PHYSICAL REVIEW D VOLUME 15, NUMBER 2 15 JANUARY 1977

Foldy-Wouthuysen transformations in an indefinite-metric space. III. Relation to Lorentz transformations for first-order wave equations and the Poincaré generators

R. A. Krajcik*

Institute of Geophysics and Planetary Physics, University of California, San Diego, La Jolla, California 92093

Michael Martin Nieto†

Theoretical Division, Los Alamos Scientific Laboratory, University of California, Los Alamos, New Mexico 87545

(Received 7 July 1976)

We show that there exists a definite relationship between a Lorentz transformation and a Foldy-Wouthuysen (FW) transformation for any relativistic wave equation in an indefinite-metric space which satisfies the following criteria: (i) The equation is first order with no external constraint equations. (ii) An adjoint equation (or, equivalently, a parity operator) exists. (iii) Lorentz transformation operators and related Poincaré generators are well defined. (iv) Any built-in subsidiary components can be decoupled. Our result allows us to obtain the explicit forms of the FW-transformed Poincaré generators from the original generators and in principle allows us to determine the exact, closed-form FW transformation.

I. INTRODUCTION

In the first two articles (FW-I, FW-II) of this series,[1,2] we demonstrated the necessary and sufficient conditions for the existence of a Foldy-Wouthuysen (FW) transformation in an indefinite-metric space,[1] and then derived some theorems for practical calculations,[2] especially useful for power-series expansions in c^{-1}. In this paper, we will establish a connection between Lorentz transformations and FW transformations for a class of relativistic wave equations in an indefinite-metric space. This connection provides the method by which an expression for the FW transformations can, in principle, be obtained, and the means by which the properties of the FW transformations can be studied. In the following paper,[3] we will use a powerful matrix theorem to derive the exact, closed-form expressions for the Lorentz transformations of this class of relativistic wave equations. That, together with the method (connection) introduced here, will then yield the exact, closed-form expressions for the FW transformations.

The wave equations we consider have the following properties: (i) They must be first-order equations (at least in the time derivative) with no external constraint equations. (ii) The adjoint equations must exist, or, equivalently, the parity operators must exist. (iii) The wave equations must be relativistically invariant, so that they possess well-defined Lorentz-transformation operators and related generators of the Poincare group. (iv) If there are built-in subsidiary components, then it must be possible to decouple them from the physics-carrying particle components by a generalized Sakata-Taketani transformation.[4]

Note that we are considering only first-order wave equations with *no* external constraint equations, so that we are *not* talking about the Rarita-Schwinger[5,6] (RS) system, for example. Our analysis does *not* rule out FW transformations for such equations; we only say that our analysis is not concerned with them even though an analogous treatment may be entirely possible.

As a positive example, our analysis *does* apply to the entire system of Bhabha equations,[7-12] which includes the Dirac and DKP (Duffin-Kemmer-Petiau) equations as special cases. In fact, the present series was undertaken as an outgrowth of the necessity to demonstrate as a matter of principle the existence of an FW transformation for the indefinite-metric Bhabha system (see Refs. 11 and 12).

Section II describes the formalism which we will later use to obtain FW transformations from Lorentz transformations. We construct the wave equations which satisfy our above criteria in Sec. II A and the implicit Lorentz transformations and Poincaré generators in Sec. II B. In Sec. II C we discuss the generalization needed to handle built-in subsidiary components, if they exist.

In Sec. III we present our method of relating the FW transformation U^{-1} to the Lorentz transformation $L(\beta)$, β the boost velocity (or rapidity operator). We begin with the observation in Sec. III A that one can construct all the eigenvectors of the Hamiltonian by Lorentz transforming the rest-state eigenvectors. Since, from FW-I,[1] the FW transformation U^{-1} has as its columns all the independent metric-orthonormalized eigenvectors, properly handling $L(\beta)$ will yield U^{-1}.

This "proper handling" of $L(\beta)$ involves two things: (i) obtaining the correct overall normalization needed to make U^{-1} metric-unitary (pseudo-unitary), and (ii) understanding the *functional* significance of β. [Since in general β will involve different mass states and a Lorentz transformation can be thought of as boosting antiparticle states in the opposite direction from particle states (the antiparticle states are the "negative energy" solutions or the charge-conjugated states), this point also must be taken care of.] In Sec. III B the above two points are resolved mathematically by three theorems, which are proved in Appendix A.

Given the above, we then derive in Sec. III C on the basis of a fourth theorem, proved in Appendix A and in part in Appendix B, the *explicit* form of the FW-transformed Poincaré generators. As we have emphasized elsewhere,[11] an FW transformation may have more than one meaning. The theorem of Sec. III C will show on general principles that our FW transformation is so defined that the transformed space-time generators (the Hamiltonian in particular) are diagonal, while the transformed rotation-boost generators are diagonal in the sense that they do not connect states of different mass (including antiparticles).

Our short discussion in Sec. IV touches upon the relation of this paper to previous work and to the physical implications for high-spin field theories.

II. FORMALISM

A. First-order and adjoint equations

Consider the class of first-order wave equations represented by[13]

$$(\partial\cdot\zeta+\chi)\psi=0. \tag{2.1}$$

In (2.1) χ is a fixed parameter and the different mass states are proportional to χ. The ζ_μ are matrices whose dimensionality must contain the particle-antiparticle spin space, $2(2S+1)$ dimensions, or combinations of such spaces, and $\zeta_\mu^\dagger = \zeta_\mu$. Also, from our criterion (i), there are no external constraint equations associated with Eq. (2.1). (Formalisms with built-in constraint equations, such as those of Duffin, Kemmer, and Petiau, are allowed.)

To have an adjoint equation, we need an adjoint matrix η which will satisfy

$$\bar\psi=\Psi^\dagger\eta, \quad \bar\psi(\partial\cdot\zeta-\chi)=0, \tag{2.2}$$

where

$$[\eta,\zeta_4]=0, \quad \{\eta,\vec\zeta\}=0. \tag{2.3}$$

The conditions for η of Eq. (2.3) are also those needed for the parity operator $e^{i\phi}\eta(\zeta_4)$, as can be shown by a standard procedure.[14]

We use the usual definitions of the norm of a state and the expectation value of an operator $\mathcal{O}$,

$$\|\psi\|^2=\bar\psi\zeta_4\psi, \quad \langle\mathcal{O}\rangle=\bar\psi\zeta_4\mathcal{O}\psi, \tag{2.4}$$

so that the metric M is

$$M=\eta\zeta_4, \tag{2.5}$$

where $M^\dagger=M$, since $\eta=\eta(\zeta_4)$ and $\zeta_4^\dagger=\zeta_4$. The expectation value is real whenever the operator $\mathcal{O}$ is metric-Hermitian (pseudo-Hermitian),

$$(M\mathcal{O})^\dagger=M\mathcal{O}, \tag{2.6}$$

and conversely.

B. Lorentz transformations and Poincare generators

For Eq. (2.1) to be Lorentz invariant, the commutation relations among the Lorentz generators $J_{\mu\nu}$ [or more properly in our notation the so(4) generators], the matrices ζ_μ, and the parity operator η must be[15]

$$[J_{\mu\nu},J_{\lambda\rho}]=i(\delta_{\mu\lambda}J_{\nu\rho}+\delta_{\nu\rho}J_{\mu\lambda}-\delta_{\nu\lambda}J_{\mu\rho}-\delta_{\mu\rho}J_{\nu\lambda}), \tag{2.7}$$

$$[J_{\mu\nu},\zeta_\lambda]=-i(\zeta_\mu\delta_{\nu\lambda}-\zeta_\nu\delta_{\mu\lambda}), \tag{2.8}$$

where $\mu,\nu,\lambda,\rho=1,2,3,4$, and,

$$\{J_{4j},\eta\}=0, \quad [J_{jk},\eta]=0, \tag{2.9}$$

where $j,k=1,2,3$. If a prime denotes a transformed quantity, then

$$\psi'(x')=L(\Lambda)\psi(x), \tag{2.10a}$$

$$L(\Lambda)=\exp(+\tfrac{1}{2}\theta_{\mu\nu}J_{\mu\nu}), \tag{2.10b}$$

$$x'_\mu=\Lambda_{\mu\nu}x_\nu, \tag{2.10c}$$

where the $\Lambda_{\mu\nu}$ and $\theta_{\mu\nu}$ are numbers which characterize the transformation.[16,17] $J_{\mu\nu}$ is self-adjoint and antisymmetric in μ and ν.

