Bhabha first-order wave equations. VII. Summary and conclusions

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We give a summary of the results we have obtained in this series on the Bhabha first-order wave equations for arbitrary spin. On the basis of these and other calculations, we present a set of conclusions about their physical implications, especially with respect to the possibility of finding and properly interpreting a mathematically consistent field theory of particles with arbitrary spin.

I. INTRODUCTION

In this paper we close our series[1-6] on the Bhabha first-order wave equations for arbitrary spin. These equations are defined by

$$(\partial\cdot\alpha+\chi)\psi=0\,, \tag{1.1}$$

where the α_μ represent the $J_{\mu 5}$ generators of the so(5) algebra[7]

$$\alpha_\mu=J_{\mu 5}=-J_{5\mu}\,,\quad J_{\mu\nu}=-i[\alpha_\mu,\alpha_\nu],\quad J_{55}=0\,, \tag{1.2}$$

$$[J_{ab},J_{cd}]=i(\delta_{ac}J_{bd}+\delta_{bd}J_{ac}-\delta_{bc}J_{ad}-\delta_{ad}J_{bc})\,, \tag{1.3}$$

$$J_{ab}=-J_{ba},\quad (a,b)=1,2,3,4,5\,. \tag{1.4}$$

For a particular equation (1.1), the matrices α_μ are taken to be a particular representation $(\mathcal{S},S)$ of the so(5) algebra. For example, the $(\frac{1}{2},\frac{1}{2})$ representation yields the Dirac equation and the $(1,0)$ and $(1,1)$ representations yield the Duffin-Kemmer-Petiau (DKP) spin-0 and spin-1 equations. Also, we define "Bhabha equations" to mean the α_μ are of necessity so(5) matrices, not any first-order wave-equations matrices, a definition that is sometimes used.

In Sec. II we will present a summary of our own results on the Bhabha first-order wave equations.[1-6] From this and the information available from other calculations[8-12] we will present in Sec. III a set of physical and mathematical conclusions about the Bhabha first-order wave equations. We have special interest in what can be said about the possibility of finding a high-spin field theory which is both mathematically consistent and has an acceptable physical interpretation.

Detailed lists of pertinent references can be found in this paper and in our previous works.[13] In Appendix A we list as a convenience the errata we have noted in our already published articles. Also see Appendix B on mass and spin.

Finally, as this paper concludes our series, elsewhere[14] we will take the opportunity to describe from published works, letters, and personal reminiscences, the historical development of these first-order wave equations. This story involves many of the great physicists of the first quarter-century of quantum mechanics, and indeed the credit for these equations does not belong only to Bhabha.

II. SUMMARY

Beginning with the mass and spin composition,[2] the general so(5) algebra of Eqs. (1.2)–(1.4) has (for a particular algebra labeled by $\mathcal{S}$) irreducible representations (irrep's) $(\mathcal{S},S)$, with $(0,\frac{1}{2})\le S\le\mathcal{S}$, of dimensions

$$d_5(\mathcal{S},S)=\tfrac{1}{6}(2\mathcal{S}+3)(2S+1)\times[(\mathcal{S}+1)(\mathcal{S}+2)-S(S+1)]\,. \tag{2.1}$$

$\mathcal{S}$ is the maximum spin of a particular representation. As discussed in paper II, simply because an irrep of so(n) can be broken up into certain sums of irrep's of so($n-1$), one knows that the so(5) irrep's have dimensions which are given by sums of quantities $2(2S+1)$, where the $(2S+1)$ are spin-S degrees of freedom and the "2" is for particle-antiparticle. That is, the system is in general multimass, multispin. In particular, an algebra contains

$$\text{spin states } S,\quad (0,\tfrac{1}{2})\le S\le\mathcal{S}\,, \tag{2.2}$$

$$\text{mass states } \pm\chi/S,\quad (1,\tfrac{1}{2})\le S\le\mathcal{S}\,. \tag{2.3}$$

The above two statements are related since, for example, the S_z generator

$$S_z=J_{12}=-i[\alpha_1,\alpha_2] \tag{2.4}$$

can be rotated into the α_4 generator. One can investigate the mass and spin composition of a particular algebra in detail, as was done in Ref. 2, with important additions in the present Appendix B.

