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Equation of motion in classical electrodynamics*

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It is proposed that an equation derived as a first-order iteration of the Lorentz-Dirac equation be considered as the exact equation of motion for classical electrodynamics. The added force term which is quadratic in the applied field is shown to be equivalent to a Poynting-like momentum being transferred to the particle. We apply this equation to the motion of a particle in a uniform magnetic field and also in a uniform electric field. In both cases the particle trajectory exhibits the essential physical behavior, while the accompanying electromagnetic radiation is supplied by the interaction with the external field.

I. INTRODUCTION

Classical electrodynamics is the theory of Maxwell's equations for the fields and an equation of motion for the charged particles. Though the electromagnetic fields obtainable from the first set of equations are universally accepted as being correct, the same cannot be said of the trajectory of the charged particle when an attempt is made to include the effect of radiation. The Lorentz-Dirac equation,[1] which is the most favored choice, is not only derived on the basis of some unphysical, or difficult to accept, ideas (advanced fields and mass renormalization), but once it is assumed as basically correct, it results in runaway solutions[1] or, alternatively, noncausal behavior in the particle motion.[1,2] This unsatisfactory situation is evidenced by the continued appearance of new equations of motion in the literature. Thus, recently Bonner[3] has put forward an equation according to which the radiated energy is supplied by a reduction in the proper mass of the particle, while Mo and Papas[4] have proposed an equation wherein an external force proportional to the four-acceleration of the particle is included.

In this paper we would like to suggest that the equation obtained by a first-order iteration of the Lorentz-Dirac equation be considered as exact. Though quadratic in the applied fields, this differential equation involves only the first derivative of the particle velocity, and therefore does not have the undesirable properties of the Lorentz-Dirac equation. In addition, as we shall show, the total force acting on the particle is easily interpreted physically, since it is the sum of the Lorentz force and a, generally smaller, force arising from the Poynting momentum of the applied field. Lastly, the solution to this equation in the case of simple fields exhibits, as one would expect, the essential physical features of the solution of the Lorentz-Dirac equation and, interestingly enough, can be expressed in closed form. We shall not consider the motion of particles under the action of gravity or other forces since this would involve other kinds of radiation.

II. EQUATION OF MOTION

The Lorentz-Dirac equation for a particle of charge e and rest mass m in an electromagnetic field, expressed by the antisymmetrical tensor $F_{\mu\nu}$, is

$$\dot{v}_\mu(\tau) - \frac{2}{3}\frac{e^2}{mc^3}\ddot{v}_\mu(\tau) + \frac{2}{3}\frac{e^2}{mc^3}v_\mu(\tau)\frac{\dot{v}_\nu\dot{v}_\nu}{c^2} = \frac{e}{mc}F_{\mu\nu}v_\nu(\tau)\,. \tag{1}$$

The units are Gaussian, and we denote differentiation with respect to the particle proper time τ by a dot over a variable. The four-velocity has the components

$$v_\mu(\tau) = (v_k = \dot{x}_k(\tau),\, ic\gamma(\tau))\,, \tag{2}$$

where the Latin subscript k takes on values 1 to 3, in contrast to the Greek subscripts which assume values 1 to 4.

If we iterate Eq. (1), assuming that the coefficients of the second and third terms are small and that the applied field is constant in space and time, we find to first order in ϵ that

$$\dot{v}_\mu(\tau) = aF_{\mu\nu}v_\nu - \epsilon\frac{v_\mu}{c^2}F_{\nu\alpha}v_\alpha F_{\nu\beta}v_\beta + \epsilon F_{\mu\nu}F_{\nu\alpha}v_\alpha\,, \tag{3}$$

in which $a = e/cm$ and $\epsilon = 2e^4/3c^5m^3$. It is this equation of motion which we now propose to be exact in classical electrodynamics for all values of ϵ and for an arbitrary applied electromagnetic field, $F_{\mu\nu}$. We note that Eq. (3) is of the Newtonian type in which the right-hand side is the total applied four-force made up of the Lorentz force linear in the applied field and two other terms quadratic in this field. As emphasized by Mo and Papas,[4] such a Newtonian equation of motion has none of the problems of the Lorentz-Dirac equation. There are no runaway solutions and no preacceleration.

