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Electron-muon puzzle and the electromagnetic coupling constant*

Herbert Jehle†
Institute for Theoretical Physics, University of Amsterdam, Valckenierstr. 65, Amsterdam C, The Netherlands
Quantum Chemistry Group, Uppsala University, Uppsala, Sweden
and Institute for Theoretical Physics, University of Munich, Theresienstr. 37, Munich 2, West Germany

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On the basis of a heuristic model we argued in an earlier paper (paper C of this series) electric field (and of course the magnetic field, too) of a lepton or of a quark may be formulated in terms of a closed loop of quantized magnetic flux whose alternative forms ("loopforms") are superposed with probability amplitudes so as to represent the electromagnetic field of that lepton or quark. The *Zitterbewegung* of a single stationary ("elementary") particle suggests a kind of quasiextension, which is assumed, in the present theory, to permit concepts of structuralization of the electromagnetic field even for leptons. Mesons and baryons may be represented by linked quantized flux loops, i.e., quark loops (as in paper B). The central problem now (in this paper D) is to formulate those probability-amplitude distributions in terms of wave functions to characterize the internal structure of the lepton or quark in question. As probability-amplitude functions one may choose bases of irreducible representations of the group with respect to which the model is to be invariant. It is seen that this implies the SO(4) group. As both the electron-muon mass ratio and the electromagnetic coupling constant depend, in this flux-quantization model, on the correct formulation of the structuralization of probability-amplitude distributions, we should expect to get an insight into both these puzzles from finding the right probability-amplitude wave functions. Furthermore, it is seen that this same structuralization of probability-amplitude distributions also permits one to estimate the rate of weak interactions, thus relating them to electromagnetic interactions.

I. INTRODUCTION

We consider a flux-quantization model whose implications for electrons and muons are being discussed in this paper. [For the basic issues of magnetic flux quantization and particle physics we refer the reader to Secs. I and II of paper C (see Ref. 1).]

The concept of a closed quantized-flux loop (which avoids introducing magnetic monopoles) leads to a theory of charge leptons, and of quarks. In order to construct a continuous magnetic field of an electron, it is assumed that the flux loop adopts a statistical distribution of alternative forms characterized by a complex probability-amplitude superposition, in a manner somewhat analogous to the superposition of path histories in Feynman's space-time approach to quantum mechanics.

On the same basis as quantized magnetic flux arises from a singularity of the gauge field, an electric field also arises when this singularity line (loop) is moving. In particular, a Coulomb field results from the spinning of the loop manifold with an angular velocity[12] equal to the *Zitterbewegung* frequency $2mc^2/\hbar$, if that loop manifold represents a Bohr magneton field.

As we shall indicate in discussing Eqs. (1.1)–(1.3) below, the calculation of the Bohr magneton from quantized flux yields the coupling constant $e^2/\hbar c$; it results in a consistent electromagnetic theory not only of magnetic and electric fields, but also of electromagnetic energy $=mc^2$ and electromagnetic angular momentum $=\hbar/2$. The calculations of mc^2 and of $\hbar/2$ are done not to "derive" these quantities but as a consistency condition; cf. paper C, Appendix B.

This calculation of the Bohr magneton from quantized flux has some very interesting aspects. We found that the source lepton may be considered as an extended object[12] of linear size with radius $\approx \hbar/2mc \approx$ the amplitude of the *Zitterbewegung*. Accordingly, a Gaussian distribution of "magnetization" to produce such an extended source field should carry an effective magnetic flux Φ_{eff} corresponding to the Bohr magneton $e\hbar/2mc$,

$$\Phi_{eff}=\frac{4\pi(e\hbar/2mc)}{3.1r_0}, \qquad (1.1)$$

where r_0 is the root mean square of the Gaussian distribution, and the factor 3.1 simply arises from calculation of the Gaussian (see paper A, Sec. V C). This Φ_{eff} cannot be directly identified with the quantized flux Φ_q which is carried by the flux loop because

$$\Phi_q=hc/e \qquad (1.2)$$

is about two orders of magnitude larger than Φ_{eff}. We postulated, therefore, that the probability-amplitude superposition (of alternative forms which a quantized-flux loop may have), when used to build up the continuous magnetic field, implies interference effects of these complex probability amplitudes, in some way analogous to interference of probability amplitudes in the superposition of alternative path histories. The effective field may thus, because of interference, be considered as a properly reduced Φ_q, the reduction being

$$\tfrac{1}{2}N = \Phi_q/\Phi_{\rm eff} = 3.1\frac{\hbar c}{e^2}\frac{r_0}{\hbar/mc}. \qquad (1.3)$$

We called N the reduction factor. [The $\frac{1}{2}$, as explained in paper C, or Sec. IV, simply arises because a quantized flux loop of winding numbers (2, 1) passes twice through the "core" of the source; it is therefore $2\Phi_q$ which is to be reduced to $\Phi_{\rm eff}$.] Such a reduction may be understood by considering the total field of a lepton as being due to N statistically independent "bundles of loopforms" (which is not to mean "bundles" in the terminology of fiber space topology) whose probability amplitudes have statistically independent phases; because of the probability-amplitude interference, the effective flux is $\Phi_{\rm eff} = 2\Phi_q/N$ (cf. paper C Appendix B, and the present paper, Sec. VI). This bundling was a crude but useful way of structuralizing the manifold of loopforms which thus made it possible to get some numerical estimates for the partial interference, i.e., for the reduction. These bundles had been considered in paper A as sets of loop neighborhoods, and will now be considered to be represented by modes of probability-amplitude waves (some eigenfunctions).

The reduction is formulated by comparing a "random-phased" situation with a "coherent" situation. We are discussing that in terms of the global (i.e., integrated over Euler angles and size of the loopforms) quantities: total magnetic flux, magnetic dipole moment, electromagnetic energy, and electromagnetic angular momentum. As the modes $D^{j_L}D^{j_K}$ [cf. Eqs. (3.3) and (4.6)] are orthogonal, the probabilities for their integrated fluxes should be considered as adding up arithmetically. This may be visualized by a photon-type statistical distribution of "excitons" [which correspond to the photons of a Planck-Bose-Einstein (PBE) distribution]. As we shall see in a more detailed analysis in Sec. VI, each exciton may be represented by a random-phased unit vector in the complex plane, denoting the probability amplitude; those vectors belonging to the same mode j may add up to a mode amplitude $\vec{r}_j$ which may be chosen to represent the number of excitons on the mode j, i.e., $n_j = |\vec{r}_j|^2 = r_j^2$. Applying a Planck distribution to those mode excitons, one obtains the expectation values of the occupancies

$$\langle|\vec{r}_j|^2\rangle_{\rm ex} = \langle n_j\rangle_{\rm ex} = (e^{\beta\omega_j} - 1)^{-1}$$

and an average occupancy for the distribution

$$\sum_j \langle n_j\rangle_{\rm ex}\langle n_j\rangle_{\rm ex}\Big/\sum_j \langle n_j\rangle_{\rm ex},$$

or, with better statistics, rather (cf. Sec. VI)

$$\sum_j \langle n_j^2\rangle_{\rm ex}\Big/\sum_j \langle n_j\rangle_{\rm ex}.$$

(For the purpose of addition of expected occupancies, mode amplitudes of different modes may be considered as random-phased vectors in a complex plane.)

If we now consider a hypothetical situation in which those excitons which belong to the same mode j are all coherent, and that this is so for every mode, that situation corresponds to what we call "in phase"; it has, for each mode, about $(\langle n_j\rangle_{\rm ex})^{1/2}$ times larger a probability amplitude as the random-phased ordinary photon-type distribution (4.5 ph). We make the assumption that this coherent situation corresponds to the occurrence of quantized flux $2\Phi_q$ (i.e., that a momentary coherence implies a momentary or virtual occurrence of the entire $2\Phi_q$), whereas the ordinary random-phased situation corresponds to the actual effective flux $\Phi_{\rm eff}$. We formulate this as

$$N = 2\Phi_q/\Phi_{\rm eff} = \sum_j \langle n_j\rangle_{\rm ex}\langle|\vec{r}_j|^2\rangle_{\rm ex}\Big/\sum_j\langle|\vec{r}_j|^2\rangle_{\rm ex} = \sum_j \langle n_j\rangle_{\rm ex}^2\Big/\sum_j \langle n_i\rangle_{\rm ex}$$

or, rather,

$$N = \sum_j \langle n_j^2\rangle_{\rm ex}\Big/\sum_j \langle n_j\rangle_{\rm ex}.$$

We recognize therefore that the reduction factor N is simply the average occupancy, the statistically weighted aforementioned average over all the modes j. A similar procedure characterizes the relationship between nonreduced and effective values of the other electromagnetic global quantities as well [cf. the equivalent statement in paper C, Eqs. (B11) and (B14)].

To be able to give a reasonable estimate or calculation of that reduction factor N is of great importance. In the preceding formula (1.3) $r_0/(\hbar/mc)$ is of the order of $\frac{1}{2}$ because that implies the only reasonable velocity, i.e., c at the equatorial region of the core, and because that also brings electromagnetic energy and angular momentum to the required values mc^2 and $\hbar/2$, respectively.

Accordingly a calculation of N is a calculation of the electromagnetic coupling constant $e^2/\hbar c$. Instead of defining some N bundles of loopforms to achieve the reduction in the heuristic model of paper C or paper A, we define now (using a PBE statistical distribution of probability-amplitude modes corresponding to a temperature $T=\beta^{-1}$) the reduction factor N as the ratio of coherent to noncoherent values of the (resultant probability amplitude)^2.

A central question with which we are concerned in this paper (D) is to find appropriate probability-amplitude wave functions in order to be able to define the reduction factor N. The heuristic model gives the clues to that and shows why we have to choose representations of SO(4). An appropriate choice of probability amplitudes for the loopform distributions will also have to give an account of the electron-muon dichotomy. With the heuristic model it was immediately obvious [in the discussion of Eq. (3) of C; and in Sec. V B of A] that the same Coulomb field arises for a point electron and for a point muon, an important feature of the theory. In trying to understand the distinction between electron and muon probability amplitudes, it was not possible to specify the issue any further in paper A, except that that paper referred to beat frequency and frequency for electron and muon, respectively.

It is this crucial electron-muon dichotomy which is first to be investigated with the probability-amplitude wave description in the present paper. It may be understood in the following way:

Isotropy of the electric field arises if we assume that loopforms are distributed over axial directions $\hat{\zeta}$ with probability amplitudes whose absolute values are proportional to $[1+\cos(\hat{\zeta},\hat{z})]^{1/2}$ and that they all spin with the same spinning angular velocity Ω, $\Omega=2mc^2/\hbar$ or $3mc^2/\hbar$ [cf. Secs. IV, V, and Eq. (4.4)]. The isotropy condition is derived in paper C, Sec. II C and Appendix A (cf. also paper A). Such a model, formulated semiclassically in terms of loopforms which have both orientation and angular velocity, is to be replaced below by a probability-amplitude wave model which implies the uncertainty relation between orientation (azimuth) and angular momentum.

The constancy of spinning angular velocity Ω of a distribution of loopforms over their Euler angles, i.e., over orientations $\hat{\zeta}\equiv\beta,\theta$ and azimuths α, implies that the distribution of the representative points $\tilde{x}_0,\tilde{x}_1,\tilde{x}_2,\tilde{x}_3$ on the S_3 sphere, Eq. (2.2), maintains invariant interdistances, i.e., performs an SO(4)-invariant motion. The SO(4) wave functions (probability-amplitude functions) should be sums [cf. Eqs. (3.8)–(3.10)] of $D^{j_L}_{n_L m_L} D^{j_K}_{n_K m_K} e^{-i(\omega_L+\omega_K)t}$ terms for a muon and of $D^{j_L}_{n_L m_L} D^{j_K}_{n_K m_K} e^{+i(\omega_L-\omega_K)t}$ terms for an electron. These functions define, depending on the wave equation which connects the j's with the ω's, the angular velocities

$$\begin{aligned}&\Omega_\mu \propto (\omega_L+\omega_K)/\tfrac{1}{2} \quad \text{(muon)},\\ &\Omega_e \propto (\omega_L-\omega_K)/\tfrac{1}{2} \quad \text{(electron)}.\end{aligned} \tag{1.4}$$

The Wigner D functions are defined as usual, cf. Eq. (5.4), and we shall see that $|j_L-j_K|=\frac{1}{2}$, $m_L=+j_L$, $n_K=-j_K$. The ratio of the expectation values of their corresponding Ω implies their mass ratio m_μ/m_e.

