Response of a disk antenna to scalar and tensor gravitational waves*

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Response of a disk antenna to a completely general Riemann tensor with six possible modes of polarization is analyzed. Only the monopole and quadrupole modes of the antenna are found to couple to an arbitrary gravitational wave. The absorption cross sections of these modes for scalar and tensor waves are calculated numerically. It is pointed out that, with two local disk detectors oriented 90° with respect to each other, one can not only determine the incident angles and polarization of the wave but also eliminate spurious non-gravitational-wave signals.

I. INTRODUCTION

A gravitational-wave antenna of a disk shape was first utilized by Weber in an attempt to measure a scalar component of gravitational radiation.[1] Interest in a disk detector has been revived recently by the Rochester group, who are considering the disk geometry for their single-crystal sapphire detector.[2] In this paper we analyze the interaction of a completely general Riemann tensor with various mechanical modes of a circular disk and specifically evaluate the absorption cross sections for a few selected modes. We find that only the monopole ($n=0$) and the quadrupole ($n=2$) modes couple to a general Riemann tensor. A similar property has been shown for a cylindrical antenna.[3] The tensor-wave cross section per unit antenna mass for the lowest quadrupole mode of a disk is found to be bigger than that for the lowest longitudinal mode of a cylinder by a factor of 2 when averaged over all directions. It is also pointed out that, with two local disk detectors oriented 90° with respect to each other, one can not only determine the incident angles and polarization of the wave but also discriminate gravitational wave bursts against spurious non-wave-originated disturbances.

The resonance integral of the cross section for the nth normal mode of the antenna to a polarized tensor wave is given[4] by

$$\int_0^\infty \sigma_n(\nu)d\nu = \frac{\pi}{4}\frac{G}{c^3}\frac{\omega_n^{\,2}}{M}\left|\bar{I}_n^{jk}e_{jk}\right|^2 , \qquad (1)$$

where M is the total mass of the antenna, ω_n is the angular eigenfrequency of the normal mode, e_{jk} is the polarization matrix for the incoming wave, and $\bar{I}_n^{jk}$ is the "reduced quadrupole factor for the nth normal mode." Here $\bar{I}_n^{jk}$ is defined as

$$\bar{I}_n^{jk} \equiv I_n^{jk} - \tfrac{1}{3}\delta^{jk}I_{nl}^{\;l} , \qquad (2)$$

where

$$I_n^{jk} \equiv \int_V \rho(u_n^j x^k + u_n^k x^j)d^3x . \qquad (3)$$

The normalization condition for eigenvectors $\vec{u}_n(\vec{x})$ becomes

$$\int_V u_n^i u_{mi} d^3x = V\delta_{nm} , \qquad (4)$$

when the density ρ is a constant throughout the entire volume of the antenna V.

It has been shown[5,6] that the Riemann tensor of the most general wave is composed of six modes of polarization:

$$R_{j0k0}(t) = \sum_{A=1}^{6} P_A(\vec{k},t)e_{jk}^A(\vec{k}) , \qquad (5)$$

where $e_{jk}^A(\vec{k})$ are the unit polarization matrices for wave direction $\vec{k}$, and $P_A(\vec{k},t)$ are the amplitudes of the wave. The formulas for $P_A(\vec{k},t)$ and $e_{jk}^A(\vec{k})$ are given in Ref. 6. Using this general Riemann tensor, Eq. (1) is modified[7] into

$$\Sigma_n^A \equiv \int_0^\infty \sigma_n^A(\nu)d\nu = \frac{\pi G}{c^3}\frac{\omega_n^{\,2}}{M}\lambda_A\left|I_n^{jk}e_{jk}^A\right|^2 , \qquad (6)$$

where λ_A is the coupling factor of the Ath polarization with matter. General relativity requires $\lambda_4=\lambda_5=1$ and $\lambda_1=\lambda_2=\lambda_3=\lambda_6=0$; the Brans-Dicke theory allows an additional nonvanishing coupling factor $\lambda_6=(2\omega+3)^{-1}$.

In Sec. II we calculate I_n^{jk} for all the extensional modes of a disk and consider their interactions with a gravitational wave of arbitrary polarization e_{jk}^A. Absorption cross sections for a few selected modes will be calculated numerically in Sec. III as a function of Poisson's ratio of the antenna material. And, finally, in Sec. IV we will evaluate a disk detector as a practical experimental apparatus.

