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Bhabha first-order wave equations. VI. Exact, closed-form, Foldy-Wouthuysen transformations and solutions

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We give a closed-form, finite-polynomial expression for the Foldy-Wouthuysen (FW) transformation U^{-1} for arbitrary-spin Bhabha fields. Our result is obtained by appropriately normalizing the Lorentz transformation operator, expressing this transformation as a finite polynomial, and then properly interpreting the energy, mass, and especially momentum operators involved. For integer-spin fields the built-in subsidiary components are projected out. An algorithm is given which allows one to easily write the expression for the FW transformation of any Bhabha field. We comment on the properties of U^{-1}. We note that the columns of the FW transformation are the metric-orthonormal eigenvectors of the Hamiltonian, $\hat{u}_k$, and provide the relation of the $\hat{u}_k$ to the solutions of the wave equation, ψ_k. Special cases up to $\mathcal{S}=3$ are listed and investigated. Some physical and mathematical applications of our method and results are also given.

I. INTRODUCTION

In our series[1-5] on the Bhabha first-order wave equations, we discussed in the previous paper[5] how to generate the power-series expansion in c^{-1} of the Foldy-Wouthuysen (FW) transformation for arbitrary-spin Bhabha fields. The power-series expansion is more useful for understanding the physics of the situation, but contrary to the Dirac and Duffin-Kemmer-Petiau (DKP) special cases, in general it is not easy to sum the series into a closed form. This is because with each additional maximum spin allowed in particular so(5) representations of the Bhabha α_μ matrices, the algebra is increasingly complicated.

However, it is possible to obtain a general closed form for the FW transformation by another technique. This technique was developed in a separate series of four papers[6-9] on Foldy-Wouthuysen transformations in an indefinite-metric space. When applied to the particular Bhabha system it yields the exact, closed-form, Bhabha FW transformation.

The idea is that since the columns of the FW transformation U^{-1} can be shown[6] to be the metric-orthonormal eigenvectors of the Hamiltonian $\hat{u}_k$, one can construct the implicit matrix U^{-1}. One uses theorems[7] to renormalize the implicit Lorentz transformation matrix[8] $L(\theta)$ to metric-unitary form and relates the parameters involved in the Lorentz transformation to the physical energies, momenta, and masses.[8,9] Finally, by using an elegant matrix theorem, the Lorentz transformation matrix can be expressed in closed form.[9]

In Sec. II we will apply the above technique to the Bhabha system. Our application will be somewhat physically motivated. A more mathematical discussion, which includes the Bhabha system as a special case, is given in Refs. 8 and 9. There the precise relationship between Lorentz and FW transformations of first-order wave equations in indefinite-metric spaces is developed. (Note that the use of the Lorentz transformation is a generalization of the observation[10-12] that the original Dirac FW transformation can be related to the Lorentz transformation of the rest-system eigenvectors.)

In Sec. III we will discuss a number of special cases, up to $\mathcal{S}=3$. These will include demonstrations that for $\mathcal{S}=\frac{1}{2}$ and 1, our results reduce to the exact Dirac and [Sakata-Taketani (ST) version] DKP forms. Also, we will show that the columns of the exact FW transformation for the special high-spin case $(\mathcal{S},S)=(\frac{3}{2},\frac{1}{2})$ are indeed the eigenvectors of the Hamiltonian, and that the exact result agrees with the power-series result calculated to order c^{-3} in Sec. VI of V^{5} for this special case.

In Sec. IV we will describe some applications of this work. We begin by studying Hepner's[13,14] use of the rotation which connects the so(5) generators J_{43} and J_{45}. He used the special-case algebras for these rotations up to $\mathcal{S}=\frac{3}{2}$ to obtain the unnormalized, Lorentz-transformed, rest-frame eigenvectors. We shall elucidate the connection to our general FW transformations. We will also discuss use of the eigenvectors for calculating kinematic factors in current divergence matrix elements and how our method is an implicit solution to the inversion of the Vandermonde matrix made up of the matrix elements of the Lorentz transformation.

The last article[15] of our series on the Bhabha first-order wave equations will present a summary of and set of conclusions on the results we have obtained.

II. EXACT, CLOSED FORM FOR U^{-1}

A. Derivation of U^{-1}

As we showed in paper FW-I,[6] the FW transformation U^{-1} which will diagonalize the Hamiltonian H in the Bhabha indefinite-metric space is given by

$$U^{-1}=[\hat{u}_1,\hat{u}_2,\ldots,\hat{u}_j], \tag{2.1}$$

$$\hat{u}_k=u_k|M_{kk}|^{1/2}, \tag{2.2}$$

$$\hat{u}_i^\dagger M\hat{u}_k=M_{kk}\delta_{ik}. \tag{2.3}$$

The u_k are the j independent eigenvectors of H, normed to ± 1, where

$$M=\eta_4\alpha_4, \tag{2.4}$$

M taken diagonal, is the metric operator. Also, the proper interpretation of the above operators for integer-spin fields is that they are the particle components, with the built-in subsidiary components having been removed by the generalized Sakata-Taketani decompositions described in paper II.[2]

Since the rest-state eigenvectors $\hat{u}_k(0)$ can be written as just a "1" in the kth row of the eigenvectors, then a particular eigenvector can be written in an arbitrary frame as

$$\hat{u}_k(\vec{p})=GL(\vec{\theta})\hat{u}_k(0). \tag{2.5}$$

G is a normalization to be given below and $L(\vec{\theta})$ is the Lorentz transformation operator

$$L(\vec{\theta})=e^{-\theta_k J_{4k}}, \tag{2.6}$$

where for the Bhabha system, J_{4k} is the Lorentz generator

$$J_{4k}=-i[\alpha_4,\alpha_k], \tag{2.7}$$

and θ_k is the boost velocity (or rapidity operator)

$$\tanh\theta=\beta,\quad \gamma\equiv(1-\beta^2)^{-1/2}=\cosh\theta\ . \tag{2.8}$$

[For ease of calculation, in Eq. (2.8) and below we have taken $\vec{p}$ parallel to the z axis so that $\theta_k\to\theta\hat{z}$ and $J_{4k}\to J_{43}$.] There is an arbitrary minus sign in the exponential (2.6) which has been chosen so as to transform particle states to positive momenta vs antiparticle states (see below).

Comparing (2.5) with (2.1) one would suspect that one can write the FW operator U^{-1} as

$$U^{-1}=GL(\theta). \tag{2.9}$$

In paper FW-IV (Ref. 9) we showed in a set of detailed calculations that, with an understanding of θ, Eq. (2.9) is correct for a wide class of first-order wave equations (which include Bhabha equations).

