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Charged particles in Einstein's unified field theory

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The structure of charged particles in Einstein's unified field theory—the theory of the nonsymmetric field—is analyzed. A charged particle is represented through a time-independent spherically symmetric solution to the field equations. Using this idealization, it is shown that the structure of a charged particle is completely determined by the field equations, the condition that the symmetric part of the field be flat at infinity, and the requirement that the particle interact with other charged particles over laboratory distances via the conventional electromagnetic interaction. The structure of a charged particle is described in detail, and the limitations of the idealization of spherical symmetry are discussed. The nonconventional electromagnetic interaction over astronomical distances and its possible empirical consequences are examined.

I. INTRODUCTION

Einstein regarded the completion of his gravitational theory in 1915 as only the first step in the development of the general theory of relativity. For over thirty years afterwards Einstein worked on the problem of extending the theory of relativity so that it would encompass not only gravitation but all physical phenomena in a natural way. Einstein regarded the usual practice of introducing a phenomenological energy-momentum tensor into general relativity as only a provisional treatment of this problem. During the last years of his life Einstein believed he had at last found the natural generalization of his gravitational theory and thus the extension of the general theory of relativity he was looking for. He called this theory the relativistic theory of the nonsymmetric field.[1] It is a unified field theory in that it attempts to describe all of nature through one continuous field. Einstein believed it to be the logically simplest relativistic field theory at all possible. It is the theory we shall be investigating.

During the first fifteen years after the theory took its final form, almost no progress was made in extracting physical consequences from the theory beyond those which could be extracted from the pure gravitational theory. In fact, what little progress was made was widely interpreted as indicating that the theory was wrong (the theory appeared to be incompatible with the observed fact that classical electrodynamics is approximately valid over a range of interaction distances), and interest in the theory waned.[2] During the last several years, however, the situation has changed. Mathematical techniques have been developed so that we now know how to extract physical consequences from the theory in a systematic way and we also know that the theory is compatible with classical electrodynamics over the appropriate range of interaction distances.[3,4]

In Einstein's theory as in nature it is found that over macroscopic interaction distances (laboratory and astronomical distances) only two types of interactions between particles are important: gravitational and electromagnetic. The gravitational interaction between particles over this range of distances is weak and is given, to a good approximation, by the conventional classical gravitational interaction. What is of more interest to us is the electromagnetic interaction between charged particles. It is found in Einstein's theory that only over laboratory and relatively small astronomical distances do charged particles interact to a good approximation through the conventional classical electromagnetic interaction. Over larger astronomical distances a long-range nonconventional electromagnetic interaction between particles becomes important, so that Einstein's theory predicts significant deviations from classical electrodynamics over such distances. A discussion of possible tests of Einstein's theory, based on this fact, can be found in the literature.[5]

At the present time little is known concerning the interaction of particles over microscopic distances in Einstein's theory. Although techniques exist for investigating such interactions, the techniques involve a great deal of labor and have not yet been fully applied to the problem.

In previous work on the physical consequences of Einstein's unified field theory, the emphasis was on investigating the interaction among charged particles. In this paper we wish to begin an investigation of the structure of charged particles in Einstein's theory. In order to do this we will at first make certain simplifications or idealizations. In this paper we shall assume that an isolated charged particle can be represented by a time-independent spherically symmetric solution to Einstein's field equations. This is clearly an idealization, as no elementary particle found in nature is both stable and spherically symmetric. All stable elementary particles possess spin.

The physical meaning and justification of the above idealization is discussed in Sec. II of the paper. In Sec. III, making use of this idealization, we show that the structure of a charged particle is completely determined by Einstein's field equations, the condition that the symmetric part of the field be flat at infinity, and the requirement that the particle interact with other charged particles over laboratory distances through the conventional classical electromagnetic interaction. In Sec. IV we discuss in some detail the structure of these charged particles in Einstein's theory, and in Sec. V we briefly describe the electrodynamics of the particles.

II. SPACE-TIME CONTINUUM

A. Field equations

In Einstein's theory of the nonsymmetric field, nature is regarded as a four-dimensional space-time continuum whose structure is described through a second-rank tensor field $g_{\mu\nu}$. The fundamental field $g_{\mu\nu}$ satisfies the general-relativistic field equations[6]

$$\Gamma^{\nu}_{[\mu\nu]}=0\,, \tag{1a}$$

$$R_{[\mu\nu,\lambda]}=0\,, \tag{1b}$$

$$R_{(\mu\nu)}=0\,, \tag{1c}$$

where the displacement field $\Gamma^{\rho}_{\mu\nu}$ and the contracted curvature tensor $R_{\mu\nu}$ are defined through the equations

$$g_{\mu\nu;\rho}(=g_{\mu\nu,\rho}-g_{\sigma\nu}\Gamma^{\sigma}_{\mu\rho}-g_{\mu\sigma}\Gamma^{\sigma}_{\rho\nu})=0, \tag{2}$$

$$R_{\mu\nu}=\Gamma^{\rho}_{\mu\nu,\rho}-\Gamma^{\rho}_{\mu\rho,\nu}-\Gamma^{\rho}_{\mu\sigma}\Gamma^{\sigma}_{\rho\nu}+\Gamma^{\rho}_{\mu\nu}\Gamma^{\sigma}_{\rho\sigma}\,. \tag{3}$$

B. Particles and physical fields

A region of the continuum is called flat if a coordinate system can be found in the region so that the fundamental tensor field is equal to the Minkowski tensor throughout the region, that is,

$$g_{\mu\nu}=\eta_{\mu\nu}\,.$$

Particles are limited portions of the continuum—limited at least in the spatial directions—which have a very nonflat structure. Portions of the continuum between the particles and possessing a nearly flat structure are known as empty space or vacuum. The slight deviations from flatness in such portions of space-time will be taken to indicate the presence of an electromagnetic field if $g_{[\mu\nu]}\neq 0$ and the presence of a gravitational field if $g_{(\mu\nu)}\neq\eta_{\mu\nu}$. Nearer the particles, where the deviations from flatness are larger, the field $g_{\mu\nu}$ may also be associated with weak and strong interactions.

In this paper we will represent an isolated charged particle through a time-independent spherically symmetric solution to Einstein's field equations. This is of course an idealization since no elementary particle in nature is both time independent and spherically symmetric. All stable particles possess spin. However, spherical symmetry should be a good approximation as long as we are not concerned with the structure of a particle too near its "center." The approximation is certainly adequate when describing the interaction of elementary particles over macroscopic distances.

One final point with respect to the structure of a charged particle: We shall assume with Einstein that only regular (nonsingular) solutions to the field equations are realized in nature. This means that the time-independent spherically symmetric solutions we choose to represent particles will be assumed to approximate regular (nonsingular) solutions to the field equations. This of course does not mean that the time-independent spherically symmetric solutions themselves will be regular. Near the center of a particle, where time independence and spherical symmetry cannot be considered a good approximation, the time-independent spherically symmetric solutions are expected to become singular. We shall find that they do in fact become singular.

III. SOLUTIONS TO THE FIELD EQUATIONS

A. Time-independent spherically symmetric solutions

Assuming spherical symmetry about the origin of coordinates, it can be shown that in polar coordinates $x^1=r$, $x^2=\theta$, and $x^3=\varphi$ the fundamental field $g_{\mu\nu}$ can be put into the form[7]

$$g_{\mu\nu}=\begin{bmatrix}-\alpha & 0 & 0 & w\\ 0 & -\beta & f\sin\theta & 0\\ 0 & -f\sin\theta & -\beta\sin^2\theta & 0\\ -w & 0 & 0 & \gamma\end{bmatrix}, \tag{4}$$

where α, β, γ, f, and w are functions of r and t. In addition, if the field $g_{\mu\nu}$ is to be regarded as approximating a regular (nonsingular) solution to Einstein's field equations we must choose $w=0$. Let us see why.