The objective of the present model is to understand the relationships of $e^2/\hbar c$ and of m_μ/m_e in the framework of electromagnetism and quantum mechanics. To specify the task in this way makes it tightly confined and more interesting because of this. What are then the premises which are introduced here in addition to electromagnetic field concepts? First, we consider the field lines as alternative forms of a line of quantized flux $\Phi_q=hc/e$, and we require the probability amplitudes (which form the quantum state function) to have the properties demanded by quantum mechanics [cf. conditions (a) and (b) in Sec. III]. Second, we assume that the magnetic field lines representing a lepton (or a quark) are closed flux loops. Third, we assume that the source region of that field is "quasiextended" corresponding to the *Zitterbewegung*; the interpolation of space into the smaller region permits (due to the different kinds of position operators connected to each other by a Pryce-Foldy-Wouthuysen transformation) a consideration of the source as if it were extended according to the particle's $\hbar/mc$. The second and third premises bring topological concepts into the model: the Seifert-Threlfall fibration. A fibration of ordinary three-space may be obtained by drawing field lines following the directions to a smooth vector field (in our case the magnetic field $\vec{B}$), provided the field is such that all these lines are nonintersecting closed lines ("flux loopforms") and that any pair of lines which are neighbors are so all along their lengths. Such fibration fills the entire three-space smoothly with field lines and these closed loops then have to have the forms of torus loops of winding numbers $(2,\pm1)$, etc., the first number telling how often the loop goes around the "dough" of the doughnut (torus), the second how often around the hole of the doughnut. Those loops are therefore, apart from topologically permitted deformations, of the form of nonintersecting lines on coaxial toruses. Dipole field lines are $(1,0)$.

This paper starts with Sec. II in which the geometrical and kinematical significance of Cayley-Klein parameters is specified with respect to the loop model.

Section III investigates the mathematical structure of the probability-amplitude waves in terms of irreducible representations of SO(4).

Section IV discusses the wave equations for probability-amplitude waves. A completion of the objective of this section, i.e., to find the appropriate wave equation, will influence the numerical results of the Secs. V and VI.

Section V specifies the electron-muon dichotomy, a calculation of their mass ratio in terms of the Hagedorn temperature.

Section VI presents the calculation of the reduction factor, also from that temperature, in the present probability-amplitude wave model, and thereby relates the coupling constant $e^2/\hbar c$ to the electron-muon mass ratio.

Section VI A specifies further details to the foregoing calculations.

The appendixes discuss the stereographic projection of the Cayley-Klein parameters (Appendix A), questions about the wave equation (Appendix B), footnotes and corrections to previous papers (Appendix C), rates of weak interactions which also may be estimated from this model (Appendix D), and a summary of the entire project (Appendix E).

The list of references is intended to draw attention to the range of issues which should be considered in the present program; several references (Refs. 18, 22–26) and others are relating not just to a particular point of the paper.

II. THE SO(4) GROUP

We would like to find out which group is admitted by the forms of flux loops and by the probability-amplitude function characterizing the loopform distribution of a lepton. We may be reminded that the quantum-mechanical field is that probability-amplitude "wave" function and that the electromagnetic field plays the role of a dynamical variable which, in turn, determines the effective magnetic moment and the equivalent electric charge, and which is compatible with an electromagnetic energy mc^2 and electromagnetic angular momentum $\hbar/2$.

If we know the group, its invariants, and its representations, we should form appropriate linear combinations of the representations of the group so as to formulate the probability amplitude functions. With those we should be able to calculate $e^2/\hbar c$ (cf. Sec. VI, and paper C, Appendix B), that was formerly only crudely estimated on the basis of the heuristic model. The electron-muon dichotomy should result from a proper formulation of probability-amplitude functions, too. The present proposal may be a first step in the direction of resolving these puzzles in more than a heuristic way.

Before we can determine the group in question, we have to describe the topological and geometrical character of loopforms in x, y, z space (1) and of their parameter distribution in the space of the Cayley-Klein parameters (2).

We assumed a manifold of lepton flux loopforms to be a superposition of submanifolds of the type of sheaves of loopforms which satisfy the criteria of a Seifert-Threlfall fibration. There are two topological singularities in such a fibrated x, y, z space; these may be chosen to be a central straight flux orientation axis ($\hat{\zeta}$) and a core equatorial axis. Let us consider the latter to be a circle of a given radius for the present discussion (Fig. 1; also see Fig. 3 of paper C, Fig. 1 of paper B, and Fig. 5 of paper A).

A particular loopform is represented by a set of parameters characterizing a torus loop of (a) a particular pair of winding numbers (the sign of the second also indicates the handedness of the loop; the winding numbers, furthermore, determine the unknotting number, i.e., the strangeness), of (b) a spinning motion (parallel or antiparallel to the resultant magnetic moment of the loopform), of (c)

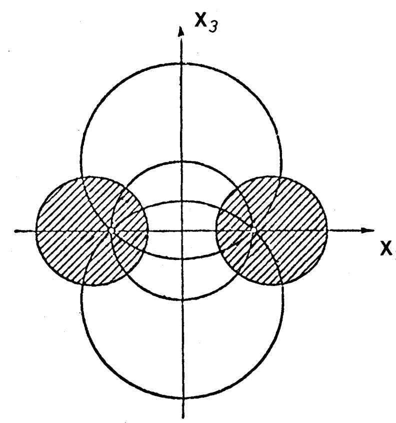

FIG. 1. Stereographic projection of a four-dimensional Euclidean space E_4 (with coordinates $\mathfrak{r}\tilde{x}_0, \mathfrak{r}\tilde{x}_1, \mathfrak{r}\tilde{x}_2, \mathfrak{r}\tilde{x}_3$) or of the points $\tilde{x}_0$, $\tilde{x}_1$, $\tilde{x}_2$, $\tilde{x}_3$ of the unit hypersphere $S_3 \subset E_4$ onto the three-space (with coordinates x_1, x_2, x_3) whose x_1, x_3 axes are shown in this figure. This implies a projection from the pole $X_0 = 1$, $X_1 = X_2 = X_3 = 0$ of the unit hypersphere $S_3 \subset E_4$ onto the equatorial hyperplane $x_0 = 0$. (For details cf. Fig. 2 and Appendix A.) If $\tilde{x}_0$, $\tilde{x}_1$, $\tilde{x}_2$, $\tilde{x}_3$ represent the Cayley-Klein parameters of Eq. (2.1), then the loci $(\beta - \alpha) =$const are the meridional planes through the x_3 axis, e.g., the plane of this figure. The surface of the toroid "doughnut" (whose cross section with the meridional plane of the figure is indicated by shading) is a locus $\theta =$const. The loci $(\beta + \alpha) =$const are spheres whose centers are on the x_3 axis and whose cross section with the meridional figure plane are the circles in Fig. 1. The nice thing about the Cayley-Klein parameters is that the distance, such as it appears in Eq. (2.6), denoted an appropriate measure for the relatedness (neighborhood) of two sets $\beta^{(1)}, \theta^{(1)}, \alpha^{(1)}$ and $\beta^{(2)}, \theta^{(2)}, \alpha^{(2)}$. Whereas, so far, we referred to the Cayley-Klein space, (2), this Fig. 1 also describes, in different interpretation, a toroidal system in ordinary three-dimensional x, y, z space, (1).

a whirling motion (when both winding numbers are given and thus, with the signature of the second winding number, the handedness of the loop is given, the whirling motion may, in relation to the spinning motion, have opposite or same handedness as that of the loop, and thus be additively or subtractively related to the spinning motion), of (d) a size σ (thickness of the torus on which it may be drawn) of (e) a particular flux orientation $\hat{\xi}$ (orientation of its central axis, or of its circular-symmetry-axis plane, which amounts to the same thing, of (f) spin azimuth α_s (angle about the axis $\hat{\xi}$), and of (g) whirl azimuth α_w (angle about the circular axis). The manifold of loopforms is characterized by the parameters of the loopform-distribution probability amplitudes, in particular the time dependence of those amplitudes.

In general, the whirling and spinning degrees of freedom may have to be counted as two degrees of freedom. For a pair or a triplet of interlinked hadronic torus loops with different winding numbers and/or spins, spin and whirl are indeed not equivalent. For a fibration in terms of one type of smooth torus loops of given winding numbers there is, however, an equivalence between spinning and whirling, and one may therefore consider the whirling motion as simply contributing, additively or subtractively, to the effective angular velocity of spinning, as was amply discussed in paper C. The amount of that contribution is determined by the winding numbers. This is one of the simplifications implied by the parametrized description of loops. Spinning and whirling, being coaxial motions, may be described by the same wave functions of the Euler angles.

We consider for electrons or muons loopforms of winding numbers $(2, \pm 1)$ and for neutrinos loopforms $(3, \pm 2)$ because those loopforms, belonging to a sheaf, are interlinked, unlike loopforms of photons, e.g., (0,1). Quark loopforms $(2, \pm 1)$, $(3, \pm 1)$, $(3, \pm 2)$, etc. are interlinked, too. The linkage of the loopforms of a lepton loop or of a quark loop among each other, and their linkage with other loops, accounts for confinement: Interlinkage of the toroidal loop(form)s implies their linkage to *one* core whose size is given as of the order of $\approx \hbar/mc$, the *Zitterbewegung* amplitude of the source of the loop(s) in question.

We may describe loopforms in two different ways:

(1) We may simply consider the topology of the flux loopforms in ordinary x, y, z space. We single out a "*sheaf*" of flux loopforms, all pertaining to the same pair of winding numbers [e.g., (2,1)] and the same flux orientation axis $\hat{\xi}$, but spanning a *continuous manifold of sizes* σ *and of azimuths* α_s; therefore the designation as a sheaf. This manifold presents a Seifert fibration of x, y, z space. It is an important submanifold of the full manifold of loopforms because the loopforms of a sheaf share the axes of the fibration (a common straight axis $\hat{\Omega} = \pm \hat{\xi}$ of spinning, and also a common corresponding circular axis of whirling), and since those loopforms do not cut across each other while moving; we may say there is no "cross-cutting." The manifold of the $\hat{\xi}$ parameters thus defines a manifold of fibrations (sheaves). We considered superpositions of those sheaves in paper C, Fig. 2. We may then map such a sheaf on the corresponding torus loops drawn on the surfaces of the toruses of a toroidal coordinate system [paper C, Fig. 3(d)]: Such a mapping is topologically significant, not metrically, because the toroidal coordinate system's meridional cross section consists of circles, whereas the flux loopforms are of a somewhat different form, and because the density of lines in a toroidal coordinate system in its dependence on the thickness of the torus could only *ad hoc* be transformed so as to represent the density of flux loopforms of the magnetic field in question (the density of Faraday lines is proportional to magnetic-field intensity).

(2) We may describe a loopform as a point in the parameter space of the above-enumerated parameters. It is the parameters $\hat{\xi}$, α_s, i.e., the three *Euler angles*, which we consider first of all. We look for a distribution of probability amplitudes over that *manifold* $\hat{\xi}, \alpha_s$ of loopforms. Following Sec. IV we deal with the distribution over α_w.

We give an analysis of description (2) now, and shall come back later to (1).

What are the invariance properties of the distribution of loopforms in the parameter space of the loopforms? The Cayley-Klein parameters[2]

$$\begin{aligned}
\tilde{x}_0 &= \cos\tfrac{1}{2}\theta \cos\tfrac{1}{2}(\beta+\alpha) \\
&= \cos\tfrac{1}{2}\phi, \\
\tilde{x}_1 &= \sin\tfrac{1}{2}\theta \cos\tfrac{1}{2}(\beta-\alpha) \\
&= \sin\tfrac{1}{2}\phi \cos a_1, \\
\tilde{x}_2 &= \sin\tfrac{1}{2}\theta \sin\tfrac{1}{2}(\beta-\alpha) \\
&= \sin\tfrac{1}{2}\phi \cos a_2, \\
\tilde{x}_3 &= \cos\tfrac{1}{2}\theta \sin\tfrac{1}{2}(\beta+\alpha) \\
&= \sin\tfrac{1}{2}\phi \cos a_3
\end{aligned} \tag{2.1}$$

define a 3-dimensional unit sphere S_3,

$$\tilde{x}_0^2 + \tilde{x}_1^2 + \tilde{x}_2^2 + \tilde{x}_3^2 = 1, \tag{2.2}$$

in a Euclidean E_4 space, which suggests that there might be rotational invariance under SO(4).[3] We denote the six generators of rotation[4,5] in this E_4 space by $J_{23}, J_{31}, J_{12}, J_{01}, J_{02}, J_{03}$.

Consider, instead of the Euler angles, the ϕ, a_1, a_2, a_3 to characterize the orientation of a rigid body, an "object" (in our case, a flux loop whose flux orientation axis $\hat{\xi}$ and azimuth α_s are obtained from a zero position of orientation in the $\hat{z}$ direction and azimuth 0, by a rotation about $\hat{a}$ with an angle $-\phi$; $\hat{a}$ lies halfway between the $\hat{\xi}$ axis and the $\hat{z}$ axis). SO(4) rotations may place this object into an arbitrarily different orientation. Indeed, an active transformation

$$e^{i\kappa_1 J_{12}} e^{i\kappa_2 J_{31}} e^{i\kappa_3 J_{12}} e^{i\kappa_4 J_{03}} e^{i\kappa_5 J_{31}} e^{i\kappa_6 J_{12}} \tag{2.3}$$

first uses the generator J_{12} to perform a rotation κ_6 in this $\tilde{x}_1\tilde{x}_2$ plane so as to make $\cos a_2$, i.e., $\tilde{x}_2$, vanish; this is a rotation of the $\hat{a}$ vector. [This makes $\frac{1}{2}(\beta-\alpha)$ vanish at fixed $\frac{1}{2}(\beta+\alpha)$ and θ.] Thereupon $e^{i\kappa_5 J_{31}}$ makes the remaining $\cos a_1$, i.e., $\tilde{x}_1$ vanish. These two operations bring the $\hat{a}$ axis, and thus also the $\hat{\xi}$ axis, into the $\tilde{x}_3$ direction. The generator J_{03} now performs a rotation in the (0, 3) plane (in terms of the $\hat{a}, \phi$ system by a change of ϕ). The subsequent J_{12}, J_{31}, J_{12} bring the object into any desired orientation. These 6 parameters of rotation accordingly characterize the original and final orientations, not only the 3-parameter rotation itself.

