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Foldy-Wouthuysen transformations in an indefinite-metric space. IV. Exact, closed-form expressions for first-order wave equations

R. A. Krajcik*
Institute of Geophysics and Planetary Physics, University of California, San Diego, La Jolla, California 92093

Michael Martin Nieto†
Theoretical Division, Los Alamos Scientific Laboratory, University of California, Los Alamos, New Mexico 87545

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We use a powerful matrix theorem to derive the closed-form, finite-polynomial, matrix expression of a Lorentz transformation for the class of relativistic, first-order wave equations described in the preceding paper. Combined with the method developed there, this theorem allows the exact, closed-form expression to be given for a Foldy-Wouthuysen (FW) transformation. An algorithm is given which is a simple procedural prescription for writing down the FW transformation results we have obtained. We discuss two specific examples which illustrate our results, and in a third example show that our method must be modified if the wave equation is not first-order in space.

I. INTRODUCTION

In the preceding article (FW-III) of this series,[1-3] we developed a method[3] by which a Foldy-Wouthuysen (FW) transformation could be derived from a Lorentz transformation. In this paper, we use an elegant matrix theorem to determine an exact, closed-form expression for the Lorentz transformation. Together with our method,[3] this then yields an exact, closed-form expression for the FW transformation of any relativistic wave equation which satisfies the following criteria: (i) It must be a first-order equation with no external constraint equations. (ii) The adjoint equation must exist, or, equivalently, the parity operator must exist. (iii) The wave equation must be relativistically invariant, so that it possesses well-defined Lorentz transformation operators. (iv) If there are built-in subsidiary components, then it must be possible to decouple them from the particle components by a generalized Sakata-Taketani transformation.[4] The exact FW transformations so constructed will have the properties determined earlier from general principles.[3]

Section II contains a quick résumé of the pertinent results[3] which relate an FW transformation to a Lorentz transformation and to the solutions.

In Sec. III we present our derivation. The use of the above matrix theorem allows us to show how the Lorentz transformation can be written as a closed-form, finite-polynomial, matrix expression. Combined with our method of FW-III, this yields the closed-form FW transformation. Our results can be combined in an algorithm which is a simple procedure for writing down the FW transformation. The algorithm is described in Sec. III B, along with a physical argument for understanding its basis; this argument complements the mathematical derivation.[3]

In Sec. IV, we give three examples. The first example illustrates our algorithm for the FW transformation of the Dirac equation in two space-time dimensions.[5-7] The second example lists specific results for the Bhabha system of equations[8-13]; the details are described elsewhere.[13] Our final example illustrates that our method must be modified if the wave equation is first-order in time but not in space. This example is the Weaver-Hammer-Good (WHG)[14] and Mathews[15] single-mass, high-spin wave equation.

This, then, concludes the present series.

II. RÉSUMÉ OF PROPERTIES

A. FW transformations

From FW-III, the class of equations we are considering is represented by[3]

$$(\partial \cdot \zeta + \chi)\psi = 0 , \qquad (2.1)$$

where the ζ_μ are matrices whose dimension must contain a particle-antiparticle spin space $2(2S+1)$, or combinations of such spaces, and there are no external constraint equations associated with (2.1). The adjoint equation exists,

$$\bar{\psi} = \psi^\dagger \eta, \quad \bar{\psi}(\partial \cdot \zeta - \chi) = 0 , \qquad (2.2)$$

or equivalently the parity operator $e^{i\phi}\eta$ exists. Equation (2.1) is relativistically invariant, and a pure Lorentz transformation along the $\hat{z}$ axis is given by

$$L(\beta) \equiv e^{-\theta J_{43}}, \quad \tanh\theta = \beta . \qquad (2.3)$$

The Poincaré generators are well defined,[3] and if there are subsidiary components then they can be decoupled.[3]