Making use of the definition (2) we arrive at the identity

$$\mathfrak{g}^{(\mu\nu)}\Gamma^{\rho}_{[\nu\rho]}=\mathfrak{g}^{[\mu\nu]}{}_{,\nu}\,, \tag{5}$$

where

$$\mathfrak{g}^{\mu\nu}=(-g)^{1/2}g^{\mu\nu} \tag{6}$$

and

$$g_{\mu\rho}g^{\nu\rho}=g_{\rho\mu}g^{\rho\nu}=\delta_\mu{}^\nu . \tag{7}$$

Thus Eqs. (1a) are equivalent to the equations

$$\mathfrak{g}^{[\mu\nu]}{}_{,\nu}=0 . \tag{8}$$

If we restrict ourselves to solutions to the field equations which are regular (nonsingular) everywhere, we see from (8) that $\mathfrak{g}^{[\mu\nu]}$ can be derived from a potential. That is,

$$\mathfrak{g}^{[\mu\nu]}=\epsilon^{\mu\nu\rho\sigma}\Phi_{\sigma,\rho} . \tag{9}$$

Over a region of space in which deviations from spherical symmetry are negligible

$$\Phi_1=\Phi_1(r,t), \quad \Phi_2=\Phi_3=0, \quad \Phi_4=\Phi_4(r,t) . \tag{10}$$

Placing (10) in (9), we see that over such a region all components of $\mathfrak{g}^{[\mu\nu]}$ vanish (or are negligible) except $\mathfrak{g}^{[23]}=-\mathfrak{g}^{[32]}$. But from (4) one finds for the components of $g^{[\mu\nu]}$ over such a region

$$\mathfrak{g}^{[23]}=-\mathfrak{g}^{[32]}=f\,\frac{(\alpha\gamma-w^2)^{1/2}}{(\beta^2+f^2)^{1/2}} ,$$
$$\mathfrak{g}^{[41]}=-\mathfrak{g}^{[14]}=w\,\frac{(\beta^2+f^2)^{1/2}}{(\alpha\gamma-w^2)^{1/2}}\sin\theta ; \tag{11}$$

all other components vanish. We conclude that if the field $g_{\mu\nu}$ in (4) is to approximate a regular (nonsingular) solution to Einstein's field equations we must choose $w=0$.

The general time-independent spherically symmetric solution to Einstein's field equations under the assumption $w=0$ was first found by Wyman.[8] He finds for α, β, γ, and f (see Ref. 9)

$$\alpha=\frac{(f^2+\beta^2)\gamma'^2}{4m_1{}^2\gamma} ,$$
$$f+i\beta=\frac{m_1{}^2(1+ih_1)e^\delta\operatorname{sech}^2[\tfrac{1}{2}(1+ih_1)^{1/2}\delta+a]}{c_1+i} , \tag{12}$$
$$\gamma=e^{-\delta} ,$$

where the integration constants m_1, c_1, and h_1 are Wyman's m, c, and h_1, and the variable δ is the negative of Wyman's x. The variable δ is an arbitrary function of r.

Since we are only interested in solutions which may represent particles, we shall restrict our study to those solutions found by Wyman for which the field $g_{(\mu\nu)}$ is flat at infinity, i.e., can take its Minkowski value $\eta_{\mu\nu}$ at infinity. For such solutions one can show that

$$\sinh^2 a=-1 ,$$

so that f and β are given by

$$f+i\beta=-\frac{m_1{}^2(1+ih_1)e^\delta}{c_1+i}\operatorname{csch}^2[\tfrac{1}{2}(1+ih_1)^{1/2}\delta] . \tag{13}$$

We place no *a priori* boundary condition on $g_{[\mu\nu]}$.

In investigating this Wyman solution we shall find it convenient to work in "standard" coordinates. Standard polar coordinates are defined as coordinates in which $g_{\mu\nu}$ takes the form (4) with $\beta=r^2$. Standard Cartesian coordinates are defined in terms of standard polar coordinates through the transformation

$$x^1=r\sin\theta\cos\varphi, \quad x^2=r\sin\theta\sin\varphi, \quad x^3=r\cos\theta .$$

In standard Cartesian coordinates Wyman's solution takes the form

$$g_{st}=-\delta_{st}-(\alpha-1)\frac{x^s x^t}{r^2}-\epsilon_{stk}v\frac{x^k}{r} ,$$
$$g_{44}=\gamma, \quad g_{4s}=0, \quad g_{s4}=0 , \tag{14}$$

where

$$\alpha=\frac{m_1{}^2(1+h_1{}^2)}{c_1{}^2+1}\frac{e^\delta}{(\cosh\xi-\cos\eta)^2}\left(\frac{d\delta}{dr}\right)^2 ,$$
$$\gamma=e^{-\delta} ,$$
$$v=-\frac{f}{r^2}=\frac{2m_1{}^2}{c_1{}^2+1}\frac{e^\delta}{(\cosh\xi-\cos\eta)^2}\times\frac{1}{r^2}[(c_1+h_1)(\cosh\xi\cos\eta-1)-(1-c_1h_1)\sinh\xi\sin\eta] , \tag{15}$$

and

$$\xi=\mu\delta, \quad \eta=\nu\delta ,$$
$$\mu=[\tfrac{1}{2}+\tfrac{1}{2}(1+h_1{}^2)^{1/2}]^{1/2}, \quad \nu=h_1[2+2(1+h_1{}^2)^{1/2}]^{-1/2} . \tag{16}$$

The variable δ in standard coordinates satisfies the equation

$$\frac{m_1{}^2}{c_1{}^2+1}\frac{2e^\delta}{(\cosh\xi-\cos\eta)^2}\times[(c_1+h_1)\sinh\xi\sin\eta+(1-c_1h_1)(\cosh\xi\cos\eta-1)]=r^2 . \tag{17}$$

At distances sufficiently far from the origin of coordinates the functions α, γ, and v can be expanded in a power series in r^{-1}. We find

$$\alpha=1+\frac{2m_1}{(c_1{}^2+1)^{1/2}}\frac{1}{r}+O(r^{-2}) ,$$
$$\gamma=1-\frac{2m_1}{(c_1{}^2+1)^{1/2}}\frac{1}{r}+O(r^{-3}) , \tag{18}$$
$$v=c_1-\tfrac{1}{3}m_1{}^2h_1\frac{1}{r^2}+O(r^{-3}) .$$

The above results suggest we replace the constants m_1, c_1, and h_1 characterizing the Wyman solution by the constants m, q, and l defined in the following way:

$$m_1 = \left(1 + \frac{1}{4}\frac{q^2}{l^2}\right)^{1/2} m\ ,$$
$$c_1 = -\frac{1}{2}\epsilon\frac{q}{l}\ , \tag{19}$$
$$h_1 = -\frac{3lq}{m^2}\left(1 + \frac{1}{4}\frac{q^2}{l^2}\right)^{-1}\ ,$$

where ϵ can take the value 1 or -1. There is no loss of generality in assuming $l \geq 0$. In terms of m, q, and l we find

$$\alpha = 1 + \frac{2m}{r} + O(r^{-2})\ ,$$
$$\gamma = 1 - \frac{2m}{r} + O(r^{-3})\ , \tag{20}$$
$$v = l\left(\frac{q}{r^2} - \frac{1}{2}\epsilon\frac{q}{l^2}\right) + O(r^{-3})\ .$$

We have defined m and q so that at large distances from the origin of coordinates, $g_{(\mu\nu)} - \eta_{\mu\nu}$ is proportional to m, and $g_{[\mu\nu]}$ is proportional to q. Thus the form of the fundamental field at large distances from a particle suggests that we identify m with the mass of the particle and q with its charge. It also suggests that the length l associated with each particle is universal, that is, the same for each particle. Finally, it suggests that the length l is an astronomical length, for we know that over laboratory distances the electric field produced by a charge falls off with distance as r^{-2}.