A particular example, which we studied in papers

V and VI, is the 16-dimensional $(\frac{3}{2}, \frac{1}{2})$ representation. It is composed of a ground state of mass $2\chi/3$ with spin $\frac{1}{2}$ and two sets of excited states of mass 2χ with spin $\frac{3}{2}$ and spin $\frac{1}{2}$. [Note in particular that a single-mass $\chi/\mathcal{S}$ single-spin-S ground state will be contained in the representation $(\mathcal{S},S)$.]

For integer spin, there are built-in subsidiary components due to the fact that the α_μ (like S_z) have zero eigenvalues. Thus the matrices do not have inverses and these components must always be handled specially. As shown in paper II, the decoupling of these components can be accomplished by a generalization of the Peirce decomposition which Sakata and Taketani[15-17] (ST) first performed for the special DKP case (which is the $\mathcal{S}=1$ Bhabha algebra).

The decomposition can be done with a set of projection operators.[2] First, there are projection operators onto the physical "particle components," $\mathcal{G}^{(P)}(\mathcal{S})$, whose dimensions are sums of quantities $2(2S+1)$. Similarly, there are projection operators, $\mathcal{G}^{(S)}(\mathcal{S})$, onto the "subsidiary components" (built-in, not external). $\mathcal{G}^{(P)}(\mathcal{S})$ can also be decomposed into projection operators onto the individual mass states, $\mathcal{G}_j^{\pm}(\mathcal{S})$. With the above operators one can then derive a set of decoupling equations to obtain only the particle components, or even only the components of a particular mass state. The procedure for this last type of calculation was set up in paper II, but actually it would physically correspond to a Foldy-Wouthuysen (FW) transformation which we discussed in a more standard vein in papers V and VI.

Coming to the space-time symmetries, in paper I we described the C, P, and T operators. They are linear products of the operators $\eta_\mu(\alpha_\mu)$ which were defined functionally in the Appendix of paper I and are such that

$$\mathcal{C}\mathcal{P}\mathcal{T}=\eta_1\eta_2\eta_3\eta_4 e^{i\phi} . \tag{2.5}$$

Specifically, the η_μ are polynomials of order $2\mathcal{S}$ in the α_μ. Therefore, one can always take a representation where a particular α_μ, and hence $\eta_\mu(\alpha_\mu)$, is diagonal. Let $\mu=4$. Then η_4 is, up to a phase, represented by a "+1" in the space $\mathcal{G}_{\mathcal{S}}^{+}(\mathcal{S})$ spanned by the ground state of mass $\chi/\mathcal{S}$, a "−1" in the space $\mathcal{G}_{\mathcal{S}-1}^{+}(\mathcal{S})$ spanned by the first excited state of mass $\chi/(\mathcal{S}-1)$, oscillating back and forth. Since η_4 is the parity or adjoint operator these properties mean two things.

First, note that the space $\mathcal{G}_j^{+}(\mathcal{S})$ of a particle of mass $\chi/\mathcal{S}$ is an odd-integer number of mass blocks [eigenvalue blocks of α_μ] from the space $\mathcal{G}_j^{-}(\mathcal{S})$ of its antiparticle states. Therefore, antifermions have opposite intrinsic parity to fermions.[4,5] Further, because of the added block of zero eigenvalues for bosons there is an added change of sign so that antibosons have the same intrinsic parity as bosons.[4,5]

Secondly, since the metric operator M is the product of the adjoint operator η_4 with the fourth component of the current operator, α_4,

$$M=\eta_4\alpha_4 , \tag{2.6}$$

M has the sign property of η_4 with the one additional change of sign in going from particles to antiparticles because of the sign change in α_4. This is the indefinite metric which has occupied so much of our discussion and which only is positive-definite for the Dirac case when $\eta_4=2\alpha_4$. Even for spin-0 and spin-1 DKP there is a negative norm for antiparticles. There one can use the Pauli-Weisskopf ansatz of saying one has a charge probability density, but for $S>1$ the norms and parities alternate with each excited state, so such an ansatz does not follow.