The Poincaré generators $\vec P$, $\vec J$, H, and $\vec K$ associated with Eq. (2.1) are then obtained in terms of the operators and parameters ζ_μ, χ, $\vec x$, $\vec p=-i\vec\partial$, $J_{\mu\nu}$, and t by adding the generators of space-time translations and rotations to the $J_{\mu\nu}$:

$$P_k=p_k=-i\partial_k, \tag{2.11}$$

$$J_k=-i\epsilon_{klm}\left(x_l\partial_m+\frac{i}{2}J_{lm}\right), \tag{2.12}$$

$$H=\zeta_4^{-1}(\vec\partial\cdot\vec\zeta+\chi), \tag{2.13}$$

$$K_k=x_kH-tP_k+iJ_{4k}, \tag{2.14}$$

where we have assumed for now that ζ_4 is nonsingular, so that its inverse ζ_4^{-1} exists and may be determined from the Cayley-Hamilton theorem by a standard procedure.[18] The Hamiltonian equation then becomes

$$-\partial_4\psi = H\psi = \zeta_4^{-1}(\vec{\partial}\cdot\vec{\zeta}+\chi)\psi = E\psi. \tag{2.15}$$

The generators defined in (2.11)–(2.14) can easily be shown to be metric-Hermitian from Eqs. (2.3), (2.8), and (2.9), and, in a more tedious calculation that also requires (2.7), can be shown to satisfy the Lie algebra.[19] Also note that since we are using the complete definition of $\vec{K}$ in (2.14), with $t\vec{P}$ on the right, if we substitute $-\partial_4$ for the right-hand side of (2.13) and for the H in (2.14), these new generators still satisfy the Lie algebra.

C. Subsidiary components

If ζ_4 does not have an inverse because of the zero eigenvalues of the spin and other matrices (as will often occur for integer-spin fields), then one will have built-in subsidiary components which are to be decoupled. By criterion (iv) we assume that this is possible.

The subsidiary components will correspond to the singular pieces of the metric, and hence of ζ_4. Removing these singular pieces allows a proper indefinite-metric FW transformation,[1] and can be done by replacing the columns of eigenvectors $\hat{u}_k$ in the FW transformation U^{-1} by columns of the particle-components eigenvectors $\hat{u}_k^{(P)}$ defined by[4,8,11]

$$\hat{u}_k^{(P)} = I^{(P)}\hat{u}_k = (I - I^{(S)})\hat{u}_k, \tag{2.16}$$

where $I^{(P)}$ and $I^{(S)}$ are the projection operators onto the particle components and subsidiary components. (If there are no subsidiary components, then $I^{(P)} \equiv I$.) This means that the particle-components FW transformation $(U^{-1})^{(P)}$ can be obtained by the substitution

$$(U^{-1}) \to I^{(P)}U^{-1}I^{(P)}. \tag{2.17}$$

This is equivalent to replacing the metric M with $I^{(P)}MI^{(P)}$, and rederiving all the results above and below. The operators $I^{(P)}$ and $I^{(S)}$ are assumed to exist for the class of relativistic wave equations being considered.

III. LORENTZ-FW TRANSFORMATIONS AND FW-POINCARÉ GENERATORS

A. Lorentz-transformed rest-state eigenvectors

Since we are dealing with free relativistic wave equations, the eigenvalues of Eq. (2.15) are the free (real) energies and the solutions of Eqs. (2.2) and (2.15) have nonzero norm. From the discussion of Sec. II C, we can assume a nonsingular metric M, so that the criteria of FW-I is satisfied. (Consult Sec. II C for the case of singular metric and the eliminations of built-in subsidiary components.) This means that an FW transformation which diagonalizes H exists, and is composed of the metric-orthonormal eigenvectors of H. The purpose of this section is to give a precise procedure for constructing this FW transformation by means of four theorems and a lemma about the relation of Lorentz transformations to FW transformations and about the form of the FW-transformed Poincaré generators.

Without loss of generality, we simplify our discussion by conveniently choosing ζ_4 diagonal and the momentum in the $\hat{z}$ direction. A pure Lorentz transformation along the $\hat{z}$ direction then becomes

$$L(\beta) = e^{-\theta J_{43}}, \tag{3.1}$$

where one can easily see from Eq. (2.10) that the velocity β of the boosted frame will be related to θ by

$$\tanh\theta = \beta \geq 0. \tag{3.2}$$

There is an arbitrary sign in the exponential of Eq. (3.1) which we have already chosen so as to have particles at rest (vs antiparticles) be transformed to positive momenta.

The eigenfunctions of H are now easily constructed. The mass spectrum associated with Eq. (2.15), $\varepsilon_k m$, is given by the rest frame of Eq. (2.15),

$$-\partial_4\psi_k(0) = \zeta_4^{-1}\chi\psi_k(0), \tag{3.3a}$$

where

$$\psi_k(0) = l_k(0)e^{-i\varepsilon_k mt} \tag{3.3b}$$

and

$$m = \min[\zeta_4^{-1}]_{kk}\chi, \tag{3.3c}$$

$$\varepsilon_k = \frac{[\zeta_4^{-1}]_{kk}}{\min[\zeta_4^{-1}]_{kk}}. \tag{3.3d}$$

$\mathrm{Min}[\zeta_4^{-1}]_{kk}$ is the smallest positive matrix element of ζ_4^{-1} and the $l_k(0)$ are column matrices which are everywhere zero except for a "1" in the kth row. Thus, m is the ground-state mass and the ε_k are the k/ground-state mass ratios. Note that whenever ζ_4^{-1} has dissimilar diagonal elements (other than a sign change for antiparticles), a multimass theory results.

Because the Lorentz transformation can only be a function of β [e.g., $\gamma \equiv 1/(1-\beta^2)^{1/2}$], each component of mass m_k must move with the same velocity

$$\beta = \frac{\varepsilon_k \mathcal{P}}{\varepsilon_k \mathcal{E}} = \frac{\mathcal{P}}{\mathcal{E}}, \quad m_k = \varepsilon_k m, \tag{3.4}$$

and we have, in the new frame,

$$-\partial_4\psi_k = H\psi_k = \varepsilon_k\mathcal{E}\psi_k, \tag{3.5a}$$

where

$$\psi_k = L(\mathcal{P})\psi_k(0)$$
$$= l_k(\mathcal{P})e^{+i\varepsilon_k(\mathcal{P}z - \mathcal{E}t)}, \tag{3.5b}$$

$$[l_k(\mathcal{P})]_j = [L(\mathcal{P})]_{jk}, \tag{3.5c}$$

so that $l_k(\mathcal{P})$ are the columns of $L(\mathcal{P})$, $\mathcal{P}$ and $\mathcal{E}$ are the ground-state momentum and energy, and the Lorentz-invariant rest-frame phase $-mt$ has been replaced by $\mathcal{P}_\mu x_\mu = \mathcal{P}z - \mathcal{E}t$. The ψ_k are eigenfunctions of H with eigenvalue $\varepsilon_k\mathcal{E}$, where $\mathcal{E} \equiv (\mathcal{P}^2 + m^2)^{1/2}$.

Observe that the $l_k(\mathcal{P})$ are clearly related to the metric-orthonormalized eigenfunctions $\hat{u}_k$ described as the columns of U^{-1} in FW-I. The connection will be explicit in Theorem III below.