It is helpful to discuss this also from the point of view of the Euler angles β, θ, α (see Appendix A). Consider the stereographic projection from a pole $X_0=1$, $X_1=X_2=X_3=0$ onto the 3-dimensional flat region $x_0=0$ (Fig. 2). Our object's (loopform's) orientation, characterized by Euler angles β, θ, α, is represented by a point on S_3, or by a point in that 3-dimensional projected space whose orthogonal coordinates are x_1, x_2, x_3. Toroidal coordinates in this stereographically projected space provide for a mapping of β, θ, α onto x_1, x_2, x_3. The toroidal surfaces (of a doughnut of thickness $2\cot\frac{1}{2}\theta$, and whose circular axis $x_1^2+x_2^2=1$ lies in the x_1x_2 plane) characterize θ: $\theta=0$, i.e., $x_1=x_2=0$, is the x_3 axis; $\theta=\pi$, i.e., $x_0=x_3=0$, characterizes the unit circle $x_1^2+x_2^2=1$. The meridional planes through the x_3 axis form an angle $\frac{1}{2}(\alpha-\beta)$ with the x_1x_3 plane; a half turn (180°) implies a change π of that angle $\frac{1}{2}(\alpha-\beta)$. Ordinary spherical surfaces which all pass through the unit circle $x_1^2+x_2^2=1$ and which are centered on the x_3 axis at $x_3=-\cot\frac{1}{2}(\alpha+\beta)$ have a radius $\{1+\cot^2[\frac{1}{2}(\alpha+\beta)]\}^{1/2}$; when the sphere's center moves from $x_3=-\infty$ to $x_3=+\infty$, $\frac{1}{2}(\alpha+\beta)$ changes by π and the intersection of the sphere with the doughnut surface whirls about by a half turn (180°).

From the above definitions of the Cayley-Klein parameters we can now recognize the geometry of SO(4). The generator J_{12} causes an increase of $\frac{1}{2}(\beta-\alpha)$, i.e., a rotation about the x_3 axis in the stereographically projected space. The generator J_{03} causes an increase of $\frac{1}{2}(\beta+\alpha)$, i.e., a whirling motion about the circular doughnut axis in the projected space.

As the spinning-whirling operations, generated by J_{12}, J_{03} commute, and as J_{23}, J_{31}, J_{12} do not commute, their commutation relations invite, in the usual fashion, the definitions

$$2L_3 = J_{03}+J_{12}, \quad 2K_3 = J_{03}-J_{12}, \quad \text{etc.} \tag{2.4}$$

As $\vec{L}$ commutes with $\vec{K}$, they are generators for SU(2) × SU(2). It is readily seen that $\vec{L}$ generates changes in β, i.e., rotations about the $\hat{z}$ axis, whereas $\vec{K}$ generates changes in α, i.e., rotations about the $\hat{\xi}$ axis. This means that one of the SU(2) factors may be considered as characterizing rotations in reference to a space-fixed system ($\vec{L}$), whereas the other SU(2) factor is spanned by operations of the angular momentum $\vec{K}$ with reference to the body-fixed system [cf. the discussion along with Eqs. (3.4)–(3.7) below]. Of special importance are the Casimir operators $\vec{K}^2$ and $\vec{L}^2$.

What is the point of considering this description (2) of flux loopforms and the most general SO(4) invariance?

In paper C, Sec. IIC and Appendix A [as well as in paper A, Eqs. (4.11)–(4.18)] we showed that the *isotropic Coulomb field* results from the assumption that *each flux loopform spins about its own flux orientation axis*. This important circumstance implies a *motion* which has the following *SO(4)-invariant property*: Consider a change of the Cayley-Klein parameters corresponding to a motion in time such that $d\beta=0$, $d\theta=0$, $d\alpha=(2mc^2/\hbar)dt$, i.e., a spinning of loopforms about their own flux orientation axis. From (2.1) follows

$$d\tilde{x}_0^2+d\tilde{x}_3^2+d\tilde{x}_1^2+d\tilde{x}_2^2=\tfrac{1}{4}(d\alpha)^2 = (mc^2/\hbar)^2 dt^2, \tag{2.5}$$

i.e., an amount independent of β, θ, α. Furthermore, consider two (infinitesimally) neighboring loopforms' distance, which is, as an elementary calculation shows,

$$\Delta\tilde{x}_0^2+\Delta\tilde{x}_3^2+\Delta\tilde{x}_1^2+\Delta\tilde{x}_2^2 = \tfrac{1}{4}[(\Delta\beta)^2+(\Delta\alpha)^2+2\cos\theta\,\Delta\beta\Delta\alpha+(\Delta\theta)^2].$$

As time proceeds, and thereby $d\beta=0$, $d\theta=0$, $d\alpha=(2mc^2/\hbar)dt$, i.e., independent of β, θ, α, for *all* loopforms, the new distance parameters $\Delta\beta$, $\Delta\theta$, and also $\Delta\alpha$ are the same as the previous ones. This result is valid not only for infinitesimal distances $\Delta\tilde{x}_0, \Delta\tilde{x}_1, \Delta\tilde{x}_2, \Delta\tilde{x}_3$. Indeed, a straightforward calculation of

$$\frac{d}{dt}[(\tilde{x}_0^{(2)}-\tilde{x}_0^{(1)})^2+(\tilde{x}_3^{(2)}-\tilde{x}_3^{(1)})^2+\cdots] \tag{2.6}$$

for any two points $\tilde{x}_0^{(1)}, \tilde{x}_3^{(1)}, \ldots$ and $\tilde{x}_0^{(2)}, \tilde{x}_3^{(2)}, \ldots$

satisfying (1) gives zero when $d\beta/dt=0$ and $d\theta/dt=0$ and $d\alpha/dt=$ same value for the two points. Accordingly, the distances $\Delta\tilde{x}_0^2+\Delta\tilde{x}_1^2+\Delta\tilde{x}_2^2+\Delta\tilde{x}_3^2$ are not changing in time. Therefore, the motion of the *loopforms' points on* $S_3\subset E_4$ is a *distance-preserving motion*, i.e., a *SO(4)-invariant motion.* [(2.6) is also zero when $d\beta/dt$ has the same nonvanishing values for the two points.]

It is most interesting to realize that the basic physical assumption of constant angular velocity of the loopforms exactly satisfies the mathematical invariance of Cayley-Klein parameters maintaining fixed distances over the sphere $S_3\subset E_4$.

In these sections we ignore, as pointed out already, the whirling (in ordinary three-dimensional space) degree of freedom and, accordingly, we characterize one loopform simply by $\hat{\xi}, \alpha_s$, i.e., by three Euler angles (a loopform being parametrized like a symmetric top), i.e., by a point on the aforementioned three-sphere $S_3\subset E_4$.

As spinning angular velocity, for additive spin-whirl motion, one has then to consider the value $3mc^2/\hbar$, which may be called the *Zitterbewegung* angular velocity of a (2,1) loop — rather than the value $2mc^2/\hbar$, which refers to spinning alone. m is the mass of the muon in $3mc^2/\hbar$ and is the mass of the electron in $3mc^2/\hbar$.

We may digress here to make a few remarks about this SO(4) invariance and about the implications of that invariance as regards representations.

To this effect we remind ourselves of the situation on the level of SO(3) invariance in ordinary three-dimensional space, i.e., one dimension less than our problem.

Bopp and Haag[6] pointed out the following: A spherical pendulum, represented by a single point on a spherical surface $S_2\subset E_3$ (i.e., in a three-dimensional Euclidean space) has Y_{lm} eigenfunctions. A pair of points on S_2, having a fixed distance from each other, i.e., a pair of spherical penduli, is characterized by three (instead of the former two) parameters, like a top or a loopform. It defines SO(3) rotations[7] by virtue of the invariance of their interdistance. The eigenfunctions of such pairs of points then turn out, of course, to be the Wigner D^j_{nm} functions. In other words, the motion of one pair of points, or a top, defines an SO(3) rotation.[6] The irreducible components of the regular representation (i.e. the probability-amplitude functions) of the SO(3) group is then provided by the SO(3) group's transformation matrices D^j_{nm} themselves.

Proceeding now to the consideration of several loopforms, characterized by several sets of $\hat{\xi}, \alpha_s$, i.e., of Euler angles, we know that their representative points on $S_3\subset E_4$ have fixed distances from each other. A first point position has three parameters $\beta_1, \theta_1, \alpha_1$, a second point has only two additional free parameters, a third point one more, and the remaining points are determined by the fixed interdistances. These six parameters obviously may be interpreted as defining a most general rotation of $S_3\subset E_4$, i.e., an SO(4) rotation. The basis for the regular representation (i.e., probability-amplitude function) of SO(4) is then given by the SO(4)-group transformation matrices which we obtain from the generators of SU(2) $\times$ SU(2), i.e., a direct product of two Wigner D^j_{nm} functions; such products are irreducible representations of SU(2) $\times$ SU(2).

As we are thus interested in a description (of the probability-amplitude distribution of loopforms) which is invariant with respect to rotations which maintain the interdistances $\Delta\tilde{x}_0^2+\Delta\tilde{x}_1^2+\Delta\tilde{x}_2^2+\Delta\tilde{x}_3^2$ of representative points on $S_3\subset E_4$ in the *Cayley-Klein* parameter space, we need an (invariant) description in terms of the irreducible components of the regular representation of the SO(4) group, not only the SO(3) group. We also come to this conclusion by considering the following remark.

The concept of bundling of loopforms in the heuristic model also demands the covariance of the theory in regard to the full SO(4) group because bundling (based on the concept of neighborhood, which was already discussed in Casimir's thesis) may reasonably be defined only on the basis of the concept of neighborhood of representative points on $S_3\subset E_4$ of the loopforms.

We would like now to come back to the description (1) of the loopforms in ordinary space-time (x,y,z,t). We may introduce abstract analogs to the "Euler angles" such that x_1, x_2, x_3 of the stereographically projected space are ordinary x,y,z position-space coordinates. Consider now the aforementioned SO(4) transformations in the $\tilde{x}_0, \tilde{x}_1, \tilde{x}_2, \tilde{x}_3$ space and, to avoid confusion, let us denote the "rotation" generators now by $I_{23}, I_{31}, I_{12}, I_{01}, I_{02}, I_{03}$. With the mapping of loopforms on torus knots placed on the toroidal surfaces [as described above, cf. Fig. 3(d), of paper (C)], we may now give a simple geometrical interpretation to an operation

$$e^{i\alpha_1 I_{12}}e^{i\alpha_2 I_{31}}e^{i\alpha_3 I_{12}}e^{i\alpha_4 I_{03}}e^{i\alpha_5 I_{31}}e^{i\alpha_6 I_{12}}. \qquad (2.7)$$

This SO(4) operator may give to an arbitrarily oriented $(\hat{\xi})$ fibration characterized by $\hat{a}$ (in the Cayley-Klein definitions) [i.e., to a manifold (σ, α_s) of torus loops, representing a sheaf of loopforms in x,y,z space, with a probability-amplitude distribution over σ, α_s] the most general spin (and whirl) motion (be it a displacement or a velocity, that does not matter for the present discussion), a motion which preserves the internal structure of that fibration of ordinary x,y,z space.

We noted that this statement is to be understood topologically, not metrically. The generator $e^{i\alpha_5 I_{31}}e^{i\alpha_6 I_{12}}$ places the axis $\hat{a}$ into the $\hat{z}$ direction; α_5, α_6 are the parameters characterizing the arbitrarily given original $\hat{a}$ direction. The generator $e^{i\alpha_3 I_{12}}e^{i\alpha_4 I_{03}}$ whirls (α_w) and spins (α_s) the fibration about its circular and central axes which have just been laid into the (x, y) plane and the z direction. The two factors of this generator commute. The generator $e^{i\alpha_1 I_{12}}e^{i\alpha_2 I_{31}}$ finally puts the central axis into the arbitrarily requested direction. The and result is a fibration again of the same internal structure as the initial one. To summarize, the full SO(4) group generates the full manifold of loopforms from one loopform, the subgroup of (whirling and) spinning generates a sheaf of loopforms. This involvement of the SO(4) group expresses the *invariance* of the underlying *topological structure* of torus-knot fibrations[8] and thus of loopforms, i.e., SO(4) represents, topologically, a congruence mapping of the fibration by torus knots.

What is done here is closely related to Finkelstein's discussion of spaces with torsion.[7]

III. STRUCTURE OF THE PROBABILITY-AMPLITUDE WAVES

The discussion of the SO(4) invariance above gives us the means of finding the wave functions for loopform distributions, by choosing basis functions of appropriate representations of SO(4). We sketch the calculation, in order to present the idea of the solution.