The Poincaré generators were given in paper III and shown to satisfy the associated Lie algebra, but their properties brought in both the ST decomposition and the indefinite metric. For half-integer-spin fields, the Bhabha Poincaré generators satisfy the Lie algebra, "algebraically." However, for integer-spin fields, this is not the case. When the integer-spin Poincaré generators are inserted in the Lie algebra there are terms left over. However, it could be shown from the "consequent equations" that these terms are zero if they operate on the fields. That is, the necessity of having operators on the fields is due to the subsidiary components. When one performs an ST decomposition on the integer-spin Poincaré generators, the particle-components Poincaré generators satisfy the Lie algebra "algebraically."

Further, because of the indefinite metric, the Poincaré generators $\mathcal{G}$ are not "Hermitian," but rather "metric-Hermitian" (pseudo-Hermitian),

$$(M\mathcal{G})^{\dagger}=(M\mathcal{G}) . \tag{2.7}$$

Equation (2.7) holds as is for half-integer-spin fields, but again only as an operator equation on the fields for integer spin. However, as before, when one does an ST decomposition, the particle components of the integer-spin Poincaré generators are metric-Hermitian "algebraically."

In paper IV (Ref. 4) we demonstrated the important result that the Bhabha fields are causal with minimal electromagnetic coupling, both in the c-number theory and in the q-number theory. Again, a special handling of the integer-spin subsidiary components is necessary. In the c-number theory this is because the integer-spin algebra matrices are singular. Thus, instead of taking the determinant of the wave equation to investigate causality one has to utilize the KG (Klein-Gordon) divisors, (which were constructed for arbitrary spin).

The q-number theory is interesting for two reasons. First, the quantization *has* to be with an indefinite metric. Without the indefinite metric the theory would have been noncausal. The extra minus signs of the indefinite metric have to be part of the theory. Secondly, the subsidiary components again necessitate a special handling. This is because when one calculates the Heisenberg fields, they each contain a new piece which apparently introduces noncausality into the theory. However, a detailed investigation showed[4] that the new pieces are multiplied by the projection operators onto the subsidiary components, $\mathcal{G}_0(\mathcal{S})$. This means that the commutation relations among the physical particle components are preserved, so those fields are causal. (This has long been known for the special DKP case.)

In coming to the FW transformation, there was a question of principle involved. Since the Bhabha system is an indefinite-metric system, an FW transformation does not have to exist *a priori*.[9] This led to a separate series of four papers (FW-I,[9] FW-II,[10] FW-III,[11] and FW-IV[12]) on the general problem of FW transformations in an indefinite-metric space.

Applied to the Bhabha system in paper V,[5] those papers in Refs. 9–12 implied that an FW transformation U^{-1} does exist, since the eigenvalues of the system are real and the norms of the eigenvectors are not zero.[9] The FW transformation U^{-1} is metric-unitary (pseudounitary),

$$(U^{-1})^\dagger M U^{-1} = M , \tag{2.8}$$

and is composed of columns of the metric-orthonormal eigenvectors $\hat{u}_k$,

$$U^{-1} = [\hat{u}_1, \hat{u}_2, \ldots, \hat{u}_n] , \tag{2.9}$$

satisfying

$$\hat{u}_j^\dagger M \hat{u}_k = M_{kk}\delta_{jk} . \tag{2.10}$$

We then pointed out,[5] on the basis of FW-III,[11] what the exact, FW-transformed Poincaré generators were. As to the transformation itself, we gave as a first calculation a method for generating the FW transformation as a power-series expansion in c^{-1}. The power-series expansion offers physical insight since it is the easiest representation with which to discuss relativistic corrections to the nonrelativistic forms, such as the *Zitterbewegung*. In particular, we demonstrated that the FW transformation we explicitly wrote in a power series up to order $(c^{-1})^3$ transformed the original Bhabha Poincaré generators into the exact, FW Poincaré generators to that same order. A detailed discussion was included on the importance of the indefinite metric in correctly generating this series-expansion FW transformation. Also, we derived the functional relationship of the particle-components, integer-spin transformations to the half-integer-spin transformations.