The object is to show that with the correct normalization and the proper interpretation of β and/or $\mathcal{P}/\mathcal{E}$, the Lorentz transformation (3.1) or (3.5c) can be equated with the FW transformation U^{-1}. For now, we only need to know that the infinite-series expression for $L(\beta)$ can, as a matter of principle, be reduced to a *single* matrix from which the $l_k(\mathcal{P})$ can actually be determined. [The exact, closed-form expression for $L(\beta)$ will be given in the following paper.[3]] In the three theorems of Sec. III B we will prove this equivalence. The theorem and lemma of Sec. III C will show the form of the FW-transformed Poincaré generators which is implied by U^{-1}.

The proofs of the theorems in Secs. III B and III C are given in Appendix A, with some of the specially detailed calculations for Theorem IV given in a separate Appendix B. Each theorem is dependent upon the previous theorem.

B. Construction of U^{-1} from $L(\beta)$

Theorem I. Let M be the (indefinite) metric given by Eq. (2.5), and let $L(\beta)$ be the Lorentz transformation (boost) given by Eq. (3.1). Then,[20]

$$L^\dagger(\beta)ML(\beta) = \gamma M + i\beta\gamma\eta\zeta_3, \tag{3.6}$$

where

$$\gamma = 1/(1-\beta^2)^{1/2}. \tag{3.7}$$

Theorem II. Let $l_k(\mathcal{P})$ be the columns of $L(\mathcal{P})$, the Lorentz transformation given above with $\beta = \mathcal{P}/\mathcal{E}$, and let H be the Hamiltonian given by Eq. (2.13). Then, replacing each $\mathcal{P}$ in $l_k(\mathcal{P})$ by the operator $-i\varepsilon_k^{-1}\partial_z \equiv \varepsilon_k^{-1}p$ yields the set of operators $l_k(\varepsilon_k^{-1}p)$ with the following properties: (i) The $l_k(\varepsilon_k^{-1}p)$ are normalized such that

$$l_k^\dagger(\varepsilon_k^{-1}p)Ml_k(\varepsilon_k^{-1}p) = \frac{E(\varepsilon_k^{-1}p)}{m}M_{kk}, \tag{3.8a}$$

where

$$E(\varepsilon_k^{-1}p) \equiv [(\varepsilon_k^{-1}p)^2 + m^2]^{1/2}, \tag{3.8b}$$

and M is the metric. (ii) The $l_k(\varepsilon_k^{-1}p)$ are metric-orthogonal,

$$l_j^\dagger(\varepsilon_j^{-1}p)Ml_k(\varepsilon_k^{-1}p) = 0, \quad j \neq k. \tag{3.9}$$

Theorem III. Let $\hat{u}_k$ and U^{-1} be defined by

$$\hat{u}_k \equiv \left[\frac{m}{E(\varepsilon_k^{-1}p)}\right]^{1/2} l_k(\varepsilon_k^{-1}p) \tag{3.10a}$$

and

$$[U^{-1}]_{jk} \equiv [\hat{u}_k]_j, \tag{3.10b}$$

where $l_k(\varepsilon_k^{-1}p)$ and $E(\varepsilon_k^{-1}p)$ are given above. Then we find the following: (i) The $\hat{u}_k$ are metric-orthonormal. (ii) The $\hat{u}_k$ are eigenvectors of the Hamiltonian H. (iii) U^{-1} is metric-unitary (pseudounitary) and has an inverse U.

Theorem III provides a method of constructing the metric-unitary operator U^{-1} for the class of relativistic wave equations considered here. The existence of an operator U^{-1} was established earlier.[1] Also, as we will see in Sec. III C, U^{-1} is indeed the transformation which diagonalizes the metric-Hermitian Hamiltonian.

At this point we reemphasize an observation which will be made clearer in FW-IV.[3] It is that, in addition to the normalization, the main change in going from the Lorentz transformation to the FW transformation is the identification

$$\beta = \left(\frac{\mathcal{P}}{\mathcal{E}}\right) \to \frac{\pm p}{(p^2 + m_j^2)^{1/2}}, \tag{3.11}$$

where the + (−) sign is for particles (antiparticles). The identification can be understood on physical grounds. Since the antiparticle equation is the charge-conjugated equation, the Lorentz transformation which boosts a particle in the positive direction "boosts" the "negative-energy" antiparticle in the reverse direction. Our three theorems have verified the validity of the above physical argument and shown that the identification (3.11) is correct.

C. FW-transformed Poincaré generators

Definition. An operator $\mathcal{O}$ is called "ε-diagonal" if $\mathcal{O}_{jk} = 0$ whenever $\varepsilon_j \neq \varepsilon_k$, where ε_k is defined in (3.3d).

Observe that in a multimass theory such as the Bhabha theory,[7-12] where the representation of ζ_4 can be taken such that all j with the same ε_j are grouped together, ε-diagonal corresponds to mass-block diagonal.

Theorem IV. Let U^{-1} be defined as in Theorem III, and let $\vec{P}$, $\vec{J}$, H, and $\vec{K}$ be defined by Eqs.

(2.11)–(2.14). Then (i) U^{-1} diagonalizes H such that

$$H^{\mathrm{FW}} \equiv UHU^{-1} = \zeta_4^{-1}E_p, \tag{3.12a}$$

$$E_p \equiv (p^2\zeta_4^{\,2} + \chi^2)^{1/2}; \tag{3.12b}$$

(ii) U^{-1} leaves $\vec{\mathrm{P}}$ diagonal and leaves $\vec{\mathrm{J}}$ ε-diagonal,

$$\vec{\mathrm{P}}^{\mathrm{FW}} \equiv U\vec{\mathrm{P}}U^{-1} = \vec{\mathrm{P}}, \tag{3.13}$$

$$\vec{\mathrm{J}}^{\mathrm{FW}} \equiv U\vec{\mathrm{J}}U^{-1} = \vec{\mathrm{J}}, \tag{3.14}$$

and (iii) U^{-1} ε-diagonalizes $\vec{\mathrm{K}}$ such that

$$\vec{\mathrm{K}}^{\mathrm{FW}} \equiv U\vec{\mathrm{K}}U^{-1} = \frac{\zeta_4^{-1}}{2}\{\vec{\mathrm{x}}, E_p\} - t\vec{\mathrm{P}} - \frac{\zeta_4(\vec{\mathrm{S}}\times\vec{\mathrm{P}})}{E_p + \chi}, \tag{3.15a}$$

where

$$S_k \equiv \tfrac{1}{2}\epsilon_{klm}J_{lm}. \tag{3.15b}$$

It is the proof of Eq. (3.15) which is especially detailed, and so also involves the calculations of Appendix B.

Lemma. The transformed generators $\vec{\mathrm{P}}^{\mathrm{FW}}$, $\vec{\mathrm{J}}^{\mathrm{FW}}$, H^{FW}, and $\vec{\mathrm{K}}^{\mathrm{FW}}$ do not connect states of different mass.

To see this, observe that if $(\text{mass})_j \neq (\text{mass})_k$, then from (3.3c) and (3.3d) $\varepsilon_j \neq \varepsilon_k$. Thus, from Theorems III and IV, $\mathcal{O}_{jk} = 0$, where $\mathcal{O}$ represents any of the transformed generators. This argument includes particle-antiparticle pairs.

IV. DISCUSSION

Through several theorems, we have developed a method by which an FW transformation can, in principle, be derived from a Lorentz transformation.[16,21-27] In the following paper (FW-IV)[3] we show that this method can in fact be explicitly implemented. With a combination of a matrix theorem and our method, an exact, closed-form FW transformation is written for our class of relativistic, first-order wave equations in an indefinite-metric space.[28]

We have also shown how any relativistic wave equation which satisfies the criteria of the introduction can be decoupled into unconnected mass states, including particles and antiparticles, by this FW transformation. In fact, we have determined the explicit form of the transformed Poincare generators.

Our method basically is the recognition that, with the proper renormalization and understanding of the meaning of the quantities we have called $\mathcal{P}$ and $\mathcal{E}$, an FW transformation U^{-1} can be related to a Lorentz transformation $L(\beta)$. It is therefore appropriate to note that our method is in fact a generalization, to a large class of multimass and multispin, first-order wave equations, of similar and related observations which have in whole or in part been used by other authors,[16,21-27] especially for the Dirac equation.