To begin, the necessity that U^{-1} be metric-unitary (pseudounitary),

$$(U^{-1})^\dagger MU^{-1}=M, \tag{2.10}$$

or equivalently that (2.5) satisfy the metric-normalization (2.3), implies a precise determination for the normalization G. Using Theorem IV of FW-II (Ref. 7) this constant was found[8] to be

$$\begin{aligned}G&=\gamma^{-1/2}\\&=(\cosh\theta)^{-1/2}.\end{aligned} \tag{2.11}$$

Specifically, (2.11) holds for the Bhabha case.

The meaning of θ was also derived in FW-III.[8] Since θ is the rapidity operator, we must have that

$$\begin{aligned}\beta&=\tanh^{-1}\theta\\&=\mathcal{P}/\mathcal{E},\end{aligned} \tag{2.12}$$

where $\mathcal{P}$ and $\mathcal{E}$ are momentum and energy quantities. Since in a particular k column U^{-1} is proportional to the Lorentz-transformed rest-state eigenvector, $l_k(\theta)$,

$$\hat{u}_k(p)=(U^{-1})_k=(\gamma^{-1/2}L(\theta))_k\equiv\gamma^{-1/2}l_k(\theta), \tag{2.13}$$

one could suspect that in the kth column of $\gamma^{-1/2}L(\theta)$, $(\beta)_k$ becomes

$$(\beta)_k\to\frac{\pm p_k}{E_k}=\frac{\pm p_k}{(p_k^{\,2}+m_k^{\,2})^{1/2}}, \tag{2.14}$$

where E_k, m_k, and p_k are the energy, rest mass, and momenta of the kth eigenvector. The added point is that since an antiparticle solution is the charge-conjugated (or "negative energy") solution of the particle solution, Eq. (2.14) should imply the plus sign in a column k which represents a particle solution and a minus sign in a column k which represents an antiparticle solution. Then all the eigenvectors so formed would represent $+p$ eigenvectors and hence would be metric-orthonormal and constitute the proper columns of U^{-1}.

In fact, the detailed examination of FW-III (Ref. 8) shows that the above arguments are in the end the correct mathematical results.

The remaining problem is to write down the closed form for the Lorentz-transformation operator $L(\theta)$. The method for doing this was described in FW-IV (Ref. 9) and uses the following theorem of matrix algebra.[16]

Theorem. Consider a matrix B with n *distinct* eigenvalues λ_j. Then there also exist n idempotents e_j,

$$e_j e_i=e_j\delta_{ij}. \tag{2.15}$$

Each idempotent is the space of eigenvectors for a particular distinct eigenvalue, and they all are defined by and have the properties

$$e_j = \prod_{\substack{k=1\\k\neq j}}^{n}\left(\frac{B-\lambda_k I}{\lambda_j-\lambda_k}\right). \tag{2.16}$$

Further, B can be represented by

$$B = \sum_{j=1}^{n} \lambda_j e_j, \tag{2.17}$$

which in turn implies that

$$f(B) = \sum_{j=1}^{n} f(\lambda_j) e_j. \tag{2.18}$$

The above theorem is, in essence, the matrix form of the Lagrange interpolation formula with zero remainder term that was used by Madhavarao *et al.*[17] to obtain the formulas for the η_μ, as discussed in the Appendix of paper I.[1] Equation (2.16) is also the algebraic form of the $\mathcal{g}_j^{\pm}(\mathcal{S})$, the idempotents of the matrix α_4. But since J_{43}, like α_4, is a generator of so(5), they have the same eigenspectra, and so the idempotents of J_{43} are algebraically the same. Thus, the idempotents of J_{43} for a particular algebra $\mathcal{S}$ are given by

$$e_j(\mathcal{S}) = \prod_{\substack{k=-\mathcal{S}\\k\neq j}}^{\mathcal{S}}\left[\frac{J_{43}-kI}{(j-k)}\right], \quad -\mathcal{S} \leq j \leq \mathcal{S}. \tag{2.19}$$

But now we just have to combine (2.6), (2.18), and (2.19) to obtain

$$L(\theta,\mathcal{S}) = e^{-\theta J_{43}} = \sum_{j=-\mathcal{S}}^{\mathcal{S}} e^{-\theta j} \prod_{\substack{k=-\mathcal{S}\\k\neq j}}^{\mathcal{S}}\left[\frac{J_{43}-kI}{j-k}\right], \tag{2.20}$$

and that is it.

A little algebraic manipulation puts (2.20) into the specific half-integer and integer-spin forms

$$L(\theta,\mathcal{S}=n+\tfrac{1}{2}) = \sum_{j=1/2}^{\mathcal{S}}\left[\cosh j\theta - \left(\frac{J_{43}}{j}\right)\sinh j\theta\right] \prod_{\substack{k=1/2\\k\neq j}}^{\mathcal{S}}\left(\frac{J_{43}{}^2-k^2}{j^2-k^2}\right), \tag{2.21}$$

$$L(\theta,\mathcal{S}=n) = \prod_{k=1}^{\mathcal{S}}(1-J_{43}{}^2/k^2) + \sum_{j=1}^{\mathcal{S}}\left[\left(\frac{-J_{43}}{j}\right)\sinh j\theta + \left(\frac{J_{43}}{j}\right)^2\cosh j\theta\right]\prod_{k=1}^{\mathcal{S}}\left(\frac{J_{43}{}^2-k^2}{j^2-k^2}\right). \tag{2.22}$$

Useful formulas for $\cosh(j\theta)$ and $\sinh(j\theta)$ are given in Eqs. (3.7) and (3.8) of FW-IV.[9]

B. Algorithm for U^{-1}

We summarize the derivation of U^{-1} in the following algorithm which gives a concise description of how to write the FW transformation for arbitrary-spin Bhabha fields.

The exact, closed-form FW transformation for arbitrary half-integer-spin and (particle-components) integer-spin Bhabha fields are

$$U^{-1}(\mathcal{S}=n+\tfrac{1}{2}) = \gamma^{-1/2}L(\theta,\mathcal{S}=n+\tfrac{1}{2}), \tag{2.23}$$

$$U^{-1}(\mathcal{S}=n) = [1-\mathcal{g}_0(\mathcal{S})]\gamma^{-1/2}L(\theta,\mathcal{S}=n)[1-\mathcal{g}_0(\mathcal{S})]. \tag{2.24}$$

$L(\theta,\mathcal{S})$ is given by Eqs. (2.21) or (2.22) for half-integer- and integer-spin fields, respectively. One also has

$$\tanh\theta = \beta, \quad \cosh\theta = \gamma, \tag{2.25}$$

$$\gamma = (1-\beta^2)^{-1/2}. \tag{2.26}$$

In $U^{-1}(\beta,\gamma)$ make the substitutions

$$\beta = \frac{\mathfrak{P}}{\mathfrak{W}}, \quad \gamma = \frac{\mathfrak{W}}{\mathfrak{M}}. \tag{2.27}$$

Finally, in each and every kth column of $U^{-1}(\mathfrak{P}/\mathfrak{W}, \mathfrak{W}/\mathfrak{M})$ make the replacements

$$\mathfrak{M} \to |m_k| = |\chi/(\alpha_4)_{kk}|, \tag{2.28a}$$

$$\mathfrak{P} \to \begin{cases} +p & \text{for particle column,} \\ -p & \text{for antiparticle column,}\end{cases} \tag{2.28b}$$

$$\mathfrak{W} \to (p^2+m_k{}^2)^{1/2} = |E_k|, \tag{2.28c}$$

where m_k is the rest mass of the state represented by the kth eigenvector. In the representation we have been using, the particle columns are the first half of the columns in U^{-1}, and the antiparticle columns are the second half.