The heuristic model provides us with excellent guidelines to set up appropriate amplitude functions. It teaches us that the correct isotropic Coulomb field results from a manifold of spinning magnetic-dipole field loops of various flux orientations $\hat{\zeta}$ (cf. Fig. 4 of paper C) if (a) these loopforms contribute to a resultant spin-$\frac{1}{2}$ state and, accordingly, (b) have probability amplitudes proportional to $[1+\cos(\hat{\zeta},\hat{z})]^{1/2}=(1+\zeta_z)^{1/2}$, if (c) each loopform spins about its flux orientation axis $\hat{\zeta}$ with (d) the same spin angular velocity of $2mc^2/\hbar$, i.e., spin and whirl add to an effective angular velocity $3mc^2/\hbar$ (see Sec. IV below and paper C, Appendix A). The heuristic model also teaches us (e) that the structuralization is to be thought of as resembling some bundling in terms of unit radian, of azimuths, and of flux orientations of the loopforms — as indicated in Appendix B of paper C and in paper A. In some manner these conditions of the heuristic model have to be reflected by a proper choice of probability-amplitude (wave) functions.

A word may be in place here about the consistency of the conditions (a), (b), (c), and (d) with each other, again in terms of the heuristic model. It was there shown how effective magnetic moments of the size of a Bohr magneton $e\hbar/2mc$ may be distributed over the various flux orientations $\hat{\zeta}$ with probabilities $(1+\zeta_z)(d\zeta_z/2)$. (One may either superpose bundles of quantized flux by means of probability amplitudes, or superpose bundles of effective flux with probabilities.) One obtained in this way the resultant magnetic moment equal to 1 Bohr magneton,

$$\int_{-1}^{+1}(e\hbar/2mc)(1+\zeta_z)(d\zeta_z/2)=(e\hbar/2mc). \qquad (3.1)$$

[The normalized probabilities $(1+\zeta_z)(d\zeta_z/2)$ for the loopforms, inclined with direction cosines $\zeta_z=\hat{\zeta}\cdot\hat{z}$ toward the $+\hat{z}$ axis, may be considered as probabilities for the contributions toward the *resultant* magnetic moments $\pm e\hbar/2mc$ of the $\pm z$ quantum states $|\mu_z^{\pm}\rangle$, respectively. One should, however, remember that, as there are many more bundles of loopforms than quantum states, the relations between the two are underdetermined.]

As it is assumed in the aforementioned conditions that spinning occurs around each flux loop orientation axis $\hat{\zeta}$ with the same angular velocity, therefore this Eq. (3.1) for resulting magnetic moment repeats itself as regards the calculation of the resulting spin: The resultant spin is obtained from spinning about the $\hat{\zeta}$ axes, corresponding to angular velocities $2mc^2/\hbar$ or $3mc^2/\hbar$).

To proceed now to the wave-function formulation of the probability amplitudes, we refer to the discussions (2), relating to rotations in the space of the four Cayley-Klein parameters of Eq. (2.1). We have seen in (2.6) that the above conditions (c), (d) imply an SO(4)-invariant description of the probability-amplitude wave functions. The well-known split of SO(4), in terms of commuting generators which we would like to call $\vec{L}$ and $\vec{K}$, into the direct product SU(2) × SU(2) implies that irreducible components of the regular representation of SO(4) may be written in terms of direct products of two Wigner D^j_{nm} functions. Bases for the SU(2) × SU(2) group, generated by the commuting pair of generators $\vec{L}\times\vec{K}$, may thus be chosen as some products

$$D^{j_L}_{n_L m_L}(\beta_L, \theta_L, \alpha_L)D^{j_K}_{n_K m_K}(\beta_K, \theta_K, \alpha_K). \qquad (3.2)$$

Because of the product form of (3.2) it is obvious that the *sum* of rotations $(\beta_L, \theta_L, \alpha_L)$ and $(\beta_K, \theta_K, \alpha_K)$ now indicate the Euler angles of the loopforms with respect to space-fixed axes (z). $(\beta_L, \theta_L, \alpha_L)$ thus represents the orientation of "bases" (system K), around which $D^{j_K}_{n_K m_K}(\beta_K, \theta_K, \alpha_K)$ marks the loopform distribution [cf. also Eqs. (3.8)–(3.10) below, with discussion of (5.9) and (5.12)].

Why is it that even though we may describe a loopform *distribution* at one moment by means of a function over only three Euler angles, we need a wave function of six Euler angles? The answer is obvious: We want to describe a general *SO(4)-invariant motion* which then can only be represented by superposition of bilinear terms (3.3). (This is the generalization from 3 to 4 dimensions of the argument made in Sec. II above.[6])

One should note that the functions $D^{j_L}_{n_L m_L}(\beta_L, \theta_L, \alpha_L)$ and $D^{j_K}_{n_K m_K}(\beta_K, \theta_K, \alpha_K)$ both cover the same 3-dimensional manifold of orientations β, θ, α, specified only with respect to different bases $[(\beta_L, \theta_L, \alpha_L)$ and $(\beta_K, \theta_K, \alpha_K)$, respectively$]$ to which the Euler angles are referred. And we have to consider the projections of this 6-dimensional space of Euler angles on a 3-dimensional one. Those projections of (3.2) in general no longer form an orthonormal set.

If we had to describe the motion of just *one* loopform, spinning like one rigid top, the bases of representation for such motion could be chosen as the well-known $D^j_{nm}(\beta, \theta, \alpha) \propto e^{(in\beta)} e^{(im\alpha)}$ representations of SU(2), n characterizing an eigenvalue of angular momentum about the z axis, and m an eigenvalue of the angular momentum, referred to body axes.

In analogy to the argument of Bopp and Haag, the manifold of loopforms (tops) implies, however, a six-parameter manifold of representative points on $S_3 \subset E_4$ (apart from an additional size parameter), the loopforms spinning about their respective base orientation axes with the same angular velocity $2mc^2/\hbar$ (or $3mc^2/\hbar$) respectively [conditions (c) and (d), above].

The six-parameter manifold of loopform distributions is thus to be represented by a six-parameter manifold of functions (3.2). The *two* generators $\vec{L}$ and $\vec{K}$ refer now to the two factors of the product (3.2), respectively, $\vec{L}$ operating on variables $(\beta_L, \theta_L, \alpha_L)$ referring to a distribution $D^{j_L}_{n_L m_L}$ in the space-fixed system, $\vec{K}$ operating on variables $(\beta_K, \theta_K, \alpha_K)$ referring to a distribution relative to "moving bases." The manifold of loopforms may be thus represented by the direct product, i.e., in terms of sums of products of the type

$$\sum D^{j_L}_{n_L m_L}(\beta_L, \theta_L, \alpha_L) e^{(\mp i\omega_L t)} D^{j_K}_{n_K m_K}(\beta_K, \theta_K, \alpha_K) e^{(\mp i\omega_K t)}. \tag{3.3}$$

(We use a summation sign to symbolize a superposition of those *DD* terms.) Such a summation of terms (3.3) and subsequent projection does evidently not lead to a simple function $D^j_{nm}(\beta, \theta, \alpha)$ such as, e.g., $D^{1/2}_{1/2\ 1/2}(\beta, \theta, \alpha)$.

We now have to remind ourselves of the geometrical significance of (one of) the two D's in Eq. (3.3), considering therefore $(\beta_L, \theta_L, \alpha_L)$ in lieu of (β, θ, α).

Referring to the discussions of rotations (2), Eqs. (21), (A15) and (A14), we consider rotations of the Cayley-Klein space E_4 in its $(\tilde{x}_0, \tilde{x}_3)$ plane, simultaneously with rotations in its $(\tilde{x}_1, \tilde{x}_2)$ plane. As (cf. Fig. 2, Appendix A)

$$\begin{aligned} \cos^2(\tfrac{1}{2}\theta_L) &= \tilde{x}_0^{\,2} + \tilde{x}_3^{\,2} \equiv \tilde{z}_{03}\tilde{z}^*_{03}, \\ \sin^2(\tfrac{1}{2}\theta_L) &= \tilde{x}_1^{\,2} + \tilde{x}_2^{\,2} \equiv \tilde{z}_{12}\tilde{z}^*_{12}, \end{aligned} \tag{3.4}$$

such rotations characterize changes of β_L and of azimuth α_L which leave θ_L *constant*. In the stereographic projections $\theta_L = 0$, i.e., $x_1^{\,2} + x_2^{\,2} = 0$, means the x_3 axis which is perpendicular to the (x_1, x_2) plane. $\theta_L = \pi$, i.e., $x_1^{\,2} + x_2^{\,2} = 1$, is the circular axis of the torus; any other value of θ_L is characterized by a toroidal surface which has $\theta_L = 0$ and $\theta_L = \pi$ as its two axes. We now specialize in rotations of the above kind for which, in addition, the |rotation angles| in those two planes are equal. The result of these simultaneous rotations is described in terms of changes of *one* set of Euler angles $\beta_L, \theta_L, \alpha_L$. The change of $\tilde{z}_{03}\tilde{z}_{12}$ is given by the change of

$$\begin{aligned} \tilde{z}_{03}\tilde{z}_{12} &= (\tilde{x}_0 + i\tilde{x}_3)(\tilde{x}_1 + i\tilde{x}_2) \\ &= |\tilde{z}_{03}| \exp[\tfrac{1}{2}(\beta_L + \alpha_L)] \, |\tilde{z}_{12}| \exp[\tfrac{1}{2}i(\beta_L - \alpha_L)] \\ &= \text{const} \times e^{i\beta_L}, \end{aligned} \tag{3.5}$$

whatever the change of α_L. The constant in (3.5) is, by (3.4), a function of θ_L which indeed is constant. $\tilde{z}_{03}\tilde{z}_{12} = \text{const}$ characterizes a change of the azimuth α_L only. With these conditions θ_L and β_L are fixed.

Such rotations about that base axis are, by (2.4), generated by

$$\tfrac{1}{2}(J_{03} - J_{12}) = K_3. \tag{3.6}$$

The change of $\tilde{z}_{03}\tilde{z}^*_{12}$ is the change of $\text{const} \times e^{i\alpha_L}$, whatever the change of β_L. $\tilde{z}_{03}\tilde{z}^*_{12} = \text{const}$ characterizes a change of β_L only, it implies a rotation (also of that base axis) about the z axis, generated by

$$\tfrac{1}{2}(J_{03} + J_{12}) = L_3. \tag{3.7}$$

Those are the well-known rotations: K_3 about the base axes, and L_3 about the space-fixed axes.

Let us now discuss the conditions (a) and (b) to represent an object whose resultant (absolute value of) spin corresponds to a lepton of spin $\frac{1}{2}$. For $\vec{L} + \vec{K}$ to correspond to spin $\frac{1}{2}$ we have

$$|j_L - j_K| = \tfrac{1}{2} \tag{3.8}$$

and

$$\text{similar direction of } \vec{L} \text{ and } \vec{K}. \tag{3.9}$$

[As regards the combination of terms (3.3) and as regards $j_L, n_L, m_L, j_K, n_K, m_K$ we speak of "modes" and "mode numbers" to avoid misuse of the terminology "states" and "quantum numbers."] We denote the "average" $j_a = \frac{1}{2}(j_L + j_K)$.

We come to the discussion of the conditions (c) and (d) concerning the spinning of flux loops about their flux orientation axes and concerning Eq. (3.9), as the spinning is supposed to occur about the flux orientation axes, i.e., essentially about the "moving base" axes, we expect

$$|m_L| = j_L, \quad |n_K| = j_K, \tag{3.10}$$

with the appropriate signatures for m_L and n_K to make $\vec{L}$ and $\vec{K}$ about parallel, not antiparallel. These signatures depend on the actual combinations of type (3.3), i.e., on the ones in Eqs. (5.8) and (5.11).

The spinning, and angular momentum about the base axes are essentially determined from terms of the type

$$e^{(im_L\alpha_L)}e^{(in_K\beta_K)}. \tag{3.11}$$

We may summarize that and nicely illustrate the above with the use of wave functions (5.9) and (5.12): We considered $\vec{L}$ as generating a rotation in the space-fixed system, defining a "base"; from there on $\vec{K}$ generated a rotation to the system of a flux loopform, orientated along the flux orientation axis. The conditions (5.9) and (5.12) tell that the spinning motions, as regards both the $\vec{L}$ motion and the $\vec{K}$ motion, represented by the wave function being proportional to

$$e^{i[j_a(\alpha_L - \beta_K)]} \exp\{i[\tfrac{1}{4}(\alpha_L + \beta_K) - (\omega_L \mp \omega_K)t]\}$$

are about the same (base) axis $\beta_L, \theta_L, \alpha_L$. The flux orientations of the loopforms are characterized by the Euler angles β, θ, α (i.e., the combined rotations $\beta_L, \theta_L, \alpha_L$ and then $\beta_K, \theta_K, \alpha_K$) and are symmetrically arranged about the $\beta_L, \theta_L, \alpha_L$ axis for a mode (3.2), (5.9) and (5.12). This wave picture has, of course, to conform to the uncertainty principle, i.e., for a given sharp value of the angular velocity, or a given angular momentum around the $\beta_L, \theta_L, \alpha_L$ directions, the directions of the (flux) orientations of the loopforms are spread out.

The particular forms of muon and of electron wave functions will be discussed in Sec. V. Quite apart from the discussions there, the question of distribution of j_L, j_K is to be handled first in Sec. IV. We shall see that it is reasonable to assume that the j_L distribution as well as the j_K distribution is given (because the wave function's frequencies ω_L and ω_K are determined by j_L and j_K, respectively) by some thermal distribution.