Throughout this paper, we have seen that the larger class of relativistic wave equations discussed here formally retains a similarity to the Bhabha equations[7-12]; a system of equations which is free from many of the problems which usually occur in high-spin theories. Whatever problems this larger class of relativistic wave equations may have,[29] we have shown that a well-defined FW transformation is not one of them. This opens the way for an interpretation and understanding of the operators in the theory and the negative-normed states which result from an indefinite metric. The latter is a remaining problem of the Bhabha equations. To this end, our results provide insight not only into the particular subset of equations which make up the Bhabha system[11,12] but also into this larger class of equations.

APPENDIX A: PROOFS OF THEOREMS

Theorem I. From Eq. (3.1) and Theorem IV of FW-II,[2] and remembering that $J_{43}^{\dagger} = J_{43}$, we have directly $L^{\dagger}(\beta) = L(\beta)$ and

$$L^{\dagger}(\beta)ML(\beta) = \sum_{n=0}^{\infty}\frac{(-\theta)^n}{n!}(\{J_{43},)^n M(\})^n, \tag{A1}$$

where M is the (indefinite) metric defined in Eq. (2.5), and again[2] our notation is that quantities in the parentheses are written out n times. Because of Eqs. (2.8) and (2.9), the nested anticommutators in (A1) close on themselves. In particular,

$$(\{J_{43},)^0 M(\})^0 \equiv M, \tag{A2a}$$

$$\{J_{43}, M\} = -i\eta\zeta_3, \tag{A2b}$$

$$\{J_{43}, \{J_{43}, M\}\} = M, \tag{A2c}$$

so that[20]

$$L^{\dagger}(\beta)ML(\beta) = M\cosh\theta + i\eta\zeta_3\sinh\theta = \gamma M + i\beta\gamma\eta\zeta_3, \tag{A3}$$

where Eq. (3.2) has been used in the final expression.

Theorem II. Parts (i) and (ii) will be proved in the same order as given in the theorem.

(i) Since ζ_4 is taken to be diagonal, the jk matrix elements of Eq. (2.8) imply that ($q = 1, 2, 3$)

$$[\zeta_q]_{jk} = +i([\zeta_4]_{jj} - [\zeta_4]_{kk})[J_{4q}]_{jk}, \tag{A4}$$

so that ζ_q is necessarily nondiagonal. Taking the diagonal matrix elements of Eq. (3.6) in Theorem I then gives [note that $\eta = \eta(\zeta_4)$ is diagonal]

$$l_k^{\dagger}(\mathcal{P})Ml_k(\mathcal{P}) = \gamma M_{kk}. \tag{A5}$$

Since only the kth column is involved, $\mathcal{P}$ is a dummy variable and may be everywhere replaced by $\varepsilon_k^{-1}p$ to yield

$$l_k^\dagger(\varepsilon_k^{-1}p)Ml_k(\varepsilon_k^{-1}p)=\frac{E(\varepsilon_k^{-1}p)}{m}M_{kk}, \tag{A6}$$

where $\gamma=\mathcal{E}/m=(\mathcal{P}^2+m^2)^{1/2}/m$ has been used, and $E(\varepsilon_k^{-1}p)\equiv[(\varepsilon_k^{-1}p)^2+m^2]^{1/2}$. This establishes the normalization of the $l_k(\varepsilon_k^{-1}p)$.

(ii) Here, there are two cases: (a) Suppose $\varepsilon_j=\varepsilon_k$, $j\neq k$. Then from Eq. (3.3d)

$$\frac{\varepsilon_k}{\varepsilon_j}=\frac{[\zeta_4^{-1}]_{kk}}{[\zeta_4^{-1}]_{jj}}=\frac{[\zeta_4]_{jj}}{[\zeta_4]_{kk}}, \tag{A7}$$

so that whenever $\varepsilon_j=\varepsilon_k$, $[\zeta_4]_{jj}=[\zeta_4]_{kk}$ and $[\zeta_q]_{jk}=0$ by (A4). Hence ζ_q is ε-nondiagonal. Since M is diagonal, the jk matrix elements of Eq. (3.6) are just

$$l_j^\dagger(\mathcal{P})Ml_k(\mathcal{P})=0,\quad \varepsilon_j=\varepsilon_k,\ j\neq k. \tag{A8}$$

Because $\varepsilon_j^{-1}p=\varepsilon_k^{-1}p$ and $\mathcal{P}$ is a dummy variable, we may replace $\mathcal{P}$ by $\varepsilon_j^{-1}p$ in $l_j^\dagger(\mathcal{P})$ and $\varepsilon_k^{-1}p$ in $l_k(\mathcal{P})$ to obtain

$$l_j^\dagger(\varepsilon_j^{-1}p)Ml_k(\varepsilon_k^{-1}p)=0,\quad \varepsilon_j=\varepsilon_k,\ j\neq k. \tag{A9}$$

(b) Suppose $\varepsilon_j\neq\varepsilon_k$, $j\neq k$. Because the replacement $\mathcal{P}\to\varepsilon_k^{-1}p$ does not change the form of $l_k(\mathcal{P})$ when $l_k(\varepsilon_k^{-1}p)$ operates on the space-time phase factor $\exp[+i\varepsilon_k(\mathcal{P}z-\mathcal{E}t)]$, the eigenvalue problem of (3.5) can be rewritten as

$$H\psi_j=\varepsilon_j\mathcal{E}\psi_j, \tag{A10a}$$

where

$$\psi_j=l_j(\varepsilon_j^{-1}p)e^{+i\varepsilon_j(\mathcal{P}z-\mathcal{E}t)}. \tag{A10b}$$

Multiplying Eq. (A10a) on the left-hand side by M, taking the adjoint, and multiplying on the right-hand side by ψ_k yields

$$\psi_j^\dagger MH\psi_k=\varepsilon_j\mathcal{E}\psi_j^\dagger M\psi_k \tag{A11a}$$

$$=\varepsilon_k\mathcal{E}\psi_j^\dagger M\psi_k, \tag{A11b}$$

where we have used $M^\dagger=M$ and $(MH)^\dagger=(MH)$. Subtracting Eq. (A11b) from (A11a) then yields

$$(\varepsilon_j-\varepsilon_k)\mathcal{E}\psi_j^\dagger M\psi_k=0, \tag{A12a}$$

or, because $\varepsilon_j\neq\varepsilon_k$,

$$e^{-i\varepsilon_j(\mathcal{P}z-\mathcal{E}t)}l_j^\dagger(\varepsilon_j^{-1}p)Ml_k(\varepsilon_k^{-1}p)e^{+i\varepsilon_k(\mathcal{P}z-\mathcal{E}t)}=0. \tag{A12b}$$

However, from Eqs. (A3), (A4), and (A7),

$$l_j^\dagger(\mathcal{P})Ml_k(\mathcal{P})=+i\beta\gamma\eta_{jj}(\zeta_3)_{jk}. \tag{A13}$$

If $(\zeta_3)_{jk}\neq 0$, (A12b) implies that

$$l_j^\dagger(\varepsilon_j^{-1}p)Ml_k(\varepsilon_k^{-1}p)=0,\quad \varepsilon_j\neq\varepsilon_k,\ j\neq k. \tag{A14a}$$

If $(\zeta_3)_{jk}=0$, then replacing $\mathcal{P}$ with $\varepsilon_j^{-1}p$ in (A13) and multiplying by $\exp[+i\varepsilon_j(\mathcal{P}z-\mathcal{E}t)]$ on the right-hand side and $\exp[-i\varepsilon_k(\mathcal{P}z-\mathcal{E}t)]$ on the left-hand side implies that

$$l_j^\dagger\left(\frac{\varepsilon_k\mathcal{P}}{\varepsilon_j}\right)Ml_k(\mathcal{P})=0,\quad \varepsilon_j\neq\varepsilon_k,\ j\neq k. \tag{A14b}$$

Now let $\mathcal{P}\to\varepsilon_k^{-1}p$.

Equations (A9) and (A14) then establish (ii) for all ε_j and ε_k, and hence for all $j\neq k$.