Through several theorems,[3] we showed that if the infinite-series expression (2.3) for the pure Lorentz transformation can be written in closed form as a single matrix with columns $l_k(\mathcal{P})$, then an exact, closed-form expression for the FW transformation U^{-1} is given by

$$[U^{-1}]_{jk} \equiv [\hat{u}_k]_j \ , \tag{2.4a}$$

$$\hat{u}_k \equiv \left[\frac{m}{E(\epsilon_k^{-1}p)}\right]^{1/2} l_k(\epsilon_k^{-1}p) \ , \tag{2.4b}$$

where

$$\epsilon_k \equiv \frac{\chi}{m}[\zeta_4^{-1}]_{kk} \ , \tag{2.4c}$$

$$E(\epsilon_k^{-1}p) \equiv [(\epsilon_k^{-1}p)^2 + m^2]^{1/2} \ , \tag{2.4d}$$

and $l_k(\epsilon_k^{-1}p)$ is determined from $l_k(\mathcal{P})$ by replacing $\mathcal{P}$ with the operator $-i\epsilon_k^{-1}(\partial/\partial z) \equiv \epsilon_k^{-1}p$. Here m, $\mathcal{P}$, and $\mathcal{E}$ represent the ground-state mass, momentum, and energy, and $\beta = \mathcal{P}/\mathcal{E}$.

As it should be, the FW transformation U^{-1} so defined is metric-unitary,

$$(U^{-1})^\dagger M U^{-1} = M \ . \tag{2.5}$$

B. Relation to wave-equation solutions

The solutions to (2.1) are given by

$$\psi_k = \left[\frac{E(\epsilon_k^{-1}p)}{m}\right]^{1/2} \hat{u}_k e^{i\epsilon_k(\mathcal{P}z - \mathcal{E}t)} \ , \tag{2.6}$$

where the $\hat{u}_k$ are metric-orthonormal spinors,

$$\hat{u}_j^\dagger M \hat{u}_k = M_{kk}\delta_{jk} \ , \tag{2.7}$$

and M is the (indefinite) metric. From (2.4a) and (2.5), the $\hat{u}_k$ satisfy the completeness relation

$$\sum_n \frac{M_{kk}}{M_{nn}} (\hat{u}_n)_j (\hat{u}_n^\dagger)_k = \delta_{jk} \ . \tag{2.8}$$

Note that ψ_k here has the Lorentz-invariant normalization.

III. FW TRANSFORMATIONS

A. Derivation of the closed-form Lorentz transformation

For the method outlined in Sec. II A to be successful, the infinite-series expansion in J_{43} for the Lorentz transformation of Eq. (2.3) must be expressed in a closed form, so that a single matrix with columns $l_k(\mathcal{P})$ can actually be constructed.

The closed form of $L(\beta)$ can be obtained from the matrix theorem[16] which states that a matrix B with n *distinct* eigenvalues λ_j can be represented in terms of the n idempotents of the spaces of the eigenvalues,

$$e_j = \prod_{\substack{k=1\\k\neq j}}^{n} \frac{B - \lambda_k I}{\lambda_j - \lambda_k} \ , \tag{3.1}$$

as

$$B = \sum_{j=1}^{n} \lambda_j e_j \ , \tag{3.2}$$

which in turn implies that

$$f(B) = \sum_{j=1}^{n} f(\lambda_j) e_j \ . \tag{3.3}$$

Since the generators J_{43} have the same eigenvalues as the spin matrices, we immediately have in terms of S, the maximum spin contained in the algebra we are considering,

$$L(\beta) = \sum_{j=-S}^{S} e^{-\theta j} \prod_{\substack{k=-S\\k\neq j}}^{S} \left[\frac{J_{43} - kI}{j - k}\right] \ , \tag{3.4}$$

$$\tanh\theta \equiv \beta, \quad \cosh\theta = \gamma \ . \tag{3.5}$$

A little algebraic manipulation puts (3.4) in the half-integer-spin and integer-spin forms

$$L(\beta, S = n + \tfrac{1}{2}) = \sum_{j=1/2}^{S} \left[\cosh j\theta - \left(\frac{J_{43}}{j}\right)\sinh j\theta\right] \times \prod_{\substack{k=1/2\\k\neq j}}^{S} \left(\frac{J_{43}{}^2 - k^2}{j^2 - k^2}\right) \ , \tag{3.6a}$$

