l_a \equiv r_b i L_{ba}$, we obtain

$$\hat{\phi}_a(r) \xrightarrow[U]{} \phi_a(r) = -i\Sigma_{ab} r_b. \tag{17}$$

It is this form ϕ_a which will be extensively used in Sec. IV for solving the ghost and small-oscillations equations in O(5).

We now make use of the methods of this section to project Eqs. (7b) and (9b) into the O(5) space. Since $d=0$ for the ghost field, $\hat{\theta}^{(\mu)}(r) = \theta^{(\mu)}(x)$, so Eq. (7b) becomes, in O(5),

$$\{[\mathfrak{D}_a(\hat{\phi})]^2 + \mu\}\hat{\theta}^{(\mu)}(r) = 0, \tag{18a}$$

where $\mathfrak{D}_a(\hat{\phi})$ is the gauge-covariant derivative in the field of the pseudoparticle solution in O(5) space: $\mathfrak{D}_a(\hat{\phi})\, G = l_a G + [\hat{\phi}_a, G]$, for an arbitrary field G. The form of Eq. (18a) suggests that we can trivially perform the gauge transformation U given by Eq. (15). Since $\mathfrak{D}_a(\hat{\phi})$ is the gauge-covariant derivative, one has $\mathfrak{D}_a(\phi) = U^{-1}\mathfrak{D}_a(\hat{\phi})U$. Thus, if $\hat{\theta}^{(\mu)}(r)$ is chosen to transform in the same way, $\hat{\theta}^{(\mu)} = U^{-1}\hat{\theta}^{(\mu)}(r)U$, we obtain the ghost equation for the O(5) gauge group,

$$\{[\mathfrak{D}_a(\phi)]^2 + \mu\}\theta^{(\mu)}(r) = 0. \tag{18b}$$

The field $\theta^{(\mu)}(r)$ is an anti-Hermitian matrix in the space of generators of the gauge group O(5), $\theta^{(\mu)}(r) = -i\Sigma_{ab}\theta^{(\mu)}_{ab}(r)$, where $\theta^{(\mu)}_{ab}(r) = -\theta^{(\mu)}_{ba}(r)$. However, the functions $\theta^{(\mu)}_{ab}(r)$ are not arbitrary; the equation relating the O(4)-gauge-group field $\hat{\theta}^{(\mu)}$ to the O(5)-gauge-group field $\theta^{(\mu)}$ which was used to derive Eq. (18b),

$$\theta^{(\mu)}(r) = U^{-1}\hat{\theta}^{(\mu)}(r)U, \tag{19a}$$

imposes a restriction upon the components $\theta^{(\mu)}_{ab}(r)$. Explicitly, the connection (19a) becomes, in terms of $\theta^{(\mu)}_{ab}$,

$$\begin{aligned} \theta^{(\mu)}_{5\alpha} &= \hat{\theta}^{(\mu)}_{\alpha\beta} r_\beta, \\ \theta^{(\mu)}_{\alpha\beta} &= \hat{\theta}^{(\mu)}_{\alpha\beta} - \hat{\theta}^{(\mu)}_{\alpha\gamma} r_\gamma x_\beta + \hat{\theta}^{(\mu)}_{\beta\gamma} r_\gamma x_\alpha. \end{aligned} \tag{19b}$$

Equations (19b) then imply the restriction

$$r_a \theta^{(\mu)}_{ab}(r) = 0. \tag{20a}$$

This restriction can also be written in matrix form,

$$[r\cdot\Gamma, \theta^{(\mu)}(r)] = 0, \tag{20b}$$

where the five matrices Γ_a, introduced in the paper of Jackiw and Rebbi[3] for the fermion problem, are given by

$$\begin{aligned} \Gamma_\mu &= i\alpha_\mu \alpha_5, \\ \Gamma_5 &= \alpha_5, \\ \alpha_5 &= \alpha_1\alpha_2\alpha_3\alpha_4. \end{aligned} \tag{21}$$

Equations (20) will take on particular significance in Sec. IV when we solve Eq. (18b).

It should be noted that Eq. (18b) can also be derived completely within the O(5) framework. The O(4) gauge condition Eq. (5), when written in O(5), takes the very simple form

$$\mathfrak{D}_a(\hat{\phi})\hat{A}_a(r) = 0. \tag{22}$$

Applying Eq. (16) to the infinitesimal gauge transformation $e^{\hat{\theta}(r)} \simeq 1 + \hat{\theta}(r)$, and letting $\hat{A}_a(r) = \hat{\phi}_a(r)$, we obtain

$$[\mathfrak{D}_a(\hat{\phi})]^2\hat{\theta}(r) = 0, \tag{23}$$

which, when converted to an eigenvalue equation, reproduces Eq. (18).