Theorem III. Parts (i) through (iii) will be proved in the same order as given in the theorem.

(i) From definition (3.10a) and Theorem II, the $\hat{u}_k$ are metric-orthonormal,

$$\hat{u}_j^\dagger M\hat{u}_k=M_{kk}\delta_{jk}. \tag{A15}$$

(ii) From (3.10a) and (A10), the $\hat{u}_k$ satisfy the equation

$$H\hat{u}_k e^{+i\varepsilon_k(\mathcal{P}z-\mathcal{E}t)}=\varepsilon_k\mathcal{E}\hat{u}_k e^{+i\varepsilon_k(\mathcal{P}z-\mathcal{E}t)} \tag{A16a}$$

$$=\frac{\varepsilon_k}{|\varepsilon_k|}E_k\hat{u}_k e^{+i\varepsilon_k(\mathcal{P}z-\mathcal{E}t)}, \tag{A16b}$$

where $E_k\equiv(p^2+m_k^2)^{1/2}$ and $m_k\equiv\varepsilon_k m$. Multiplying (A16b) on the left-hand side by $\hat{u}_j^\dagger M$ then yields

$$\left[\hat{u}_j^\dagger MH\hat{u}_k-\frac{\varepsilon_k}{|\varepsilon_k|}E_kM_{kk}\delta_{jk}\right]e^{+i\varepsilon_k(\mathcal{P}z-\mathcal{E}t)}=0. \tag{A17}$$

Since the expression in square brackets of (A17) can only be a scalar function of the operator p, it must be identically zero. Dropping the phase factor and rewriting, one has

$$\hat{u}_j^\dagger M\left[H\hat{u}_k-\frac{\varepsilon_k}{|\varepsilon_k|}E_k\hat{u}_k\right]=0, \tag{A18}$$

for all j. Because the $\hat{u}_j$ span the space,

$$H\hat{u}_k=\frac{\varepsilon_k}{|\varepsilon_k|}E_k\hat{u}_k,\quad \text{all } k \tag{A19}$$

where $E_k\equiv(p^2+m_k^2)^{1/2}$ and $m_k\equiv\varepsilon_k m$. Hence, the $\hat{u}_k$ are eigenvectors of H.

(iii) From (3.10b) and (A15), U^{-1} satisfies

$$[U^{-1}]^\dagger MU^{-1}=M, \tag{A20}$$

so that U^{-1} is metric-unitary.

If M is nonsingular, then M has an inverse M^{-1}. Multiplying (A20) on the left by M^{-1} shows that

$$U\equiv M^{-1}(U^{-1})^\dagger M \tag{A21}$$

is the left inverse of U^{-1}. (If M is singular, then see Sec. II C.) Since U^{-1} represents an automorphism (i.e., 1-to-1 and onto) given by

$$\hat{u}_k(\varepsilon_k^{-1}p)=U^{-1}\hat{u}_k(0), \tag{A22}$$

and U is the left inverse of U^{-1}, it is clear from left multiplying (A22) by U that U represents the inverse automorphism given by

$$\hat{u}_k(0) = U\hat{u}_k(\varepsilon_k^{-1}p), \tag{A23}$$

where $\hat{u}_k(0) = l_k(0)$ and $UU^{-1} = I$ have been used. Conversely, U^{-1} must represent the inverse automorphism of U, which implies that $U^{-1}U = I$; so that U is also the right inverse of U^{-1}. This establishes $U = M^{-1}(U^{-1})^\dagger M$ as the inverse of U^{-1}.

Theorem IV. Again parts (i) through (iii) will be proved in the same order as stated in the theorem.

(i) From Eqs. (3.3c), (3.3d), and (A19), it follows that

$$H\hat{u}_k = [\zeta_4]_{kk}^{-1}[p^2(\zeta_4)_{kk}^2 + \chi^2]^{1/2}\hat{u}_k, \tag{A24a}$$

which, when written in matrix form, becomes

$$HU^{-1} = U^{-1}\zeta_4^{-1}(p^2\zeta_4^2 + \chi^2)^{1/2}, \tag{A24b}$$

where $(p^2\zeta_4^2 + \chi^2)^{1/2}$ represents a power series in $p^2\zeta_4^2/\chi^2$. Using the results of (iii) of Theorem III,

$$H^{\mathrm{FW}} \equiv UHU^{-1}$$
$$= \zeta_4^{-1}E_p, \tag{A25a}$$

$$E_p \equiv (p^2\zeta_4^2 + \chi^2)^{1/2}, \tag{A25b}$$

so that U^{-1} diagonalizes H as given above.

(ii) Since U^{-1} is a scalar function of ζ_μ, p_μ, and χ, it necessarily commutes with both $\vec{P}$ and $\vec{J}$, leaving

$$\vec{P}^{\mathrm{FW}} \equiv U\vec{P}U^{-1}$$
$$= UU^{-1}\vec{P} + U[\vec{P}, U^{-1}]$$
$$= \vec{P}, \tag{A26}$$

and, similarly,

$$\vec{J}^{\mathrm{FW}} \equiv U\vec{J}U^{-1}$$
$$= UU^{-1}\vec{J} + U[\vec{J}, U^{-1}]$$
$$= \vec{J}. \tag{A27}$$

[Recall that in general, $\beta = (\mathcal{P}_x^2 + \mathcal{P}_y^2 + \mathcal{P}_z^2)^{1/2}/(\mathcal{P}_x^2 + \mathcal{P}_y^2 + \mathcal{P}_z^2 + m^2)^{1/2}$, a scalar.]

(iii) The discussion of $\vec{K}^{\mathrm{FW}} \equiv U\vec{K}U^{-1}$ is complicated, and will be broken down into 4 subsections labeled "(a)" through "(d)", with many of the details left to the reader.

(*a*) *K_i^{FW} is ε-diagonal*. Since $\vec{P}$, $\vec{J}$, H, and $\vec{K}$ satisfy the Lie algebra (see Sec. II B and footnote 19), so do $\vec{P}^{\mathrm{FW}}$, $\vec{J}^{\mathrm{FW}}$, H^{FW}, and $\vec{K}^{\mathrm{FW}}$. (A list of the commutation relations may be found in Ref. 9, for example.) Taking the jk matrix elements of

$$[K_i^{\mathrm{FW}}, H^{\mathrm{FW}}] = iP_i^{\mathrm{FW}}, \tag{A28}$$

we find that

$$(K_i^{\mathrm{FW}})_{jk}\left(\frac{\varepsilon_k}{|\varepsilon_k|}E_k - \frac{\varepsilon_j}{|\varepsilon_j|}E_j\right) + \left[(K_i^{\mathrm{FW}})_{jk}, \frac{\varepsilon_j}{|\varepsilon_j|}E_j\right] = iP_i\delta_{jk}. \tag{A29a}$$

Since $\vec{x}^{\mathrm{FW}} \equiv U\vec{x}U^{-1} = UU^{-1}\vec{x} + U[\vec{x}, U^{-1}] = \vec{x} + \Delta\vec{x}$, and U^{-1} is a function only of ζ_μ, p_μ, and χ, it is clear that the *Zitterbewegung* term $\Delta\vec{x}$ can also only be a function of ζ_μ, p_μ, and χ, and hence commutes with the jth component E_j. Thus,

$$\left[(K_i^{\mathrm{FW}})_{jk}, \frac{\varepsilon_i}{|\varepsilon_i|}E_j\right] = \left[x_i\delta_{jk}\frac{\varepsilon_k}{|\varepsilon_k|}E_k, \frac{\varepsilon_j}{|\varepsilon_j|}E_j\right]$$
$$= iP_i\delta_{jk},$$

implying that $\vec{K}^{\mathrm{FW}}$ is ε-diagonal,

$$(K_i^{\mathrm{FW}})_{jk} = 0, \quad \varepsilon_j \neq \varepsilon_k. \tag{A29b}$$

Hence, $\vec{K}^{\mathrm{FW}}$ can only be a function of x_μ, p_μ, χ, ζ_4, and $\vec{S}$. Using (2.8) and (A7), the jk matrix elements of $[S_i, \zeta_4] = 0$. This shows that S_i is ε-diagonal:

$$(S_i)_{jk}([\zeta_4]_{jj} - [\zeta_4]_{kk}) = 0. \tag{A30}$$

(*b*) *$\vec{K}^{\mathrm{FW}} = \vec{K}' + \delta\vec{K}$, where*

$$\vec{K}' \equiv \frac{\zeta_4^{-1}}{2}\{\vec{x}, E_p\} - t\vec{p} - \frac{\zeta_4(\vec{S}\times\vec{p})}{E_p + \chi}, \tag{A31a}$$

$$E_p \equiv (p^2\zeta_4^2 + \chi^2)^{1/2}, \tag{A31b}$$

with $\delta\vec{K}$ determined by the solutions to Eqs. (A33)–(A36).