$$L(\beta, S = n) = \prod_{k=1}^{S} (1 - J_{43}{}^2/k^2) + \sum_{j=1}^{S} \left[\left(\frac{-J_{43}}{j}\right)\sinh j\theta + \left(\frac{J_{43}}{j}\right)^2 \cosh j\theta\right] \times \prod_{\substack{k=1\\k\neq j}}^{S} \left(\frac{J_{43}{}^2 - k^2}{j^2 - k^2}\right) \ . \tag{3.6b}$$

For $S = n + \frac{1}{2}$ one can write

$$\cosh j\theta = \left(\frac{\gamma+1}{2}\right)^{1/2} \cosh[j]\theta + \left(\frac{\gamma-1}{2}\right)^{1/2} \sinh[j]\theta \ , \tag{3.7a}$$

$$\sinh j\theta = \left(\frac{\gamma-1}{2}\right)^{1/2} \cosh[j]\theta + \left(\frac{\gamma+1}{2}\right)^{1/2} \sinh[j]\theta \ , \tag{3.7b}$$

and for $S = n$ one can write

$$\cosh j\theta = \gamma^j \sum_{k=0}^{[j/2]} \binom{j}{2k} \beta^{2k} \ , \tag{3.8a}$$

$$\sinh j\theta = \gamma^j \sum_{k=0}^{[(j-1)/2]} \binom{j}{2k+1} \beta^{2k+1} \ , \tag{3.8b}$$

where $[x]$ denotes the largest integer $\leq x$. Equations (3.8) for $\cosh n\theta$ and $\sinh n\theta$ follow from using De Moivre's theorem,

$$\cosh n\theta + \sinh n\theta = (\cosh\theta + \sinh\theta)^n \ , \qquad (3.9)$$

letting $\theta \to -\theta$, adding and subtracting, respectively, and using (3.5).

It is clear from Eqs. (3.6)–(3.8) that a single matrix for $L(\beta)$ can be calculated in a straightforward way for any representation of J_{43}. With $\gamma \equiv (1-\beta^2)^{-1/2}$ and $\beta = \mathcal{P}/\mathcal{E}$ the columns of $L(\beta)$ can be explicitly determined as a function of $\mathcal{P}$,

$$[l_k(\mathcal{P})]_j = [L(\mathcal{P})]_{jk} \ . \qquad (3.10)$$

Then, as outlined in Sec. II, $\hat{u}_k$, U^{-1}, and ψ_k can be constructed.

B. Algorithm for writing an exact, closed-form FW transformation

The following interpretation provides a convenient algorithm for working out exact FW transformations and hence solutions. Since the Lorentz transformation depends only on $\beta = \mathcal{P}/\mathcal{E}$, the replacement $\mathcal{P} \to \epsilon_k^{-1} p$ in each column $l_k(\mathcal{P})$ implies that each column depends only on

$$\beta = \frac{\mathcal{P}}{\mathcal{E}} \to \frac{\epsilon_k^{-1} p}{E(\epsilon_k^{-1} p)} = \frac{(|\epsilon_k|/\epsilon_k) p}{(p^2 + m_k^2)^{1/2}} \ , \qquad (3.11)$$

where $m_k \equiv m\epsilon_k$ and (2.4d) have been used. For example,

$$\gamma = \frac{\mathcal{E}}{m} \to \frac{E(\epsilon_k^{-1} p)}{m} = \frac{(p^2 + m_k^2)^{1/2}}{|m_k|} \ . \qquad (3.12)$$

Because ϵ_k changes sign with respect to the ground state only for antiparticles (the ground-state mass is positive by definition), the replacement $\mathcal{P} \to \epsilon_k^{-1} p$ is identical with the following algorithm.