However, only by investigating the interaction among particles represented by the Wyman solutions can we interpret the solutions physically and properly relate the arbitrary constants appearing in the solutions to the mass and charge of a particle. When this is done we will find that the above suggestions are correct.

B. Equations of motion

Approximation procedure. In order to investigate the interaction among particles in a nonlinear field theory one must in general use an approximation procedure to solve the field equations. In this paper we shall use a fast-motion approximation procedure developed by one of the authors in a previous series of papers.[3] By a fast-motion approximation procedure we mean a procedure which does not assume slow variation of the field. The approximation procedure we shall be using is similar to the conventional slow-motion approximation procedure of Einstein, Infeld, and Hoffmann (the EIH procedure) in that one expands the field $g_{\mu\nu}$ in a power series in a parameter which measures the strength of the singularities associated with particles (the parameter will parameterize mass and charge), but the procedure differs from the EIH procedure in that one does not consider time variation to be necessarily small and thus does not choose the parameter to also order time variation. The procedure leads to Lorentz-covariant equations of motion at each order of approximation. For further discussion see the papers of Ref. 3.

In the following discussion of the approximation procedure, unless otherwise stated, all indices will be raised and lowered with the Minkowski metric $\eta_{\mu\nu} = \eta^{\mu\nu}$. The subscript (k) to the left of a field will indicate order. We will be using the notation $\Box^2\psi = \eta^{\mu\nu}\psi_{,\mu\nu}$.

If we assume the expansion

$$g_{\mu\nu} = \eta_{\mu\nu} + \sum \kappa^k{}_{(k)}g_{\mu\nu} \tag{21}$$

for the fundamental field $g_{\mu\nu}$ (κ is the expansion parameter) the field equations (1) can be put into the form[3]

$$\Box^2\gamma^*_{[\mu\nu]}{}^{,\nu} = s_\mu\ ,$$
$$\gamma^*_{[\mu\nu,\lambda]} = 0\ , \tag{22}$$
$$\Box^2\gamma_{(\mu\nu)} - \gamma_{(\mu\rho)}{}^{,\rho}{}_{,\nu} - \gamma_{(\nu\rho)}{}^{,\rho}{}_{,\mu} + \eta_{\mu\nu}\gamma_{(\rho\sigma)}{}^{,\rho\sigma} = t_{\mu\nu}\ ,$$

where

$$\gamma^*_{[\mu\nu]} = \tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}\mathfrak{g}^{[\rho\sigma]}\ ,$$
$$\gamma_{(\mu\nu)} = \eta_{\mu\rho}\eta_{\nu\sigma}\mathfrak{g}^{(\rho\sigma)} - \eta_{\mu\nu}\ ,$$
$$\gamma_{\mu\nu} = \eta_{\mu\rho}\eta_{\nu\sigma}\mathfrak{g}^{\rho\sigma} - \eta_{\mu\nu}\ , \tag{23}$$
$$\mathfrak{g}^{\mu\nu} = (-g)^{1/2}g^{\mu\nu}\ ,$$
$$g = \det g_{\mu\nu}\ ,$$
$$g_{\mu\rho}g^{\nu\rho} = \delta_\mu{}^\nu\ ,$$

and

$$s_\mu = -\tfrac{1}{3}\eta_{\mu\rho}\epsilon^{\rho\sigma\kappa\lambda}R^N_{[\kappa\lambda,\sigma]}\ , \tag{24}$$
$$t_{\mu\nu} = -2(R^N_{(\mu\nu)} - \tfrac{1}{2}\eta_{\mu\nu}\eta^{\rho\sigma}R^N_{(\rho\sigma)})\ . \tag{25}$$

The field $R^N_{\mu\nu}$ is that part of the tensor $R_{\mu\nu}$ which is nonlinear in $\gamma_{\mu\nu}$.

When investigating the physical consequences of Eqs. (22) it will be understood that we are investigating the relevant fields only at points which are sufficiently "distant" from the world lines of particles so that (21) is valid.

For convenience we will also impose the coordinate conditions

$$\gamma_{(\mu\nu)}{}^{,\nu} = 0 \tag{26}$$

on the field at each order of approximation. Coordinates for which (26) are valid are known as harmonic coordinates. It is important to note that we do not impose (26) on the exact solutions, only on the solutions up to the order of approximation in which we choose to investigate the fields. It can be shown that under these conditions the use of harmonic coordinates will not restrict the set of invariantly distinct solutions to the field equations (22).[10] In harmonic coordinates the field equations (22) take the form

$$\Box^2\gamma^*_{[\mu\nu]},{}^{\nu}=s_\mu\,, \tag{27a}$$

$$\gamma^*_{[\mu\nu,\lambda]}=0\,, \tag{27b}$$

$$\Box^2\gamma_{(\mu\nu)}=t_{\mu\nu}\,, \tag{27c}$$

$$\gamma_{(\mu\nu)},{}^{\nu}=0\,. \tag{27d}$$

Application of approximation procedure. To lowest order (first order) we have from (24) and (25)

$$s_\mu=0,\quad t_{\mu\nu}=0\,. \tag{28}$$

Because we do not want gravitational interaction to appear in the lowest-order interaction terms (second order), we will choose mass and therefore $\gamma_{(\mu\nu)}$ to be a second-order quantity. In this way we avoid having to investigate gravitational interaction in second order. Thus to lowest nontrivial order (second order) we find from (24) and (25)

$$s_\mu=0\,, \tag{29}$$

$$\begin{aligned}t_{\mu\nu}=&\tfrac{1}{2}\gamma_{[\rho\sigma],\mu}\gamma^{[\rho\sigma]}{}_{,\nu}+\gamma_{[\mu\rho]},{}^{\sigma}\gamma_{[\nu\sigma]},{}^{\rho}-\gamma_{[\mu\rho],\sigma}\gamma_{[\nu}{}^{\rho],\sigma}\\&-\tfrac{1}{4}\eta_{\mu\nu}\gamma_{[\rho\sigma],\kappa}\gamma^{[\rho\sigma],\kappa}-\tfrac{1}{2}\eta_{\mu\nu}\gamma_{[\rho\sigma],\kappa}\gamma^{[\rho\kappa],\sigma}\\&+\gamma^{[\rho\sigma]}\gamma_{[\mu\sigma],\nu\rho}+\gamma^{[\rho\sigma]}\gamma_{[\nu\sigma],\mu\rho}\\&-\tfrac{1}{2}\eta_{\mu\nu}\gamma^{[\rho\sigma]}\Box^2\gamma_{[\rho\sigma]}\,.\end{aligned} \tag{30}$$

In a harmonic coordinate system, and keeping only terms linear in m and q, one finds from the solution (14)–(17), representing an isolated particle,

$$\gamma_{[st]}=-l\epsilon_{stk}\left(\frac{q}{r^2}-\frac{1}{2}\epsilon\frac{q}{l^2}\right)\frac{x^k}{r},\quad \gamma_{[4s]}=0\,, \tag{31a}$$

$$\gamma_{(44)}=\frac{4m}{r},\quad \gamma_{(4s)}=0,\quad \gamma_{(st)}=0\,. \tag{31b}$$