Finally, in paper VI, we derived the exact, closed-form FW transformation and the solutions to the wave equation. A separate method was necessary to find the exact FW transformation since, contrary to the simple Dirac and DKP cases, the power-series expansion in (c^{-1}) does not lend itself to summation. This is because the Bhabha algebra $\mathcal{S}$ has $2\mathcal{S}$ terms before it starts to close on itself. For $\mathcal{S}>1$, the closure properties rapidly become horrendous.

The method of obtaining the exact FW transformation was discussed in detail in FW-III and FW-IV. It is based on the observation that the solutions to the wave equation are the Lorentz-transformed rest-state eigenvectors. Thus even in an indefinite-metric space the FW transformation is related to the Lorentz transformation $L(\theta)$ by[11] ($\vec{p}\,\|\hat{z}$)

$$U^{-1}(\beta) = GL(\theta) = Ge^{-\theta J_{43}} . \tag{2.11}$$

Using a theorem of FW-II, the normalization G was found,[5] and a matrix theorem discussed in FW-IV allowed $L(\theta)$ to be written as a *finite polynomial* in J_{43} of order $2\mathcal{S}$, instead of an infinite power series. With the proper identification of the θ in terms of energy, momentum, and mass quantities, this gives the exact, closed-form FW transformation for arbitrary-spin Bhabha fields. Therefore, one also has the form of the eigenvectors.

Special case FW examples were discussed at length in papers V and VI. In VI we also used the eigenvectors to study expectation values of physical currents. These are of interest since other work[7,18-29] has shown that there can be a difference in the expectation values of currents in symmetry-breaking situations when calculated with first-order fields instead of with second-order fields.

Throughout the entire series, we have given numerous physical examples, and described in detail our results as specifically applied to special representations (Dirac, DKP, $\mathcal{S}=\frac{3}{2}$, etc.).

III. CONCLUSIONS

Many of the troubles afflicting certain high-spin field theories do not occur in the Bhabha case. For example, the theory is casual with minimal electromagnetic coupling. Furthermore, since one can make physical and mathematical arguments in favor of first-order field theories, a unification of high-spin fields in a theory such as Bhabha theory would be esthetically pleasing. (Again, we are not declaring that Bhabha theory is the answer but rather that the study we have undertaken can provide insight as to what direction the answer might lie.)

But a theory like Bhabha's demands its dues. The first price one must pay is that the theory is multimass and multispin. However, as such spectroscopical concepts are part of the entire framework of modern particle physics this represents no conceptual impediment, at least to the present authors. (Indeed, Hietarinta has developed a supersymmetry formalism using the Bhabha first-order fields.[8])

The second and ultimately real price is the indefinite metric. Except for problems of renormalization, the whole question of this type of theory lies in understanding what physical meaning, if any, there is to an indefinite metric.

Through the indefinite metric, the adjacent mass states of, in general, different norm are coupled. (In fact, this coupling of adjacent mass states physically exists in the only positive-normed space, the Dirac case. There it is the *Zitterbewegung* and, of course, in quantum field theory, the virtual coupling to antiparticles.) One could hope to take the negative-normed states out of the theory,[30] but as the FW discussion indicates (see below) this does not appear to be possible.

Given the indefinite metric, then, what are the interpretations that it and the other work in high-spin field theories could be suggesting?

The first possibility is simply that there does not exist a mathematically and physically consistent high-spin field theory because nature does not have (massive, quantum) fundamental high-spin fields. The problems with high-spin field theories would then be a mathematical indication of this fact. Physically one can observe that there are no stable high-spin fields (ignoring the hypothetical graviton[31,32]). Perhaps, then, the only stable and free high-spin fields[33] allowable in nature are either composite fields (such as nuclei) or massless, fundamental fields (also see the next paragraph). Fundamental high-spin fields with an indefinite metric might be allowed in confined situations, if one could show probability problems are avoided.