In order to study in more detail the nature of this equation, it is desirable to write it explicitly in terms of the component electric E and magnetic H fields. To facilitate manipulation, we initially express the field tensor as

$$F_{\mu\nu}=\delta_{\mu j}\delta_{\nu k}\epsilon_{jkl}H_l+(\delta_{\mu 4}\delta_{\nu j}-\delta_{\mu j}\delta_{\nu 4})iE_j\,, \tag{4}$$

where ϵ_{jkl} is the total antisymmetric tensor with $\epsilon_{123}=1$, and the repeated Latin indices denote a summation from 1 to 3. It is well to note that for the Kronecker δ's $\delta_{\mu j}\delta_{\mu k}=\delta_{jk}$, while $\delta_{\mu j}\delta_{\nu j}\neq\delta_{\mu\nu}$. Equation (3) now takes the form

$$\dot v_j(\tau)=\frac{e}{m}\left(\gamma E_j+\epsilon_{jkl}\frac{v_k}{c}H_l\right)-\epsilon v_j\left[\gamma^2(E_kE_k+H_kH_k)-\left(\frac{v_kE_k}{c}\right)^2-\left(\frac{v_kH_k}{c}\right)^2-2\gamma\epsilon_{klm}\frac{v_k}{c}E_lH_m\right]$$
$$+\epsilon(E_jv_kE_k+H_jv_kH_k+c\gamma\epsilon_{jkl}E_kH_l) \tag{5a}$$

and

$$\dot\gamma(\tau)=\frac{e}{mc^2}v_kE_k-\epsilon\gamma\left[(\gamma^2-1)(E_kE_k+H_kH_k)-\left(\frac{v_kE_k}{c}\right)^2-\left(\frac{v_kH_k}{c}\right)^2\right]+\epsilon(2\gamma^2-1)\epsilon_{jkl}\frac{v_j}{c}E_kH_l\,. \tag{5b}$$

We complete this section by presenting the rate of radiated four-momentum which is consistent with the equation of motion. Since Maxwell's equations are unchanged, this rate is given in covariant form as[5]

$$\left(\frac{dP_\mu}{d\tau}\right)^{\text{rad}}=\frac{\epsilon m}{a^2}v_\mu\frac{\dot v_\alpha\dot v_\alpha}{c^2}\,. \tag{6}$$

According to Eq. (3), this becomes

$$\left(\frac{dP_\mu}{d\tau}\right)^{\text{rad}}=m\epsilon\frac{v_\mu}{c^2}F_{\nu\alpha}v_\alpha F_{\nu\beta}v_\beta+\frac{m\epsilon^3}{a^2}\frac{v_\mu}{c^2}\left[F_{\nu\rho}F_{\rho\alpha}v_\alpha F_{\nu\sigma}F_{\sigma\beta}v_\beta+(1/c^2)(F_{\nu\alpha}v_\alpha F_{\nu\beta}v_\beta)^2\right], \tag{7}$$