II. INTERACTION OF THE EXTENSIONAL MODES WITH A GENERAL RIEMANN TENSOR

The coordinate system chosen for our calculation is shown in Fig. 1. The center of mass is taken to be the origin, and the middle plane of the disk lies on the x^1x^2 plane. The radius and thickness of the disk are denoted by a and h, respectively, and a uniform density is assumed. The spatial eigenfunctions of extensional modes of a circular disk are given[8] as

$$u^r_{np} = v^r_n(r)(\delta_{p1}\cos n\theta + \delta_{p2}\sin n\theta)\ , \tag{7a}$$

$$u^\theta_{np} = v^\theta_n(r)(\delta_{p1}\sin n\theta - \delta_{p2}\cos n\theta)\ , \tag{7b}$$

$$u^z_{np} = 0\ , \tag{7c}$$

where $p = 1, 2$ represent the two polarization states of the nth mode and

$$v^r_n(r) \equiv A\,\frac{dJ_n(kr)}{dr} + nB\,\frac{J_n(k'r)}{r}\ , \tag{8a}$$

$$v^\theta_n(r) \equiv -\left[nA\,\frac{J_n(kr)}{r} + B\,\frac{dJ_n(k'r)}{dr}\right]. \tag{8b}$$

The eigenvalues k and k' are defined by

$$k^2 = (1-\sigma^2)\left(\frac{\omega}{v}\right)^2\ , \tag{9a}$$

$$k'^2 = 2(1+\sigma)\left(\frac{\omega}{v}\right)^2\ , \tag{9b}$$

where ω is the angular eigenfrequency, $v = (E/\rho)^{1/2}$ is the sound velocity, and σ is Poisson's ratio.

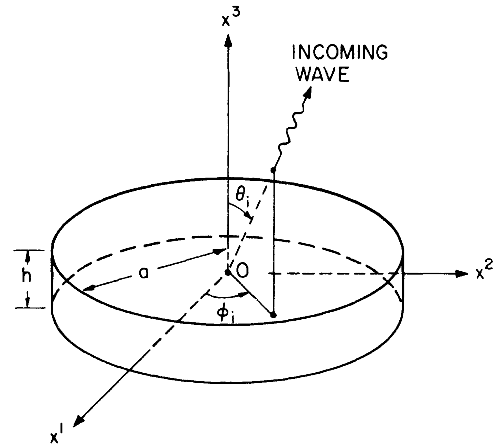

FIG. 1. The coordinate system used for calculation. The center of mass is taken to be the origin.

From Eqs. (7) one can see that, when $n = 0$, the motion is decoupled into a purely radial one ("radial" mode, $p = 1$) and a purely angular one ("torsional" mode, $p = 2$).

The Cartesian components of $\vec{u}_{np}$ are obtained using Eqs. (7):

$$\begin{pmatrix} u^1_{np} \\ u^2_{np} \\ u^3_{np} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u^r_{np} \\ u^\theta_{np} \\ u^z_{np} \end{pmatrix}. \tag{10}$$

We now calculate the moment of inertia factor I^{jk}_{np}. Substituting Eq. (10) into Eq. (3), we find

$$I^{jk}_{np} = \rho\int_0^a r^2 dr \int_0^{2\pi} d\theta \int_{-h/2}^{h/2} dz \begin{bmatrix} u^r_{np}(1+\cos 2\theta) - u^\theta_{np}\sin 2\theta & u^r_{np}\sin 2\theta + u^\theta_{np}\cos 2\theta & \frac{z}{r}(u^r_{np}\cos\theta - u^\theta_{np}\sin\theta) \\ * & u^r_{np}(1-\cos 2\theta) + u^\theta_{np}\sin 2\theta & \frac{z}{r}(u^r_{np}\sin\theta + u^\theta_{np}\cos\theta) \\ * & * & 0 \end{bmatrix}, \tag{11}$$

where we have used $u^z_{np} = 0$, and elements represented by an asterisk are the same as their symmetric counterparts. The third column drops out identically upon z integration. Next we substitute Eqs. (7) into Eq. (11) and carry out the θ integration to obtain

$$I^{jk}_{np} = \delta_{n0}\delta_{p1}\tfrac{2}{3}MaV_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \delta_{n2}MaV_2 \left[\delta_{p1}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \delta_{p2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right], \tag{12}$$

where the dimensionless quantities V_0 and V_1 are defined as

$$V_0 \equiv \int_0^1 \xi^2 d\xi\, v^r_0(a\xi)\ , \tag{13a}$$

$$V_2 \equiv \int_0^1 \xi^2 d\xi\,[v^r_2(a\xi) - v^\theta_2(a\xi)]\ . \tag{13b}$$