To mention other possibilities, one perhaps should consider the space-time structure,[34] and/or include gravity directly,[35] and/or consider the gauge field forms. Any of the above programs might be necessary in finding a consistent high-spin field theory, be it Bhabha's or another set of fields.

Along this line, an interesting proposal has been raised in the context of supergravity theories. In a recently discussed version,[36-39] a massless, Rarita-Schwinger spin-$\frac{3}{2}$ field is coupled to a massless, spin-2 graviton. Because the fields are massless many of the *ad hoc* causality problems are absent. (A Higgs mechanism could be called upon to yield the physical mass.) However, partially because the theory is no longer in flat space-time, proving that there are no causality problems in the fully interacting theory will be difficult to do.[40,41] Whether or not this particular idea eventually works, it does emphasize the obvious (but usually ignored) fact that a totally consistent theory would have to include gravity. But if the origin of all the problems involving high-spin field theories stems from the necessity for a "fundamental" theory to involve the most fundamental physical interaction, gravity, then why does the Dirac equation work?

The last, and admittedly most radical, possibility suggested by our results is that the indefinite metric is physically meaningful and must be understood via a new interpretive link. One would want to change the normal probability interpretation in such a way as to understand the indefinite metric yet keep standard quantum mechanics[42-44] in the regimes where it has proved so successful.

The FW transformation, which decouples the mass states, goes to the heart of the situation. By decoupling the mass states, the Poincaré generators, and in particular the Hamiltonian, become infinite polynomials. Thus when minimal electromagnetic coupling is introduced in this representation, the theory becomes noncausal. This is a manifestation, in our language, of Wightman's observation[45] that for a wide class of high-spin theories, the theory is either noncausal, or if it is causal, it can be shown to be unstable; this related to ghost states.[46] (Note that the same comments hold for Dirac, but there one does not need the FW representation in a q-number theory because the metric is positive-definite.)

The choices appear to be either that (i) in some way the correct physics lies along other lines (such as there are no fundamental high-spin fields or the space-time structure must be modified), or (ii) the indefinite metric is physical and necessitates an as-yet-not-obtained generalization of quantum mechanics for spin $\neq \frac{1}{2}$ fields; perhaps even both.[47] The above is for speculation. The hard content is that for the Bhabha system to be causal with minimal electromagnetic coupling, the indefinite metric is necessary in this multimass, multispin field theory.

Ultimately we return to the nagging question: Why is there such a beautiful first-order field theory for a massive interacting particle of spin $\frac{1}{2}$, the Dirac theory in QED, and none for high spin?

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In ending this series, we wish to once again acknowledge all the people whom we have thanked individually in our papers. However, there are two people who deserve special recognition for aiding us to achieve whatever of value has come from this work.

Our colleague, J. D. Louck, often gave of his valuable time to discuss calculational problems on this series[1-6] and the associated FW series.[9-12] As the careful reader will have seen, his knowledge of group and function theory often was the key to doing a particularly nasty calculation.

Also, A. S. Wightman's kind interest, even in the early stages,[1] should be noted. His shared knowledge and wisdom on field theory have been deeply appreciated.

APPENDIX A: CORRECTIONS TO THE SERIES

Paper I.[1] In Eq. (3.7) insert a "$+m$" inside the square brackets.

In Eqs. (5.1) and (5.2) change the normalization in front of the spinors from $(4mE)^{-1}$ to $(4mE)^{-1/2}$.

Paper II.[2] In Table I, in the $(\mathcal{S}, S)=(\frac{7}{2}, \frac{1}{2})$ representation, in the row labeled by $(l_{4,1}, l^L_{4,2}) =(\frac{3}{2}, \frac{1}{2})$ and column labeled by d^L_4, there is a "14." Change that to "12."

Paper III.[3] In Eq. (5.31), the term $[\alpha_4, \alpha_i]$ should be $[\alpha_4, \alpha_j]$.

In Ref. 36, the citation to Ref. 24 should be to Ref. 35.

Paper IV.[4] In the first paragraph and in Ref. 9, Lubański was spelled with the accent over the "a" instead of the "n".

In Eq. (A1), the "0 =" on the left can be deleted.