The Yang-Mills field equations and action can be written in O(5) by defining the field strengths, which in O(5) form a totally antisymmetric third-rank tensor $\hat{F}_{abc}$. In terms of the gauge fields $\hat{A}_a$, one has

$$\begin{aligned} \hat{F}_{abc}(r) = {} & iL_{ab}\hat{A}_c(r) + r_c[\hat{A}_a(r), \hat{A}_b(r)] \\ & + (\text{cyclic permutations of } a, b, c). \end{aligned} \tag{24}$$

It is possible to show that the O(4) and O(5) field strengths are related as

$$F_{\mu\nu}(x) = \left(\frac{2}{1+x^2}\right)^2 (\hat{F}_{5\mu\nu} + x_\lambda \hat{F}_{\lambda\mu\nu}). \tag{25}$$

When this equation is used, along with the other rules for projection into O(5), one finds the O(5) field equations

$$iL_{ab}\hat{F}_{abc}(r) + [r_a\hat{A}_b(r) - r_b\hat{A}_a(r), \hat{F}_{abc}(r)] = 0. \tag{26}$$

Performing a similar projection upon the small oscillations Eq. (9b), one finds in O(5)

$$\{[\mathfrak{D}_b(\hat{\phi})]^2 + \mu - 2\}\hat{a}^{(\mu)}_a(r) + 2r_b[\hat{a}^{(\mu)}_c(r), \hat{f}_{bca}(r)] = 0, \tag{27a}$$

where $\hat{f}_{abc}(r)$ is the field tensor constructed of $\hat{\phi}_a(r)$. Equation (27a) can easily be placed into the O(5) gauge group by defining the gauge-transformed $a^{(\mu)}_a(r) \equiv U^{-1}\hat{a}^{(\mu)}_a(r)U$. Then Eq. (27a) is seen to be gauge invariant and one has

$$\{[\mathfrak{D}_b(\phi)]^2 + \mu - 2\}\, a^{(\mu)}_a(r) + 2r_b[a^{(\mu)}_c(r), f_{abc}(r)] = 0, \tag{27b}$$

where

$$\begin{aligned} f_{abc}(r) &= U^{-1}\hat{f}_{abc}(r)U \\ &= i(r_a\Sigma_{bc} + r_b\Sigma_{ca} + r_c\Sigma_{ab}). \end{aligned}$$

The necessity of eventual reduction back to the O(4) gauge group, as in the ghost case, imposes restrictions upon $a^{(\mu)}_a(r)$. In O(5), $a^{(\mu)}_a(r)$ is an anti-Hermitian matrix in the space of generators of O(5): $a^{(\mu)}_a(r) = -ia^{(\mu)a}_{bc}(r)\Sigma_{bc}$, where $a^{(\mu)a}_{bc} = -a^{(\mu)a}_{cb}$. The condition of reducibility to O(4) is then

$$r_b a_{bc}^{(\mu)a} = 0, \tag{28a}$$

or

$$[r\cdot\Gamma, a_a^{(\mu)}] = 0. \tag{28b}$$

Of course, Eq. (26) may also be obtained directly in the O(5) formalism from the action

$$I[\hat{A}] = \frac{1}{24}\int d\Omega\, \mathrm{Tr}\hat{F}_{abc}(r)\hat{F}_{abc}(r), \tag{29}$$

where the integration is over the surface of the unit hypersphere in O(5); this action is identical to the O(4) action Eq. (1).

In turn, Eqs. (27) may be obtained from Eq. (26) by writing

$$\hat{A}_a(r) = \hat{\phi}_a(r) + \hat{a}_a(r), \tag{30}$$

and linearizing the resulting equation in $\hat{a}_a(r)$.

What we have succeeded in proving in this section is the equivalence of the O(4) and O(5) formalisms, provided the O(4) gauge group is used in both spaces. If the O(5) gauge group is used, one must also satisfy Eqs. (20) and (28). All equations in O(4) directly correspond to equations in O(5), and the O(5) equations, which will be straightforward to solve, possess solutions which can be easily converted back to the O(4) space. We proceed, therefore, to the solution of Eqs. (18b) and (27b) subject to constraints (20) and (28).

IV. ANALYSIS OF O(5) EQUATIONS

The eigenvalue equations derived in Sec. III, Eqs. (18b) and (27b), have an important symmetry which aids in their solution and, when properly interpreted, allows one to see that the underlying gauge group is really SU(2), as expected. In particular, this symmetry makes it possible to exhibit the SU(2)-gauge-group, O(4)-space solutions to the ghost and small-oscillations equations, which are the results one actually seeks in this research.

To exhibit this symmetry, we shall, for definiteness, study the ghost equation (18b), though these considerations will hold for the small oscillations Eq. (27b) as well. The differential operator which appears in Eq. (18b) can be shown to satisfy[12]

$$(r\cdot\Gamma)[\mathfrak{D}_a(\phi)]^2(r\cdot\Gamma) = [\mathfrak{D}_a(\phi)]^2, \tag{31}$$

where Eq. (31) should be thought of as an operator equation. If we multiply Eq. (18b) on the left by $(r\cdot\Gamma)$ and use Eq. (31), we obtain

$$\{[\mathfrak{D}_a(\phi)]^2 + \mu\}(r\cdot\Gamma)\theta^{(\mu)}(r) = 0, \tag{32}$$