Equation (12) implies that only the (radial) *monopole* modes ($n = 0$) and the two polarization states of the *quadrupole* modes ($n = 2$) couple to an entirely general Riemann tensor. Hier and Rasband[3] have shown that, in addition to these, the bending modes also couple to gravitational waves in the case of a cylinder ($h/a \gg 1$). By analogy, it is expected that certain transverse modes of a disk ($h/a \ll 1$) can also be excited by gravitational waves. In this paper we confine ourselves to only extensional modes.

When the incident angles of the wave are θ_i and ϕ_i as shown in Fig. 1, the unit polarization matrices $e^A_{jk}(\vec{k})$ are obtained from the basis polarization matrices given in Eq. (30) of Ref. 6 using a rotation matrix

$$\vec{R} = \begin{pmatrix} \cos\phi_i & -\sin\phi_i & 0 \\ \sin\phi_i & \cos\phi_i & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta_i & 0 & \sin\theta_i \\ 0 & 1 & 0 \\ -\sin\theta_i & 0 & \cos\theta_i \end{pmatrix}. \tag{14}$$

Substituting the results into Eq. (6) along with Eq. (12), we find

$$\Sigma^1_{np} = \frac{36\pi G}{c^3} Ma^2\lambda_1 \sin^4\theta_i[4\omega_0{}^2V_0{}^2\delta_{n0}\delta_{p1} + \omega_2{}^2V_2{}^2\delta_{n2}(\delta_{p1}\cos^2 2\phi_i + \delta_{p2}\sin^2 2\phi_i)], \tag{15a}$$

$$\Sigma^2_{np} = \frac{16\pi G}{c^3} Ma^2\lambda_2 \cos^2\theta_i \sin^2\theta_i[4\omega_0{}^2V_0{}^2\delta_{n0}\delta_{p1} + \omega_2{}^2V_2{}^2\delta_{n2}(\delta_{p1}\cos^2 2\phi_i + \delta_{p2}\sin^2 2\phi_i)], \tag{15b}$$

$$\Sigma^3_{np} = \frac{16\pi G}{c^3} Ma^2\lambda_3\omega_2{}^2V_2{}^2 \sin^2\theta_i\delta_{n2}(\delta_{p1}\sin^2 2\phi_i + \delta_{p2}\cos^2 2\phi_i), \tag{15c}$$

$$\Sigma^4_{np} = \frac{\pi G}{4c^3} Ma^2\lambda_4[4\omega_0{}^2V_0{}^2\sin^4\theta_i\delta_{n0}\delta_{p1} + \omega_2{}^2V_2{}^2(1+\cos^2\theta_i)^2\delta_{n2}(\delta_{p1}\cos^2 2\phi_i + \delta_{p2}\sin^2 2\phi_i)], \tag{15d}$$

$$\Sigma^5_{np} = \frac{\pi G}{c^3} Ma^2\lambda_5\omega_2{}^2V_2{}^2\cos^2\theta_i\delta_{n2}(\delta_{p1}\sin^2 2\phi_i + \delta_{p2}\cos^2 2\phi_i), \tag{15e}$$

$$\Sigma^6_{np} = \frac{\pi G}{4c^3} Ma^2\lambda_6[4\omega_0{}^2V_0{}^2(1+\cos^2\theta_i)^2\delta_{n0}\delta_{p1} + \omega_2{}^2V_2{}^2\sin^4\theta_i\delta_{n2}(\delta_{p1}\cos^2 2\phi_i + \delta_{p2}\sin^2 2\phi_i)]. \tag{15f}$$

While Eqs. (15) show how a wave with particular polarization interacts with the disk, it is clear that the various modes of polarization will in general interfere with one another to excite each antenna mode. Decoding the responses of the modes of several antennas to determine the amplitudes of various polarization modes of the incoming wave and its direction of propagation will be a nontrivial task. We will consider this problem in Sec. IV under some simplifying assumptions.