An alternative method to (2.28b) for dealing with $\mathfrak{P}$ is to take every algebraic factor involving $\mathfrak{P}$ to the right of all matrix operators, and then to make the substitution

$$\mathfrak{P} \to p\tau_3, \quad \tau_3 = \sum_{j=(1/2,1)}^{\mathcal{S}}[\mathcal{g}_j^{+}(\mathcal{S}) - \mathcal{g}_j^{-}(\mathcal{S})]. \tag{2.28b$'$}$$

The $\mathcal{g}_j^{\pm}(\mathcal{S})$ are the projection operators defined in

Eq. (II 3.9c). Thus, in our representation, τ_3 is the diagonal "Pauli-like" operator with $+1$ in the upper left corner block and -1 in the lower right corner block. [For integer spin, τ_3 can be considered surrounded by $1-\mathcal{G}_0(\mathcal{S})$.] As we observed in FW-IV,[9] this exactly corresponds to the extra γ_0 in the Dirac FW transformation with respect to the Dirac Lorentz transformation.

C. Comments on the properties of U^{-1}

Before proceeding to specific examples, there are a number of comments we wish to make on the properties of U^{-1}.

The first is that care must be taken with the normalizations ± 1 and M_{kk} for the eigenvectors u_k and $\hat{u}_k$ which are the starting point for constructing U^{-1}. For example, the normalization $+1$ is usually desired to yield a probabilistic interpretation for quantum mechanics, and generalized to ± 1 for an indefinite-metric space. This means that one has already taken the measure into account. Here the metric operator M does not have eigenvalues of ± 1, so that one is implying a different measure for different mass states if one demands the ± 1 normalization. For example, in the special Dirac case, one should use the $\hat{u}_k$ eigenvectors, which correspond to the usual probabilistic normalization of the Dirac γ matrix representation. Using the u_k eigenvectors would mean a different measure. We mention this to make sure that the reader is conscious of the delicate bookkeeping which is necessary to keep track of the measure. It is the $\hat{u}_k$ notation which corresponds to the more usual quantum-mechanical normalization. Ultimately that is why the columns of U^{-1} are composed of the j eigenvectors $\hat{u}_k$. In principle, one could choose for one's theory the u_k, but the physical meaning would be different.

The next observation is that the FW transformation U^{-1} which we have constructed is indeed the transformation which diagonalizes the original Bhabha Poincaré generators of Eqs. (III 1.5)–(III 1.9) and (III 5.28)–(III 5.31) to the Bhabha FW-Poincaré generators we gave in Eqs. (V4.2)–(V4.6) and (V4.44)–(V4.47). The change of forms between original Poincaré generators and FW-Poincaré generators was derived in FW-III (Ref. 8) for a large class of first-order wave equations. When those general results are restricted to the Bhabha system, our above statement ensues.

Further, the connection between Lorentz transformation and FW transformation allows one to construct the solutions ψ to the wave equation

$$(\partial \cdot \alpha + \chi)\psi = 0 \tag{2.29}$$

viz the eigenvectors of the Hamiltonian *matrix*. When specified to the Bhabha system, the results of FW-III (Ref. 8) allows one to write

$$\psi_k \sim \hat{u}_k(p) e^{+i\varepsilon_k(\mathcal{P}z - \mathcal{E}t)}, \tag{2.30}$$

where $\mathcal{E}$ and $\mathcal{P}$ represent the ground state 4-momenta,

$$\mathcal{E} \equiv (\mathcal{P}^2 + \chi^2/\mathcal{S}^2)^{1/2}, \tag{2.31}$$

and ε_k converts the exponential into the appropriate excited state

$$\varepsilon_k = \mathcal{S}(\alpha_4^{-1})_{kk}. \tag{2.32}$$

The relationship of ψ_k to $\hat{u}_k$ gives another way to look at the crucial signs in the relationship between the Lorentz and FW transformations. U^{-1}, composed of the $\hat{u}_k$, is a matrix which diagonalizes another matrix, H, via

$$H^{\mathrm{FW}} = UHU^{-1}. \tag{2.33}$$

That is, the matrix elements of U^{-1} and hence the objects inside can be considered "numbers." However, the Lorentz transformation acts on the solutions to the wave equation (2.30), including the exponential in the wave equation. Hence, in that sense the objects in the matrix elements of the Lorentz transformation are "operators." It is this difference in going between the FW and Lorentz transformations which, when studied in detail,[8] mathematically accounts for the replacements (2.28), which we have argued for on the basis of Lorentz-transformed particle and antiparticle state properties.

Finally, a comment is in order on what one's chances would be of explicitly summing the expanded infinite series for $L(\theta)$ if one did know of our theorem. At least in principle, this would be simpler than trying to sum the infinite power series in c^{-1} generated in paper V (Ref. 5) because here we have each term of the sum trivially generated by expanding the exponential. But just as in V, *summing* the generated series for specific increasing $\mathcal{S}$ rapidly becomes horrendous. For $\mathcal{S}=\frac{1}{2}$ and $\mathcal{S}=1$ (Dirac and DKP) the series can be summed to yield answers equivalent to these known special cases (see Sec. III). Already for $\mathcal{S}=\frac{3}{2}$, where the characteristic equation has four terms, summing the series becomes monumental. We have done it for $\mathcal{S}=\frac{3}{2}$, but the method involves summing staggered binomial expansions. For $\mathcal{S}>\frac{3}{2}$ obtaining special cases by explicitly summing becomes a semi-infinite process.