Algorithm for the FW transformation. Recalling that

$$\cosh\theta = \gamma, \quad \sinh\theta = \beta\gamma, \quad \gamma = (1-\beta^2)^{-1/2} \ , \qquad (3.13)$$

determine $L(\beta)$ from Eqs. (3.6)–(3.8) and explicitly write out

$$U^{-1} = \gamma^{-1/2} L(\beta) \equiv U^{-1}(\beta, \gamma) \ . \qquad (3.14)$$

In $U^{-1}(\beta, \gamma)$ make the replacements

$$\beta = \frac{\mathfrak{P}}{\mathfrak{W}} , \quad \gamma = \frac{\mathfrak{W}}{\mathfrak{M}} \ . \qquad (3.15)$$

Then in each and every kth column of $U^{-1}(\mathfrak{P}/\mathfrak{W}, \mathfrak{W}/\mathfrak{M})$ make the substitutions

$$\mathfrak{M} \to |m_k| \ , \qquad (3.16a)$$

$$\mathfrak{P} \to \begin{cases} +p \text{ for particles } (\epsilon_k/|\epsilon_k| = +1) \ , \\ -p \text{ for antiparticles } (\epsilon_k/|\epsilon_k| = -1) \ , \end{cases} \qquad (3.16b)$$

$$\mathfrak{W} \to |(p^2 + m_k^2)^{1/2}| = |E_k| \ , \qquad (3.16c)$$

where m_k is the rest mass of the eigenvector which represents the kth column. The above then is the FW transformation U^{-1} which diagonalizes the Hamiltonian.

An alternative method to (3.16b) for dealing with $\mathfrak{P}$ is to take every algebraic factor involving $\mathfrak{P}$ to the right of all matrix operators, and then make the substitution

$$\mathfrak{P} \to p\tau_3 \ , \qquad (3.16b')$$

where τ_3 is a "Pauli-like" matrix. In a representation where ζ_4 is diagonal and the particle states are ordered to precede the antiparticle states, τ_3 will be completely diagonal with $+1$ in the upper left block and -1 in the lower right block. As an example, observe that this alternative description of the algorithm exactly gives the algebraic difference between the normal forms of the Lorentz and FW transformations in the Dirac case. Thus, if one compares Eqs. (3.7) and the equation for e^{iS} on p. 47 of Ref. 17, then besides the normalization, it is the extra γ_0 (there β) in the second term of the equation on p. 47 of Ref. 17 which corresponds to our τ_3.

What the above algorithm says is that, with the correct overall normalization, the FW transformation is composed of columns which are rest-state eigenvectors that have been Lorentz-transformed in the same direction. However, since the antiparticle states are charge-conjugate states, the Lorentz-transformation matrix which takes a particle from energy m_k to energy $E_k = (p^2 + m_k^2)^{1/2}$ and momentum p will transform an antiparticle to momentum $-p$. The above algorithm reverses this extra minus sign in the three-momentum of the Lorentz-transformation matrix. [Note that if the matrix ζ_4 is such that the first half of the (diagonal) elements are positive while the remaining half are negative, then for Eq. (3.16b) simply replace $\mathfrak{P}$ with $+p$ in the first half of $\gamma^{-1/2} L(\beta)$ in (3.14), and with $-p$ in the second half.]

Finally, the solutions to (2.1) are given by

$$\psi_k = \left(\frac{E_k}{m_k}\right)^{1/2} \hat{u}_k e^{i\epsilon_k(\mathcal{P}z - \mathcal{E}t)} \ , \qquad (3.17)$$

where $\hat{u}_k$ is the kth column of U^{-1} determined from Eqs. (3.14)–(3.16).