To investigate the electromagnetic interaction among these particles one must choose as the lowest-order solution to Eqs. (27)

$$\gamma^*_{[\mu\nu]}=\tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}\gamma^{[\rho\sigma]}=\gamma_{\mu,\nu}-\gamma_{\nu,\mu},\quad \gamma_{(\mu\nu)}=0\,, \tag{32}$$

where

$$\gamma_{[\mu\nu]}=\epsilon_{\mu\nu\rho\sigma}\gamma^{\sigma,\rho}\,,$$

$$\gamma_\mu=\sum_p {}^{(p)}l^{(p)}\left[qu_\mu(r_\rho u^\rho)^{-1}+\frac{1}{2}\epsilon\frac{q}{l^2}r_\mu\right]_{\text{ret}}. \tag{33}$$

We are using the notation

$${}^{(p)}r^\mu=x^\mu-{}^{(p)}\xi^\mu\,,$$

$${}^{(p)}u^\mu={}^{(p)}\dot{\xi}^\mu\,,$$

$$d^{(p)}\tau^2=\eta_{\mu\nu}d^{(p)}\xi^\mu d^{(p)}\xi^\nu\,.$$

A superscript (p) to the left of an expression means that the quantities in the expression which are associated with a particle are associated with the pth particle. A dot over a quantity associated with the pth particle means the derivative of that quantity with respect to ${}^{(p)}\tau$. The subscript ret means that in the expression in brackets those quantities associated with the pth particle are to be evaluated at the "retarded point"

$${}^{(p)}r_\rho{}^{(p)}r^\rho=0\,,\quad r^4>0\,.$$

The coordinates of the pth particle have been denoted by ${}^{(p)}\xi$. The quantities ${}^{(p)}q$ and ${}^{(p)}l$ are time independent. Let us investigate the interaction among such particles.

To find the equations of motion to second order satisfied by these particles one must study the solutions to Eqs. (27) to second order. The solutions to Eqs. (27a) and (27b) to second order are identical to the solutions to first order. No equations of motion are involved. In order to obtain equations of motion we must investigate Eqs. (27c) and (27d) to second order.

Placing the field (33) into (30) one can solve Eqs. (27c) to second order. From (31b) we see that the solution will take the form

$$\gamma_{(\mu\nu)}=\sum_p {}^{(p)}[4mu_\mu u_\nu(r_\rho u^\rho)^{-1}]_{\text{ret}}+\gamma^I_{(\mu\nu)}\,, \tag{34}$$

where

$$\Box^2\gamma^I_{(\mu\nu)}=t_{\mu\nu}\,. \tag{35}$$

From (34), (35), (30), and (33), one finds for $\gamma_{(\mu\nu)},{}^{\nu}$

$$\gamma_{(\mu\nu)},{}^{\nu}=\sum_p {}^{(p)}[C_\mu(r_\rho u^\rho)^{-1}]_{\text{ret}}\,, \tag{36}$$

where

$${}^{(p)}C_\mu={}^{(p)}\left[\frac{d}{d\tau}(4mu_\mu)-lq\Box^2\gamma^{*\text{ext}}_{[\mu\nu]}u^\nu+\epsilon\frac{q}{l}\gamma^{*\text{ext}}_{[\mu\nu]}u^\nu-\tfrac{4}{3}\epsilon q^2(\ddot{u}_\mu+\dot{u}_\rho\dot{u}^\rho u_\mu)\right], \tag{37}$$

with

$$^{(p)}\gamma^{*\,\mathrm{ext}}_{[\mu\nu]} = {}^{(p)}\gamma^{\mathrm{ext}}_{\mu,\nu} - {}^{(p)}\gamma^{\mathrm{ext}}_{\nu,\mu}\,, \tag{38}$$

$$^{(p)}\gamma^{\mathrm{ext}}_{\mu} = \sum_{p'\neq p} {}^{(p')}l^{(p')}\left[qu_\mu(r_\rho u^\rho)^{-1} + \tfrac{1}{2}\epsilon\frac{q}{l^2}r_\mu\right]_{\mathrm{ret}}. \tag{39}$$

The procedure used to find $\gamma_{(\mu\nu)}{}^{,\nu}$ is discussed in previous papers.[3] Since Eqs. (27d) must also be satisfied we must have

$$^{(p)}C_\mu = 0\,. \tag{40}$$

Particle motion is restricted by (40). The equations of motion satisfied by the particles to second order are

$$m\dot{u}_\mu = \tfrac{1}{4}lq\Box^2\gamma^{*\,\mathrm{ext}}_{[\mu\nu]}u^\nu - \tfrac{1}{4}\epsilon\frac{q}{l}\gamma^{*\,\mathrm{ext}}_{[\mu\nu]}u^\nu + \tfrac{1}{3}\epsilon q^2(\ddot{u}_\mu + \dot{u}_\rho\dot{u}^\rho u_\mu)\,. \tag{41}$$

These equations follow from (37) and (40). If we introduce the effective electromagnetic field produced by a particle $^{(p)}\tilde{\gamma}_{[\mu\nu]}$,

$$^{(p)}\tilde{\gamma}_{[\mu\nu]} = \tfrac{1}{2}\,{}^{(p)}(\gamma^*_{[\mu\nu]} - \epsilon l^2\Box^2\gamma^*_{[\mu\nu]})\,, \tag{42}$$

and the effective external field

$$^{(p)}\tilde{\gamma}^{\mathrm{ext}}_{[\mu\nu]} = \tfrac{1}{2}\,{}^{(p)}(\gamma^{*\,\mathrm{ext}}_{[\mu\nu]} - \epsilon l^2\Box^2\gamma^{*\,\mathrm{ext}}_{[\mu\nu]})\,, \tag{43}$$

equations of motion (41) reduce to

$$m\dot{u}_\mu = \frac{1}{2}\epsilon\frac{q}{l}\tilde{\gamma}^{\mathrm{ext}}_{[\nu\mu]}u^\nu + \frac{1}{3}\epsilon q^2(\ddot{u}_\mu + \dot{u}_\rho\dot{u}^\rho u_\mu)\,. \tag{44}$$

If we introduce the mass M, charge e, and effective electromagnetic field $F_{\mu\nu}$ in practical units,

$$m = \frac{GM}{c^2}\,, \tag{45a}$$

$$q = \left(\frac{G}{2\pi\epsilon_0 c^4}\right)^{1/2} e\,, \tag{45b}$$

$$\tilde{\gamma}_{[\mu\nu]} = \frac{(8\pi\epsilon_0 G)^{1/2}}{c}\,lF_{\mu\nu}\,, \tag{45c}$$

Eqs. (44) can be put into the form

$$M\dot{u}_\mu = \epsilon\frac{e}{c}F^{\mathrm{ext}}_{\nu\mu}u^\nu + \frac{2}{3}\epsilon\left(\frac{e^2}{4\pi\epsilon_0 c^2}\right)(\ddot{u}_\mu + \dot{u}_\rho\dot{u}^\rho u_\mu)\,, \tag{46}$$

where

$$^{(p)}F^{\mathrm{ext}}_{\mu\nu} = {}^{(p)}A^{\mathrm{ext}}_{\mu,\nu} - {}^{(p)}A^{\mathrm{ext}}_{\mu,\nu}\,,$$

$$^{(p)}A^{\mathrm{ext}}_\mu = \frac{1}{4\pi\epsilon_0}\sum_{p'\neq p}\frac{1}{2}\frac{^{(p')}l}{^{(p)}l}\left(1 + \frac{^{(p)}\epsilon\,^{(p)}l^2}{^{(p')}\epsilon\,^{(p')}l^2}\right) \times {}^{(p')}\left[\frac{e}{c}u_\mu(r_\rho u^\rho)^{-1}\right]_{\mathrm{ret}} + \frac{1}{4\pi\epsilon_0}\sum_{p'\neq p}\frac{^{(p')}l}{^{(p)}l}\,{}^{(p')}\left[\epsilon\frac{e}{c}\left(\frac{1}{4l^2}\right)r_\mu\right]_{\mathrm{ret}}. \tag{47}$$