which has the obvious meaning that if $\theta^{(\mu)}(r)$ is a solution of Eq. (18b) with eigenvalue μ, then so is $\tilde{\theta}^{(\mu)}(r) = (r\cdot\Gamma)\theta^{(\mu)}(r)$. Thus, we expect to find pairs of degenerate eigenfunction solutions to (18b). One can write $\tilde{\theta}_{ab}^{(\mu)}(r)$ in terms of $\theta_{ab}^{(\mu)}(r)$ as follows. Since $[\theta^{(\mu)}, r\cdot\Gamma] = 0$,

$$\begin{aligned}\tilde{\theta}^{(\mu)}(r) &= \tfrac{1}{2}\{r\cdot\Gamma, \theta^{(\mu)}(r)\}\\ &= \frac{i}{2}\epsilon_{abcde}r_a\theta_{bc}^{(\mu)}(r)\Sigma_{de}\\ &= -i\Sigma_{ab}[-\tfrac{1}{2}\epsilon_{abcde}r_c\theta_{de}^{(\mu)}(r)],\end{aligned} \tag{33a}$$

one has

$$\tilde{\theta}_{ab}^{(\mu)}(r) = -\tfrac{1}{2}\epsilon_{abcde}r_c\theta_{de}^{(\mu)}(r). \tag{33b}$$

We need only mention at this point that, by similar arguments, one deduces that if $a_a^{(\mu)}(r)$ is a solution of Eq. (27b) with eigenvalue μ, then $\tilde{a}_a^{(\mu)}(r) = (r\cdot\Gamma)a_a^{(\mu)}(r)$ is also a solution with the same eigenvalue μ. Further,

$$\tilde{a}_{bc}^{(\mu)a}(r) = -\tfrac{1}{2}\epsilon_{bcdef}r_d a_{ef}^{(\mu)a}(r). \tag{34}$$

A. Solutions to the ghost equation

Using Eqs. (18b) and (20), we can write an equation which must be satisfied by $\theta_{ab}^{(\mu)}(r)$:

$$(L^2 + 2 - \mu)\theta_{ab}^{(\mu)}(r) - 2i[L_{ac}\theta_{cb}^{(\mu)}(r) - L_{bc}\theta_{ca}^{(\mu)}(r)] = 0, \tag{35}$$

where $L^2 = -l_a l_a = \frac{1}{2}L_{ab}L_{ab}$.

The details of the analysis and solution of this equation may be found in Appendix A. Here, we merely state the results. One obtains two degenerate unnormalized solutions

$$\begin{aligned}\theta_1^{(\mu)}(r) &= i(r_a - l_a)\mathcal{Y}_b^{nm}(r)\Sigma_{ab},\\ \theta_2^{(\mu)}(r) &= \tilde{\theta}_1^{(\mu)}(r)\\ &= (r\cdot\Gamma)\theta_1^{(\mu)}(r),\end{aligned} \tag{36}$$

with $\mu = n(n+3) - 2$, where $\mathcal{Y}_a^{nm}(r)$ are the vector harmonics, whose properties have been given by Adler.[5] We briefly discuss these harmonics in Appendix A.

B. Solutions to the small-oscillations equations

Equation (27b) may be analyzed in a manner very similar to that given above for Eq. (18b). Indeed, one may write equations for the functions $a_{bc}^{(\mu)a}(r)$:

$$\begin{aligned}(L^2 + 8 - \mu)a_{bc}^{(\mu)a} &- 2(iL_{ad}a_{bc}^{(\mu)d} + iL_{bd}a_{dc}^{(\mu)a} + iL_{cd}a_{bd}^{(\mu)a})\\ &- 2(\delta_{ab}a_{dc}^{(\mu)d} - \delta_{ac}a_{db}^{(\mu)d})\\ &- 2(a_{bc}^{(\mu)a} + a_{ca}^{(\mu)b} + a_{ab}^{(\mu)c}) = 0,\end{aligned} \tag{37a}$$

which is equivalent to (27b), and

$$l_a a_{bc}^{(\mu)a} + r_b a_{ac}^{(\mu)a} - r_c a_{ab}^{(\mu)a} = 0, \tag{37b}$$

which is the gauge condition $\mathfrak{D}_a(\phi)a^{(\mu)a} = 0$, obtained from Eq. (22) by the usual gauge transformation U of Eq. (15). The tensor $a_{bc}^{(\mu)a}$ must also satisfy the constraints imposed by the O(5) projection method

$$r_a a_{bc}^{(\mu)a} = 0 \ ,\quad r_b a_{bc}^{(\mu)a} = 0 \ . \tag{38}$$

Equation (37a), subject to constraints (37b) and (38), may be solved by considering the complete set of irreducible representations contained in the decomposition of the reducible representation appropriate to tensors of the form A_{bc}^a, where $A_{bc}^a = -A_{cb}^a$, i.e., the direct product $(1,0)\otimes(1,1)\otimes(n,0)$. This procedure is outlined in Appendix B. Here, we quote the results of that study. We obtain the following (unnormalized) eigenfunction solutions of Eqs. (37) and (38):