III. SPECIAL CASES

The special cases of Eqs. (2.21) and (2.22) up to $\mathcal{S}=3$ are

$$L(\theta,\tfrac{1}{2})=\cosh\left(\frac{\theta}{2}\right)-2J_{43}\sinh\left(\frac{\theta}{2}\right)\,, \tag{3.1}$$

$$L(\theta,1)=(1-J_{43}{}^2)-J_{43}\sinh\theta+J_{43}{}^2\cosh\theta, \tag{3.2}$$

$$\begin{aligned}L(\theta,\tfrac{3}{2})=&\left[\tfrac{9}{8}\cosh\left(\frac{\theta}{2}\right)-\tfrac{1}{8}\cosh\left(\frac{3\theta}{2}\right)\right]\\&+J_{43}\left[-\tfrac{9}{4}\sinh\left(\frac{\theta}{2}\right)+\tfrac{1}{12}\sinh\left(\frac{3\theta}{2}\right)\right]+J_{43}{}^2\left[-\tfrac{1}{2}\cosh\left(\frac{\theta}{2}\right)+\tfrac{1}{2}\cosh\left(\frac{3\theta}{2}\right)\right]+J_{43}{}^3\left[\sinh\left(\frac{\theta}{2}\right)-\tfrac{1}{3}\sinh\left(\frac{3\theta}{2}\right)\right],\end{aligned} \tag{3.3}$$

$$\begin{aligned}L(\theta,2)=&(I-\tfrac{5}{4}J_{43}{}^2+\tfrac{1}{4}J_{43}{}^4)+J_{43}[-\tfrac{4}{3}\sinh\theta+\tfrac{1}{6}\sinh2\theta]+J_{43}{}^2[\tfrac{4}{3}\cosh\theta-\tfrac{1}{12}\cosh2\theta]\\&+J_{43}{}^3[\tfrac{1}{3}\sinh\theta-\tfrac{1}{6}\sinh2\theta]+J_{43}{}^4[-\tfrac{1}{3}\cosh\theta+\tfrac{1}{12}\cosh2\theta],\end{aligned} \tag{3.4}$$

$$\begin{aligned}L(\theta,\tfrac{5}{2})=&\left[\tfrac{75}{64}\cosh\left(\frac{\theta}{2}\right)-\tfrac{25}{128}\cosh\left(\frac{3\theta}{2}\right)+\tfrac{3}{128}\cosh\left(\frac{5\theta}{2}\right)\right]+(J_{43})\left[-\tfrac{75}{32}\sinh\left(\frac{\theta}{2}\right)+\tfrac{25}{192}\sinh\left(\frac{3\theta}{2}\right)-\tfrac{3}{320}\sinh\left(\frac{5\theta}{2}\right)\right]\\&+(J_{43})^2\left[-\tfrac{17}{24}\cosh\left(\frac{\theta}{2}\right)+\tfrac{13}{16}\cosh\left(\frac{3\theta}{2}\right)-\tfrac{5}{48}\cosh\left(\frac{5\theta}{2}\right)\right]+(J_{43})^3\left[\tfrac{17}{12}\sinh\left(\frac{\theta}{2}\right)-\tfrac{13}{24}\sinh\left(\frac{3\theta}{2}\right)+\tfrac{1}{24}\sinh\left(\frac{5\theta}{2}\right)\right]\\&+(J_{43})^4\left[\tfrac{1}{12}\cosh\left(\frac{\theta}{2}\right)-\tfrac{1}{8}\cosh\left(\frac{3\theta}{2}\right)+\tfrac{1}{24}\cosh\left(\frac{5\theta}{2}\right)\right]+(J_{43})^5\left[-\tfrac{1}{6}\sinh\left(\frac{\theta}{2}\right)+\tfrac{1}{12}\sinh\left(\frac{3\theta}{2}\right)-\tfrac{1}{60}\sinh\left(\frac{5\theta}{2}\right)\right],\end{aligned} \tag{3.5}$$

$$\begin{aligned}L(\theta,3)=&(1-J_{43}{}^2)\left(1-\frac{J_{43}{}^2}{4}\right)\left(1-\frac{J_{43}{}^2}{9}\right)\\&+(J_{43})[-\tfrac{3}{2}\sinh\theta+\tfrac{3}{10}\sinh(2\theta)-\tfrac{1}{30}\sinh(3\theta)]+(J_{43})^2[\tfrac{3}{2}\cosh\theta-\tfrac{3}{20}\cosh(2\theta)+\tfrac{1}{90}\cosh(3\theta)]\\&+(J_{43})^3[\tfrac{13}{24}\sinh\theta-\tfrac{1}{3}\sinh(2\theta)+\tfrac{1}{24}\sinh(3\theta)]+(J_{43})^4[-\tfrac{13}{24}\cosh\theta+\tfrac{1}{6}\sinh(2\theta)-\tfrac{1}{72}\cosh(3\theta)]\\&+(J_{43})^5[-\tfrac{1}{24}\sinh\theta+\tfrac{1}{30}\sinh(2\theta)-\tfrac{1}{120}\sinh(3\theta)]+(J_{43})^6[\tfrac{1}{24}\cosh\theta-\tfrac{1}{60}\cosh(2\theta)+\tfrac{1}{360}\cosh(3\theta)].\end{aligned} \tag{3.6}$$

As an illustrative exercise, the reader can convince himself that Eqs. (3.1)–(3.6) are indeed the correct, exact expressions. Take Eq. (1.6), giving $L(\theta)$ as $e^{(-\theta J_{43})}$, and expand the exponential as a power series in J_{43}. [Expand up to $J_{43}{}^{2\mathcal{S}}$ for the easiest verification. If one expands higher, then the characteristic equation (I 2.31) for J_{43} should be used to reduce the maximum power in J_{43} to be $2\mathcal{S}$.] Then compare this result with Eqs. (3.1)–(3.6) with the sinh and cosh terms expanded as power series in θ. The numerical coefficients for any product $\theta^n J_{43}{}^m$, $m\leq 2\mathcal{S}$, will be the same.

Now we illustrate the cases $\mathcal{S}=\frac{1}{2}$, 1, and $\frac{3}{2}$ in detail.

First, take $\mathcal{S}=\frac{1}{2}$, the Dirac case. Using the representation

$$\alpha_3=\tfrac{1}{2}\gamma_3=-\frac{i}{2}\begin{bmatrix}0&0&1&0\\0&0&0&-1\\-1&0&0&0\\0&1&0&0\end{bmatrix},\qquad \alpha_4=\tfrac{1}{2}\gamma_4=\tfrac{1}{2}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{bmatrix}, \tag{3.7}$$

one has

$$U^{-1}=\gamma^{-1/2}L(\theta,\tfrac{1}{2})$$
$$=\left(\frac{\mathcal{W}+\mathfrak{M}}{2\mathcal{W}}\right)^{1/2}\begin{bmatrix} 1 & 0 & \dfrac{\mathfrak{P}}{\mathcal{W}+\mathfrak{M}} & 0 \\ 0 & 1 & 0 & \dfrac{-\mathfrak{P}}{\mathcal{W}+\mathfrak{M}} \\ \dfrac{\mathfrak{P}}{\mathcal{W}+\mathfrak{M}} & 0 & 1 & 0 \\ 0 & \dfrac{-\mathfrak{P}}{\mathcal{W}+\mathfrak{M}} & 0 & 1 \end{bmatrix}. \tag{3.8}$$

If one uses the interpretation of Eqs. (2.28), especially $\mathfrak{P}=-p$ in columns 3 and 4, then Eq. (3.8) is exactly the FW transformation U^{-1} defined by Eq. (V3.9), taking care not to confuse that Dirac notation with the present Bhabha notation.