or, as a function of the E and H fields,

$$\left(\frac{dP_\mu}{d\tau}\right)^{\text{rad}}=m\epsilon v_\mu\left[(\gamma^2-1)H_jH_j+\gamma^2E_jE_j-\left(\frac{v_jE_j}{c}\right)^2-\left(\frac{v_jH_j}{c}\right)^2-2\gamma\epsilon_{jkl}\frac{v_jE_kH_l}{c}\right]$$
$$+\frac{m\epsilon^3}{a^2}v_\mu\Bigg\{\gamma^2(\gamma^2-1)(H_jH_j)^2+\gamma^2(2\gamma^2-1)(E_jE_j)H_kH_k+\gamma^2(\gamma^2-1)(E_jE_j)^2-\gamma^2(E_jH_j)^2$$
$$+\left[\left(\frac{v_jE_j}{c}\right)^2+\left(\frac{v_jH_j}{c}\right)^2\right]^2-(2\gamma^2-1)H_jH_j\left(\frac{v_kH_k}{c}\right)^2-2\gamma^2H_jH_j\left(\frac{v_kE_k}{c}\right)^2$$
$$-2\gamma^2E_jE_j\left(\frac{v_kH_k}{c}\right)^2+2(E_jH_j)\left(\frac{v_kE_k}{c}\right)\left(\frac{v_lH_l}{c}\right)-(2\gamma^2-1)E_jE_j\left(\frac{v_kE_k}{c}\right)^2$$
$$-2\gamma(2\gamma^2-1)(E_jE_j+H_jH_j)\epsilon_{klm}\frac{v_k}{c}E_lH_m+4\gamma\left[\left(\frac{v_jE_j}{c}\right)^2+\left(\frac{v_jH_j}{c}\right)^2\right]\epsilon_{klm}\frac{v_k}{c}E_lH_m$$
$$+(4\gamma^2-1)\left(\epsilon_{jkl}\frac{v_j}{c}E_kH_l\right)^2\Bigg\}\,. \tag{8}$$

III. PHYSICAL INTERPRETATION

The equation of motion as developed in the preceding section is referred to an arbitrary inertial system. In order to see the significance of the various expressions, it is best to consider the equation with respect to the instantaneous rest system of the particle, that is, one in which the particle has $v_k=0$ and $\gamma=1$. Thus, in this proper frame, Eqs. (5a) and 5(b) are represented by

$$m\dot v_j=eE_j+\left(\frac{8\pi}{3}\frac{e^4}{m^2c^4}\right)\left(\frac{1}{4\pi}\epsilon_{jkl}E_kH_l\right) \tag{9a}$$

and

$$\dot\gamma=0 \tag{9b}$$

The first part of the force term on the right-hand side of Eq. (9a) is evidently the Lorentz force dependent only on the electric field in the rest frame. Instead, the second part of the force term has been written in such a way that it equals the product of the Thomson cross section (for a particle of mass m and charge e) and the equivalent Poynting momentum per unit area and time of the applied field in the rest frame. With this interpretation, one might say that the particle acquires momentum due to the electric field and the Poynting momentum which it intercepts in its rest frame. The equation of motion, Eq. (3), is then the covariant generalization of Eq. (9).

The rate of radiated four-momentum also takes on a simple aspect when viewed in the instantaneous rest frame, for Eq. (8) becomes

$$\left(\frac{dP_k}{d\tau}\right)^{\text{rad}} = 0 \tag{10a}$$

and

$$\left(\frac{dP_4}{d\tau}\right)^{\text{rad}} = icm\,\epsilon E^2\left(1 + \frac{\epsilon^2}{a^2}H^2\sin^2\theta\right). \tag{10b}$$

Here the variable θ is the angle between the direction of the electric and magnetic fields in the proper frame.

IV. MOTION IN A UNIFORM MAGNETIC FIELD

The relativistic motion of a particle in a field specified by $E_k = 0$ and $H_k = (0, 0, H)$ has been the subject of some discussion of late,[6-8] and, consequently, it is a good testing ground for applying an equation of motion. Writing $\omega = eH/mc$ and $\epsilon H^2 = \lambda\omega$, we find that Eqs. (5a) and (5b) are then expressed as

$$\dot{v}_1 = \omega v_2 - \lambda\omega v_1\left(\gamma^2 - \frac{v_3^{\,2}}{c^2}\right), \tag{11a}$$

$$\dot{v}_2 = -\omega v_1 - \lambda\omega v_2\left(\gamma^2 - \frac{v_3^{\,2}}{c^2}\right), \tag{11b}$$

$$\dot{v}_3 = -\lambda\omega v_3\left(\gamma^2 - 1 - \frac{v_3^{\,2}}{c^2}\right), \tag{11c}$$

and

$$\dot{\gamma} = -\lambda\omega\gamma\left(\gamma^2 - 1 - \frac{v_3^{\,2}}{c^2}\right). \tag{11d}$$