$$a_1^{(\mu)a} = i\epsilon_{abcde}\Sigma_{bc}L_{de}Y_{nm},\quad a_2^{(\mu)a} = r\cdot\Gamma a_1^{(\mu)a}, \tag{39a}$$

with $\mu = n(n+3)-4$, $n=1,2,\ldots$, $m=1,2,\ldots,d(n,0)$,

$$a_3^{(\mu)a} = i(r_a + l_a)\epsilon_{bcdef}\Sigma_{bc}L_{de}\mathcal{Y}_{nm}^f - 2\epsilon_{abcde}\Sigma_{bc}[(\mu-1)r_d + l_d]\mathcal{Y}_{nm}^e \ , \tag{39b}$$

$$a_4^{(\mu)a} = r\cdot\Gamma a_3^{(\mu)a},$$

with $\mu = n(n+3)-2$, $n=1,2,\ldots$, $m=1,2,\ldots,d(n,1)$,

$$a_5^{(\mu)a} = i\epsilon_{bcdef}\Sigma_{bc}L_{de}Z_{nm}^{fa} \ ,\quad a_6^{(\mu)a} = r\cdot\Gamma a_5^{(\mu)a}, \tag{39c}$$

with $\mu = n(n+3)+2$, $n=2,3,\ldots$, $m=1,2,\ldots,d(n,2)$, where Y_{nm} are scalar harmonics, with magnetic quantum number m ranging from 1 through $d(n,0) = (\frac{1}{6})(n+1)(n+2)(2n+3)$, and Z_{nm}^{ab} are second-rank tensor harmonics, with m ranging from 1 through $d(n,2) = (\frac{5}{6})(n-1)(n+4)(2n+3)$. The properties of these harmonics are listed in the Appendixes.

C. Zero-frequency modes

If one examines the eigenvalue spectra of the ghosts and small oscillations, Eqs. (36) and (39), one readily sees that the only solutions whose eigenvalues can vanish are the $n=1$ solutions of Eq. (39a). Since the (1, 0) representation of O(5) has dimension $d(1,0)=5$, there are five pairs of zero-frequency modes present in the small oscillations. The five spherical harmonics $Y_1^{(m)}$, $m=1,\ldots,5$, may be chosen to be

$$Y_1^{(m)} = r_m, \tag{40}$$

so that the zero-frequency modes of Eq. (39) are

$$a_{(m)}^{(0)a}(r) = i(\Sigma_{am} - r_a r_c\Sigma_{cm} + r_m r_c\Sigma_{ca}),\quad \tilde{a}_{(m)}^{(0)a}(r) = (r\cdot\Gamma)a_{(m)}^{(0)a}(r). \tag{41}$$

We shall prove in Sec. IV D that the degeneracy produced by the $(r\cdot\Gamma)$ symmetry corresponds precisely to the fact that there are two SU(2)'s in the decomposition of O(4) = SU(2)×SU(2), so that we may focus our attention solely upon $a_{(m)}^{(0)a}(r)$.

These zero-frequency modes may be understood in the following way. The massless Yang-Mills field theory we are here considering is invariant under the action of the full O(5, 1) conformal group, a group consisting of 15 infinitesimal generators, which may be taken to be $M_{\mu\nu}$, $R_\mu \equiv \frac{1}{2}(K_\mu + P_\mu)$, $S_\mu \equiv \frac{1}{2}(K_\mu - P_\mu)$, and D.[13] Jackiw and Rebbi[3] have shown in addition that the pseudoparticle solution is invariant under the O(5) subgroup of the conformal group which is generated by $M_{\mu\nu}$ and R_μ. Thus, there are five remaining generators, S_μ, D, which can be formed into a five-vector S_a in the O(5) formalism,[13] and which commute with the field equations but do not annihilate $\phi_a(r)$. The five vectors $A_b^{(a)}(r) = \delta^a\phi_b(r) \equiv i[S_a, \phi_b]$, where the commutator is taken in the field-theoretic sense, must then be zero-frequency solutions of the small-oscillations equation (27b).[14]

The action[15] of S_a upon an O(5) vector field $A_b(r)$, ignoring for the moment gauge group indices, is

$$\delta_a^{(1)}A_b(r) \equiv i[S_a, A_b(r)] = -(l_a - r_a)A_b(r) - r_bA_a(r). \tag{42a}$$

The effects of the gauge-group structure may be included in (42a) by performing, in addition, the infinitesimal gauge transformation $e^{C_a\phi_a(r)} \simeq 1 + C_a\phi_a(r)$, where C_a is infinitesimal. Using Eq. (16) and afterward dropping C_a, we obtain

$$\delta_a^{(2)}A_b(r) = l_b\phi_a(r) - [\phi_a(r), A_b(r)], \tag{42b}$$

as the change in $A_b(r)$ due to the gauge transformation alone. The sum of Eqs. (42a) and (42b) gives, then, the complete change in $A_b(r)$ due to the generator $\tilde{S}_a$ which includes[16] both space-time and gauge transformations:

$$\delta_aA_b(r) \equiv i[\tilde{S}_a, A_b(r)] = -[\mathfrak{D}_a(\phi) - r_a]A_b(r) + l_b\phi_a(r) - r_bA_a(r). \tag{42c}$$

Applying $\tilde{S}_a$ to $\phi_b(r)$, finally, gives

$$\delta_a\phi_b(r) = a_{(a)}^{(0)b}(r), \tag{43}$$

indicating that the existence of five zero-frequency modes is consistent with the requirements of conformal invariance of the theory.