IV. EXAMPLES

A. Dirac equation in two dimensions

The Dirac equation in one time and one space dimensions was originally investigated[5–7] as a case study in model field theories. There is now a revival of interest in it as a tool to study abnormal states of nuclear matter and quark confinement.[18] With only one space direction in the energy square-root operator $E=(p^2+m^2)^{1/2}$, the method of breaking the square root which yields the Dirac algebra in four dimensions yields the Pauli algebra in two dimensions, so that

$$(\partial\cdot\tau+m)\psi=(\partial_4\tau_3+ip\tau_2+m)\psi=0\ , \tag{4.1}$$

which means that the Hamiltonian is

$$-\partial_4\sim H=\tau_3{}^{-1}(ip\tau_2+m)=p\tau_1+\tau_3 m\ . \tag{4.2}$$

It is to be noted that Iachello[19] has derived a very similar equation for spin-0 particles in four dimensions. The Hamiltonian is the same as in Eq. (4.2) except that p in two dimensions becomes $|\vec{p}|$ in four dimensions.[19] Iachello also obtained an FW transformation for his spin-0 equation. Modulo the subtle absolute value in his equation, this FW transformation is functionally the same as the one we now derive.

From the free equation (4.1) one can see that the generator "J_{43}," analogous to the Dirac four-dimensional J_{43}, has the commutator of the τ matrices multiplied by the same $\frac{1}{4}$ which multiplies the commutator of the γ matrices in the original. This gives

$$\begin{aligned}J_{43}&=-i(\tfrac{1}{4})[\tau_3,\tau_2]\\&=-\tfrac{1}{2}\tau_1\ .\end{aligned} \tag{4.3}$$

Thus, $L(\beta)$ is

$$\begin{aligned}L(\beta)&=\exp\left(\frac{\theta}{2}\tau_1\right)\\&=\cosh\frac{\theta}{2}+\tau_1\sinh\frac{\theta}{2}\ .\end{aligned} \tag{4.4}$$

Combining (4.4) with the algorithm (3.14)–(3.16) yields

$$\cosh\frac{\theta}{2}=\left(\frac{\mathcal{W}+\mathfrak{M}}{2\mathfrak{M}}\right)^{1/2},\quad \sinh\frac{\theta}{2}=\frac{\mathfrak{P}}{[2\mathfrak{M}(\mathcal{W}+\mathfrak{M})]^{1/2}}\ , \tag{4.5}$$

and thus

$$U^{\mp1}=\left(\frac{E+m}{2E}\right)^{1/2}\begin{bmatrix}1 & \mp\dfrac{p}{(E+m)}\\ \dfrac{\pm p}{(E+m)} & 1\end{bmatrix}, \tag{4.6a}$$

or

$$U^{\mp1}=\left(\frac{E+m}{2E}\right)^{1/2}\left[I\mp i\tau_2\frac{p}{(E+m)}\right]\ . \tag{4.6b}$$

From (3.17), the solutions to (4.1) are

$$\psi_1=\left(\frac{E+m}{2m}\right)^{1/2}\begin{bmatrix}1\\ \dfrac{p}{E+m}\end{bmatrix}e^{i(\mathcal{P}x-\mathcal{E}t)}\ , \tag{4.7a}$$

and

$$\psi_2=\left(\frac{E+m}{2m}\right)^{1/2}\begin{bmatrix}\dfrac{-p}{E+m}\\ 1\end{bmatrix}e^{-i(\mathcal{P}x-\mathcal{E}t)}\ , \tag{4.7b}$$

B. Bhabha fields

Elsewhere[13] we have discussed in great detail the application of our method to the Bhabha system

$$(\partial\cdot\alpha+\chi)\psi=0\ . \tag{4.8}$$

For a given so(5) algebra of order $2\mathcal{S}$, the method applies with $S\to\mathcal{S}$, $\eta\to\eta_4$, $\zeta_\mu\to\alpha_\mu=J_{\mu5}$, $J_{43}=-i[\alpha_4,\alpha_3]$, and $I^{(S)}\to\mathcal{J}_0(\mathcal{S})$ for integer spin, where the above quantities are described completely in Refs. 8–13. The Bhabha system is a multimass, multispin system where, for a given $\mathcal{S}$, the mass states are given by $\pm\chi/j$, $(\frac{1}{2}\text{ or }1)\le j\le\mathcal{S}$, the energy states E_j by $\pm(p^2+\chi^2/j^2)^{1/2}$, and the possible spin S by $(\frac{1}{2}\text{ or }0)\le S\le\mathcal{S}$.