On p. 929, first paragraph, lines 3 and 4, respectively, the factors $(2\mathcal{S})$ and $(2\mathcal{S}+1)$ should read $(2\mathcal{S}-1)$ and $(2\mathcal{S})$, respectively.

Paper V.[5] In the first paragraph and in Ref. 10, Lubański was spelled with the accent over the "a" instead of the "n".

In Ref. 110, the second page number should be 447(E), not 451.

Paper FW-II.[8] On the right-hand side of Eq. (2.19) an η_4 should be inserted after the "$-i$." (Note that the minus sign there is due to the opposite convention for θ used in that paper.)

APPENDIX B: PHYSICAL SPIN AND MASS DECOMPOSITION

In Tables I and II of Ref. 2 the table headings should start with "The Lorentz and so(3) spin decomposition...," instead of "The mass and spin decomposition...," and the "Mass" columns should be deleted (with appropriate changes elsewhere). The *physical spin* and mass decomposition is given below and comes about because of the following:

Although the so(3) decomposition listed in those tables is correct, the so(3) spin representations so obtained are not in general the physical spin-σ representations.[48] This is because, as R. K. Loide has kindly emphasized to us,[49] α_4 mixes the Lorentz representations, so that the Lorentz representations, and hence the so(3) representations obtained from them, contain different mass states. This yields so(3) representations which are mixed-mass and not the physically interesting single-mass so(3) representations usually associated with particle spin. Thus, although the number and type of so(3) spin representations obtained from so(5) via the Lorentz representations are the same number and type as the physical spin-σ representations obtained from so(5) directly, it is the σ representations which always can be associated with the mass.

To obtain the σ and mass decomposition, one can use the results of Lubański,[50] who showed that the number of physical spin-σ representations for a particle of mass $\pm\chi/j$, in the so(5) representation $(\mathcal{S}, S)$, is $(j \geq 0)$

$$Z(\mathcal{S}, S, \pm j, \sigma)=\min\{[\tfrac{1}{2}(\mathcal{S}-j-|S-\sigma|)+1], [\sigma-\epsilon+1], [S-\epsilon+1], \mathcal{S}-\sigma+1, \mathcal{S}-S+1\}, \quad (\mathrm{B1})$$

where the value of (B1) is taken to be zero if the formula yields a negative number, $[x]$ denotes the greatest integer $\leq x$, and

$$\epsilon=\tfrac{1}{2}(\mathcal{S}-j-S+\sigma)-[\tfrac{1}{2}(\mathcal{S}-j-S+\sigma)]. \quad (\mathrm{B2})$$

One also can use as a check the formula[50] for the multiplicity of the eigenvalues $\pm j$ of α_4 in the representation $(\mathcal{S}, S)$, which is

$$M(\mathcal{S}, S, \pm j)=\begin{cases}\tfrac{1}{2}(\mathcal{S}-j+1)(\mathcal{S}-j+2)(2S+1), & S \leq j \leq \mathcal{S} \\ \tfrac{1}{2}(\mathcal{S}-S+1)((\mathcal{S}-S+2)(2S+1)+(S^2-j^2)), & 0 \leq j \leq S,\end{cases} \quad (\mathrm{B3})$$

Tables I and II below give the so(5) decomposition into physical spin-σ and mass states.

Note, however, that all of our general calculations and conclusions remain the same. Further, all of the detailed special-case calculations of

TABLE I. The mass and physical spin decomposition of half-integer-spin Bhabha fields up to maximum spin $\frac{7}{2}$. The notation is as before, and we include both the particle and antiparticle states in the counting.