With the aid of Eqs. (11c) and (11d), it is readily shown that the three-velocity ($u_3 = v_3/\gamma$) along the direction of the magnetic field is a constant, as expected. Under this condition, the equations reduce to

$$\dot{w} + \lambda\omega w\frac{\gamma^2}{\gamma_L^{\,2}} = -i\omega w \tag{12a}$$

and

$$\dot{\gamma} + \lambda\omega\gamma\,\frac{\gamma^2 - \gamma_L^{\,2}}{\gamma_L^{\,2}} = 0\,. \tag{12b}$$

In these relations we have introduced the complex velocity variable

$$w(\tau) = v_1(\tau) + iv_2(\tau)\,, \tag{13}$$

and the constant longitudinal energy γ_L (in units of mc^2) given by

$$\gamma_L^{\,2} = 1/[1 - u_3^{\,2}(0)/c^2]\,. \tag{14}$$

For a particle with an initial transverse four-velocity of $w(0)$ and a total initial energy specified by $\gamma(0)$, Eqs. (12) now yield the solution in closed form:

$$\gamma(\tau)/\gamma(0) = \left\{\frac{\gamma^2(0)}{\gamma_L^{\,2}} - \left[\frac{\gamma^2(0)}{\gamma_L^{\,2}} - 1\right]\exp(-2\lambda\omega\tau)\right\}^{-1/2} \tag{15a}$$

and

$$w(\tau) = w(0)\frac{\gamma(\tau)}{\gamma(0)}\exp[-(i+\lambda)\omega\tau]\,. \tag{15b}$$

These results should be compared with Eq. I(5) of Ref. (7). It is clear that the present solution possesses the essential physical properties of the perturbation solution obtained there for the Lorentz-Dirac equation. This is particularly true for realistic magnetic fields presently achievable, since the expansion parameter λ is very small for such fields ($|\lambda| \approx 10^{-16}$ H for an electron). However, we must emphasize that the physical interpretation of the solutions of Eq. (3) is very different from that of the Lorentz-Dirac equation. This is best illustrated in this instance by considering the energy balance for the motion in a uniform magnetic field. Applying Eq. (8), we find that the rate of radiated four-momentum is

$$\left(\frac{dP_\mu}{d\tau}\right)^{\text{rad}} = mv_\mu\lambda\omega\frac{ww^*}{c^2}\left[1 + \lambda^2\left(\frac{ww^*}{c^2} + 1\right)\right]. \tag{16}$$

The fourth component of this four-vector can now be integrated from zero proper time to infinite proper time, that is, over the complete trajectory of the particle. One thereby arrives at the total radiated energy:

$$\frac{1}{icm}\int dP_4^{\text{rad}} = [\gamma(0) - \gamma_L] \times\left\{1 + \frac{\lambda^2}{3}\left[\frac{\gamma^2(0)}{\gamma_L^{\,2}} + \frac{\gamma(0)}{\gamma_L} + 1\right]\right\}. \tag{17}$$

This relationship shows that the radiated energy consists not only of the total change in energy of the radiating particle, $[\gamma(0) - \gamma_L]$, but also of an additional energy depending on the magnetic field strength through the parameter λ. While here the added energy is contributed by the extra force term appearing in Eq. (3), the analogous energy radiated during the motion according to the Lorentz-Dirac equation is due to the unphysical Schott energy[7,9-11] term which depends on the particle acceleration.