If one measures only the total energy deposited into the nth mode without regard to its polarization, the resonance integrals must be summed over p so that the ϕ_i dependence drops out. In this case the resonance integral for an unpolarized tensor wave[9] becomes

$$\begin{aligned}\Sigma^T_{n,\text{unpolarized}} &= \tfrac{1}{2}(\Sigma^4_n + \Sigma^5_n) \\ &= \delta_{n0}\frac{\pi G}{2c^3} Ma^2\omega_0{}^2V_0{}^2\sin^4\theta_i + \delta_{n2}\frac{\pi G}{2c^3} Ma^2\omega_2{}^2V_2{}^2\left[\left(\frac{1+\cos^2\theta_i}{2}\right)^2 + \cos^2\theta_i\right],\end{aligned} \tag{16}$$

where we have used $\lambda_4 = \lambda_5 = 1$. This is different from the angular dependence of the cross section calculated for a cylindrical antenna in Ref. 3. The nonvanishing interference term proportional to $\sin 4\phi_i$ appears in Eq. (7c) of Ref. 3 because the summation over p was done *before* squaring the reduced quadrupole factors. However, it is impossible for an *axially symmetric* detector to have an angular-dependent cross section.[10] The orthogonality of eigenfunctions for different p states would prevent one from adding their amplitudes before squaring them.

III. EVALUATION OF CROSS SECTIONS

In this section we will concern ourselves with cross sections averaged over all directions. Taking an average of the angular factors in Eq. (16), we find

$$\langle \Sigma^T_n \rangle = \frac{G}{c^3} Mv^2(\delta_{n0}S^T_0 + \delta_{n2}S^T_2), \tag{17}$$

where S^T_0 and S^T_2 are the reduced cross sections defined by

$$S^T_0 = \frac{4\pi}{15}\frac{(ka)^2}{1-\sigma^2}V_0{}^2, \tag{18a}$$

$$S_2^T = \frac{2\pi}{5} \frac{(ka)^2}{1-\sigma^2} V_2^{\,2} . \tag{18b}$$

Similarly, one can write down average cross sections for scalar waves:

$$\langle \Sigma_n^S \rangle = (2\omega+3)^{-1} \frac{G}{c^3} Mv^2 (\delta_{n0} S_0^S + \delta_{n2} S_2^S) , \tag{19}$$

where

$$S_0^S = 7 S_0^T , \quad S_2^S = \tfrac{1}{3} S_2^T . \tag{20}$$

The same can be done for the remaining three modes of polarization of the Riemann tensor.

The volume integrals of eigenfunctions, V_0 and V_2, can be evaluated analytically using differential properties of the Bessel functions:

$$V_0 = kA \int_0^1 \xi^2 d\xi [-J_1(ka\xi)] = -\frac{A}{a} J_2(ka) , \tag{21a}$$

$$V_2 = kA \int_0^1 \xi^2 d\xi [J_1(ka\xi) + (B/A)(k'/k) J_1(k'a\xi)] = \frac{A}{a} [J_2(ka) + (B/A) J_2(k'a)] . \tag{21b}$$

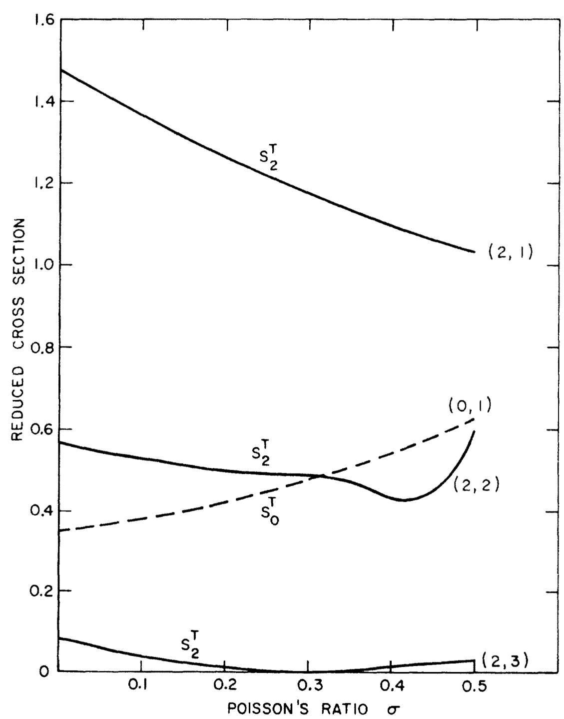

FIG. 2. Reduced resonant cross sections of a disk antenna as functions of Poisson's ratio σ. The numbers in parentheses correspond to the multipole and harmonic numbers of the antenna modes, respectively.