Now consider the DKP $(\mathcal{S},S)=(1,0)$ spin-0 case. Taking the representation of the $\alpha_\mu=\beta_\mu$ matrices as

$$\alpha_3=\beta_3=\begin{bmatrix} 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \end{bmatrix},$$
$$\alpha_4=\beta_4=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}, \tag{3.9}$$

one obtains

$$\gamma^{-1/2}L(\theta,1)=\frac{1}{2(\mathfrak{M}\mathcal{W})^{1/2}}\begin{bmatrix} (\mathcal{W}+\mathfrak{M}) & 0 & i\sqrt{2}\,\mathfrak{P} & 0 & (-\mathcal{W}+\mathfrak{M}) \\ 0 & 2\mathfrak{M} & 0 & 0 & 0 \\ -i\sqrt{2}\,\mathfrak{P} & 0 & 2\mathcal{W} & 0 & i\sqrt{2}\,\mathfrak{P} \\ 0 & 0 & 0 & 2\mathfrak{M} & 0 \\ (-\mathcal{W}+\mathfrak{M}) & 0 & -i2\mathfrak{P} & 0 & \mathcal{W}+\mathfrak{M} \end{bmatrix}. \tag{3.10}$$

Surrounding the above with the projection operators onto the particle components $[I-\mathcal{I}_0(1)]$ removes the middle three columns and rows, and yields

$$U^{-1}=\frac{1}{2(\mathfrak{M}\mathcal{W})^{1/2}}\begin{bmatrix} (\mathcal{W}+\mathfrak{M}) & (-\mathcal{W}+\mathfrak{M}) \\ (-\mathcal{W}+\mathfrak{M}) & (\mathcal{W}+\mathfrak{M}) \end{bmatrix}. \tag{3.11}$$

Again this is exactly the FW transformation for the ST version of DKP given in Eq. (V3.40). In other words, with the algorithm of Sec. II B, the columns of (3.11) are the two ST eigenvectors given in Eqs. (5.1) and (5.4) of Ref. 18, or the properly normalized[19] version of Eqs. (I5.1) and (I5.2).

Note that we did not use the original Kemmer[20] representation of the β matrices, but rather one similar to the ρ representation of Dirac for the γ matrices.[21] The reason is that, as observed in paper I,[1] the Kemmer representation mixes up particle and antiparticle states, whereas we want a representation which keeps them separate, as comes out in the ST two-component form.

Finally we consider the $(\mathcal{S},S)=(\frac{3}{2},\frac{1}{2})$ high-spin case. This representation is 16-dimensional. If we consider only momenta in one (z) direction, the α_3 and α_4 matrices can conveniently be written in the $(8\times 8)\otimes I(2\times 2)$ form given in Eqs. (V6.2) and (V6.1), which again is analogous to the Dirac ρ matrices. [$I(2\times 2)$ is a two-dimensional unit matrix which multiplies every element of an (8×8) matrix or eigenvector.] Using this representation for the α_μ matrices of Sec. II B yields

$$U^{-1}=\left[\frac{1}{2\mathcal{M}}\left(\frac{\mathcal{W}+\mathcal{M}}{2\mathcal{W}}\right)^{1/2}\right]$$

$$\times\begin{bmatrix}
(\mathcal{W}+\mathcal{M}) & 0 & i\sqrt{3}\,\mathcal{P} & 0 & 0 & -\sqrt{3}(\mathcal{W}-\mathcal{M}) & 0 & -i\mathcal{P}\left(\frac{\mathcal{W}-\mathcal{M}}{\mathcal{W}+\mathcal{M}}\right) \\
0 & 2\mathcal{M} & 0 & 0 & \frac{i2\mathcal{P}}{(\mathcal{W}+\mathcal{M})} & 0 & 0 & 0 \\
-i\sqrt{3}\,\mathcal{P} & 0 & (3\mathcal{W}-\mathcal{M}) & 0 & 0 & i\mathcal{P}\left(\frac{3\mathcal{W}+\mathcal{M}}{\mathcal{W}+\mathcal{M}}\right) & 0 & -\sqrt{3}(\mathcal{W}-\mathcal{M}) \\
0 & 0 & 0 & 2\mathcal{M} & 0 & 0 & \frac{i2\mathcal{P}}{(\mathcal{W}+\mathcal{M})} & 0 \\
0 & \frac{-i2\mathcal{M}\mathcal{P}}{(\mathcal{W}+\mathcal{M})} & 0 & 0 & 2\mathcal{M} & 0 & 0 & 0 \\
-\sqrt{3}(\mathcal{W}-\mathcal{M}) & 0 & -i\mathcal{P}\left(\frac{3\mathcal{W}+\mathcal{M}}{\mathcal{W}+\mathcal{M}}\right) & 0 & 0 & (3\mathcal{W}-\mathcal{M}) & 0 & i\sqrt{3}\,\mathcal{P} \\
0 & 0 & 0 & \frac{-i2\mathcal{M}\mathcal{P}}{(\mathcal{W}+\mathcal{M})} & 0 & 0 & 2\mathcal{M} & 0 \\
i\mathcal{P}\left(\frac{\mathcal{W}-\mathcal{M}}{\mathcal{W}+\mathcal{M}}\right) & 0 & -\sqrt{3}(\mathcal{W}-\mathcal{M}) & 0 & 0 & -i\sqrt{3}\,\mathcal{P} & 0 & (\mathcal{W}+\mathcal{M})
\end{bmatrix}$$

$$\otimes I(2\times 2). \tag{3.12}$$

The interpretation of Eq. (2.28) applied to (3.12) is

$$\mathcal{W}(8\times 8)=\begin{cases}(p^2+\frac{4}{9}\chi^2)^{1/2}, & \text{cols. } 1, 8\\ (p^2+4\chi^2)^{1/2}, & \text{cols. } 2\text{–}7\end{cases} \tag{3.13}$$

$$\mathcal{M}(8\times 8)=\begin{cases}\frac{2}{3}\chi, & \text{cols. } 1, 8\\ 2\chi, & \text{cols. } 2\text{–}7\end{cases} \tag{3.14}$$

$$\mathcal{P}(8\times 8)=\begin{cases}+p, & \text{cols. } 1\text{–}4\\ -p, & \text{cols. } 5\text{–}8.\end{cases} \tag{3.15}$$