Using the techniques discussed in Sec. II B, one can show that, together with $\vec{P}^{\mathrm{FW}}$, $\vec{J}^{\mathrm{FW}}$, and H^{FW}, $\vec{K}'$ is a particular solution of the Lie algebra. Hence

$$\vec{K}^{\mathrm{FW}} = \vec{K}' + \delta\vec{K}. \tag{A32}$$

Since both $\vec{K}'$ and, of necessity, $\vec{K}^{\mathrm{FW}}$ satisfy the Lie algebra, $\delta\vec{K}$ satisfies the following commutation relations:

$$[J_i, \delta K_j] = i\epsilon_{ijk}\delta K_k, \tag{A33}$$

$$[P_i, \delta K_j] = 0, \tag{A34}$$

$$[H^{\mathrm{FW}}, \delta K_j] = 0, \tag{A35}$$

$$[\delta K_i, K_j'] - [\delta K_j, K_i'] + [\delta K_i, \delta K_j] = 0. \tag{A36}$$

From the results of (a), (A31), (A33), and (A34),

$$\delta\vec{K} = \vec{p}A + \tfrac{1}{2}[\vec{S}, B] + \tfrac{1}{2}[(\vec{S}\times\vec{p}), C], \tag{A37}$$

where A, B, and C are functions of p^2 and $(\vec{p}\cdot\vec{S})$. (Terms like S^2 and ζ_4, which commute with everything, will not be explicitly discussed.) Since $\vec{K}$ is metric-Hermitian and U^{-1} is metric-unitary, $\vec{K}^{\mathrm{FW}}$ is metric-Hermitian. Since $[\vec{K}^{\mathrm{FW}}, M] = 0$, $\vec{K}^{\mathrm{FW}}$ is self-adjoint and A, B, and C are real.

One can also observe that $\delta\vec{K}$ can be at most a linear function of spin. While S_i, $i=1,2,3$ is from (A30) ε-diagonal, terms like $(\vec{p}\cdot\vec{S})^n$, where $n\neq 0$ or 1, would not be ε-diagonal. Since we have shown in (a) that $\delta\vec{K}$ is ε-diagonal, it cannot contain such terms. Hence, we set

$$A \equiv (\vec{p}\cdot\vec{S})a(p^2)+d(p^2), \tag{A38}$$

$$B \equiv p^2 b(p^2), \tag{A39}$$

$$C \equiv c(p^2), \tag{A40}$$

to obtain,

$$\delta K_i = p_i(\vec{p}\cdot\vec{S})a + S_i p^2 b + (\vec{S}\times\vec{p})_i c + p_i d. \tag{A41}$$

Putting (A41) into (A36) now yields 3 independent equations,

$$\epsilon_{ijk}p_k(\vec{p}\cdot\vec{S})f=0, \tag{A42}$$

$$\epsilon_{ijk}S_k g=0, \tag{A43}$$

$$\epsilon_{ijk}(\vec{S}\times\vec{p})_k h=0, \tag{A44}$$

where f, g, and h are functions of a, b, and c. Since i, j, and k are arbitrary,

$$f=-2\frac{\partial c}{\partial p^2}H^{\mathrm{FW}}-2c\phi^{\mathrm{FW}}+c^2-p^2ab=0, \tag{A45}$$

$$g=+2p^2\frac{\partial c}{\partial p^2}H^{\mathrm{FW}}+2cH^{\mathrm{FW}}+p^4(a+b)b=0, \tag{A46}$$

$$h=-2p^2\frac{\partial b}{\partial p^2}H^{\mathrm{FW}}+(a-2b)H^{\mathrm{FW}}+p^2(a+b)(c-\Phi^{\mathrm{FW}})=0, \tag{A47a}$$

where

$$\phi^{\mathrm{FW}}\equiv\zeta_4/(E_p+\chi). \tag{A47b}$$

The solution of these nonlinear coupled equations is given in (c) and (d).

(*c*) $a=-b$, *and* $d=0$. Using the results of Appendix B, we can directly calculate

$$\psi_k = e^{i\theta K_3}\psi_k(0) \tag{A48a}$$

$$= e^{-\theta J_{43}}l_k(0)e^{+i\varepsilon_k(\mathcal{P}z-\mathcal{E}t)} \tag{A48b}$$

$$= \left[\frac{E(\varepsilon_k^{-1}p)}{m}\right]^{1/2}\hat{u}_k(\varepsilon_k^{-1}p)e^{+i\varepsilon_k(\mathcal{P}z-\mathcal{E}t)} \tag{A48c}$$

$$= \left(\frac{\mathcal{E}}{m}\right)^{1/2}U^{-1}l_k(0)e^{+i\varepsilon_k(\mathcal{P}z-\mathcal{E}t)}, \tag{A48d}$$

where (2.14), (3.1), and (3.10) have been used. Left multiplying by U yields

$$\psi_k^{\mathrm{FW}}\equiv U\psi_k \tag{A49a}$$

$$= \left(\frac{\mathcal{E}}{m}\right)^{1/2}l_k(0)e^{+i\varepsilon_k(\mathcal{P}z-\mathcal{E}t)} \tag{A49b}$$

$$= Ue^{i\theta K_3}U^{-1}U\psi_k(0) \tag{A49c}$$

$$= e^{i\theta K_3}\psi_k(0), \tag{A49d}$$

where $U\psi_k(0)=\psi_k(0)$ has been used. Because $\psi_k(0)$ is independent of z, $[U(p=0)]_{jk}=[l_k(0)]_j=\delta_{jk}$.

Again using the results of Appendix B (A49d) becomes

$$\psi_k^{\mathrm{FW}}=\left(\frac{\mathcal{E}}{m}\right)^{1/2}\exp[+i\varepsilon_k(\mathcal{P}z-\mathcal{E}t)]\exp\left(+i\int_0^\theta \varepsilon_k m\sinh\rho\{\varepsilon_k m\sinh\rho S_z[a(\rho)+b(\rho)]+d(\rho)\}d\rho\right)l_k(0). \tag{A49e}$$

Comparing (A49e) with (A49b) implies that

$$\int_0^\theta \varepsilon_k m\sinh\rho\{\varepsilon_k m\sinh\rho S_z[a(\rho)+b(\rho)]+d(\rho)\}d\rho=2\pi n,\quad n=0,1,2,\ldots. \tag{A49f}$$

Since θ is continuous and arbitrary, $a=b\equiv 0$, $d=0$, or $a=-b$, $d=0$. We assume the latter.

(*d*) $\delta\vec{K}=0$. Putting $a=-b$ into (A46) and (A47) implies that $c\equiv 0$, or $c=\hat{c}/p^2$, and $b\equiv 0$, or $b=3\hat{b}/2p^2$, where $\hat{c}\neq\hat{c}(p^2)$ and $\hat{b}\neq\hat{b}(p^2)$. Of the 4 possibilities, only $b=0$, $c=0$ satisfies (A45). Together with the results of (b) and (c),

$$\delta\vec{K}=0 \tag{A50}$$

and

$$\vec{K}^{\mathrm{FW}}\equiv U\vec{K}U^{-1} = \frac{\zeta_4^{-1}}{2}\{\vec{x},E_p\}-t\vec{p}-\frac{\zeta_4(\vec{S}\times\vec{p})}{E_p+\chi}, \tag{A51a}$$

where

$$E_p\equiv(p^2\zeta_4{}^2+\chi^2)^{1/2}. \tag{A51b}$$

This completes the proof.