We refer the reader to Ref. 13 for the explicit FW transformation results and solutions, mentioning here that they are given in closed form for arbitrary spin and special cases up to $\mathcal{S}=3$ are studied as examples. These cases include detailed discussions of the $(\frac{1}{2},\frac{1}{2})$ representation, which reduces to the Dirac equation, the DKP (Duffin-Kemmer-Petiau) spin-0 equation (1, 0) representation in both the 5×5 and the Sakata-Tahetani "particle components" 2×2 forms, and the high-spin 16-dimensional $(\frac{3}{2},\frac{1}{2})$ representation, where it is demonstrated that the 16 columns of the FW U^{-1} matrix obtained by our method are indeed the metric-orthonormal eigenvectors of the Hamiltonian. We also show that the exact FW expressions agree with the power-series expansions in c^{-1} discussed in Ref. 12.

C. Wave equation of Weaver, Hammer, and Good

Some time ago, Weaver, Hammer, and Good (WHG)[14] developed a single-mass, single-spin, relativistic wave equation in terms of the $2(2S+1)$-dimensional matrices

$$\beta=\begin{pmatrix}0 & I\\ I & 0\end{pmatrix},\quad \vec{\alpha}=\frac{1}{S}\begin{pmatrix}\vec{S} & 0\\ 0 & -\vec{S}\end{pmatrix}, \tag{4.9}$$

where $\vec{S}$ are the usual $(2S+1)$-dimensional spin matrices and I is a $(2S+1)$-dimensional unit matrix. The idea was to Lorentz-transform a Hamiltonian which was given by $m\beta$ in the rest system, and thereby find a single-mass, single-spin Hamiltonian in the boosted frame. Later, from a different point of view, Mathews[15] derived a polynomial expansion of the form

$$H = \sum_{\nu} \frac{(E+p)^{4\nu} - m^{4\nu}}{(E+p)^{4\nu} + m^{4\nu}} E C_\nu + \beta \sum_{\nu} \frac{2Em^{2\nu}(E+p)^{2\nu}}{(E+p)^{4\nu} + m^{4\nu}} B_\nu \ , \tag{4.10}$$

where the C_ν and B_ν are certain projection operators. Mathews observed that[15] the "agreement [of the above Hamiltonian with that of WHG] is, in fact, to be expected for any spin since their starting point is the assumption that the rest-system Hamiltonian is $H_0 = \beta m \equiv \sigma m$ (which we have shown to be the only one consistent with covariance and regularity conditions), and they utilize an integrated form of [Mathews's Eq.] (3d) to pass to an arbitrary reference frame."

Weaver, Hammer, and Good[14] proposed that their equation would, in principle, have an FW transformation related to the Lorentz transformation in a manner analogous to the Dirac case, which their Hamiltonian reduces to for $S=\frac{1}{2}$. In particular, they considered the operator

$$Y(\theta) = \cosh(z\theta) - \beta \sinh(z\theta) \ , \tag{4.11a}$$

$$z = S\vec{\alpha}\cdot\hat{p}, \quad \theta = \tan^{-1}(p/E) \ , \tag{4.11b}$$

which in the Dirac case transforms all the eigenvectors in the same direction. Thus, in *our set of definitions*, vs that of WHG, Y is proportional to the FW transformation U, not the Lorentz transformation L, in the Dirac case.

For all spin Y has the property

$$YHY^{-1} = E\beta \ , \tag{4.12}$$

yet for higher spin a great deal of sometimes seemingly contradictory literature has arisen concerning these transformations.[20–29] The confusion has been clarified by recent work,[27,28] and it can be further elucidated with the language of our series. This is because the crux of the problem involves the nonunitarity of Y for $S>\frac{1}{2}$, and this amounts to having a metric operator.