Let us now consider a particular pair j_L, j_K of that statistical distribution of $D^{j_L}_{n_L m_L} D^{j_K}_{n_K m_K}$ pairs of functions. The degeneracy is $(2j_L+1)^2(2j_K+1)^2$ (referring to the 4-parameter manifold of n_L, m_L, n_K, m_K). The conditions (3.10) select a submanifold of that degenerate set, a submanifold of "multiplicity" less than the "degeneracy." We may readily realize that the conditions (3.10) still permit a multiplicity

$$M''_{j_L j_K} = (2j_L + 1)(2j_K + 1).$$

[The double prime on M'' is used to indicate the two factors $(2j+1)$ on the right-hand side of this expression.]

It might be instructive to comment that if we had to discuss, instead of $D^{j_L}_{n_L m_L} D^{j_K}_{n_K m_K}$, a multiplicity of $Y^{j_L}_{m_L} Y^{j_K}_{m_K}$ under similar conditions, i.e., given j_L and j_K and m_L and m_K, instead of the given j_L and j_K with Eq. (3.8) and given (3.10), we would then have no multiplicity. The additional two-mode numbers in $D^{j_L}_{n_L m_L} D^{j_K}_{n_K m_K}$ bring about the above-stated degeneracy.

The condition (b) demands the formation of particular linear combinations of modes (just like a linear transformation to effect a change of basis in the Hilbert space of quantum state functions). That reduces the multiplicity from $M''_{j_L j_K}$ to

$$M_{j_L j_K} = 1. \tag{3.12}$$

We proceed here with the discussion of a level scheme of type Eq. (5.2), i.e., $(j_K, j_L) = (0, \frac{1}{2})$, $(\frac{1}{2}, 1)$, $(1, \frac{3}{2})$, $(\frac{3}{2}, 2)$, etc. A more detailed discussion of the "paired" level scheme $(0, \frac{1}{2})$, $(\frac{1}{2}, 0)$, $(\frac{1}{2}, 1)$, $(1, \frac{1}{2})$, $(1, \frac{3}{2})$, $(\frac{3}{2}, 1)$, $(\frac{3}{2}, 2)$, $(2, \frac{3}{2})$, etc., which implies the use of *all* j_K, j_L pairs whose $|j_L - j_K| = \frac{1}{2}$ will be given in Sec. VI; the pairs $(0, \frac{1}{2})$, $(\frac{1}{2}, 0)$ and the higher pairs may have degenerate frequencies and thus the multiplicity

$$M = 2. \tag{3.13}$$

This is a matter of importance when coherence of excitations of those degenerate mode pairs is to be taken into consideration in calculating the reduction factor N; it will not affect the electron-muon mass ratio.

IV. STATISTICS AND WAVE EQUATION FOR LOOPFORM AMPLITUDES

A. Wave equation[9,12]

How does the wave equation enter into the picture? With the recognition of the requirement of SO(4) invariance we have already come to know the essential properties of the probability-wave amplitudes $D^{j_L}D^{j_K}$. They are characterized by mode numbers $(j_K, j_L) = (0, \frac{1}{2})$, $(\frac{1}{2}, 1)$, $(1, \frac{3}{2})$, etc., or the more complete set (j_K, j_L). But we need also to know the ω as functions of j [cf. remarks before Eq. (4.5)],

$$\omega = \text{``}\omega\text{''}(j_K, j_L) \approx \omega\left(\tfrac{1}{2}(j_L + j_K)\right) = \omega(j_a) = \omega_j \ , \tag{4.1}$$

in order to relate the ω_j to the j_a both in Eqs. (5.13), (5.14), (4.10), and (6.2), and then, to evaluate a statistical distribution (4.5) with an assumed temperature T, (4.8).

Let us give here only a brief sketch of the issue of the wave equation, and refer questions about details to Appendix B. Questions about the exact form of the wave equation are only a secondary issue in the present *parametrized* version of the flux-quantization model of particles, as long as we are interested in approximate results only. That is, of course, not the case in a version in which one attempts a description of flux loopforms in terms of functionals.

In a reasonable approximation one may argue as follows in regard to the relationship $\omega_j = \omega(j_a)$. In view of the fact that dependence of the wave function on the Euler angles is characterized by values j_a which are fairly high, $\langle j_a\rangle_{av} \approx 51$ by Eq. (5.19), and the wave function's dependence on the size parameter σ will be seen to be characterized by a low power κ of σ, i.e., σ^κ [cf. Appendix B, Eqs. (B9)], we may, as a first step, simply ignore the latter and remember the familiar Casimir result that the angular momentum operator $\mathcal{J}^2$ for the spinning top has the eigenvalues $4j(j+1)$ [cf. Appendix B, Eq. (B7)]. Indeed, if a wave equation of the type

$$(\mathcal{J}^2 + \partial^2/\partial t^2)\psi = 0 \tag{4.2}$$

were assumed, the eigenvalues would be

$$\begin{aligned}\omega &= [4j(j+1)]^{1/2}\\ &\approx 2j+1.\end{aligned} \tag{4.3}$$

We might assume such a relation to hold for ω_L as well as for ω_K, and thus for ω_j. Here is ω in units of $m_e c^2/\hbar$ as seen from (5.6) or (5.14).

B. Spin and whirl

The magnetic dipole field of an electron or a muon is represented by a fibration, i.e., represented in terms of loopform manifolds of quantized flux. It is assumed to be a superposition of a right-handed and a left-handed set of torus-loopforms of winding numbers $(2,+1)$ and $(2,-1)$, respectively, so that the superposed sets do not have any handedness. Indeed, a manifold of $(2,+1)$ loopforms hangs together, i.e., is interlinked and forms a Seifert fibration, and the same holds for the one of $(2,-1)$, whereas the ordinary dipole field lines $(1,0)$ are apt to "fall apart" if we consider the Maxwell stress tensor as applicable to the situation. These loopforms $(2,\pm 1)$ are then assumed to spin about their common straight central axis and whirl about their common circular torus axis, both with angular velocities $2mc^2/\hbar$ (*Zitterbewegung*).

$D^{j_L}D^{j_K}$ modes of types $(2,+1)$ and those of type $(2,-1)$ are substates which are to be superposed to form a quantum state, in a manner analogous to superposition of alternative path histories. Spin and whirl are simultaneous motions of any one mode, one does not superpose spin and whirl.

Loopform manifolds of type $(2,1)$ rather than of type $(1,2)$ are considered for the electron or muon because the magnetic-field lines of type $(2,1)$ are considerably less densely spaced than those of $(1,2)$. Inasfar as a requirement for minimal electromagnetic energy may be applicable, the $(2,1)$ have a distinct advantage.

The $(1,2)$ loops imply the same electric charge and half of the magnetic moment of the $(2,1)$ loops; because of the higher flux-line density, they might represent heavy leptons. Still another type of heavy leptons was listed in paper C, Table II, with winding numbers $(5,1)$.

Actually we have not only spinning but also whirling. As the whirling motion (cf. paper C) of a "smooth" torus loop of winding numbers $(2,1)$ is equivalent to a spinning motion of half its angular velocity, the whirling degree of freedom may be mapped on the spinning degree of freedom. If whirling is to be taken into consideration besides spinning, that implies a multiplication of the angular velocity of lepton loops by the factor $(1+\frac{1}{2})=\frac{3}{2}$. [In paper C it was shown that this additive motion of spin and whirl implies the Coulomb field e/r for the muon or the electron, while the subtractive motion $(1-\frac{1}{2})=\frac{1}{2}$ implies the $\mathfrak{N}$ quark field $e/3r$.] Let us now translate the simplified model (spinning only) into the model of "effective" spinning, i.e., inclusion of the effect of whirling, too, implying the angular velocity $3mc^2/\hbar$.

Spinning and whirling, which, in general, are independent degrees of freedom, share, of course, the same two singular lines (central straight axis and circular axis) of the Seifert fibration, and are formulated in terms of the same set of Euler angles β, θ, α (and one size parameter σ). In our parametrized model, whirling may be mapped on spinning motion, involving only one set of Euler angles, and the *same* wave functions DD for the probability amplitudes for spinning and for whirling motion. These DD functions (3.3) or (5.7)–(5.9) and (5.10)–(5.12), which formulate the probability amplitudes of the Euler angles, show, in reference to the spinning motion $(\omega_L \pm \omega_K)$ and the spin angular momentum $\frac{1}{4}+\frac{1}{4}=\frac{1}{2}$.

The effective spinning angular velocities of the flux loopforms, however, have contributions from spinning and from whirling motion; in the present parametrized model the whirling contribution is to be accounted for by merely multiplying the angular velocities due to spinning by the factor $(1+\frac{1}{2})=\frac{3}{2}$, resulting in [cf. Eq. (5.14)]

$$(\Omega_e)_{\rm eff}=3m_e c^2/\hbar. \tag{4.4}$$

As regards electron and muon neutrinos we remarked that they are to be represented by right-handed trefoils $(3,-2)$, which, as is readily visualized, move in left-hand helical manner through space like coasting three-bladed propellers. Their handedness being invariant with regard to any Lorentz transformation, they have no rest system, and thus zero rest mass. In any field-theoretical representation of a $\mu\to e$ decay we would have to see to it that the combination of (internal) high frequencies, high wave numbers of the μ^- and $\bar{\nu}_\mu$ in some way match the combination of (internal) low frequencies, low wave numbers of the e^- and $\bar{\nu}_e$, a simple 4-fermion interaction.

Spin and whirl are of special interest in the case of quarks where the whirling motion "subtractively" detracts from the spinning motion. In determining the magnetic moment of a hadron, one applies the SU(6) function in the usual way. But instead of assuming a quark's contribution to magnetic moment as proportional to its electric charge, it is assumed as being proportional to the number of "core traverses" of the quark loop times the quantized flux times $\hbar/m_{\rm meson}c$ or $\hbar/m_{\rm baryon}c$. One may therefore determine the equivalent electric charges of quark loops of winding numbers $(2,1),(3,1),(3,2)$ which have, apart from whirling, the spinning angular velocities $2m_{\rm meson}c^2/\hbar$ or $2m_{\rm baryon}c^2/\hbar$, respectively. Because of subtractive spinning-whirling motion, there pass, per spinning period, $2-1=1$ of the $(2,1)$ wings, $3-1=2$ of the $(3,1)$ wings, $3-2=1$ of the $(3,2)$ wings over a fixed point, resulting in the charges e times $\frac{1}{3},\frac{2}{3},\frac{1}{3}$. The masses cancel out again as they do in the lepton calculations.

One may add here two supplementary remarks in regard to quark confinement and quark localizability. In the flux loop model, meson or baryon quarks are two or three interlinked torus loopform manifolds which share the same central spin axis and the same circular whirl axis of the torus; they are, therefore, prevented from falling apart (cf. papers B or C, and Appendix D of the present paper). (The size of the circular torus axis is again assumed to be of the size of the *Zitterbewegung*, of the order of magnitude $\hbar/m_{\rm hadron}c$.) Their interlinkage also makes the quarks localizable objects which do not need to be represented by antisymmetric functions; there is thus no need to introduce color.

C. Statistical distribution of modes

We pointed out already that the structuralization of the distribution of loopforms is essential to effect, by interference, a reduction of quantized flux to effective flux. [This concerns the condition (e).] Such structuralization, which was accomplished by bundling of loopforms in the heuristic model, may now be formulated by means of superposition of a sizable number of mode excitations (excitons) (3.3), similar to photon excitations over a given frequency (ω_j) spectrum. Coherence compared to random-phasedness of excitations characterizes $2\Phi_q$ compared to $\Phi_{\rm eff}$, i.e., N, which is also the number designating the average occupancy per mode. Corresponding to the bundling we expect now the participation of wave functions (3.3) coming from j_K, j_L values $(0,\frac{1}{2}),(\frac{1}{2},1),(1,\frac{3}{2})$, etc. or, as pointed out in Eq. (6.25), from values $(0,\frac{1}{2}),(\frac{1}{2},0)$, $(\frac{1}{2},1),(1,\frac{1}{2})$, respectively, the contributions tapering off at higher j values.

What possibilities are there to effect such a distribution? Even though the $D^{j_L}_{n_L m_L}D^{j_K}_{n_K m_K}$, for a permitted combination of $j_L, n_L, m_L, j_K, n_K, m_K$ are not quantum states, but modes which, superposed, are to represent a quantum state, we may assume that these modes are statistically excited corresponding to a temperature T, to be denoted also by β^{-1}. We assume that this *internal* temperature T is in some way related to the concept of the Hagedorn[10] temperature $T_H\approx 130$ to 190 MeV, the latter however referring to a *limit* temperature in hadronic reactions (due to meson production). The circumstance that a wave equation relates the j to the frequencies ω, where approximately $\omega\approx 2j+1$ for ω_L as well as for ω_K, permits us to make obvious assumptions about such a temperature distribution.