We see that the particles can interact to a good approximation over laboratory distances through the conventional classical electromagnetic interaction if and only if $^{(p)}l$ is a universal astronomical length and $^{(p)}\epsilon = 1$. In this case (46) takes the form

$$M\dot{u}_\mu = \frac{e}{c}F^{\mathrm{ext}}_{\nu\mu}u^\nu + \frac{2}{3}\left(\frac{e^2}{4\pi\epsilon_0 c^2}\right)(\ddot{u}_\mu + \dot{u}_\rho\dot{u}^\rho u_\mu)\,, \tag{48}$$

and (47) takes the form

$$^{(p)}F^{\mathrm{ext}}_{\mu\nu} = {}^{(p)}A^{\mathrm{ext}}_{\mu,\nu} - {}^{(p)}A^{\mathrm{ext}}_{\nu,\mu}\,,$$

$$^{(p)}A^{\mathrm{ext}}_\mu = \frac{1}{4\pi\epsilon_0}\sum_{p'\neq p}{}^{(p')}\left[\frac{e}{c}u_\mu(r_\rho u^\rho)^{-1}\right]_{\mathrm{ret}} + \frac{1}{4\pi\epsilon_0}\sum_{p'\neq p}{}^{(p')}\left[\frac{e}{c}\left(\frac{1}{4l^2}\right)r_\mu\right]_{\mathrm{ret}}. \tag{49}$$

We must still answer the question of over what range of interaction distances we can expect the approximations used in arriving at (48) and (49) to be valid. This has been discussed in the literature, where it is shown that as long as l is a moderate astronomical length Eqs. (48) and (49) should be valid over both laboratory and astronomical distances.[11] We shall discuss in Secs. IV and V reasons for believing that l satisfies this criterion.

We see from the above results that our tentative identifications of m and q, based on the long-range form of the field associated with solution (14)–(17), were correct. The parameter m represents the mass of the particle and the parameter q represents the charge. We also find that the length l associated with the solution must be a universal (moderate) astronomical length, and ϵ must be equal to 1.

Finally, we note that according to (49), in Einstein's theory, in an approximation which should be valid over laboratory and astronomical distances, the effective electromagnetic field produced by an arbitrary moving charged particle consists of two parts: the conventional Maxwell field derived from the Liénard-Wiechert potentials, and a long-range nonconventional field given by

$$\vec{\mathrm{E}} = \frac{e}{4\pi\epsilon_0}\left(-\frac{1}{4l^2}\right)[(\vec{\mathrm{n}} - \vec{\beta})(1 - \vec{\mathrm{n}}\cdot\vec{\beta})^{-1}]_{\mathrm{ret}}\,,$$

$$\vec{\mathrm{B}} = \frac{\mu_0 e}{4\pi}\left(-\frac{1}{4l^2}\right)[(\vec{\mathrm{v}}\times\vec{\mathrm{n}})(1 - \vec{\mathrm{n}}\cdot\vec{\beta})^{-1}]_{\mathrm{ret}}\,, \tag{50}$$

where

$$\vec{\mathrm{n}} = \frac{\vec{\mathrm{r}}}{|\vec{\mathrm{r}}|}\,,\quad \vec{\mathrm{r}} = \vec{\mathrm{x}} - \vec{\xi}\,,\quad \vec{\mathrm{v}} = \frac{d\vec{\xi}}{dt}\,,\quad \vec{\beta} = \frac{\vec{\mathrm{v}}}{c}\,. \tag{51}$$

The vector $\vec{\xi}$ describes the position of the particle. Because l is an astronomical length, the long-range fields (50) are extremely weak and would be undetectable at the present time except perhaps in some astronomical phenomena. A brief discussion of the implications of these long-range fields will be given in Sec. V. A discussion can also be found in the literature.[5]

IV. PARTICLE STRUCTURE

A. General solution

We have shown that the most general time-independent spherically symmetric solution to Einstein's field equations which can represent a charged particle at rest[12] is characterized by three parameters: m, q, and l. The parameter m is the mass of the particle and the parameter q is the charge. The parameter l is a universal length—the same for each particle—bounded but not determined by existing experimental evidence. The evidence suggests

$$10^8 \text{ m} \lesssim l \lesssim 10^{12} \text{ m}, \tag{52}$$

and will be discussed later.

The solution $g_{\mu\nu}$ is given in standard coordinates by

$$\begin{aligned} g_{st} &= -\delta_{st} - (\alpha - 1)\frac{x^s x^t}{r^2} - \epsilon_{stk} v \frac{x^k}{r}, \\ g_{44} &= \gamma, \quad g_{4s} = 0, \quad g_{s4} = 0, \end{aligned} \tag{53}$$

where, introducing the definition

$$r_0 = (|q|\, l)^{1/2} = \left(\frac{Ge^2}{2\pi\epsilon_0 c^4}\right)^{1/4} l^{1/2}, \tag{54}$$

we can write

$$\begin{aligned} \alpha &= \frac{6e^\delta}{(\cosh\xi - \cos\eta)^2}\left[\left(1 + \frac{1}{4}\frac{q^2}{l^2}\right)^{-1} + \frac{1}{9}\left(1 + \frac{1}{4}\frac{q^2}{l^2}\right)\left(\frac{m}{r_0}\right)^4\right]\left(r_0\frac{d\lambda}{dr}\right)^2, \\ \gamma &= e^{-\delta}, \\ v &= \frac{q}{|q|}\frac{6e^\delta}{(\cosh\xi - \cos\eta)^2}\left\{\left[\left(1 + \frac{1}{4}\frac{q^2}{l^2}\right)^{-1} + \frac{1}{6}\frac{|q|}{l}\left(\frac{m}{r_0}\right)^2\right](1 - \cosh\xi\cos\eta) \right. \\ &\qquad \left. - \left[\frac{1}{2}\frac{|q|}{l}\left(1 + \frac{1}{4}\frac{q^2}{l^2}\right)^{-1} - \frac{1}{3}\left(\frac{m}{r_0}\right)^2\right]\sinh\xi\sin\eta\right\}\left(\frac{r_0}{r}\right)^2, \end{aligned} \tag{55}$$

with

$$\begin{aligned} \delta &= \left(\frac{2}{3}\right)^{1/2}\left(1 + \frac{1}{4}\frac{q^2}{l^2}\right)^{1/2}\left(\frac{m}{r_0}\right)\lambda, \\ \xi &= \left\{\left[1 + \frac{1}{9}\left(1 + \frac{1}{4}\frac{q^2}{l^2}\right)^2\left(\frac{m}{r_0}\right)^4\right]^{1/2} + \frac{1}{3}\left(1 + \frac{1}{4}\frac{q^2}{l^2}\right)\left(\frac{m}{r_0}\right)^2\right\}^{1/2}\lambda, \\ \eta &= \left\{\left[1 + \frac{1}{9}\left(1 + \frac{1}{4}\frac{q^2}{l^2}\right)^2\left(\frac{m}{r_0}\right)^4\right]^{1/2} + \frac{1}{3}\left(1 + \frac{1}{4}\frac{q^2}{l^2}\right)\left(\frac{m}{r_0}\right)^2\right\}^{-1/2}\lambda. \end{aligned} \tag{56}$$

The parameter λ in (56) is a function of r defined through the equation

$$\frac{6e^\delta}{(\cosh\xi - \cos\eta)^2}\left\{\left[\left(1 + \frac{1}{4}\frac{q^2}{l^2}\right)^{-1} + \frac{1}{6}\frac{|q|}{l}\left(\frac{m}{r_0}\right)^2\right]\sinh\xi\sin\eta - \left[\frac{1}{2}\frac{|q|}{l}\left(1 + \frac{1}{4}\frac{q^2}{l^2}\right)^{-1} - \frac{1}{3}\left(\frac{m}{r_0}\right)^2\right](\cosh\xi\cos\eta - 1)\right\} = \left(\frac{r}{r_0}\right)^2. \tag{57}$$

The solution is a special case of a more general time-independent spherically symmetric solution to Einstein's field equations first found by Wyman. For $q = 0$ the solution reduces to the well-known Schwarzschild solution of Einstein's gravitational equations.