D. Projection back to SU(2) gauge group

Our final task is to indicate how the results of this section may be brought back to the physically interesting O(4) space and SU(2) gauge group.

It is clear that one may bring oneself from the O(5) to O(4) gauge groups simply by applying the gauge transformation $U^{-1} = U^\dagger$, where U is given by Eq. (15). Then, using the projection methods of Sec. III, one can move from the O(5) space back into O(4) space.

All that remains is to prove that what one obtains by these operations is two versions of the SU(2) gauge theory in O(4) space, one of which is expanded about the pseudoparticle solution $\varphi_\mu(x)$ and the other of which is expanded about the antipseudoparticle solution $\bar{\varphi}_\mu(x)$.

The double degeneracy of solutions we have found in the O(5) theory, which is completely accounted for by the $(\Gamma\cdot r)$ symmetry, can be understood as follows. If two degenerate solutions of one of the equations of this section (we need not distinguish here between ghosts and small oscillations, since the only objects of importance are the Σ matrices), $\theta^{(\mu)}(r)$ and $\tilde{\theta}^{(\mu)}(r)=(\Gamma\cdot r)\theta^{(\mu)}(r)$, are projected back to the O(4) gauge group by the transformation U^{-1}, one obtains

$$\begin{aligned}\hat{\theta}^{(\mu)}(r)&=U\theta^{(\mu)}(r)U^{-1},\\ \hat{\tilde{\theta}}^{(\mu)}(r)&=U(\Gamma\cdot r)U^{-1}U\theta^{(\mu)}(r)U^{-1}\\ &=\alpha_5\hat{\theta}^{(\mu)}(r),\end{aligned} \tag{44}$$

where α_5 is the matrix defined by Eq. (21), and is explicitly given by

$$\alpha_5=\begin{pmatrix}-I & 0\\ 0 & I\end{pmatrix}. \tag{45}$$

Thus, the combinations $\frac{1}{2}(\hat{\theta}^{(\mu)}-\hat{\tilde{\theta}}^{(\mu)})=\frac{1}{2}(1-\alpha_5)\hat{\theta}^{(\mu)}(r)$ and $\frac{1}{2}(\hat{\theta}^{(\mu)}+\hat{\tilde{\theta}}^{(\mu)})=\frac{1}{2}(1+\alpha_5)\hat{\theta}^{(\mu)}(r)$ are solutions for the upper and lower SU(2) gauge groups, respectively, and are derived for the pseudoparticle and antipseudoparticle SU(2) Yang-Mills theory, respectively.

The $(\Gamma\cdot r)$ symmetry apparent in the O(5)-gauge-group theory is therefore completely accounted for by the fact that two SU(2)'s appear in the decomposition of O(4) = SU(2) × SU(2). It is simply a manifestation of the mathematical device used to treat simultaneously the pseudoparticle and antipseudoparticle; it has no physical content.

The final step in this study is to find the SU(2)-gauge-group solutions to the O(4)-space equations. Using the procedures outlined in this paper, this last step is trivial to accomplish, and we do not carry out the details.

V. CONCLUSIONS

We have presented the details of two calculations pertaining to the expansion of the quantum Yang-Mills theory about the pseudoparticle solution; namely, the solution of the lowest-order gauge ghost and small-oscillations differential equations. The method used, the O(5) formalism, has made it possible to obtain analytic solutions to these equations, which can then be transformed back into ordinary Euclidean four-space by a well-defined procedure.

It is clear that the O(5) formalism will also allow for simplifications of further calculations, notably regularization and counting of states in the theory. In addition, it should be possible to solve directly for the ghost propagator using methods similar to those developed by Jackiw and Rebbi[3] for the fermion problem.

These further investigations of the theory, however, we defer to a separate publication.

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APPENDIX A: THE GROUP THEORY OF THE GHOSTS

In this appendix we discuss the details of the solution of Eq. (35) of Sec. III, and some aspects of the group theory of the solutions. Our treatment of the latter topic follows closely that given by Adler[5] in his study of the free photon propagator in O(5) QED.

The group O(5) is a simple Lie group of rank 2. The irreducible representations of O(5) are specified by two integers[17] (α_1,α_2), with $\alpha_1\geq\alpha_2$. The dimension of the representation (α_1,α_2), denoted by $d(\alpha_1,\alpha_2)$, is

$$d(\alpha_1,\alpha_2)=\tfrac{1}{6}(\alpha_1-\alpha_2+1)(\alpha_1+\alpha_2+2)(2\alpha_1+3)(2\alpha_2+1). \tag{A1}$$

For example, the $(n,0)$ representation is spanned by the O(5) spherical harmonics[18] $Y_{nm}(r)$, which are eigenfunctions of L^2 with eigenvalue $n(n+3)$,

$$L^2Y_{nm}(r)=n(n+3)Y_{nm}(r), \tag{A2}$$

where the magnetic quantum number m satisfies $1\leq m\leq d(n,0)$.