The reader can now explicitly verify for himself that with the above interpretation the 16 columns of Eq. (3.12) are the metric-orthonormalized eigenvectors $\hat{u}_j$.

$$\hat{u}_i M\hat{u}_j = M_{ii}\delta_{ij}, \tag{3.16}$$

$$H\hat{u}_j = E_j\hat{u}_j, \tag{3.17}$$

$$E_j=\begin{cases}+(p^2+\frac{4}{9}\chi^2)^{1/2}, & 1\le j\le 2\\ +(p^2+4\chi^2)^{1/2}, & 3\le j\le 8\\ -(p^2+4\chi^2)^{1/2}, & 9\le j\le 14\\ -(p^2+\frac{4}{9}\chi^2)^{1/2}, & 15\le j\le 16\end{cases} \tag{3.18}$$

of the Hamiltonian

$$H=\alpha_4^{-1}(ip\alpha_3+\chi)$$

$$=\begin{bmatrix} \frac{2}{3}\chi & 0 & \frac{ip}{\sqrt{3}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 2\chi & 0 & 0 & ip & 0 & 0 & 0 \\ i\sqrt{3}p & 0 & 2\chi & 0 & 0 & i2p & 0 & 0 \\ 0 & 0 & 0 & 2\chi & 0 & 0 & ip & 0 \\ 0 & -ip & 0 & 0 & -2\chi & 0 & 0 & 0 \\ 0 & 0 & -i2p & 0 & 0 & -2\chi & 0 & -i\sqrt{3}p \\ 0 & 0 & 0 & -ip & 0 & 0 & -2\chi & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{-ip}{\sqrt{3}} & 0 & -\frac{2}{3}\chi \end{bmatrix} \otimes I(2\times 2). \tag{3.19}$$

Further, if each element of (3.12) is expanded as a power series in c^{-1} (substitute $p\to pc$, $\chi\to\chi c^2$), then (3.12) agrees element by element with the FW transformation generated as a power series to order $(c^{-1})^3$ for this special case in Eq. (V6.5).

IV. APPLICATIONS

A. Hepner's transformation

Some of our specific results are related to the work of Hepner,[13,14] who used the canonical transformation [rotation in the (x_μ, x_5) plane] between the Bhabha so(5) generators $J_{4\mu}$ and J_{45} to help obtain the unnormalized Lorentz transformed eigenvectors for particular so(5) algebras. The connection between his work and our general method provides insight into the Bhabha formalism.

Using our notation, Hepner observed that the canonical transformation S [not to be confused with either the spin S or the power series FW coefficients S_j of V (Ref. 5)], which relates J_{43} to J_{45}, is given by

$$J_{43}=SJ_{45}S^{-1}$$
$$=S\alpha_4S^{-1}, \tag{4.1a}$$

$$SJ_{43}S^{-1}=J_{53}=-\alpha_3. \tag{4.1b}$$

Hepner showed that the solution to (4.1) is defined by

$$S^4=e^{(2\pi i\alpha_4)} \tag{4.2}$$

$$=\begin{cases}+1, & \mathcal{S}=\text{integer}\\ -1, & \mathcal{S}=\text{half-integer}.\end{cases} \tag{4.3}$$

Equation (4.3) is another manifestation of the fact that rotations of 2π produce a (-1) multiplicative factor for spinor wave functions, something which has been emphasized for Dirac particles,[22-25] and recently observed experimentally.[26]

The matrix theorem of Eqs. (2.15)–(2.18) combined with the definition of η_4 given in Eqs. (I3.47) immediately tells one that

$$S^2=\begin{cases}\eta_4, & \mathcal{S}=\text{integer}\\ i\eta_4, & \mathcal{S}=\text{half-integer}.\end{cases} \tag{4.4}$$

Equation (4.4) could be solved for S by using the polynomical expressions given in Eqs. (I3.47). The more direct method, though, is to use the matrix theorem of Eqs. (2.15)–(2.18) to give S in closed form,

$$S=\sum_{j=-\mathcal{S}}^{\mathcal{S}}\exp\left(i\frac{\pi}{2}j\right)\prod_{\substack{k=-\mathcal{S}\\k\neq j}}^{\mathcal{S}}\frac{\alpha_4-kI}{j-k}. \tag{4.5}$$

Looking at the free wave equation

$$(-\mathcal{W}\alpha_4+i\mathfrak{P}\alpha_3+\chi)\psi=0, \tag{4.6}$$

Hepner observed that if one took the unrenormalized Lorentz transformation of the rest state,

$$(-\mathfrak{M}\alpha_4+\chi)\psi(0)=0, \tag{4.7}$$

$$\psi=T\psi(0), \quad \psi(0)=T^{-1}\psi, \tag{4.8}$$

Eq. (4.7) could be multiplied on the left-hand side by T^{-1} and combined with (4.6) and (4.8) to yield

$$0=[-\mathfrak{M}T\alpha_4T^{-1}-i\mathfrak{P}\alpha_3+\mathcal{W}\alpha_4]\psi. \tag{4.9}$$

The quantity in the square brackets in Eq. (4.9) was taken as an operator equation to be solved for T.

Up to this point, Hepner's work was general. Here he specialized, in particular, to the $\mathcal{S}=\frac{3}{2}$ case of T, which we will now connect to our $L(\theta,\frac{3}{2})$. Because of the characteristic equation of J_{43} for $\mathcal{S}=\frac{3}{2}$, Hepner knew that he could write T in the form

$$T=1+AJ_{43}+BJ_{43}{}^2+CJ_{43}{}^3, \tag{4.10}$$

where A, B, and C were to be determined. Putting (4.10) into (4.9), multiplying on the left-hand side by S and on the right-hand side by S^{-1}, and using (4.1) eliminates the $J_{43}{}^2$ and $J_{43}{}^3$ terms and gives

$$0=F\alpha_4+-\left(\frac{\mathcal{W}}{\mathfrak{M}}\right)\alpha_4+i\left(\frac{\mathfrak{P}}{\mathfrak{M}}\right)J_{43}F, \tag{4.11}$$

$$F=1-A\alpha_3+B\alpha_3{}^2-C\alpha_3{}^3. \tag{4.12}$$

Then Hepner used the fact[17] that for a particular so(5) algebra the α_μ can be written as the product

$$\alpha_\mu=\eta_\mu\xi_\mu=\xi_\mu\eta_\mu \quad \text{(no sum)} \tag{4.13}$$

where by virtue of the algebra of the η_μ (upper sign for half-integer spin)