In the Appendix we have calculated the eigenvalues ka and the corresponding values of B/A and A/a. Using these and Eqs. (21), one can obtain numerical values for S_0^T and S_2^T.

The result is summarized in Fig. 2. The two numbers in the parentheses correspond to the multipole and harmonic numbers of the modes, respectively. The same notation is used in Figs. 3 through 5. The fundamental quadrupole mode (2, 1) has the biggest cross section for tensor waves as might be expected. Its second harmonic[11] (2, 2) and the lowest monopole mode (0, 1) are about half as efficient, and the third quadrupole harmonic (2, 3) is almost entirely inefficient in absorbing energy from gravitational waves. The resonant cross section for scalar waves is biggest for the (0, 1) mode as expected; the (2, 1) mode has about $\frac{1}{10}$ of this cross section.

IV. CONCLUSION

The reduced cross sections for tensor waves calculated in Sec. III are to be compared with

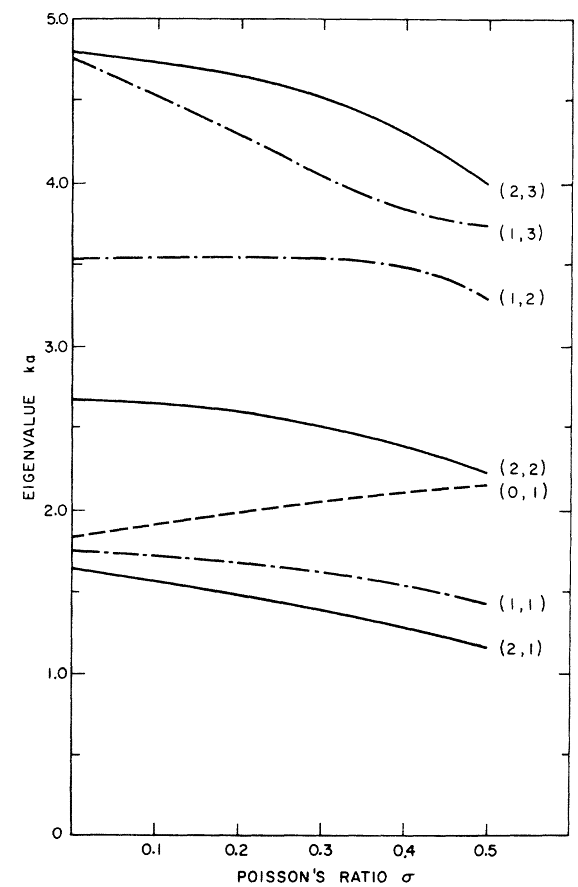

FIG. 3. Eigenvalue ka as a function of Poisson's ratio σ for various extensional modes of a disk. Eigenfrequencies can be obtained from this using Eq. (9a) in the text.