To solve Eq. (35), one might look for all possible antisymmetric, second-rank tensors which are eigenfunctions of L^2. These must all be contained in the set

$$\theta_{ab}^{(kl)nm}(r)=v_{ab}^{(kl)}Y_{nm}(r), \tag{A3}$$

where $v_{ab}^{(kl)}=\frac{1}{2}(\delta_a^k\delta_b^l-\delta_b^k\delta_a^l)$, $k,l=1,2,3,4,5$, is a set of 10 constant tensors, since

$$\sum_{n=0}^{\infty}\sum_{m=1}^{d(n,0)}\sum_{k=1}^{5}\sum_{l=1}^{5}\theta_{ab}^{(kl)nm}(r)\theta_{cd}^{(kl)nm*}(r')=\tfrac{1}{2}(\delta_c^a\delta_d^b-\delta_d^a\delta_c^b)\delta_S(r-r'), \tag{A4}$$

where $\delta_S(r-r')$ is the surface δ function on the unit sphere. However, the tensors $\theta^{(kl)nm}_{ab}(r)$ transform according to the reducible product representation $(1,1)\otimes(n,0)$, so we must deduce the irreducible representation content of $(1,1)\otimes(n,0)$:

$$(1,1)\otimes(n,0)=(n+1,1)\oplus(n,1)\oplus(n-1,1)\oplus(n,0). \tag{A5}$$

We may check the correctness of (A5) by computing the dimension of each side. Using Eq. (A1), we find

$$\begin{aligned} d[(1,1)\otimes(n,0)] &= [d(1,1)][d(n,0)] \\ &= 10\times\tfrac{1}{6}(n+1)(n+2)(2n+3) \\ &= d(n+1,1)+d(n,1)+d(n-1,1)+d(n,0), \end{aligned} \tag{A6}$$

so that (A5) checks dimensionally.

We exhibit explicitly tensors transforming according to the four irreducible representations of the direct sum Eq. (A5) in Table I. We also specify there an additional eigenvalue λ, which serves to split the different representations.

The eigenvalue λ is defined as

$$\tfrac{1}{2}[L_{ab},L_{cd}]\,Y^{(\lambda)nm}_{cd}=\lambda Y^{(\lambda)nm}_{ab}, \tag{A7}$$

where the eigenvalue equation (A7) is relevant to the splitting of irreducible representations since it is rotationally invariant.

The vector harmonics $\mathcal{Y}^{nm}_a$ which appear in Table I span the $(n,1)$ representation of O(5).[5] They have the properties

$$\begin{aligned} &r_a\mathcal{Y}^{nm}_a(r)=0, \\ &iL_{ab}\mathcal{Y}^{nm}_b(r)=\mathcal{Y}^{nm}_a(r), \\ &L^2\mathcal{Y}^{nm}_a(r)=n(n+3)\mathcal{Y}^{nm}_a(r). \end{aligned} \tag{A8}$$

We may now study the ghost equation (35) in terms of this group-theoretical apparatus. A unique property of Eq. (35) makes this analysis very simple. This property is the following. We consider the first Casimir operator of O(5) for the set of tensors of the type (A3), which is explicitly

$$\begin{aligned} &(C_1)^{ab,cd}=\tfrac{1}{2}J^{ab,gh}_{ef}J^{gh,cd}_{ef}, \\ &J^{ab,cd}_{ef}=\tfrac{1}{2}\delta^{ab}_{cd}L_{ef}+S^{ab,cd}_{ef}, \\ &S^{ab,cd}_{ef}=-\frac{i}{2}(\delta^{ef}_{ag}\delta^{gb}_{cd}+\delta^{ef}_{bg}\delta^{ag}_{cd}), \end{aligned} \tag{A9}$$

where $\delta^{ab}_{cd}=\delta^a_c\delta^b_d-\delta^a_d\delta^b_c$, and $S^{ab,cd}_{ef}$ is the spin matrix appropriate to tensors of the type (A3). Equations (A9) may be rewritten in the form

$$(C_1)^{ab,cd}=\tfrac{1}{2}\delta^{ab}_{cd}(L^2+6)-[L_{ab},L_{cd}], \tag{A10}$$

which implies that the ghost eigenvalue equation (35), when written in terms of C_1, takes a particularly simple form:

$$(C_1)^{ab,cd}\,\theta^{(\mu)}_{cd}=(\mu+4)\,\theta^{(\mu)}_{ab}. \tag{A11}$$

The eigenvalue of C_1 in an irreducible representation (n,s) is well known in group theory:

$$C_1(n,s)=n(n+3)+s(s+1), \tag{A12a}$$

so, for the representations listed in Table I, we have

$$\begin{aligned} &C_1(n,0)=n(n+3), \\ &C_1(n+1,1)=(n+1)(n+4)+2, \\ &C_1(n,1)=n(n+3)+2, \\ &C_1(n-1,1)=(n-1)(n+2)+2. \end{aligned} \tag{A12b}$$