$$\eta_\mu\eta_\nu\pm\eta_\nu\eta_\mu=0,\quad \mu\neq\nu \tag{4.14}$$

$$\eta_\mu\alpha_\nu+\alpha_\nu\eta_\mu=0,\quad \mu\neq\nu \tag{4.15}$$

one has

$$\eta_\mu\xi_\nu\mp\xi_\nu\eta_\mu=0. \tag{4.16}$$

With (4.13), one can write Eqs. (4.11) and (4.12) as

$$0=(a_1\xi_4+b_1\xi_1\xi_4+c_1\xi_4\xi_1)\eta_4 \\ +(a_2\xi_4+b_2\xi_1\xi_4+c_2\xi_4\xi_1)\eta_1\eta_4, \tag{4.17}$$

$$a_1=\left(1-\frac{\mathcal{W}}{\mathfrak{M}}\right)(1+\tfrac{3}{4}B)-\frac{1}{2}\frac{\mathfrak{P}}{\mathfrak{M}}(A-\tfrac{7}{4}C), \tag{4.18a}$$

$$b_1=B+\frac{1}{2}\frac{\mathfrak{P}}{\mathfrak{M}}(A+\tfrac{1}{4}C), \tag{4.18b}$$

$$c_1=-\frac{\mathcal{W}}{\mathfrak{M}}B-\frac{1}{2}\frac{\mathfrak{P}}{\mathfrak{M}}(A-\tfrac{13}{4}C), \tag{4.18c}$$

$$a_2=-\tfrac{3}{4}C\left(1+\frac{\mathcal{W}}{\mathfrak{M}}\right)-\frac{1}{2}\frac{\mathfrak{P}}{\mathfrak{M}}B, \tag{4.19a}$$

$$b_2=-A-\tfrac{7}{4}C-\frac{\mathfrak{P}}{\mathfrak{M}}(1-\tfrac{1}{4}B), \tag{4.19b}$$

$$c_2=-\frac{\mathcal{W}}{\mathfrak{M}}(A-\tfrac{7}{4}C)-\frac{\mathfrak{P}}{\mathfrak{M}}(1-\tfrac{5}{4}B). \tag{4.19c}$$

However, for half-integer spin, Eq. (4.16) implies that (4.13) is a direct-product algebra, and thus the ξ_μ and the η_μ are all independent. This means that all the Eqs. (4.18) and (4.19) must equal zero, and hence that Eqs. (4.18) and (4.19) separately form two sets of three equations in three unknowns: A, B, and C. The two solutions are identical, and specifically are

$$A=\frac{2}{3}\left(\frac{\mathfrak{P}}{\mathcal{W}+\mathfrak{M}}\right)\left(\frac{\mathcal{W}-13\mathfrak{M}}{5\mathfrak{M}-\mathcal{W}}\right), \tag{4.20a}$$

$$B=4\left(\frac{\mathcal{W}-\mathfrak{M}}{5\mathfrak{M}-\mathcal{W}}\right), \tag{4.20b}$$

$$C=-\frac{8}{3}\left(\frac{\mathfrak{P}}{\mathcal{W}+\mathfrak{M}}\right)\left(\frac{\mathcal{W}-\mathfrak{M}}{5\mathfrak{M}-\mathcal{W}}\right). \tag{4.20c}$$

Putting Eqs. (4.20) into (4.10) and comparing to (3.3) with θ defined by Eqs. (2.25)–(2.27), one finds

$$L(\theta,\tfrac{3}{2})=\left(\frac{5\mathfrak{M}-\mathcal{W}}{4\mathfrak{M}}\right)\left(\frac{\mathcal{W}+\mathfrak{M}}{2\mathfrak{M}}\right)^{1/2}T, \tag{4.21}$$

which completes the connection between the two methods.

Clearly, because of its generality and ease of calculation, our method is preferable for obtaining general FW transformations and normalized eigenvectors. However, Hepner's method deserves credit for being the first to try this mode of attack, and the canonical transformation S does yield insight into the algebra.

B. Current divergence matrix elements

In using the DKP spin-0 formalism to describe the symmetry-breaking meson-decay processes

$$K(\vec{\mathrm{p}},m)\to\pi(\vec{\mathrm{p}}',\mu)l\nu, \tag{4.22}$$

it was found[27-29] that the matrix elements of the divergence of the meson currents β_λ and $q_\lambda=(p_\lambda-p_\lambda')$ had "kinematic" zeros in momentum transfer squared (t) at the unphysical point t_0,

$$t\equiv-q\cdot q=t_0-2(EE'-\vec{\mathrm{p}}\cdot\vec{\mathrm{p}}'), \tag{4.23}$$

$$t_0\equiv(m+\mu)^2. \tag{4.24}$$

In particular,[29]

$$\partial_\lambda\left\langle\bar{\psi}^{\mathrm{DKP}}(E',\vec{\mathrm{p}}',\mu)\left|\beta_\lambda g_V^{\mathrm{DKP}}(t)+\frac{iq_\lambda}{m+\mu}g_S^{\mathrm{DKP}}(t)\right|\psi^{\mathrm{DKP}}(E,\vec{\mathrm{p}},m)\right\rangle=-(m-\mu)\left[\frac{1}{4m\mu}\left(\frac{m}{E}\right)^{1/2}\left(\frac{\mu}{E'}\right)^{1/2}\right]g_0^{\mathrm{DKP}}(t)Z^{\mathrm{DKP}}, \tag{4.25}$$

$$g_0^{\mathrm{DKP}}(t)=g_V^{\mathrm{DKP}}(t)-\frac{t}{m^2-\mu^2}g_S^{\mathrm{DKP}}(t), \tag{4.26}$$

$$Z^{\mathrm{DKP}}=(m+\mu)^2\left(1-\frac{t}{t_0}\right), \tag{4.27}$$

where the g's are form factors. (Such zeros have also been derived in a current-algebra formalism.[30])

The above observation generated a fair amount of controversy,[31-34] essentially on the questions of how many independent form factors are involved and what their smoothness properties are. (We refer the reader to Ref. 34 for a detailed mathematical discussion of all the "possible" currents.) We wish to emphasize here that complicated "kinematic" factors are a characteristic feature of current divergence matrix elements taken with the eigenvectors of Bhabha first-order wave equations; this includes the Dirac equation, although it is not generally realized.

The analogous Dirac nonconserved currents are to be found in the description of the semileptonic decays of baryons,

$$B(\vec{p}, m) \to B'(\vec{p}', \mu) l\nu. \tag{4.28}$$

However, there one standardly does *not*[35] calculate the current divergence matrix elements, and so it is perhaps of some surprise to note that

$$\partial_\lambda \left\langle \gamma_\lambda g_V^D(t) + \frac{iq_\lambda}{m+\mu} g_S^D(t) \right\rangle = -(m-\mu)\left[\frac{1}{4m\mu}\left(\frac{E'+\mu}{2\mu}\right)^{1/2}\left(\frac{E+m}{2m}\right)^{1/2}\right] g_0^D(t) Z^D, \tag{4.29}$$

$$g_0^D(t) = g_V^D(t) - \frac{t}{m^2-\mu^2} g_S^D(t), \tag{4.30}$$

$$Z^D = 4m\mu\left[1 - \frac{\vec{p}\cdot\vec{p}'}{(E'+\mu)(E+m)}\right] \tag{4.31}$$

In the above equations we have gone to the γ matrix formulation in Sakurai's notation,[36] used a frame where $\vec{p}\,||\,\vec{p}'\,||\,\hat{z}$, and taken the expectation value between spin-up states (the "1" eigenvector). The reader will note the similarity to the DKP result, although now, with higher spin, the answer for Z is more complicated. (Of course for nonconserved spin-$\frac{1}{2}$ currents there are, in principle, three independent form factors,[35] more still for higher spin.[34] For comparison, we will use only the α_λ and q_λ current form factors in this discussion.)