If one starts with what WHG call the "Foldy wave functions" ϕ, which are solutions to the wave equation

$$i\frac{\partial}{\partial t}\phi = E\beta\phi \ , \tag{4.13}$$

then the WHG equations and solutions are given by

$$i\frac{\partial}{\partial t}\psi = H\psi, \quad \psi = Y\phi \ . \tag{4.14}$$

Thus, in the ψ basis we are dealing with a metric space, since

$$\phi^\dagger\phi = \psi^\dagger M\psi, \quad M = (Y^\dagger)^{-1}Y^{-1} = (YY^\dagger)^{-1} = \cosh 2z\theta \ . \tag{4.15}$$

For $S=\frac{1}{2}$, M is just a normalization constant, but for $S>\frac{1}{2}$ it is a nonconstant matrix. In our language this means that allowed transformations, V, of this basis will not be unitary; rather they will be metric-unitary, satisfying the equation (as Y does)

$$V^\dagger MV = M \ . \tag{4.16}$$

Now looking at expectation values of the Hamiltonian, one finds

$$\phi^\dagger E\beta\phi = \phi^\dagger[Y^\dagger(Y^\dagger)^{-1}](Y^{-1}Y)\beta E(Y^{-1}Y)\phi = \psi^\dagger MH\psi \ . \tag{4.17}$$

Then since $(MH)^\dagger = MH$, H is not only self-adjoint; it is metric-Hermitian in the ψ basis. This is the origin of comments often found in the literature which state that the FW transformations for the WHG Hamiltonians are not unitary. In the ψ basis they are not, they are metric-unitary.

However, since this metric is *positive-definite*, it can be removed by going to the proper basis. In particular consider a transformation X, which commutes with β and M such that

$$[\beta, X] = [M, X] = 0, \quad XX^\dagger = X^\dagger X = M \ . \tag{4.18}$$

Then one can obtain a *unitary* transformation U which transforms H to $E\beta$ and which has the unitary basis Φ given by

$$U = Y^{-1}X^{-1}, \quad UHU^{-1} = E\beta, \quad U\Phi = \phi \ . \tag{4.19}$$

The discussions of unitary FW transformations of the WHG system are therefore using the Φ basis, or an equivalent one. Each basis has its advantages. The ψ basis has the FW transformation as the simple form given in Eq. (4.11), but it must deal with metric-Hermitian and metric-unitary operators. The Φ basis, on the other hand, has the advantages of dealing with a normal Hilbert space and with Hermitian and unitary operators. Further, in this basis there is a connection to our Lorentz transformation L which yields a further understanding of the nature of the FW transformation U, compared to Y.

In the unitary representation, the FW transformation should be related to $L(J_{4j})$ as before. However, because for $S>\frac{1}{2}$ the Hamiltonian is not first-order in space and the $\vec{\alpha}$ matrices cannot be rotated into the β matrix (as one would have for a first-order wave equation), J_{4j} is no longer simple. For higher spin J_{4j} must be modified. This is the origin of the form of the unitary FW transformation for $S=\frac{3}{2}$ which Weaver has proposed,

$$U = \exp[c_1(z\beta\theta) + c_2(z\beta\theta)^3] \ . \tag{4.20}$$

One can equivalently write things in the form of

(4.19) as Tekumalla and Santhanam[27] have explicitly done for up to $S=\frac{3}{2}$, a form which Jayaraman[28] observed was implicit in Mathews's work.[15] Note that the polynomial form of the exponential in (4.20) as well as the fact that X is a polynomial of certain order in z^2 is related to the fact that the $\vec{\alpha}$ matrices have eigenvalues from $+S$ to $-S$ in integer steps, whereas β has eigenvalues $+1$ and -1.

It would be interesting to see if the generalized forms for the U of Eq. (4.19) and that of (4.20) are equivalent, and if they can therefore be given in terms of $L(J_{4j})$ operators. One could perhaps use the spin-matrix polynomial formalism,[20,21] and should carefully understand the transformation properties of the Poincaré generators and wave functions.