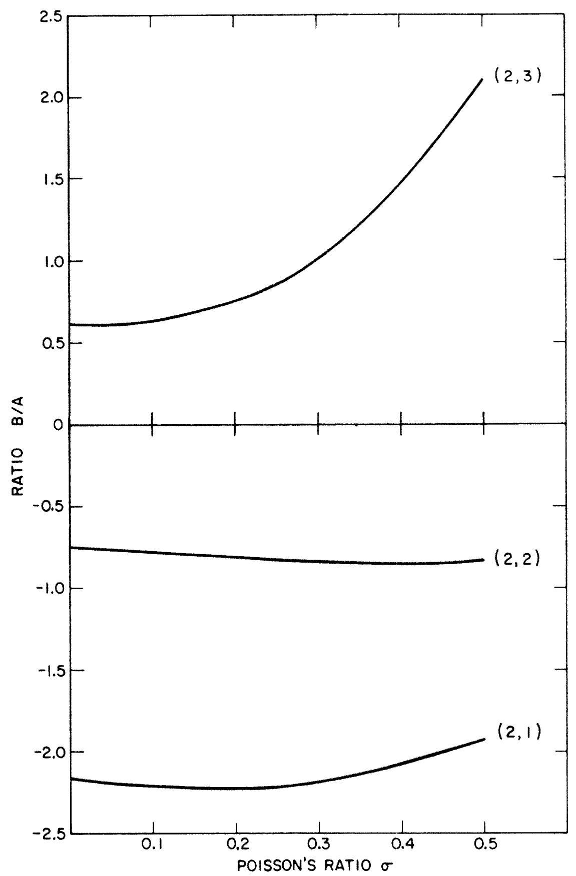

FIG. 4. The ratio B/A of eigenfunctions as a function of Poisson's ratio for the lowest three quadrupole modes and the lowest monopole mode of a disk.

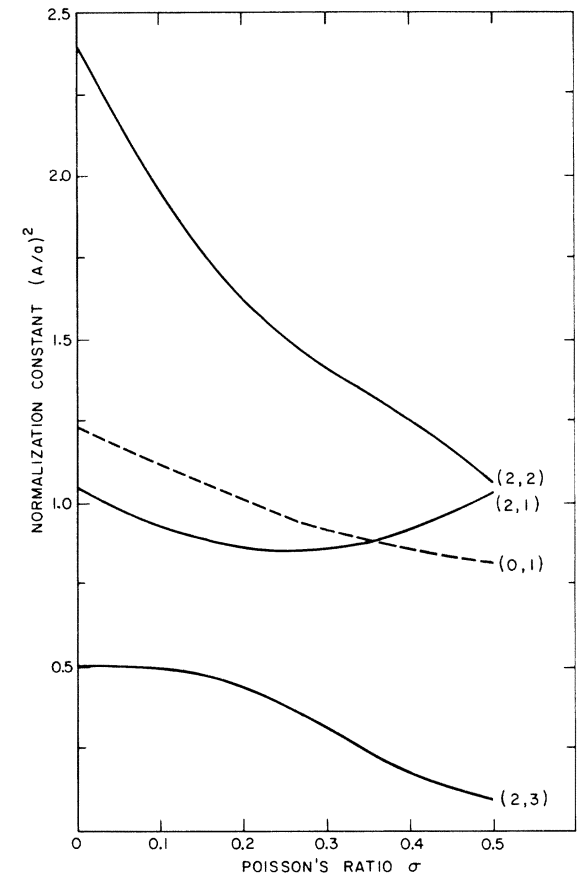

FIG. 5. Normalization constants $(A/a)^2$ of eigenfunctions as a function of Poisson's ratio for some extensional modes of a disk.

$32(15\pi)^{-1}n^{-2}$, the value for the nth axisymmetric longitudinal mode $(0, n)$ of a cylinder.[12] For antennas with $\sigma = \frac{1}{3}$ and the same Mv^2, the absorption cross section of the (2, 1) mode of a disk is almost twice as big as that of the (0, 1) mode of a cylinder. Their second harmonics have a cross-section ratio of almost 3 in favor of a disk antenna. This enhancement in cross section for a disk is partly due to the fact that its extensional modes are sensitive to both polarizations of an incoming wave, whereas a cylindrical antenna in axisymmetric modes responds to only one polarization.

The angular dependence of a disk for gravitational waves with various polarizations, as shown in Eqs. (15), can be utilized to determine the *exact polarization* of an incoming wave. The wave-induced driving force for each normal mode can be written[4,6] as

$$R_{np}(t) = -\frac{c^2}{2M}\sum_{A=1}^{6} P_A(\vec{k}, t) I_{np}^{jk} e_{jk}^{A}(\vec{k}) . \qquad (22)$$

Let us assume that only gravitational waves with definite Lorentz-invariant helicities are allowed in nature. Then only the two polarization states of helicity 2 (tensor wave: P_4, P_5) and one helicity-1 state (scalar wave: P_6) remain in Eq. (22). Using Eqs. (12) and the computed expressions for $e_{jk}^{A}(\vec{k})$, one obtains

$$R_{01} = \tfrac{1}{2} c^2 a V_0[-P_4 \sin^2\theta_i + P_6(1 + \cos^2\theta_i)] , \qquad (23a)$$

$$R_{21} = \tfrac{1}{4} c^2 a V_2\{[P_4(1+ \cos^2\theta_i) - P_6 \sin^2\theta_i]\cos 2\phi_i + 2P_5 \cos\theta_i \sin 2\phi_i\} , \qquad (23b)$$

$$R_{22} = \tfrac{1}{4} c^2 a V_2\{[P_4(1 + \cos^2\theta_i) - P_6 \sin^2\theta_i]\sin 2\phi_i - 2P_5 \cos\theta_i \cos 2\phi_i\} . \qquad (23c)$$