Equations (A11) and (A12) now permit us to deduce the correct μ eigenvalues for these representations; however, not all these representations are admissible solutions to the physical problem, since they do not all satisfy the constraint $r_a\theta^{(\mu)}_{ab}=0$. Only the $(n,1)$ representation in Table I satisfies this constraint without modification. To find other solutions of the physical problem, we must form linear combinations of the remaining representations in Table I. These combinations must not mix different representations; thus, since there is only one $(n,0)$-type representation, we may not mix it with anything else, and, since the $(n,0)$ functions in Table I are not orthogonal to r_a, we conclude that there can be no $(n,0)$-type solutions to the physical problem. This is not true of the remaining $(n,1)$-type solutions since, by shifting the eigenvalue n, we may bring their μ eigenvalues into coincidence, according to Eqs. (A11) and (A12). Indeed, if we form

$$\begin{aligned} \theta^{nm}_{ab} &= (n+1)\,Y^{(n+4)\,n+1,m}_{ab}+(n+2)\,Y^{(1-n)\,n-1,m}_{ab} \\ &= (2n+3)\left[(l_a-r_a)\mathcal{Y}^b_{nm}-(l_b-r_b)\mathcal{Y}^a_{nm}\right], \end{aligned} \tag{A13}$$

we may easily verify that θ^{nm}_{ab} is an admissible solution to the physical problem. This is the first solution quoted in Eq. (36). The second solution given in Eq. (36) is just the $(n,1)$ eigenfunction in

TABLE I. Irreducible representations of the product $(1,1)\otimes(n,0)$.

(α_1,α_2)	$Y^{(\lambda)nm}_{ab}(r)$	λ
$(n,0)$	$iL_{ab}Y_{nm}$	$+3$
$(n-1,1)$	$[l_a-(n+2)r_a]\mathcal{Y}^{n-1\,m}_b-(a\leftrightarrow b)$	$+(n+3)$
$(n,1)$	$\epsilon_{abcde}\,iL_{cd}\mathcal{Y}^{nm}_e$	$+2$
$(n+1,1)$	$[l_a+(n+1)r_a]\mathcal{Y}^{n+1\,m}_b-(a\leftrightarrow b)$	$-n$

Table I which, as we noted above, does not need to be mixed with anything else to be orthogonal to r_a.

APPENDIX B: THE GROUP THEORY OF THE SMALL OSCILLATIONS

In this appendix, we apply the same techniques that were used to solve the ghost problem in the previous appendix to analyze the small oscillations equations (37) and (38). Thus, we study tensors of the form A^a_{bc}, with $A^a_{bc} = -A^a_{cb}$. These tensors are contained in the set

$$A^{a(jkl)}_{bc} = v^{a(jkl)}_{bc}\, Y_{nm}\,, \tag{B1}$$

where $v^{a(jkl)}_{bc} = \frac{1}{2}\delta_{aj}\,\delta^{kl}_{bc}$, $j, k, l = 1, 2, 3, 4, 5$, is a set of 50 constant tensors, since

$$\sum_{n=0}^{\infty}\sum_{m=1}^{d(n,0)}\sum_{j=1}^{5}\sum_{k=1}^{5}\sum_{l=1}^{5} A^{a(jkl)}_{bc}(r)\,A^{d(jkl)*}_{ef}(r') = \tfrac{1}{2}\,\delta_{ad}\,\delta^{bc}_{ef}\,\delta_S(r - r')\,. \tag{B2}$$

The tensor (B1) transforms according to the reducible product representation $(1, 0)\otimes(1, 1)\otimes(n, 0)$, which can be decomposed into a direct sum of irreducible representations as follows:

$$\begin{aligned}(1, 0)\otimes(1, 1)\otimes(n, 0) &= (n+1, 2)\oplus(n, 2)\oplus(n-1, 2)\\ &\quad\oplus(n+2, 1)\oplus 2(n+1, 1)\oplus 4(n, 1)\oplus 2(n-1, 1)\\ &\quad\oplus(n-2, 1)\oplus 2(n+1, 0)\oplus(n, 0)\oplus 2(n-1, 0)\,.\end{aligned} \tag{B3}$$

The new features that appear in the decomposition (B3) are the $(n, 2)$-type representations. These are the second-rank tensor harmonics, which are symmetric tensors $Z^{ab}_{nm}(r)$ satisfying

$$\begin{aligned}&L^2 Z^{ab}_{nm} = n(n+3)\,Z^{ab}_{nm}\,,\\ &i\,L_{ab}\,Z^{bc}_{nm} = Z^{ac}_{nm}\,,\\ &Z^{aa}_{nm} = 0\,,\\ &r_a\,Z^{ab}_{nm} = 0\,.\end{aligned} \tag{B4}$$

We do not explicitly exhibit these harmonics; however, we note that all completeness relations and the like satisfied by Z^{ab}_{nm} may be deduced in principle from the decomposition of $(2, 0)\otimes(n, 0) = (n+1, 1)\oplus(n-1, 1)\oplus(n+2, 0)\oplus(n, 0)\oplus(n-2, 0)\oplus(n, 2)$, being the set of all traceless, symmetric second-rank tensors, of which Z^{ab}_{nm} is the $(n, 2)$ member.