Whether one might choose a Maxwell-Boltzmann distribution or, better, a photon distribution law (Bose-Einstein), a preliminary question presents itself:

Which frequency should be distributed by such a thermal distribution? Should it be $\omega_L+\omega_K$ or $\omega_L-\omega_K$ or, in view of $|j_L-j_K|=\frac{1}{2}$, i.e., $\omega_L\approx\omega_K$, simply an average $\omega_j=\frac{1}{2}(\omega_L+\omega_K)\approx\omega_L\approx\omega_K$? As the electron and the muon are to arise from the same statistical distributions of the (ω_L,ω_K) pair, the internal temperature T should not be assumed to imply distributions $e^{-(\omega_L+\omega_K)/T}$ or $(e^{(\omega_L+\omega_K)/T}-1)^{-1}$ and $e^{-(\omega_L-\omega_K)/T}$ or $(e^{(\omega_L-\omega_K)/T}-1)^{-1}$ for the muon and the electron, respectively; instead, one should let $\omega_L\approx\omega_K$ be distributed according to that temperature T. To simplify, we may consider the average, i.e., $\omega_j\approx 2j_a+1$ to be thermally distributed.

The following is an independent argument to the same effect: Because the condition $|j_L-j_K|=\frac{1}{2}$ implies $\omega_L\approx\omega_K$, the ω_L distribution and the ω_K distribution are not independent; a Boltzmann distribution $e^{(-\beta\omega_L)}e^{(-\beta\omega_K)}$ or a corresponding product distribution of photon statistics, $(e^{\beta\omega_L}-1)^{-1}(e^{\beta\omega_K}-1)^{-1}$ are therefore not admissible with restricted ω_L, ω_K. Instead, we assume that the probability

$$P_\lambda = P_{\omega_j}$$
$$= P((D^{j_L}_{n_L m_L} D^{j_K}_{n_K m_K})^*(D^{j_L}_{n_L m_L} D^{j_K}_{n_K m_K})) \qquad (4.5)$$
$$= e^{(-\beta\omega_j)} \text{ or} \qquad (4.5\text{MB})$$
$$= (e^{\beta\omega_j}-1)^{-1} \qquad (4.5\text{ph})$$

(where MB indicates the Maxwell-Boltzmann distribution) characterizes the statistical distribution of (j_L, j_K) modes, labeled by ω_j or by λ. These $P_\lambda = P_{\omega_j}$ [λ, standing for some loopform "bundle" of paper A (see paper C, is now characterized by j_L, n_L, m_L, j_K, n_K, m_K] indicate the expectation values for occupancies of the modes. For photon-type distributions (4.5ph) $P_{\omega_j} = \langle n_j\rangle_{\text{ex}}$, i.e., the expectation value for the number of random-phased excitons on the modes (cf. Sec. VI). With

$$(P_\lambda =) P_{\omega_j} = \langle |\vec{r}_j|^2\rangle_{\text{ex}}$$

the mode superposition may be denoted by

$$\sum_j \vec{r}_j D^{j_L}_{n_L m_L} e^{-i\omega_L t} D^{j_K}_{n_K m_K} e^{\mp i\omega_K t}, \qquad (4.6)$$
$$\omega_j = \tfrac{1}{2}(\omega_L + \omega_K)$$
$$\approx \tfrac{1}{2}(2j_L + 1 + 2j_K + 1).$$

We note that for the distribution of actual quanta such as photons, over *quantum states* ω_j, the energy of the ensemble of quanta has the expectation value $\langle n_j\rangle_{\text{ex}}\omega_j$ for the state j, and $\sum_{\text{states}}\langle n_j\rangle_{\text{ex}}\omega_j$ all in all. For the present distribution of excitons (not quanta) over *modes* ω_j (i.e., substates j), the energy is assumed as

$$\langle\omega_j\rangle_{\text{av}} = \frac{\sum_{\text{modes}}\langle n_j\rangle_{\text{ex}}\omega_j}{\sum_{\text{modes}}\langle n_j\rangle_{\text{ex}}}.$$

The statistical average of the frequencies ω_j is then given by

$$\langle\omega_j\rangle_{\text{av}} = \sum_{\text{permitted modes}} \omega_j e^{-\beta\omega_j} \Big/ \sum e^{-\beta\omega_j}$$
$$= \sum_{\text{frequencies}} \omega_j M e^{-\beta\omega_j} \Big/ \sum M e^{-\beta\omega_j} \qquad (4.7\text{MB})$$

or

$$\langle\omega_j\rangle_{\text{av}} = \sum_{\text{frequencies}} \omega_j M(e^{\beta\omega_j}-1)^{-1} \Big/ \sum M(e^{\beta\omega_j}-1)^{-1}. \qquad (4.7\text{ph})$$

We should remember that, as the electron frequency

$$\langle\omega_L - \omega_K\rangle_{\text{av}} \approx \langle(2j_L+1)-(2j_K+1)\rangle_{\text{av}}$$
$$= 2(j_L - j_K) = 1$$

i.e., in units of $m_e c^2/\hbar$, the muon frequency is $\langle\omega_L + \omega_K\rangle_{\text{av}} = 2\langle\omega_j\rangle$ in the same units. The multiplicity factor $\mathfrak{M}$ defined in (3.12) and (3.13), cancels out.

We prefer the photon distribution rather than the Maxwell-Boltzmann (MB) distribution for the following reason: Eq. (4.5MB) gives for the muon-mass to electron-mass ratio a value $\langle 2\omega_j\rangle_{\text{av}} = 2/\beta$ which, with

$$\beta^{-1} = T$$
$$= \frac{190\text{ MeV}}{0.511\text{ MeV}}$$
$$\approx 370, \qquad (4.8)$$

is 740, i.e., three times too high. More important still, as the noncoherent excitons occupying any mode j are indistinguishable, and are not individualizable nor localizable particle, MB statistics does not apply to them. In FD (Fermi-Dirac) statistics, a reduction cannot be defined at all by a comparison of random-phased excitations with coherent excitations. We use Planck-Bose-Einstein statistics; only the concept of indefinite number of excitons is meaningful, and only this indefiniteness permits us to apply the arguments of coherence and of fluctuations.

When we adopt photon statistics for the statistical distribution of modes which are superposed to represent a quantum state of a lepton, or of a quark, such choice of photon statistics does not imply any statement as regards the statistics which the resulting lepton or the resulting quark is to obey.

The photon statistics now results in [cf. (5.2)]

$$2\langle\omega_j\rangle_{\text{av}} \approx 2\beta^{-1}\sum_{x=3\beta/2}^{\infty} x(e^x-1)^{-1} \Big/ \sum_{x=3\beta/2}^{\infty}(e^x-1)^{-1}, \qquad (4.9)$$

where

$$x = \beta\omega_j$$
$$\approx \beta(2j_a+1)$$
$$= \tfrac{3}{2}\beta, \tfrac{5}{2}\beta, \tfrac{7}{2}\beta, \ldots \qquad (4.10)$$

and where

$$\sum_{\omega_j} \equiv \sum_x. \qquad (4.11)$$

Inserting (4.10) into (4.9) gives the expectation value $\langle\beta\omega_j\rangle_{\text{av}} = \langle x\rangle_{\text{av}}$ in terms of the limits ($x = \frac{3}{2}\beta$) and steps ($\Delta x = \beta$) of the right-hand-side sums. With an assumed value for $T \approx 190$ MeV, $\beta^{-1} \approx 370$, one gets

$$m_\mu/m_e \approx \langle 2\omega_j \rangle_{av} = 206 \quad (4.12)$$

very crudely, because the relation (4.10) is only an approximation. Towards the end of Sec. VI it is indicated that a close relationship between T_H and the mass of the muon (both in terms of the mass of the electron) is to be expected simply because of the the close relationship between pions and muons in the flux loop model.

This same value $\beta^{-1} \approx 370$ [Eq. (4.12)] also yields, in Sec. VI, a value of the reduction number N and thereby the electromagnetic coupling constant $e^2/\hbar c$. It is essentially with this one parameter β that one is able to estimate three important numbers. The flux-quantization model shows that the question of the value of m_μ/m_e and the question of the value of $e^2/\hbar c$ are part of one and the same issue, i.e., the structuralization of the leptons. Sections V and VI show the interdependence of these two numbers, and with an internal temperature T, and Appendix D shows how the corresponding structuralization of quarks, i.e., the consideration of their probability-amplitude functions as a superposition of a certain number of mode functions, permits one to estimate the rate of weak interactions.

The wave equation as well as the ω spectra are *common* to the electron and the muon. A strict evaluation of the statistical distribution of the ω's and of their average necessarily needs the knowledge of the wave equation to obtain the spectrum $\omega(j)$ of its solutions. In order to get *accurate* results in the following sections, the wave equation and the corresponding spectrum are indeed needed, both for Eqs. (5.13) and (5.14) in Sec. V as well as for Eq. (6.2) in Sec. VI.

The solutions of the wave equation have, however, in their essential features, already been obtained because we have seen that they have to be representations of the SO(4) group. These solutions may therefore be written down explicitly in Sec. V for muon and electron waves. We may thus obtain the essential results, approximately only, without detailed knowledge of the wave equation and its exact solutions.

V. MUON-ELECTRON DICHOTOMY

One may consider the ratio of the mass of a muon to that of an electron to be given by the two different frequencies $\omega_L + \omega_K$ and $\omega_L - \omega_K$ of corresponding waves of types $D^{j_L} D^{j_K}$ [cf. Eqs. (5.7) and (5.10) below]. More precisely, by the statistical average of the distribution of $\omega_L + \omega_K$ and that of $\omega_L - \omega_K$, with $\omega \approx (2j+1)$ and $|j_L - j_K| = \frac{1}{2}$, the ratio, to be statistically averaged, is

$$(\omega_L + \omega_K)/(\omega_L - \omega_K) \approx (2j_L + 1 + 2j_K + 1)/1 \approx 2(2j_a + 1). \quad (5.1)$$

A. Choice of muon and electron wave functions

Let us consider, for the pairs of j values [cf. Eq. (3.8) and Sec. VI],

$$(j_K, j_L) = (0, \tfrac{1}{2}), (\tfrac{1}{2}, 1), (1, \tfrac{3}{2}), (\tfrac{3}{2}, 2), \text{etc.} \quad (5.2)$$

the modes

$$D^{j_L} D^{j_K}, \quad (5.3)$$

with the notation

$$D^j_{nm}(\beta, \theta, \alpha) = \alpha^j_{nm}(\cos\theta) e^{i(n\beta + m\alpha)}, \quad (5.4)$$

$$|m_L| = j_L, \quad |n_K| = j_K, \quad (3.10)$$

$$j_a = \tfrac{1}{2}(j_L + j_K),$$

$$j_a - j_L = j_K - j_a = \tfrac{1}{2}(j_K - j_L) = \tfrac{1}{4}, \quad (5.5)$$

$$\omega_K + \omega_L \approx 2(2j_a + 1), \quad \omega_K - \omega_L = 1, \quad \omega > 0. \quad (5.6)$$

We suggest, for the muon wave function, modes like

$$D^{j_L}_{n_L j_L} e^{-\omega_L t} D^{j_K}_{j_K m_K} e^{-i\omega_K t} \quad (5.7)$$

$$\propto e^{i(j_L \alpha_L - j_K \beta_K)} e^{-i(\omega_L + \omega_K)t} \quad (5.8)$$

$$= e^{i[j_a(\alpha_L - \beta_K)]} \exp\{i[\tfrac{1}{4}(\alpha_L + \beta_K) - (\omega_L + \omega_K)t]\}, \quad (5.9)$$

and, for the electron wave functions, modes like

$$D^{j_L}_{n_L j_L} e^{-i\omega_L t} D^{j_K}_{j_K m_K} e^{+i\omega_K t} \quad (5.10)$$

$$\propto e^{i(j_L \alpha_L - j_K \beta_K)} e^{-i(\omega_K - \omega_K)t} \quad (5.11)$$

$$= e^{i[j_a(\alpha_L - \beta_K)]} \exp\{i[\tfrac{1}{4}(\alpha_L + \beta_K) - (\omega_L - \omega_K)t]\}. \quad (5.12)$$

As the third Euler angle α_L in the space-fixed system (L) and the first Euler angle β_K in the base system (K) amount to the same kind of rotation (the resultant rotation being $\alpha_L + \beta_K$), the resultant angular *momentum* about the base (flux loop bundle) axes (for both the muon and the electron) is equal to $\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$; also the total angular momentum $|j_L - j_K| = \frac{1}{2}$.

One may note that the angular *velocities* of the L and of the K contributions add in the case of the muons and subtract in the case of the electrons, because the frequencies ω_L, ω_K of the muons have the same signatures, whereas those $\omega_L, -\omega_K$ for the electrons are opposite. Angular momentum and angular velocity have a different relationship to each other when both frequencies have equal signatures (5.7) than when they have opposite signatures (5.10).

[A similar situation is encountered and is essential for the understanding of the Pryce-Foldy-Wouthuysen-Tani transformation to Lorentz transformations.[11] It was recognized that the most general $\vec{p}\neq 0$ Dirac wave function of a free electron is obtained from the $\vec{p}=0$ wave function by simultaneously transforming the positive-frequency (or upper two) components by a Lorentz transformation with $\vec{v}_{(+)}=-\vec{p}/|E|$, and the negative-frequency (or lower two) components by a Lorentz transformation with $\vec{v}_{(-)}=+\vec{p}/|E|$. The result amounts to a FW (Foldy-Wouthuysen) transformation.]