APPENDIX B: BOOST OPERATORS

Let $\vec{K}$ be the infinitesimal Poincaré generator of velocity translations (the boost operator). Then, if a prime denotes a boosted quantity referred to the *same* reference frame,

$$\psi'(x)=e^{i\theta\hat{v}\cdot\vec{K}}\psi(x), \tag{B1}$$

$$\vec{p}'=\vec{p}+(\cosh\theta-1)\hat{v}\cdot\vec{p}\,\hat{v}+\sinh\theta\,\mathcal{E}\hat{v}, \tag{B2a}$$

$$\mathcal{E}' = \mathcal{E}\cosh\theta + \hat{v}\cdot\vec{p}\sinh\theta, \tag{B2b}$$

$$\beta = \tanh\theta, \tag{B2c}$$

where β is the velocity of the boost. Using a technique of Osborn,[30] Eq. (B1) can be directly calculated by solving the differential equation,

$$\frac{d\psi(\theta)}{d\theta} = i\hat{v}\cdot\vec{K}\psi(\theta), \tag{B3a}$$

where

$$\psi(\theta) \equiv e^{i\theta\hat{v}\cdot\vec{K}}\psi(0). \tag{B3b}$$

Since $\hat{v}\cdot\vec{K}$ commutes with each term in the expansion of $e^{i\theta\hat{v}\cdot\vec{K}}$, this equation is well defined.

Let $K(\theta)$ be the result of operating on $\psi(\theta)$ with $i\hat{v}\cdot\vec{K}$. Then, the differential equation

$$\frac{d\psi(\theta)}{d\theta} = K(\theta)\psi(\theta) \tag{B4a}$$

has the solution

$$\psi(\theta) = \exp\left[\int_0^\theta K(\rho)d\rho\right]\psi(0), \tag{B4b}$$

provided $K(\theta)$ commutes with each term in the expansion of $\exp[\int_0^\theta K(\rho)d\rho]$.

(a) Using (2.14) for $\vec{K}$, $\vec{\mathcal{P}}=0$, $\mathcal{E}=m$, $\hat{v}=\hat{z}$, and (3.3b) for $\psi_k(0)$,

$$\begin{aligned}\frac{d\psi_k(\theta)}{d\theta} &= i(zH - tp_3 + iJ_{43})\psi_k(\theta)\\ &= i(z\varepsilon_k m\cosh\theta - t\varepsilon_k m\sinh\theta + iJ_{43})\psi_k(\theta),\end{aligned} \tag{B5}$$

so that

$$\begin{aligned}\psi_k &= \exp\left[i\varepsilon_k\int_0^\theta d\rho(zm\cosh\rho - tm\sinh\rho) - \theta J_{43}\right]\psi_k(0)\\ &= e^{-\theta J_{43}}l_k(0)e^{i\varepsilon_k(\mathcal{P}z-\mathcal{E}t)}\\ &= l_k(\mathcal{P})e^{i\varepsilon_k(\mathcal{P}z-\mathcal{E}t)},\end{aligned} \tag{B6}$$

where $\tanh\theta = \beta \equiv \varepsilon_k\mathcal{P}/(\varepsilon_k\mathcal{E}) \equiv \mathcal{P}_k/\mathcal{E}_k = \mathcal{P}/\mathcal{E}$ has been used.

(b) Using (A31) for $\vec{K}$ and (B2),

$$\begin{aligned}\frac{d\psi_k(\theta)}{d\theta} &= i\left[\vec{x}\cdot\hat{v}\mathcal{E}_k' - \frac{i}{2}\frac{\vec{\mathcal{P}}_k'\cdot\hat{v}}{\mathcal{E}_k'} - t\vec{\mathcal{P}}_k'\cdot\hat{v} - \frac{(\vec{S}\times\vec{\mathcal{P}}_k')\cdot\hat{v}}{\mathcal{E}_k'+\zeta_4^{-1}\chi}\right]\psi_k(\theta)\\ &= \left\{i[\vec{x}\cdot\hat{v}(\mathcal{E}_k\cosh\theta + \vec{\mathcal{P}}_k\cdot\hat{v}\sinh\theta) - t(\vec{\mathcal{P}}_k\cdot\hat{v}\cosh\theta + \mathcal{E}_k\sinh\theta)]\right.\\ &\quad \left. + \frac{1}{2}\left(\frac{\vec{\mathcal{P}}_k\cdot\hat{v}\cosh\theta + \mathcal{E}_k\sinh\theta}{\mathcal{E}_k\cosh\theta + \vec{\mathcal{P}}_k\cdot\hat{v}\sinh\theta}\right) - \frac{i\vec{S}\cdot\vec{\mathcal{P}}_k\times\hat{v}}{\mathcal{E}_k\cosh\theta + \vec{\mathcal{P}}_k\cdot\hat{v}\sinh\theta + \zeta_4^{-1}\chi}\right\}\psi_k(\theta).\end{aligned} \tag{B7}$$

This integrates to yield

$$\psi_k(\vec{\mathcal{P}}_k') = \left(\frac{\mathcal{E}_k'}{\mathcal{E}_k}\right)^{1/2} e^{i(\vec{x}\cdot\vec{\mathcal{P}}_k'-\mathcal{E}_k't)}e^{-i(\vec{x}\cdot\vec{\mathcal{P}}_k-\mathcal{E}_kt)}(e^{-i\vec{S}\cdot\vec{\mathcal{P}}_k\times\hat{v}\Theta(\vec{\mathcal{P}}_k',\vec{\mathcal{P}}_k)})\psi_k(\vec{\mathcal{P}}_k), \tag{B8a}$$

$$\Theta(\vec{\mathcal{P}}_k', \vec{\mathcal{P}}_k) = \frac{2}{[\mathcal{P}_k^2 - (\hat{v}\cdot\vec{\mathcal{P}}_k)^2]^{1/2}}\arctan\left\{\frac{\tanh(\frac{1}{2}\theta)[\mathcal{P}_k^2 - (\hat{v}\cdot\vec{\mathcal{P}}_k)^2]^{1/2}}{\mathcal{E}_k + \hat{v}\cdot\vec{\mathcal{P}}_k\tanh(\frac{1}{2}\theta) + m_k}\right\}, \tag{B8b}$$

$$\tanh\frac{\theta}{2} = \frac{\hat{v}\cdot\vec{\mathcal{P}}_k'\mathcal{E}_k - \hat{v}\cdot\vec{\mathcal{P}}_k\mathcal{E}_k'}{\mathcal{E}_k(\mathcal{E}_k+\mathcal{E}_k') - \hat{v}\cdot\vec{\mathcal{P}}_k(\hat{v}\cdot\vec{\mathcal{P}}_k + \hat{v}\cdot\vec{\mathcal{P}}_k')}, \tag{B8c}$$

$$\psi_k(\vec{\mathcal{P}}_k) = \left(\frac{\mathcal{E}_k}{m_k}\right)^{1/2}(e^{i(\vec{x}\cdot\vec{\mathcal{P}}_k-\mathcal{E}_kt)})l_k(0), \tag{B8d}$$

where

$$\zeta_4^{-1}\chi\psi_k(\vec{\mathcal{P}}_k) = m_k\psi_k(\vec{\mathcal{P}}_k) \tag{B8e}$$

has been used in (B8a) after the integration. (B8e) follows from (3.3c), (3.3d), and (B8d).

Clearly the effect of $e^{i\theta\hat{v}\cdot\vec{K}}$ on $\psi_k(\vec{\mathcal{P}}_k)$ is to replace $(\mathcal{E}_k/m_k)^{1/2}$ with $(\mathcal{E}_k'/m_k)^{1/2}$, to replace $e^{i(\vec{x}\cdot\vec{\mathcal{P}}_k-\mathcal{E}_kt)}$ with $e^{i(\vec{x}\cdot\vec{\mathcal{P}}_k'-\mathcal{E}_k't)}$, and to make a Wigner rotation of the spinor of $\psi_k(\vec{\mathcal{P}}_k)$.