B. Field at large distances

If at distances sufficiently far from a charged particle we expand the field $g_{\mu\nu}$ in a power series in r^{-1} we find in standard coordinates, neglecting terms of third or higher order in r^{-1},

$$\begin{aligned} \alpha &= 1 + \frac{2m}{r} + \frac{4m^2}{r^2} + \frac{1}{2}\frac{q^2}{1 + \frac{1}{4}(q^2/l^2)}\frac{1}{r^2}, \\ \gamma &= 1 - \frac{2m}{r}, \\ v &= l\left(\frac{q}{r^2} - \frac{1}{2}\frac{q}{l^2}\right). \end{aligned} \tag{58}$$

Two properties of solution (53)–(57), and of Einstein's theory, are evident from the above asymptotic expansion.

First, if we compare the above results to those obtained from the Reissner-Nordström solution of the Einstein-Maxwell equations, where to the same order in r^{-1} one finds

$$\alpha = 1 + \frac{2m}{r} + \frac{4m^2}{r^2} - \frac{1}{2}\frac{q^2}{r^2}, \qquad \gamma = 1 - \frac{2m}{r} + \frac{1}{2}\frac{q^2}{r^2}, \tag{59}$$

we see that even asymptotically—at large distances from a particle—the effect of electric charge on the field $g_{(\mu\nu)}$ is significantly different in Einstein's theory and in the Einstein-Maxwell theory of electromagnetism and gravitation.

Second, we note that at infinity one finds from (53) and (58) for the field $g_{\mu\nu}$ surrounding an isolated time-independent spherically symmetric charged particle in Einstein's theory

$$g_{(\mu\nu)} = \eta_{\mu\nu}, \qquad g_{[st]} = \frac{1}{2}\frac{q}{l}\epsilon_{stk}\frac{x^k}{r}, \quad g_{[4s]} = 0. \tag{60}$$

We see that the symmetric part of the field becomes the Minkowski metric at infinity, while the antisymmetric part, although it does not actually vanish at infinity, becomes very weak. If the charge on a particle is the electron charge one finds from (45), (52), and (60) for the field $g_{[\mu\nu]}$ at infinity

$$|g_{[\mu\nu]}| \lesssim 10^{-44}. \tag{61}$$

That the symmetric part of the field $g_{\mu\nu}$ takes its flat-space values at large distances from an isolated particle is of course consistent with experience. That is why we imposed it on the solution in the first place. That the antisymmetric part of the field becomes very weak at large distances from a particle is also consistent with experience. The antisymmetric part of the field $g_{\mu\nu}$ is related to the electric field, a field which is known observationally to become very weak at large distances from an isolated particle. However, whether the electric field surrounding an isolated charged particle at rest actually approaches the limiting value implied by (60) is not known. If the particle has a charge equal to the electron charge and l lies in the range (52), this limiting field is of the order of 10^{-26} V/m or less. A discussion of the physical consequences of the long-range electromagnetic field associated with charged particles in Einstein's theory can be found in the literature,[5] and in Sec. V of this paper.

C. General structure

We are examining the structure of solution (53)–(57) viewed as describing a model of an electrically charged elementary particle. As discussed earlier, this model will be adequate only to the extent that one can ignore deviations from spherical symmetry and any time dependence which might be associated with the elementary particle.

The expression for this field is quite complicated in general, but turns out to be manageable when values of m and q appropriate to an elementary particle are inserted in (53)–(57). For an electron one has

$$m = \frac{GM}{c^2} = 6.8\times 10^{-58}\ \text{m}, \qquad |q| = \left(\frac{e^2 G}{2\pi\epsilon_0 c^4}\right)^{1/2} = 2.0\times 10^{-36}\ \text{m}. \tag{62}$$

These values permit an expansion of the field with high accuracy in powers of the parameters m/r_0 and $|q|/l$. For an electron we find, assuming that l lies in the range (52),

$$\frac{m}{r_0} \lesssim 10^{-44}, \qquad \frac{|q|}{l} \lesssim 10^{-44}. \tag{63}$$

Although the above orders of magnitude are estimated for an electron, the expansion technique is clearly reasonable for any charged particle of microscopic mass.

To lowest order in m/r_0 and $|q|/l$ we find from (55)–(57)

$$\alpha = \frac{4(\cosh\lambda - \cos\lambda)^2 \sinh\lambda \sin\lambda}{[\sinh\lambda \sin\lambda(\sinh\lambda + \sin\lambda) + (1 - \cosh\lambda \cos\lambda)(\sinh\lambda - \sin\lambda)]^2}, \qquad \gamma = 1, \qquad v = \frac{q}{|q|}\frac{6(1-\cosh\lambda\cos\lambda)}{(\cosh\lambda - \cos\lambda)^2}\left(\frac{r_0}{r}\right)^2, \tag{64}$$

where the parameter λ is defined through the equation

$$\frac{6\sinh\lambda\sin\lambda}{(\cosh\lambda-\cos\lambda)^2}=\left(\frac{r}{r_0}\right)^2. \tag{65}$$

The parameter λ varies from π to 0 as r varies from 0 to ∞.

If one is attempting to model a charged particle of microscopic mass, Eqs. (64) and (65) are a very good approximation to (55)–(57) everywhere. They neglect only the contribution of the mass m to the field (a contribution which in this case is always small) and the contribution of the long-range electric field to the total field (a contribution which is also very small).

In Fig. 1 we have plotted $\alpha-1$, $\gamma-1$, and $|v|$ as functions of r. We have made use of (64) and (65) since we are interested in solution (53)–(57) as a model of a charged elementary particle. An examination of Fig. 1 shows that the field associated with such a particle changes its character in a qualitative way as one goes from the asymptotic region far from the particle into the region $r\lesssim r_0$. In this latter region the dependence of the field on r is seen to depart significantly from its form far from the particle. In addition to this change in form, the field in the region $r\lesssim r_0$ becomes large in an absolute sense; this is an indication that no linear approximation to the field equations can successfully describe the behavior of the field in this region. For the above reasons the distance r_0 is a measure of the size of a particle. The largest plausible particle size with which to associate r_0 would seem to be the electron Compton wavelength. This correlation provides an upper bound on l of

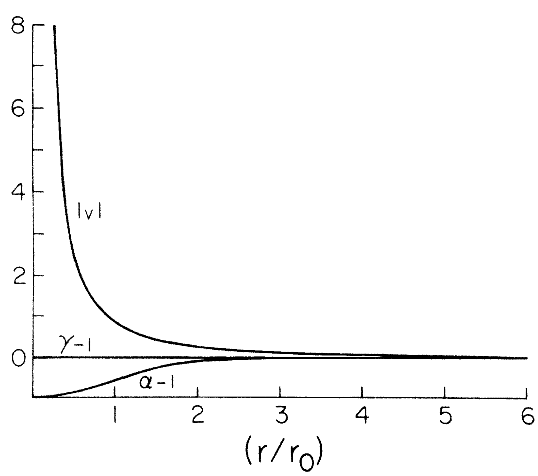

FIG. 1. The functions α-1, γ-1, and $|v|$, describing the fundamental field.

$$l\lesssim 10^{12}\ \mathrm{m}, \tag{66}$$

a few astronomical units, and implies a modification of Maxwell's equations for astronomical use if Einstein's theory is correct. Note that to the extent that the solution (53)–(57) can be regarded as describing an elementary charged particle, the structure and size of such a particle is, to a very good approximation, independent of its mass.