V. MOTION IN A UNIFORM ELECTRIC FIELD

Another example in which the radiated energy, according to the Lorentz-Dirac equation, is drawn from the Schott energy is that of motion in a uniform electric field.[2,11,12] In contrast, though Eq. (3) necessarily predicts a similar motion, the radiated energy is derived from the applied field. Since the general solution is in this case also obtainable in closed form, we shall briefly outline the pertinent steps.

Given the electric field $E_k=(0,0,E)$, we have for the equation of motion

$$\dot{v}_1 = -\epsilon v_1 E^2(1+\zeta)\,, \tag{18a}$$

$$\dot{v}_2 = -\epsilon v_2 E^2(1+\zeta)\,, \tag{18b}$$

$$\dot{v}_3 = \frac{e}{m}\gamma E - \epsilon v_3 E^2 \zeta\,, \tag{18c}$$

and

$$\dot{\gamma} = \frac{e}{mc^2} v_3 E - \epsilon\gamma E^2\zeta\,. \tag{18d}$$

In writing these relations, we have introduced the abbreviation $\zeta=(v_1{}^2+v_2{}^2)/c^2$ corresponding to the square of the transverse four-velocity. The exact solution with the proper time τ as the independent variable is

$$v_1(\tau) = v_1(0)\left[\frac{\zeta(\tau)}{\zeta(0)}\right]^{1/2}, \tag{19a}$$

$$v_2(\tau) = v_2(0)\left[\frac{\zeta(\tau)}{\zeta(0)}\right]^{1/2}, \tag{19b}$$

and

$$v_3(\tau) = \left[\frac{\zeta(\tau)}{\zeta(0)}\right]^{1/2}\exp(\epsilon E^2\tau) \times\left[v_3(0)\cosh(k\tau)+c\gamma(0)\sinh(k\tau)\right]\,, \tag{19c}$$

$$\gamma(\tau) = \left[\frac{\zeta(\tau)}{\zeta(0)}\right]^{1/2}\exp(\epsilon E^2\tau) \times\left\{\gamma(0)\cosh(k\tau)+\left[v_3(0)/c\right]\sinh(k\tau)\right\}\,, \tag{19d}$$

wherein $k=eE/mc$ and

$$\zeta(\tau)/\zeta(0) = \left\{\left[1+\zeta(0)\right]\exp(2\epsilon E^2\tau)-\zeta(0)\right\}^{-1}\,. \tag{20}$$

The accompanying rate of momentum-energy radiated is, according to Eq. (8),

$$\left(\frac{dP_\mu}{d\tau}\right)^{\mathrm{rad}} = m\epsilon v_\mu E^2(\zeta+1)\left(1+\frac{\epsilon^2}{a^2}E^2\zeta\right). \tag{21}$$

For the special initial condition when the motion is lined up with the electric field, that is, when $\zeta(0)=0$, we see from Eqs. (19) and (20) that the motion is hyperbolic and corresponds to motion under the Lorentz force alone with no radiation.[11] However, consistent with Eq. (3), a radiated momentum is predicted [Eq. (21) with $\zeta=0$], and the requisite energy is supplied by the applied external field.

VI. DISCUSSION

A theory of classical electrodynamics based on Maxwell's equations and the equation of motion proposed in this paper would be a consistent and satisfactory one as regards specifying both the trajectories of particles and the associated electromagnetic fields. The property that each particle moves only under the action of the external field due to the other particles can be viewed as an asset, since, under these circumstances, the concept of self-field interaction does not arise. Whether this equation of motion or, for that matter, the Lorentz-Dirac equation is in agreement with experiment is an open question.[6,13] In conclusion, it is of interest to mention that our interpretation of the added force term in the equation of motion is reminiscent of the fundamental idea employed by Weizsäcker and Williams[14] in their method of virtual quanta in which the perturbing field is treated as an incident electromagnetic radiation.

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