Finally, we recall[12] that the term FW transformation is used in many senses, including not only transforming the Hamiltonian to βE but also completely diagonalizing it. Note that the WHG representation is "Majorana-like" so that β is not diagonal, contrary to the "Pauli-Dirac-like" representation where β is diagonal. The transformation connecting the two algebra representations is

$$u\beta^{M}u^{-1}=\beta^{PD}\,,\quad u=\frac{1}{\sqrt{2}}\begin{bmatrix} I & \sigma_x \\ \sigma_x & -I \end{bmatrix}. \tag{4.21}$$

The additional transformation (4.21) can be used to completely diagonalize the Hamiltonian.

One can, however, perform the complete diagonalization by brute force with the aid of few tricks. We will now give the $S=\frac{3}{2}$ case as an example. The $S=\frac{3}{2}$ WHG Hamiltonian is

$$H_{3/2}=\frac{(9W^2-7m^2)m\beta+2(13W^2-10m^2)\vec{\alpha}\cdot\vec{p}-9m(\vec{\alpha}\cdot\vec{p})^2\beta-18(\alpha\cdot\vec{p})^3}{2(4W^2-3m^2)}\,, \tag{4.22}$$

where

$$W^2=p^2+m^2\,. \tag{4.23}$$

Note that W^2 has to be interpreted as Eq. (4.23), and not as E^2. Otherwise the wave equation would be third order in t, and there would be 24 solutions, not eight. In addition to the two solutions

$$E=\pm(p^2+m^2)^{1/2}\,, \tag{4.24}$$

each with multiplicity 4, direct calculation would yield eight other solutions, each with multiplicity 2:

$$E=\pm[3(p^2)+m^2]^{1/2},\quad E=\pm[(\tfrac{1}{3}p)^2+m^2]^{1/2}\,,$$
$$E=\pm[(\tfrac{3}{4}p)^2+(\tfrac{7}{8}m)^2]^{1/2},\quad E=\pm[(\tfrac{1}{4}p)^2+(\tfrac{7}{8}m)^2]^{1/2}\,. \tag{4.25}$$

On the other hand, taking $W^2=p^2+m^2$ does indeed give only correct solutions (4.24). [Remember, however, that because the equations for high spin are polynomials in p but the matrices have only $2(2S+1)$ components, minimal electromagnetic interaction will make the solutions noncausal.]

Now one observes that by making $\vec{p}\|\hat{z}$ and then rearranging the rows and columns of (4.9) one can obtain the representation

$$\alpha=\begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & \frac{1}{3} & 0 & 0\\ 0 & 0 & -\frac{1}{3} & 0\\ 0 & 0 & 0 & -1 \end{bmatrix}\otimes I(2\times 2),$$
$$\beta=\begin{bmatrix} 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0 \end{bmatrix}\otimes I(2\times 2)\,, \tag{4.26}$$

so that effectively one may work with 4×4 matrices. Further, when these are put in the Hamiltonian, the first row and column are coupled only to the fourth row and column; similarly for the second and third rows and columns. That means that to obtain the eigenvectors one only has to solve two 2×2 matrix equations. When one does that and normalizes the eigenvectors, and then makes them the columns of the matrix U^{-1}, by inspection this U^{-1} can be written in terms of the new matrices α and β. Specifically, the answer, in terms of the quantities

$$W=+(p^2+m^2)^{1/2},\quad T=8p^2+2m^2\,,$$
$$R=(8p^2+6m^2)p+WT\,, \tag{4.27}$$
$$a=[2W(W+p)]^{-1/2},\quad b=(2WR)^{-1/2}\,,$$

is

$$U^{-1} = \frac{a}{8}[27(W+p)\alpha + 9m\beta](1-\alpha^2) + \frac{b}{8}(R\alpha + 2m^3\beta)(9\alpha^2 - 1) . \quad (4.28)$$

By explicit multiplication one can verify that (4.28) is unitary and diagonalizes the Hamiltonian (4.22). The above expression, then, is a special case which can be compared with any general formulation which may be obtained.

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*Present and permanent address: Theoretical Design Division, Los Alamos Scientific Laboratory, Los Alamos, New Mexico 87545.

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