D. Field near the origin

For an examination of the field in the vicinity of the origin it is convenient to express the field as a power series in r/r_0. One finds from (55)–(57), in the vicinity of the origin,

$$\begin{aligned}\alpha&=\left[\frac{2}{3}\left(\frac{1+\cosh\pi}{\sinh\pi}\right)^2+\cdots\right]\left(\frac{r}{r_0}\right)^2+O\left(\left(\frac{r}{r_0}\right)^4\right),\\ \gamma&=\left[1-\left(\frac{2}{3}\right)^{1/2}\pi\left(\frac{m}{r_0}\right)+\cdots\right]\\ &\quad+\left[\frac{1}{6}\left(\frac{2}{3}\right)^{1/2}\frac{(1+\cosh\pi)^2}{\sinh\pi}\left(\frac{m}{r_0}\right)+\cdots\right]\left(\frac{r}{r_0}\right)^2\\ &\quad+O\left(\left(\frac{r}{r_0}\right)^4\right),\\ v&=\frac{q}{|q|}\left[\frac{6}{1+\cosh\pi}+\cdots\right]\left(\frac{r_0}{r}\right)^2+\frac{q}{|q|}[1+\cdots]\\ &\quad+O\left(\left(\frac{r}{r_0}\right)^2\right).\end{aligned} \tag{67}$$

In (67) we have written out only the lowest-order terms in the power-series expansion in m/r_0 and $|q|/l$ of the coefficients of $(r/r_0)^n$.

We see from (67) and Fig. 1 that the only singularity in the field in standard coordinates is at $r=0$.

The fact that the solution is singular at $r=0$ can be given a physical interpretation. This has been discussed in the Introduction.

E. Related fields

In Einstein's theory we are often interested in the fields $g^{\mu\nu}$ and $\mathfrak{g}^{\mu\nu}$ which are related to $g_{\mu\nu}$ through (6) and (7). Associated with the particles we are investigating we find for these fields, in standard coordinates,

$$g^{st}=\frac{1}{1+v^2}\left[-\delta_{st}-\left(\frac{1+v^2}{\alpha}-1\right)\frac{x^sx^t}{r^2}-\epsilon_{stk}v\frac{x^k}{r}\right], \tag{68}$$

$$g^{44}=\frac{1}{\gamma},\quad g^{4s}=0,\quad g^{s4}=0,$$

$$\mathfrak{g}^{st} = \left(\frac{\alpha\gamma}{1+v^2}\right)^{1/2}\left[-\delta_{st} - \left(\frac{1+v^2}{\alpha}-1\right)\frac{x^s x^t}{r^2} - \epsilon_{stk} v \frac{x^k}{r}\right],$$

$$\mathfrak{g}^{44} = \left(\frac{\alpha\gamma}{1+v^2}\right)^{1/2}\frac{1+v^2}{\gamma}, \quad \mathfrak{g}^{4s}=0, \quad \mathfrak{g}^{s4}=0. \tag{69}$$

These fields are also singular at $r=0$.

F. Electromagnetic field

From the work of Sec. III, involving the interaction of particles in Einstein's theory, it follows that the simplest field related to electromagnetic interaction in the theory is the pseudotensor field $\gamma^*_{[\mu\nu]}$, where

$$\gamma^*_{[\mu\nu]} = \tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}\mathfrak{g}^{[\rho\sigma]}. \tag{70}$$

We therefore define $\gamma^*_{[\mu\nu]}$ to be the electromagnetic field in Einstein's theory.[13] We find from (69) and this definition that the electromagnetic field associated with the solution (53)–(57) takes the form

$$\gamma^*_{[s4]} = \frac{q}{|q|}u\frac{x^s}{r}, \quad \gamma^*_{[st]} = 0, \tag{71}$$

where

$$u = \frac{q}{|q|}\left(\frac{\alpha\gamma}{1+v^2}\right)^{1/2} v. \tag{72}$$

We also see from Sec. III that the electromagnetic field in practical units would be given by

$$\left(\frac{c^2}{8\pi\epsilon_0 l^2 G}\right)^{1/2}\gamma^*_{[\mu\nu]}.$$

Thus we have for the electric field $\vec{E}$ and the magnetic field $\vec{B}$ in practical units

$$\vec{E} = \frac{e}{4\pi\epsilon_0 r_0^{\,2}}u\vec{n}, \quad \vec{B} = 0. \tag{73}$$

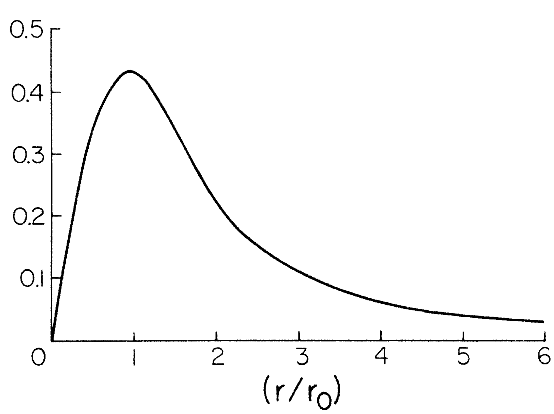

FIG. 2. The function u, proportional to the magnitude of the electric field.

In Fig. 2 we have plotted u as a function of r. Note that the electric field vanishes at the origin of coordinates. As is to be expected, r_0 is a measure of the electromagnetic size of the particle.

One word of caution: The simple form of the electric field in the vicinity of a particle should not be interpreted to mean that the force between two nearby particles is at all simple. In Einstein's theory it is only over moderate macroscopic interaction distances that the Lorentz force law, in terms of the field (70), can be considered valid,[13] and thus only over such distances can the electric field be regarded as the force per unit charge acting on a test charge at rest. Over interaction distances of the order of r_0 (the size of a particle)—and perhaps also over atomic and molecular distances—the interaction between particles is not expected to follow from the Lorentz force law and may be very complex. For a discussion of this point see the works mentioned in Ref. 11.

V. ELECTRODYNAMICS

A. Einstein electrodynamics

Equations (48) and (49) describe the electrodynamics of charged particles interacting over macroscopic distances in Einstein's unified field theory. We shall call this electrodynamics Einstein electrodynamics. Equations (48) and (49) were obtained on the basis of approximations which should be valid over both laboratory and astronomical distances, a distance range where classical Maxwell electrodynamics is usually assumed valid. The force law (48) of Einstein electrodynamics is the same as that found in Maxwell electrodynamics, but the effective field produced by a collection of particles, as given by (49), can differ significantly from the Maxwell field at astronomical distances $\gtrsim l$. At such distances from the particles the weak long-range fields (50), which are particular to Einstein's theory and not present in Maxwell's theory, can dominate the interaction with other particles. Thus, if Einstein's theory is correct, it may be possible to find evidence for the theory, and determine l, by examining astronomical electromagnetic fields.