These three forces can be measured with three accelerometers located 45° apart on the periphery of the disk. By using a wide enough detection bandwidth and mixing the three signals with appropriate weighting factors, one should be able to read out the three driving forces as functions of time. When the wave propagation vector $\vec{k}$ is known, Eqs. (23) can then be solved for $P_4(t)$, $P_5(t)$, and $P_6(t)$, respectively.

Another interesting possibility is determining the incident angles of a wave using two *local* detectors of disk shape. Let us locate another disk near the first shown in Fig. 1, and rotate the second disk 90° around the x^1 axis so that its axis of symmetry lies along the x^2 axis. Instrumenting the second detector in the same way as the first, one will obtain $R'_{01}(t)$, $R'_{21}(t)$, and $R'_{22}(t)$, which are also given by Eqs. (23) with (θ_i, ϕ_i) replaced by (θ'_i, ϕ'_i). These new incident angles are related to the first by

$$\cos\theta'_i = \sin\theta_i \sin\phi_i , \tag{24a}$$

$$\tan\phi'_i = \tan\theta_i \cos\phi_i . \tag{24b}$$

One now has six independent equations (three force equations for each detector) to determine five unknowns (θ_i, ϕ_i, P_4, P_5, and P_6). Therefore, one can use the first five of them to determine the five unknowns,[13] and the consistency required by the remaining equation can be used to eliminate spurious signals which are not of gravitational origin. If scalar waves do not exist, one will have two extra equations that should be satisfied simultaneously. This is a strong test for the identity of the received signals because it will be very improbable for a seismic or any other disturbance to have the same quadrupole signature as gravitational waves. Thus one does not have to rely on a coincidence experiment between several widely separated detectors to obtain the source location and eliminate nongravitational disturbances. However, for such a local experiment to be possible, one will need a receiver with lower noise than in a conventional multiple-detector coincidence experiment, inasmuch as one has to use a wide bandwidth to determine the pulse shape as a function of time.

The requirement that one should obtain an entire frequency spectrum of a pulse arises because the resonant frequencies ω_0 and ω_2 are different so that $P_A(\omega_0) \neq P_A(\omega_2)$ in general. This will not be necessary if all the normal modes of the antenna under observation have sufficiently close resonant frequencies. Such a system is realized in spherical geometry. From the familiar property of spherical harmonics, one obtains five degenerate states for each quadrupole mode of a sphere. Consequently, with a *single* spherical antenna properly instrumented, one should be able to determine not only the location of the source but also the signature of the pulse. More details on a spherical gravitational wave detector will be discussed in a separate paper.[14]

ACKNOWLEDGMENTS

I wish to acknowledge useful discussions with Robert Wagoner and Steve Boughn. I would also like to thank Dave Douglass for pointing out a numerical error in my original note.

APPENDIX: EIGENFUNCTIONS FOR EXTENSIONAL MODES OF A DISK

Here we calculate the eigenfrequencies and eigenfunctions of the monopole and quadrupole extensional modes of a disk as a function of Poisson's ratio σ.

The frequency equations for multipole modes are obtained by eliminating the constants A and B from Eqs. (109) of Ref. 8. Numerical solutions of these equations are plotted in Fig. 3 as functions of σ for the first few harmonics of monopole, dipole, and quadrupole modes. This result is substituted back into Eqs. (109) of Ref. 8 to obtain the ratios B/A. Figure 4 is the plot of B/A for quadrupole modes as functions of σ. For monopole radial modes the terms proportional to B in the eigenfunctions drop out identically as can be seen from Eq. (8a).