		Number of physical spin representations with dimensions $(2\sigma+1)$ for $\sigma=$				
$(\mathcal{S}, S)$	d_5	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$	$\frac{7}{2}$	Mass
$(\frac{1}{2},\frac{1}{2})$	4	2				2χ
$(\frac{3}{2},\frac{3}{2})$	20		2			$2\chi/3$
		2	2			2χ
$(\frac{3}{2},\frac{1}{2})$	16	2				$2\chi/3$
		2	2			2χ
$(\frac{5}{2},\frac{5}{2})$	56			2		$2\chi/5$
			2	2		$2\chi/3$
		2	2	2		2χ
$(\frac{5}{2},\frac{3}{2})$	64		2			$2\chi/5$
		2	2	2		$2\chi/3$
		2	4	2		2χ
$(\frac{5}{2},\frac{1}{2})$	40	2				$2\chi/5$
		2	2			$2\chi/3$
		2	2	2		2χ
$(\frac{7}{2},\frac{7}{2})$	120				2	$2\chi/7$
				2	2	$2\chi/5$
			2	2	2	$2\chi/3$
		2	2	2	2	2χ
$(\frac{7}{2},\frac{5}{2})$	160			2		$2\chi/7$
			2	2	2	$2\chi/5$
		2	2	4	2	$2\chi/3$
		2	4	4	2	2χ
$(\frac{7}{2},\frac{3}{2})$	140		2			$2\chi/7$
		2	2	2		$2\chi/5$
		2	4	2	2	$2\chi/3$
		2	4	4	2	2χ
$(\frac{7}{2},\frac{1}{2})$	80	2				$2\chi/7$
		2	2			$2\chi/5$
		2	2	2		$2\chi/3$
		2	2	2	2	2χ

this series remain the same. Technically this latter observation is because the mass-spin-state counting from the original Table I would differ from that of the present Table I only in the $(\frac{7}{2},\frac{5}{2})$ and $(\frac{7}{2},\frac{3}{2})$ representations, which were not discussed as special cases. Thus, for example, the $(\frac{5}{2},\frac{3}{2})$ decomposition of α_4 in Eq. (II4.15) is correct. Also, realizing that our original integer-spin discussion was from the viewpoint of mixing subsidiary components and physical components, the integer-spin

TABLE II. The mass and physical spin decomposition of integer spin Bhabha fields up to maximum spin 3. The notation is as before, and for mass $\neq \chi/0=\infty$ subsidiary components we include both the particle and antiparticle states in the counting.

		Number of physical spin representations with dimensions $(2\sigma+1)$ for $\sigma=$				
$(\mathcal{S}, S)$	d_5	0	1	2	3	Mass
(0, 0)	1	1				∞
(1, 1)	10		2			χ
		1	1			∞
(1, 0)	5	2				χ
			1			∞
(2, 2)	35			2		$\chi/2$
			2	2		χ
		1	1	1		∞
(2, 1)	35		2			$\chi/2$
		2	2	2		χ
			2	1		∞
(2, 0)	14	2				$\chi/2$
			2			χ
		1		1		∞
(3, 3)	84				2	$\chi/3$
				2	2	$\chi/2$
			2	2	2	χ
		1	1	1	1	∞
(3, 2)	105			2		$\chi/3$
			2	2	2	$\chi/2$
		2	2	4	2	χ
			2	2	1	∞
(3, 1)	81		2			$\chi/3$
		2	2	2		$\chi/2$
			4	2	2	χ
		1	1	2	1	∞
(3, 0)	30	2				$\chi/3$
			2			$\chi/2$
		2	2	2		χ
			1		1	∞

special cases discussed had the correct mass-spin counting.

We now give as Tables I and II, the so(5) decomposition into physical spin-σ and mass states. (Loide[49] also calculated Table II.) Note that the tables have taken for the mass states χ/j both the particle and antiparticle components. Thus, to get just the particle-components mass $=+\chi/j$ states, divide the σ representation numbers by 2, except for $j=0$. The patterns of representation numbers in the tables can be considered "tableaus," of the type Lubański[50] touched upon.

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[3]R. A. Krajcik and M. M. Nieto, Phys. Rev. D 11, 1459 (1975), paper III of this series, on the Poincaré generators.

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[41]We wish to acknowledge private communications with D. Z. Freedman and A. S. Goldhaber on this topic.

[42]There are, of course, many, many works on quantum mechanics and possible modifications of it. We wish to recommend two thorough and unusual volumes. Reference 43 deals with the connection of quantum to classical theory. Reference 44 is a discussion of hidden-variables theories proposed to do away with the

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