If $\vec{\mathcal{P}}_k \to 0$, and $\hat{v}=\hat{z}$, then

$$\psi_k = \left(\frac{\mathcal{E}}{m}\right)^{1/2} l_k(0)e^{i\varepsilon_k(\mathcal{P}z-\mathcal{E}t)}, \tag{B9}$$

where

$$\tanh\theta = \beta \equiv \varepsilon_k\mathcal{P}/(\varepsilon_k\mathcal{E}) = \mathcal{P}/\mathcal{E} \tag{B10}$$

has been used.

*Present and permanent address: Theoretical Design Division, Los Alamos Scientific Laboratory, Los Alamos, New Mexico 87545.

†Work supported by the United States Energy Research and Development Administration.

[1]R. A. Krajcik and M. M. Nieto, Phys. Rev. D 13, 2245 (1976), hereafter known as FW-I, the first in the present series.

[2]R. A. Krajcik and M. M. Nieto, Phys. Rev. D 13, 2250 (1976), hereafter known as FW-II, the second in the present series.

[3]R. A. Krajcik and M. M. Nieto, Phys. Rev. D 15, 426 (1977), hereafter known as FW-IV, the following and concluding article of the present series.

[4]M. Taketani and S. Sakata, Proc. Phys. Math. Soc. Jpn. 22, 747 (1940); S. Sakata and M. Taketani, Sci. Pap. Inst. Phys. Chem. Res. 38, 1 (1940). The above two articles have been reprinted in Prog. Theor. Phys. Suppl. [No. 1, 84 (1955); No. 1, 98 (1955)].

[5]W. Rarita and J. Schwinger, Phys. Rev. 60, 61 (1941). But also see M. L. Nack, Nuovo Cimento 68A, 89 (1970), on the RS formalism for boson fields.

[6]P. A. Carruthers, *Spin and Isospin in Particle Physics* (Gordon and Breach, New York, 1971).

[7]R. A. Krajcik and M. M. Nieto, Phys. Rev. D 10, 4049 (1974), paper I of our series on Bhabha first-order wave equations.

[8]R. A. Krajcik and M. M. Nieto, Phys. Rev. D 11, 1442 (1975), paper II of our Bhabha series.

[9]R. A. Krajcik and M. M. Nieto, Phys. Rev. D 11, 1459 (1975), paper III of our Bhabha series.

[10]R. A. Krajcik and M. M. Nieto, Phys. Rev. D 13, 924 (1976), paper IV of our Bhabha series.

[11]R. A. Krajcik and M. M. Nieto, Phys. Rev. D 14, 418 (1976), paper V of our Bhabha series, on the indefinite metric and FW transformations.

[12]R. A. Krajcik and M. M. Nieto, this issue, Phys. Rev. D 15, 433 (1977), paper VI of our Bhabha series, on exact, closed-form FW transformations and solutions.

[13]We use the metric $a \cdot a = a_\lambda a_\lambda = \vec{a} \cdot \vec{a} + a_4 a_4 = \vec{a} \cdot \vec{a} - a_0 a_0$.

[14]The general derivation follows as for the Bhabha case. See Eqs. (2.45), (2.46), and Sec. III C of Ref. 7. In particular, Eq. (A5) of Ref. 7 is analogous to our Eq. (A4) of this paper.

[15]I. M. Gel'fand, R. A. Minlos, and Z. Y. Shapiro, *Representations of the Rotation and Lorentz Groups and their Applications* (Pergamon, New York, 1963). See p, 187, p. 208, and Sec. VII of Chap. 2, p. 269.

[16]J. J. Giambiagi, Nuovo Cimento 16, 202 (1960); C. G. Bollini and J. J. Giambiagi, *ibid.* 21, 107 (1961).

[17]S. Gasiorowicz, *Elementary Particle Physics* (Wiley, New York, 1966). See Chap. 4, p. 65.

[18]The use of the Cayley-Hamilton theorem to find ζ_4^{-1} proceeds exactly as in the Bhabha case, demonstrated in Ref. 8.

[19]Given the commutation relations (2.7)–(2.9), the proof that the Poincaré generators (2.11)–(2.14) satisfy the associated Lie algebra also proceeds as in the Bhabha case, described in Ref. 9. The preceding statement also holds for the case where there are built-in subsidiary components (see Sec. II C below), as there are for integer-spin Bhabha fields. Observe that here ζ_4^{-1} is worked out of a commutator with $J_{\mu\nu}$ by multiplying by $\zeta_4^{-1}\zeta_4 = \zeta_4\zeta_4^{-1} = 1$,

$$[\zeta_4^{-1}, J_{\mu\nu}] = \zeta_4^{-1}(\zeta_4[\zeta_4^{-1}, J_{\mu\nu}]\zeta_4)\zeta_4^{-1}$$
$$= \zeta_4^{-1}[J_{\mu\nu}, \zeta_4]\zeta_4^{-1},$$

and then using (2.8). Otherwise, the procedure is the same as in Ref. 9. Recall that ζ_4^{-1} is a function only of ζ_4, and so commutes with the metric.

[20]In the special-case result of Eq. (3.6) quoted in Eq. (2.19) of FW-II (Ref. 2), the η_4 was inadvertently left out after the "*i*" in the second term on the right-hand side. The minus sign for that term is due to the opposite phase convention for θ that was used there. On this last point, see our comment after Eq. (3.2) of this paper.

[21]M. E. Rose and R. H. Good, Jr., Nuovo Cimento 22, 565 (1961).

[22]R.-K. R. Loide, Teor. Mat. Fiz. 23, 42 (1975) [Theor. Math. Phys. 23, 336 (1975)].

[23]P. M. Mathews and A. Sankaranarayanan, Nuovo Cimento 34, 101 (1964).

[24]J. E. Johnson and K. K. Chang, Phys. Rev. D 10, 2421 (1974).

[25]W. A. Hepner, Phys. Rev. 84, 744 (1951).

[26]I. Saavedra, Nucl. Phys. 74, 677 (1965).

[27]G. Alagar Ramanujam, Nuovo Cimento 20A, 27 (1974).

[28]W. Cox (private communication) has observed that certain aspects of our theorem in Ref. 1, as applied to relativistic, first-order wave equations, have been studied. Specifically, with the formalism of our Ref. 15 above, I. M. Gel'fand and A. M. Yaglom, Zh. Eksp. Teor. Fiz 18, 703 (1948); 18, 1096 (1948); 18, 1105 (1948), showed that complex eigenvalues cannot have nonzero-norm eigenvectors, which therefore leaves only the eigenvalue-eigenvector criteria discussed in Ref. 1. Also the work of three papers [F. I. Fedorov, Doklady Akad. Nauk. SSSR 79, 787 (1951); H. J. Bhabha, Philos. Mag. 43, 33 (1952); B. M. Udgaonkar, Proc. Indian Acad. Sci. 36A, 482 (1952)] implies that the necessary and sufficient condition that the charge density and the energy density be nonzero is that the minimal equation for ζ_4 not have repeated factors other than itself. Cox expressed the opinion that since a consistent quantization with positive-definite metric is only possible if ζ_4 is nondiagonalizable (see Ref. 15), this probably drew attention away from the situation discussed in our Ref. 1. Of mathematical interest, he feels our theorem can readily be generalized to a nondiagonal metric which has a diagonalizable part, and indeed one should see J. Williamson, Am. J. Math. 60, 355 (1938). Also, see W. Cox, J. Phys. A 7, 1 (1974); 7, 665 (1974).

[29]We are especially thinking of noncausality problems which can occur when interactions are introduced into high-spin field theories. The work of G. Velo and D. Zwanziger [Phys. Rev. 186, 1337 (1969); 188, 2218 (1969] focused interest in this problem.

[30]H. Osborn, Phys. Rev. 176, 1514 (1968).