We now determine A from the normalization condition, Eq. (4). For $n=0$, this becomes

$$\begin{aligned} 1 &= 2\int_0^1 \xi d\xi [v_0^r(a\xi)]^2 \\ &= 2\left(\frac{A}{a}\right)^2 \int_0^{ka} \alpha\, d\alpha\, J_1{}^2(\alpha) . \end{aligned} \tag{A1}$$

The integration can be done analytically so that

$$\left(\frac{A}{a}\right)^2 = \left[4 \sum_{l=0}^{\infty} (2+2l) J_{2+2l}{}^2(ka) \right]^{-1} . \tag{A2}$$

Likewise, for $n=2$, Eq. (4) becomes

$$\begin{aligned} 1 &= \frac{1}{2}\int_0^1 \xi d\xi \{[v_2^r(a\xi) - v_2^\theta(a\xi)]^2 + [v_2^r(a\xi) + v_2^\theta(a\xi)]^2\} \\ &= \frac{1}{2}\left(\frac{A}{a}\right)^2 \int_0^{ka} \alpha\, d\alpha \left\{ \left[J_1(\alpha) + \frac{B}{A}\frac{k'}{k} J_1\left(\frac{k'}{k}\alpha\right)\right]^2 \right. \\ &\qquad \left. + \left[-J_3(\alpha) + \frac{B}{A}\frac{k'}{k} J_3\left(\frac{k'}{k}\alpha\right)\right]^2 \right\} . \end{aligned} \tag{A3}$$

Upon integration and some manipulation, one obtains

$$\begin{aligned} \left(\frac{A}{a}\right)^2 = \Bigg\{ & 2\left[J_2(ka) + \frac{B}{A} J_2(k'a)\right]^2 \\ & + 2\sum_{l=0}^{\infty} (4+2l)\left[J_{4+2l}{}^2(ka) + \left(\frac{B}{A}\right)^2 J_{4+2l}(k'a)\right] \Bigg\}^{-1} . \end{aligned} \tag{A4}$$

Combination of Eqs. (A2) and (A4) with Figs. 3 and 4 yields Fig. 5 for the normalization constant $(A/a)^2$. In this calculation we have ignored Bessel functions of order higher than or equal to 8. We show in Fig. 5 only the lowest four extensional modes of oscillation that couple to gravitational waves.

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[1]J. Weber, Nuovo Cimento 4, 197 (1971).

[2]D. H. Douglass (private communication).

[3]R. G. Hier and S. N. Rasband, Astrophys. J. 195, 507 (1975).

[4]C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973), Chap. 37. The abbreviated expression "(absorption) cross section" refers to this quantity in this paper unless specified otherwise. Further, it is assumed that the antenna has zero background oscillation at the time the wave arrives.

[5]D. M. Eardley, D. L. Lee, A. P. Lightman, R. V. Wagoner, and C. M. Will, Phys. Rev. Lett. 30, 884 (1973).

[6]D. M. Eardley, D. L. Lee, and A. P. Lightman, Phys. Rev. D 8, 3308 (1973).

[7]It is to be noted that the polarization matrices e^A_{jk} defined in Ref. 6 are smaller in magnitude by a factor of 2 than e_{jk} defined in Ref. 4 for tensor waves. Note also that one should not take the transverse-traceless part of I_n^{jk} here.

[8]A. E. Love, *A Treatise on the Mathematical Theory of Elasticity*, 4th ed. (Dover, New York, 1927), Chap. XXII.

[9]By an "unpolarized wave," we mean a wave which belongs to an *ensemble* of random polarization. Each pulse of short duration will of course have a well-defined polarization.

[10]One could obtain information on ϕ_i by *proper instrumentation* which distinguishes the two p states of a quadrupole mode. In this case the ϕ_i dependence of the absorbed energy by each p states is given by Eq. (15) of this paper.

[11]The dip in the curve was unexpected. However, it could not be attributed to any approximation in the calculation.

[12]R. Ruffini and J. A. Wheeler, in *Relativistic Cosmology and Space Platforms*, edited by V. Hardy and H. Moore (ESRO, Paris, 1971), Chap. 6.

[13]The incident wave direction is determined within its sense of propagation because Eqs. (23) and (24) are invariant under the interchange $(\theta_i, \phi_i) \rightarrow (\pi - \theta_i, \pi + \phi_i)$ and $(\theta_i', \phi_i') \rightarrow (\pi - \theta_i', \pi + \phi_i')$.

[14]R. V. Wagoner and H. J. Paik, in Proceedings of the Accademia Nazionale die Lindei International Symposium on Experimental Gravitation, Pavia, Italy, 1976 (unpublished).