These angular velocities are

$$\begin{aligned}\Omega_\mu &= \tfrac{3}{2}(\omega_L+\omega_K)/(\tfrac{1}{4}+\tfrac{1}{4})\\ &\simeq 2\times\tfrac{3}{2}\times(2j_L+1+2j_K+1)\\ &=2\times 2\times\tfrac{3}{2}\times(2j_a+1)\\ &=12j_a+6,\end{aligned}\tag{5.13}$$

$$\begin{aligned}\Omega_e &= \tfrac{3}{2}(\omega_L-\omega_K)/(\tfrac{1}{4}+\tfrac{1}{4})\\ &\simeq 2\times\tfrac{3}{2}\times(2j_L-2j_K)\\ &=2\times 2\times\tfrac{3}{2}\times\tfrac{1}{2}\\ &=3.\end{aligned}\tag{5.14}$$

They are, of course, in units of $m_e c^2/\hbar$, as is seen from $\Omega_e=3$ and because consistency conditions (see paper A, Sec. VI, and paper C, Appendix B) demanded that the electromagnetic energy be $=mc^2$ if $\Omega=3$ in units of $mc^2/\hbar$. The electron wave functions have the frequency $\omega_L-\omega_K=1$ [cf. (5.6)].

One should note that it is the same spectrum of $\omega\approx(2j+1)$ values which applies to the muon as well as to the electron.

B. Statistical distribution of $\omega \approx 2j+1$

The irreducible representations (j_L, j_K) which make up the regular representation of the SO(4) form a manifold of probability-amplitude modes from which we form a statistical distribution. These irreducible representations (j_L, j_K) are $(2j_L+1)^2(2j_K+1)^2$-fold degenerate because the frequencies ω_L, ω_K depend only on j_L, j_K, not on all the labels $j_L, n_L, m_L, j_K, n_K, m_K$. The degeneracy of a (j_L, j_K) set disappears because of the supplementary conditions

$$|m_L|=j_L,\quad |n_K|=j_K,$$

and

$$|\text{probability amplitude}|\,\text{distribution} \to [1+\cos(\hat{\xi},\hat{z})]^{1/2}. \tag{5.15}$$

Accordingly, the statistical distribution of (j_L, j_K) values is to be taken over the permitted modes, which, because of $|j_L-j_K|=\frac{1}{2}$, are [if as a first step we disregard the pairings such as $(0,\frac{1}{2})$, $(\frac{1}{2},0)$]

$$(j_K, j_L)=(0,\tfrac{1}{2}),(\tfrac{1}{2},1),(1,\tfrac{3}{2}), \tag{5.2}$$

thus

$$j_a=\tfrac{1}{2}(j_L+j_K)=\tfrac{1}{4},\tfrac{3}{4},\tfrac{5}{4}. \tag{5.16}$$

The permitted modes correspond to the frequencies ω_j,

$$\begin{aligned}(j_L, j_K)&\to\omega_j\\ &\approx(2j_a+1)\\ &=\tfrac{1}{2}(2j_L+1+2j_K+1).\end{aligned}\tag{5.17}$$

With the assumed statistical distribution of (j_L, j_K), i.e., of ω_j, one obtains $\langle 2\omega_j\rangle_{\rm av}\approx 206$. The notation ω_j designates average between L and K, and the subscript "av" refers to the statistical distribution over the modes j. The partition function is

$$Z=\prod_{1/4,\,3/4,\,5/4}(1-e^{-\beta(2j_a+1)})^{-1}. \tag{5.18}$$

The temperature

$$T\approx 190\text{ MeV},\quad\text{i.e.,}\quad \beta^{-1}\approx 370, \tag{4.8}$$

implies

$$(2\langle j_a\rangle_{\rm av}+1)\approx\langle\omega_j\rangle\approx 103,\quad \langle j_a\rangle_{\rm av}\approx 51; \tag{5.19}$$

$$\langle\Omega_e\rangle_{\rm av}=3,\quad \langle\Omega_\mu\rangle_{\rm av}\approx 12\langle j_a\rangle_{\rm av}+6\approx 620. \tag{5.20}$$

The muon-electron mass ratio is thereby again

$$\begin{aligned}m_\mu/m_e&=\langle\Omega_\mu\rangle_{\rm av}/\langle\Omega_e\rangle_{\rm av}\\ &=\langle\omega_L+\omega_K\rangle_{\rm av}/\langle\omega_L-\omega_K\rangle_{\rm av}\\ &\approx\langle 4j_a+2\rangle_{\rm av}\\ &\approx 206,\end{aligned}\tag{5.21}$$

which is, of course, simply due to the assumed temperature T by a very crude calculation.

VI. ELECTROMAGNETIC INTERACTION CONSTANT

At the same time, this internal temperature permits an estimate of the fine-structure constant $e^2/\hbar c$. We had

$$\begin{aligned}(e^2/\hbar c)[(\hbar/mc)/6.2r_0]&=\Phi_{\rm eff}/2\Phi_q\\ &=N^{-1}.\end{aligned}\tag{6.1}$$

The concept of reduction in a theory based on flux loops has to define reduction in terms of an operation on the manifold of loopforms, i.e., loopforms of parameters $\beta,\theta,\alpha,\sigma$ being the elements of the manifold. Only indirectly does that operation state something about reduction of field quantities such as $\vec{B}(x,y,z,ct)$ and others. Directly, the reduction makes statements about the effect of random-phased probability-amplitude superpositions (of loopform excitations), excitations of modes of frequencies ω_j, and functions of the parameters $\beta, \theta, \alpha, \sigma$. The reduction states then in which manner global, integrated (over the parameters) quantities such as total magnetic flux get reduced, explaining such ratios as $\Phi_{eff}/2\Phi_q$ in Eq. (6.1).

Instead of basing the reduction formalism on some probability-amplitude distributions of loopform parameters $|(\beta,\theta,\alpha,\sigma))\equiv|(\lambda))$ and superposing them as done in Appendix B of paper C, we consider now, more appropriately, the mode (ω_j) excitations (excitons) which add up to mode amplitudes $\vec{r}_j D^{jL} D^{jK}$, (4.6), similar to photon excitations of the respective electromagnetic modes of frequency ω_j, and compare noncoherent with coherent superposition.

The expectation value of the occupancy of a mode in a Planck-Bose-Einstein distribution of an unspecified number of excitons is

$$\begin{aligned}\langle|\vec{r}_j|^2\rangle_{ex} &= \langle n_j\rangle_{ex}\\ &= (e^{\beta\omega_j}-1)^{-1}. \end{aligned} \tag{6.2}$$

The expectation values for all the kth moments of such a PBE distribution are given by

$$\langle n_j^k\rangle_{ex} = Z^{-1}(-\beta^{-1}\partial/\partial\omega_j)^k Z, \tag{6.3}$$

so that

$$\begin{aligned}\langle n_j^2\rangle_{ex} &= 2\langle n_j\rangle_{ex}^2 + \langle n_j\rangle_{ex}\\ &\approx 2\langle n_j\rangle_{ex}^2. \end{aligned} \tag{6.4}$$

The last approximation is good for the important low modes at the temperature in question; similarly

$$\langle n_j^3\rangle_{ex} \approx 6\langle n_j\rangle_{ex}^3. \tag{6.5}$$

It will be advantageous to express the statistical distribution of the occupancies n_j of a mode j in terms of distributions of corresponding probability amplitudes $\vec{r}_j$, using Gaussians, for two reasons: first, to simplify the formalism by which the statistical distribution of n_j over all the different modes j may be calculated by a Markoff random walk in the complex plane, and second, to compare noncoherent with coherent exciton amplitudes. We start with that problem.

Noncoherence versus coherence may be formulated in terms of probability-amplitude vectors $\vec{r}_j$ (vectors in the complex plane) of the excitons of any one mode j, those being added with random phases versus those being virtually added with equal phases.

We consider first a nondegenerate level scheme such as (5.2) and consider later, on the basis of Sec. VI, coherences between degenerate modes such as $(0,\frac{1}{2})$ and $(\frac{1}{2},0)$, cf. Eq. (6.24).

We therefore should start to sketch what results if the unit vectors representing the excitons of one mode j get added with random phases. A given, large number of those unit vectors add then to give approximately a Gaussian distribution of the resultant vector $\vec{r}_j$ [cf. S. Chandrasekhar's presentation of Markoff's method in random-walk problems, Rev. Mod. Phys. 15, 8 (1943), where, however, the denominator in Eq. (63) should be $2\pi l_j$]. If, in addition to the random-phase distribution, we consider not only a fixed number of such unit vectors, but a statistical distribution of numbers n_j of unit vectors, we still find the resultant to be a Gaussian. Let its variance be denoted by l_j^2. The Gaussian distribution of those resultant mode amplitude vectors in the complex plane, of length $r_j\equiv|\vec{r}_j|$, has the expectation values

$$\langle r_j^k\rangle_{ex} = \int_0^\infty r_j^k l_j^{-2} e^{-(r_j/l_j)^2} 2r_j\,dr_j, \tag{6.6}$$

$$\langle r_j^2\rangle_{ex} = l_j^2,\quad \langle r_j^4\rangle_{ex} = 2l_j^4,\quad \langle r_j^6\rangle_{ex} = 6l_j^6, \tag{6.7}$$

which means that we may approximate the moments $\langle n_j\rangle_{ex}, \langle n_j^2\rangle_{ex}, \langle n_j^3\rangle_{ex}$ by the Gaussian moments $\langle r_j^2\rangle_{ex}, \langle r_j^4\rangle_{ex}, \langle r_j^6\rangle_{ex}$ and correspondingly, for the respective distribution functions. The high dispersion of the photon distribution is properly taken care of by the Gaussian distribution in the complex plane.

This refers to any one mode. The superposition of vectors $\vec{r}_j$ from different modes, if performed randomly, should again give an estimate of

$$\sum_j \langle n_j\rangle_{ex} = \sum_j \langle r_j^2\rangle_{ex}, \tag{6.8}$$

but we should be careful when it comes to studying fluctuations of $\sum_j n_j$ (which is a problem different from that of studying fluctuations of n_j inside a mode), and when, in Appendix D, it comes to calculating energy fluctuations due to fluctuations of all the n_j.

Leaving questions of energy fluctuation to Appendix D, we return now to estimate the reduction factor N, using the computed values

$$\sum_j \langle n_j\rangle_{ex} = 2177,\quad \sum_j \langle n_j^2\rangle_{ex} = 259\,900.$$

The *random*-phased addition of unit exciton vectors leads to the mode amplitudes $\vec{r}_j$. They, in turn getting added with random phases, result in

$$\left\langle\left\langle\sum_j \vec{r}_j\right\rangle_{ex}^2\right\rangle_{av} \approx \sum_j \langle|\vec{r}_j|^2\rangle_{ex} \equiv \sum_j \langle r_j^2\rangle_{ex} = \sum_j \langle n_j\rangle_{ex}. \tag{6.9}$$

The "ex" denotes the expectation value for the distribution in any one mode, and the "av" denotes the averaging (mainly phase averaging) of the distribution over j of the $\vec{r}_j$ vectors.

In order to find the statistical distribution of the resultant vector

$$\vec{R}=\sum_j \vec{r}_j \tag{6.10}$$

the Markoff method readily gets the answer. We start with the normalized probability distribution where

$$l_j^2=\langle r_j^2\rangle_{ex}=\langle n_j\rangle_{ex}, \tag{6.11}$$

$$\begin{aligned}(d^2\vec{r}_j)\tau_j &= r_j\,d\varphi_j\,dr_j(2/2\pi l_j^2)e^{-(r_j/l_j)^2}\\ &= d(r_j/l_j)^2 e^{-(r_j/l_j)^2}.\end{aligned} \tag{6.12}$$

The product of the Fourier transforms of those distributions,

$$\begin{aligned}A(\vec{\rho}) &= \prod_j \int\int r_j\,d\varphi_j\,dr_j\,\tau_j\,e^{(i\vec{\rho}\cdot\vec{r}_j)}\\ &= \prod_j \left\{\pi^{-1/2} l_j^{-1} \times\int_{-\infty}^{+\infty} dx_j\,\exp[-(x_j/l_j)^2+i\rho_x x_j]\right\}\{(y)\}\\ &= \prod_j \exp(-l_j^2\rho^2/4)\\ &= \exp\left(-\tfrac{1}{4}\rho^2\sum_j l_j^2\right)\\ &= \exp\left(-\tfrac{1}{4}\rho^2\sum_j \langle n_j\rangle_{ex}\right)\\ &\approx e^{-2177\rho^2/4},\end{aligned} \tag{6.13}$$

gives, by the inverse Fourier transformation, the distribution of the resultant $\vec{R}$,

$$\begin{aligned}W(\vec{R})d\vec{R} &\to (d\vec{R}/4\pi^2)\int\int d\vec{\rho}A(\vec{\rho})e^{-i\vec{\rho}\cdot\vec{R}}\\ &= (2\pi R dR/2177\pi)e^{-R^2/2177}.\end{aligned} \tag{6.14}$$