For higher spin, where the eigenvectors can be obtained from the columns of the FW transformation, the situation becomes more complicated, but remains similar. Consider the $(\mathcal{S}, S) = (\frac{3}{2}, \frac{1}{2})$ representation where, with the interpretation of (2.28), the columns of (3.12) are the metric-orthonormalized eigenvectors. Taking, in particular, the expectation value between $i=1$ states [first column of Eq. (3.12)], one finds $(\vec{p}\,||\,\vec{p}'\,||\,\hat{z})$

$$\partial_\lambda \left\langle \alpha_\lambda g_V^B(t) + \frac{iq_\lambda}{m+\mu} g_S^B(t) \right\rangle = -(m-\mu)\left[\frac{1}{4m\mu}\left(\frac{E'+\mu}{2\mu}\right)^{1/2}\left(\frac{E+m}{2m}\right)^{1/2}\right] g_0^B(t) Z^B, \tag{4.32}$$

$$g_0^B(t) = 3g_V^B(t) - \frac{2t}{m^2-\mu^2} g_S^B(t), \tag{4.33}$$

$$Z^B = [E'(2E-M) + \mu(2M-E)] \times\left[1 - \frac{\vec{p}\cdot\vec{p}'}{(E'+\mu)(E+m)}\left(\frac{E'(2E+m)+\mu(2m+E)}{E'(2E-m)+\mu(2m-E)}\right)\right]. \tag{4.34}$$

The coefficients 3 and 2 in the Bhabha $\mathcal{S}=\frac{3}{2}$ scalar form factor (4.33) are due to the metric. If we had used the Bhabha notation for the Dirac case, there would have been a factor of 2 inside $g_0(t)$.

For even higher spin, one can at will get an increasingly complicated structure. One can consider matrix elements between different high-spin eigenvectors which are coupled by the current [such as the first and third columns of the (8×8) matrix in Eq. (3.12)]. Next, other independent form factors from other independent currents can be used. Finally, one can consider form factors from axial-vector currents, obtained mathematically by simply multiplying the analogous vector currents by the pseudoscalar operator of Eq. I2.18),

$$A_\lambda = \epsilon_{\mu\nu\rho\sigma}\alpha_\mu \alpha_\nu \alpha_\rho \alpha_\sigma J_\lambda. \tag{4.35}$$

The point we are making is that "kinematic" factors in current divergence matrix elements occur naturally in the Bhabha first-order formalism. The surprise at their discovery in the DKP spin-0 case was due to two causes: (i) Current divergence matrix elements had only been widely studied in the spin-0 case; (ii) those studies had only been done in the second-order Klein-Gordon formalism, where the kinematic zero does *not* occur (assuming always that the form factors involved are smooth).

C. Vandermonde matrix

An interesting sidelight of the matrix method described in Sec. II is that its use gives us the solution for the inverse of a particular Vandermonde matrix. A Vandermonde matrix is one of the form[16,37]

$$V = \begin{bmatrix} x_1^{n-1} & x_1^{n-2} & \cdots & x_1 & 1 \\ x_2^{n-1} & x_2^{n-2} & \cdots & x_2 & 1 \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ x_{n-1}^{n-1} & x_{n-1}^{n-2} & \cdots & x_{n-1} & 1 \\ x_n^{n-1} & x_n^{n-2} & \cdots & x_n & 1 \end{bmatrix}, \qquad (4.36)$$

and is usually discussed as a determinant.[37]

Refering back to Sec. II, one knows immediately from the characteristic equation for the $J_{\mu\nu}$ that one can write

$$e^{-\theta J_{43}} = \sum_{n=0}^{2\mathcal{S}} A_n(\theta) J_{43}{}^n, \qquad (4.37)$$

with the $A_n(\theta)$ to be determined. Choosing a representation where J_{43} is diagonal allows one to write the diagonal matrix elements of (4.37) in the Vandermonde form

$$[e^{-\theta\{n\}}] = \begin{bmatrix} e^{-\theta\mathcal{S}} \\ e^{-\theta(\mathcal{S}-1)} \\ \cdot \\ \cdot \\ \cdot \\ e^{+\theta(\mathcal{S}-1)} \\ e^{+\theta\mathcal{S}} \end{bmatrix}$$

$$= \begin{bmatrix} \mathcal{S}^{2\mathcal{S}} & \mathcal{S}^{2\mathcal{S}-1} & \cdots & \mathcal{S} & 1 \\ (\mathcal{S}-1)^{2\mathcal{S}} & (\mathcal{S}-1)^{2\mathcal{S}-1} & \cdots & (\mathcal{S}-1) & 1 \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ (-\mathcal{S}+1)^{2\mathcal{S}} & (-\mathcal{S}+1)^{2\mathcal{S}-1} & \cdots & (-\mathcal{S}+1) & 1 \\ (-\mathcal{S})^{2\mathcal{S}} & (-\mathcal{S})^{2\mathcal{S}-1} & \cdots & (-\mathcal{S}) & 1 \end{bmatrix} \begin{bmatrix} A_{2\mathcal{S}}(\theta) \\ A_{2\mathcal{S}-1}(\theta) \\ \cdot \\ \cdot \\ \cdot \\ A_1(\theta) \\ A_0(\theta) \end{bmatrix}$$

$$\equiv V[A_{\{n\}}]. \qquad (4.38)$$

One can now see that our general solution, Eq. (2.20), in effect gives the inverse Vandermonde matrix V^{-1}, to that in (4.38). The coefficients multiplying the $J_{43}^{\{n\}}$ in Eq. (2.20) are the $A_{\{n\}}$, and these are expressed as numerical factors multiplying the $e^{-\theta\{n\}}$. Thus, these numerical factors are the matrix elements of the inverse Vandermonde matrix V^{-1} of

$$[A_{\{n\}}] = V^{-1}[e^{-\theta\{n\}}]. \qquad (4.39)$$

Observe that because of their product form, the matrix elements of V^{-1} are related to the symmetric functions discussed in the Appendix of paper IV.[4] This property is known.

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