B. Empirical tests

At present the best evidence known to us on the determination of l comes from an examination of the approximately dipolar magnetic fields of the Earth and Jupiter. This determination makes use of the fact that in Einstein electrodynamics any localized current distribution which produces a conventional magnetic dipole field will also produce a long-range nonconventional magnetic field, particular to Einstein's theory, which falls off inversely with distance and dominates the conventional field at distances greater than $2l$. The particular form of the nonconventional field can be found in earlier papers.[14]

At the present time the evidence from the Earth and Jupiter data is not sufficient to determine l, but is sufficient to place a lower bound on l. The lower bound from the Earth data is[15]

$$l \gtrsim 10^8 \text{ m}. \tag{74}$$

A careful analysis of the Earth data is now in progress,[16] making use of a general harmonic analysis of the Earth's field, to see if this bound may be improved and if there is any evidence for Einstein's theory in the data. The lower bound from the Jupiter data of Pioneers 10 and 11 is

$$l \gtrsim 10^9 \text{ m}. \tag{75}$$

This bound is based on the fact that out to about 20 Jupiter radii from the planet the conventional dipole field appears to dominate.[17] This implies that $2l$ is greater than 20 Jupiter radii, and thus we have (75).[18] This bound is higher but might be considered less firm than that obtained from the Earth data. It comes from data taken along two paths through the field, each traversed once, and effects due to plasma currents are uncertain. Plasma currents seem to significantly distort the field at distances greater than about 20 Jupiter radii. If Einstein's theory is correct, it is possible that a more thorough analysis of data from Jupiter might reveal Einstein effects in the field and yield an estimate of l.

Evidence for Einstein electrodynamics may also arise from its application to a number of astrophysical problems of current interest. These include

(1) the structure of the solar magnetic field and interplanetary field,
(2) the galactic magnetic field, and
(3) the structure of radio galaxies and quasars.

The long-range magnetic fields particular to Einstein electrodynamics could be important in all of these problems. In connection with radio galaxies and quasars another possibility, that the long-range repulsion between unlike charges (and attraction between like charges) due to the long-range field (50) might lead to charge-separation processes over astronomical distances, should be studied.

C. Field equations

In the investigation of astrophysical phenomena using Einstein electrodynamics it is convenient to have available the field equations satisfied by the electric and magnetic fields expressed in terms of the sources of the fields, the charge density ρ, and the current density $\vec{J}$, where

$$\rho = \sum_p {}^{(p)}e\, \delta(\vec{x} - {}^{(p)}\vec{\xi}), \tag{76}$$

$$\vec{J} = \sum_p {}^{(p)}e\, {}^{(p)}\vec{v}\, \delta(\vec{x} - {}^{(p)}\vec{\xi}). \tag{77}$$

Such field equations are[19]

$$\begin{gathered}\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0, \\ \vec{\nabla} \cdot \vec{D} = \rho - \frac{1}{8\pi l^2} \int \frac{[\rho]_{\text{ret}}}{|\vec{x} - \vec{x}'|} d^3x'. \\ \vec{\nabla} \cdot \vec{B} = 0, \\ \vec{\nabla} \times \vec{H} - \frac{\partial \vec{D}}{\partial t} = \vec{J} - \frac{1}{8\pi l^2} \int \frac{[\vec{J}]_{\text{ret}}}{|\vec{x} - \vec{x}'|} d^3x',\end{gathered} \tag{78}$$

where

$$\vec{D} = \epsilon_0 \vec{E}, \quad \vec{B} = \mu_0 \vec{H}. \tag{79}$$

These equations are satisfied by the effective electromagnetic field (49) and are the replacements for Maxwell's equations in Einstein electrodynamics, expressed in conventional, three-dimensional notation. They may be applied to physical phenomena using a phenomenological fluid model for the charge density ρ and the current density $\vec{J}$.

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[1]A. Einstein, *The Meaning of Relativity* (Princeton Univ. Press, Princeton, New Jersey, 1955), Appendix II, pp. 133–166.

[2]See for example J. Callaway, Phys. Rev. 92, 1567 (1953).

[3]C. R. Johnson, Phys. Rev. D 4, 295 (1971); 4, 318 (1971); 4, 3555 (1971); 5, 282 (1972); 5, 1916 (1972); 7, 2825 (1973); 7, 2838 (1973); 8, 1645 (1973). In subsequent references we shall refer to these papers as papers I–VIII, respectively.

[4]G. W. Gaffney, Phys. Rev. D 10, 374 (1974). In this work Gaffney is investigating the interaction of particles in a version of Einstein's theory known as the "Hermitian version." Einstein's final version and the Hermitian version are closely related, and results from one version can often be readily translated into the other.

[5]See Ref. 3 (paper II, Appendixes C and D) and Ref. 4.

See also C. R. Johnson, Nuovo Cimento 8B, 391 (1972).

[6]The notation

$$A_{(\mu\nu)}=\tfrac{1}{2}(A_{\mu\nu}+A_{\nu\mu}),$$

$$A_{[\mu\nu]}=\tfrac{1}{2}(A_{\mu\nu}-A_{\nu\mu}),$$

$$A_{[\mu\nu,\lambda]}=A_{[\mu\nu],\lambda}+A_{[\nu\lambda],\mu}+A_{[\lambda\mu],\nu}$$

will be used in this paper. Greek indices take the values 1–4: Latin indices take the values 1–3. The Levi-Civita symbols $\epsilon^{\mu\nu\rho\sigma}$ and ϵ_{stk} will be chosen so that $\epsilon^{1234}=1$ and $\epsilon_{123}=1$.

[7]A. Papapetrou, Proc. Roy. Irish Acad. A52, 69 (1948). Papapetrou's analysis can be generalized to include nonstatic systems.

[8]M. Wyman, Can. J. Math. 2, 427 (1950). By a time-independent spherically symmetric solution we mean a solution which is time-independent in a coordinate system in which the field takes the form (4).

[9]Two additional solutions found by Wyman, corresponding to $m_1=0$, cannot represent charged particles and will not be discussed here.

[10]See Ref. 3 (paper III).

[11]See Ref. 3 (paper II, Appendix C and paper VIII). See also Refs. 4 and 5.

[12]We are restricting ourselves to solutions for which the symmetric part of the fundamental field is flat at infinity.

[13]Over moderate macroscopic distances from a charged elementary particle this field will differ insignificantly from the effective electromagnetic field defined in Sec. III, and thus, like the effective field, it can be identified with the observed electromagnetic field over such distances. Over larger distances, i.e. astronomical distances, differences between the two fields become significant and only the effective electromagnetic field can be identified with the observed electromagnetic field. For further discussion see Ref. 3, paper II. The relationship of field (70) to interactions over microscopic distances has not yet been fully analyzed. See the Introduction.

[14]See Ref. 3 (paper II, Appendix D). See also Ref. 4, Sec. IV C.

[15]This bound is discussed in Ref. 4.

[16]R. L. Wilson, private communication.

[17]E. J. Smith, L. Davis, Jr., D. E. Jones, P. J. Coleman, Jr., D. S. Colburn, P. Dyal, and C. P. Sonett, Science 188, 451 (1975).

[18]The Jupiter data taken by Pioneer 10 have been analyzed by L. Davis, Jr., A. S. Goldhaber, and M. M. Nieto [Phys. Rev. Lett. 35, 1402 (1975)] as a test of Maxwell's equations. When applied to our problem their results are found to be consistent with the bound (75).

[19]Equations (78) can only be expected to be valid over those regions of space where $\gamma^*_{[\mu\nu]}\ll 1$. This means that the electric and magnetic fields appearing in (78) cannot be arbitrarily strong. One finds as the restrictions on these fields

$$E\ll\left(\frac{c^4}{8\pi\epsilon_0 l^2 G}\right)^{1/2},\quad B\ll\left(\frac{c^2}{8\pi\epsilon_0 l^2 G}\right)^{1/2}.$$

For $l=10^{10}$ m one finds

$$E\ll 10^{17}\ \text{V/m},\quad B\ll 10^{8}\ \text{Tesla}.$$