Just as in the ghost case, we can construct the Casimir operator appropriate to tensors of the type (B1). We have

$$\begin{aligned}&(C_1)^{a,d}_{bc,ef} = \tfrac{1}{2} J^{a,bc;g,hi}_{jk}\, J^{g,hi;d,ef}_{jk}\,,\\ &J^{a,bc;g,hi}_{jk} = \tfrac{1}{2}\,\delta_{ag}\,\delta^{bc}_{hi}\,L_{jk} + S^{a,bc;g,hi}_{jk}\,,\\ &S^{a,bc;g,hi}_{jk} = -\frac{i}{2}\,(\delta^{jk}_{ag}\,\delta^{bc}_{hi} + \delta_{ag}\,\delta^{jk}_{bl}\,\delta^{lc}_{hi} + \delta_{ag}\,\delta^{jk}_{cl}\,\delta^{bl}_{hi}),\end{aligned} \tag{B5}$$

where $S^{a,bc;g,hi}_{jk}$ is the spin matrix appropriate to tensors of the type (B1). Explicitly,

$$\begin{aligned}(C_1)^{a,d}_{bc,ef}A^d_{ef} &= (L^2+12)A^a_{bc} - 2(i\,L_{ad}\,A^d_{bc} + i\,L_{bd}\,A^a_{dc} + i\,L_{cd}\,A^a_{bd})\\ &\quad - 2(\delta_{ab}A^d_{dc} - \delta_{ac}A^d_{db}) - 2(A^a_{bc} + A^b_{ca} + A^c_{ab})\,,\end{aligned} \tag{B6}$$

for an arbitrary tensor A^a_{bc} of the proper symmetry. A glance at the small-oscillations equation (37a) reveals that it may be written

$$(C_1)^{a,d}_{bc,ef}\,a^{(\mu)d}_{ef} = (\mu + 4)\,a^{(\mu)a}_{bc}\,, \tag{B7}$$

so that the eigenvalues μ may be directly deduced from those of the Casimir operator, which [cf. Eq. (A12)] depend only upon which irreducible representation is being considered:

$$C_1(n, s) = n(n+3) + s(s+1)\,. \tag{B8}$$

We may find functions transforming according to the irreducible representations appearing in the decomposition (B3) by simultaneously diagonalizing the two operators L^2 and Λ, where

$$(\Lambda A)^a_{bc} = i\,L_{ad}\,A^d_{bc} + i\,L_{bd}\,A^a_{dc} + i\,L_{cd}\,A^a_{bd}\,. \tag{B9}$$

We do not exhibit this rather lengthy set of eigenfunctions. Instead, we describe the procedure one uses to find functions satisfying the small-oscillations equation (37a) and the constraints (37b) and (38). The procedure is the same one used in the ghost case: Form linear combinations of eigenfunctions which, if only the same irreducible representations are mixed, are automatically solutions of (B6), and which are chosen in such a way that they satisfy (37b) and (38). In this manner, one finds the eigenfunction solutions listed in Eqs. (39).

Finally, by these group-theoretical arguments, it is clear that there can be no other solutions satisfying all the necessary conditions.

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[8]One is actually interested in an SU(2) gauge group; the extension to SU(2) × SU(2) = O(4) is merely a convenient device for studying simultaneously the quantum theories expanded about the pseudoparticle and antipseudoparticle solutions. See the discussion following Eq. (3).

[9]Invariance for a gauge field means that any change in the field may be compensated for by a gauge transformation.

[10]For a review of the transformations of the conformal group, see S. Treiman, R. Jackiw, and D. Gross, *Lectures on Current Algebra and its Applications* (Princeton Univ. Press, Princeton, 1972), p. 97.

[11]The factor is usually written $(x^2+\lambda^2)/2\lambda$, λ being a parameter with units of length, but we uniformly set λ to unity in this paper.

[12]This symmetry has already been used by Jackiw and Rebbi, Ref. 3, to solve the fermion problem.

[13]Cf. S. Fubini, CERN report, 1976 (unpublished), for a complete discussion.

[14]This argument for the existence of zero-frequency modes is merely the extension to the conformal group of the usual argument given for solitons; e.g., J. Goldstone and R. Jackiw, Phys. Rev. D 11, 1486 (1975).

[15]The action of S_a upon a scalar field has been derived by Fubini, Ref. 13. The extension to fields with spin is straightforward.

[16]The field-theoretic operators $\tilde{S}_a$, $\tilde{M}_{ab}$, which include the effects of the gauge transformations, can be shown to close upon the O(5, 1) conformal algebra. Thus, for example, $[\tilde{S}_a, \tilde{S}_b] = i\tilde{M}_{ab}$.

[17]There are also spinor (half-integer) representations, but we do not consider these in this paper.

[18]The standard text for these functions is N. J. Vilenkin, *Special Functions and the Theory of Group Representations*, Translations of Mathematical Monographs (American Mathematical Society, Providence, 1968).