For the *coherent* addition of these same unit vectors, we define such addition by assuming that the mode occupancies are not changed in comparison with the coherent occupancies. The n_j unit vectors of any one mode are added in phase, so that the expectation value of the square of the arithmetic sum, i.e., $(\sum_{excitons}|\vec{r}_{unit}|)^2$, is now no longer equal to $\langle n_j\rangle_{ex}$ but equal to $\langle n_j^2\rangle_{ex}$. The sum of the coherent occupancies $\langle n_j^2\rangle_{ex}$ may now again be approximated by a random-phased addition of the mode vectors whose (lengths)^2 are now $=n_j^2$ so that the expectation values l_j^2 are now $=\langle n_j^2\rangle_{ex}$ (instead of the $\langle n_j\rangle_{ex}$ for noncoherent superposition. The Markoff method then leads to

$$\begin{aligned}A(\vec{\rho}) &= \exp\left(-\tfrac{1}{4}\rho^2\sum_j \langle n_j^2\rangle_{ex}\right)\\ &\approx e^{-259\,900\rho^2/4}\end{aligned} \tag{6.15}$$

and

$$W(\vec{R})d\vec{R}\to(2\pi R dR/259\,900\pi)e^{-R^2/259\,900}. \tag{6.16}$$

The reduction factor N is then simply the ratio of the $\langle R^2\rangle$ of the coherent distribution to that of the noncoherent distribution, which is equal to

$$\tfrac{259\,900}{2177}\approx 119.4. \tag{6.17}$$

So far, we have considered only (j_K, j_L) values $(0,\frac{1}{2}), (\frac{1}{2},1)$, etc. But we should now consider the degenerate pairs $(j_K, j_L)=(0,\frac{1}{2})$ and $(\frac{1}{2},0)$, $(\frac{1}{2},1)$ and $(1,\frac{1}{2})$, etc., which do not affect the average frequency

$$\begin{aligned}\langle\omega_\mu\rangle_{av} &= \langle\omega_L+\omega_K\rangle_{av}\\ &= \langle 2\omega_j\rangle_{av}\\ &= \sum_j 2\omega_j\langle n_j\rangle_{ex}\Big/\sum_j\langle n_j\rangle_{ex}.\end{aligned} \tag{6.18}$$

But, with the degeneracies introduced with the pairing modes (cf. end of Sec. VI), a virtual coherent excitation now implies coherence of all excitons which belong to the same frequency ω_j, i.e., in both $(0,\frac{1}{2})$ and $(\frac{1}{2},0)$ modes, and, similarly, implies coherence of all excitons in $(1,\frac{1}{2})$ and $(\frac{1}{2},1)$ modes, etc. Thus, the reduction factor

$$N=\frac{\sum_{\text{levels}\,j}\langle n_j^2\rangle_{ex}}{\sum_{\text{levels}}\langle n_j\rangle_{ex}} \tag{6.19}$$

(where a level means a mode pair, cf., end of Sec. VI) becomes twice as large, i.e.,

$$N=2\times 119.4=239. \tag{6.20}$$

The radius of the core equator (i.e., of the circular "axis" of the Seifert-Threlfall fibration) is $1.23r_0$, where r_0 is the root mean square of the Gaussian distribution of "magnetization" (cf. Sec. V of A, and C). This $1.23r_0$ may be taken to be equal to $\frac{1}{3}\hbar/mc$. In the parametrized description, the whirl motion is completely mapped onto the spin motion, and consequently a (2, 1) loop has the effective spinning angular velocity $3mc^2/\hbar$, Eq. (4.4). The linear velocity owing to effective spinning, at the core equator, is assumed to be

$$(\tfrac{1}{3}\hbar/mc)(3mc^2/\hbar)=c. \tag{6.21}$$

That obvious choice for r_0, i.e.,

$$r_0=(\hbar/mc)/3\times 1.23, \tag{6.22}$$

therefore leads approximately to

$$e^2/\hbar c = N^{-1}\times 6.2\, r_0/(\hbar/mc)$$
$$= 6.2/239\times 3\times 1.23$$
$$= \tfrac{1}{142}. \qquad (6.23)$$

As the accurate formulation of the model, in particular, the form of the wave equation, has not yet been given, the essential task consisted in the formulation of physically meaningful assumptions, and the development of the model, in particular the checks on its consistency. Accurate numerical results would, at present, be a poor criterion of the achievements of the flux loop model.

Even though we have here to deal with a superposition of internal modes, giving rise to one quantum state, i.e., the electron or the muon, so that the possibility of using the concept of temperature is not obvious, it is most interesting that this temperature may be understood in relation to the Hagedorn temperature.

Indeed, in the present model, both the leptons and their electromagnetic interactions, as well as the quarks of the hadrons and their strong interactions, are formulated in terms of quantized flux loops (i.e., entirely in terms of electromagnetic phenomena). Both the internal relationship between leptonic flux loops, as well as the strong interactions of hadrons in Hagedorn's theory,[10] imply time spans which permit the application of concepts of thermodynamic equilibrium. Because of this close relationship between leptonic and hadronic particles, it makes sense that similar temperatures should apply to both types of flux loop manifolds, leptonic as well as hadronic.

One may therefore state that the statistical distribution (of the leptonic internal modes) which accounts for $e^2/\hbar c$ and for m_μ/m_e refers to a temperature of the order of the Hagedorn temperature[10] which is derived from the statistical bootstrap model of hadronic interactions. The concept, and the numerical value of the Hagedorn temperature, being based on particle reactions, was determined to be of the order of the low-lying meson masses, $T_H \approx 160$ MeV, in accord with experimental data which indicate 130 to 190 MeV. The temperature which fits the observed muon-electron mass ratio, in the rough approximation $\omega = 2j+1$, lies at the very high end of the experimental range of values for that limit temperature.

The temperature T which we use is an internal temperature characteristic of leptons, not identical with that universal limit temperature T_H.

The above temperature T is not calculated by the present theory; it is introduced into it, specifically in the form of *assuming*, for leptons, $\beta^{-1}\approx 370$ [Eq. (4.8)], i.e., in units of the electron mass. m_μ/m_e as an obvious consequence [Eqs. (5.19)–(5.2)] of this number 370 follows.

That some meson and muon masses should be roughly of the same order of magnitude could, however, have been anticipated by the following independent, simple consideration: comparing some low-lying mesons with muons. Indeed, as these mesons and the muon are, in the present flux loop model, the low-lying energy states of two-loop and of one-loop distributions, respectively, they should be expected to have roughly the same masses, the two-loop mesons somewhat higher than the one-loop muon. We thereby assume that the quark loops of a meson, as does the loop of a muon, have probability-amplitude modes proportional to $e^{-i(\omega_L+\omega_K)t}$; mesons and muon corresponding to each other in that respect. The interactions of the two quark loops of a meson presumably do not permit the low-frequency $e^{-i(\omega_L-\omega_K)t}$ modes of electrons; the electron thus seems to have no counterparts among mesons.

It is reassuring to know that the concept of a statistical distribution temperature, necessary for a selection of a finite number N of probability amplitude modes in the calculation of $e^2/\hbar c$, and for the selection of averages for $\langle\Omega_\mu\rangle_{\text{av}} = 3m_\mu c^2/\hbar$, Eq. (5.13), is already needed in a very different context with Hagedorn's T_H.

What is astonishing, however, is that the dichotomy of muon and electron states should so simply and obviously arise from the basic requirement of the Euler-angle independence of spinning angular velocity (and thus isotropy of the electric field).

And it is particularly interesting to realize that the two important constants $e^2/\hbar c$ and m_μ/m_e, which should be both a matter of electromagnetic theory and quantum mechanics, are interdependent, both arising simply as a consequence of the structuralization of leptons which is demanded by the flux loop model. And, furthermore, it is interesting to realize that an understanding of the rates of weak interactions (Appendix D) follows from the flux loop model with the same quantitative aspect of structuralization, i.e., with the reduction factor N.

A. Further details about the mode distributions

We presented the model in a simplified form in order not to overload the main arguments with details. Some of those may be brought up now.

The discussions became particularly simplified as we assumed the modes j, i.e., the $D^{j_L}_{n_L m_L} D^{j_K}_{n_K m_K}$, to be restricted to multiplicity $\mathfrak{M}=1$, i.e., that from among the many degenerate modes of the same frequency, only one is admissible, as originally pointed out in Eq. (3.12).

We want now to suggest that because of the similar role which the operators $\vec{L}$ and $\vec{K}$ play, there are always pairs of permitted degenerate modes which then indeed form a complete set of functions; thus

$$(j_K, j_L) = (0, \tfrac{1}{2}), (\tfrac{1}{2}, 0), (1, \tfrac{1}{2}), (\tfrac{1}{2}, 1), \text{etc.}, \quad (6.24)$$

i.e., multiplicity

$$\mathfrak{M} = 2, \quad (3.13)$$

instead of

$$(j_K, j_L) = (0, \tfrac{1}{2}), (1, \tfrac{1}{2}), \text{etc.} \quad (5.2)$$

for which $\mathfrak{M} = 1$, as in (3.12). Let us discuss the consequences of this pairing of modes. We may assume that, in the case of (virtual) coherence of mode amplitudes of excitons, coherence is not only shown by the excitons which belong to the same mode, but by the excitons which belong to the same degenerate pair of modes $(0, \frac{1}{2}), (\frac{1}{2}, 0)$ and similarly for the other mode pairs. Accordingly, the reduction factor N then gets doubled because of that mode pair structure (6.24).

More detailed attention needs to be given to the concept of "average occupancy." Before even discussing the definitions of average occupancy when degenerate modes are involved, we need to shortly state the definition in the case of absence of degeneracies.

One may (a) form an average occupancy from a collective (population) whose members are the modes j, providing for a (normalized) statistical weight $\langle n_j \rangle_{ex} / \sum_j \langle n_j \rangle_{ex}$. The quantity to be weighted with that is the expected occupancy of the mode j, i.e., $\langle n_j \rangle_{ex}$, resulting in an average occupancy

$$\frac{\sum_j \langle n_j \rangle_{ex}^2}{\sum_j \langle n_j \rangle_{ex}}. \quad (6.25)$$

But, more reasonably (b) one may form an average occupancy from a collective (population) whose members are the various distributions of $n_1, n_2, \ldots, n_j, \ldots$. The contribution of one of the n_j to the expectation value $\langle n_j \rangle_{ex}$ is

$$\frac{n_j e^{-n_j \beta \omega_j}}{\sum_{n_j} e^{-n_j \beta \omega_j}} = \frac{n_j e^{-n_j \beta \omega_j}}{(1 - e^{-\beta \omega_j})^{-1}}. \quad (6.26)$$

This quantity, when normalized by dividing it by

$$\sum_j \langle n_j \rangle_{ex} = \sum_j (e^{\beta \omega_j} - 1)^{-1}, \quad (6.27)$$

may be used as the statistical weight. The quantity to be weighted is now the individual occupancy

$$n_j \quad (6.28)$$

itself, and the average occupancy is obtained by combining these three factors,

$$\frac{n_j n_j e^{-n_j \beta \omega_j} (1 - e^{-\beta \omega_j})}{\sum_j (e^{\beta \omega_j} - 1)^{-1}}, \quad (6.29)$$

and summing over n_j and over j, resulting in

$$\frac{\sum_j \langle n_j^2 \rangle_{ex}}{\sum_j \langle n_j \rangle_{ex}}, \quad (6.30)$$

cf. also Eq. (6.4).

Returning now to the question of what the introduction of degeneracy implies to the concept of average occupancy, because of the relation (photon statistics)

$$\langle n_j^2 \rangle_{ex} = 2 \langle n_j \rangle_{ex}^2 + \langle n_j \rangle_{ex} \quad (6.4)$$

the average occupancy per mode, i.e., $\sum_j \langle n_j^2 \rangle_{ex} / \sum_j \langle n_j \rangle_{ex}$, does not depend on whether or not modes are degenerate.

If we ask, however, for the average occupancy per mode pair (i.e., per level in our level scheme), that average occupancy per level doubles up through the introduction of degenerate pairs of modes.

The interesting point is that this average occupancy behaves as does the reduction factor N, both of them doubling up when going from the level scheme (5.2) to the (6.24) scheme.

B. Question of *ab initio* calculation of the reduction number N

In paper A (cf. also Appendix B of paper C, and Appendixes C and E of the present paper) an attempt was made to present an *ab initio* calculation of N and thereby of $e^2/\hbar c$. This was achieved by making simple assumptions about bundling loopform manifolds (the bundling was performed by grouping loopforms together into bundles in such a manner that the orientations of neighboring bundles differ by about one radian). Although the reduction factor N was, in the present Sec. VI, properly calculated, for electron and muon, from an inner temperature T, it is important to realize that the ratio of that temperature to the mass of the electron is a number which was inserted into the calculation. The question is therefore whether it may be possible to calculate that ratio of T/m_{electron}.

It should be noted that the reduction factor N is to be a universal factor, not only for leptons but for quarks and all hadrons which are formed from coaxial quark loops: The universality of N indeed implies that the ratios of electric charges of the quarks have, because of simple topological properties of quark loop fields, the correct fractional charge values as required by the original SU(3) model (cf. paper C and the fourth paragraph in Sec. IV B). This remark underlines the obvious fact that a key issue is an *ab initio* calculation of N and thereby of $e^2/\hbar c$, m_μ/m_e, and weak interaction rates, implying again the concept of an internal temperature which for leptons was $\approx T_H$.

In this program which is tightly confined to quantum mechanics and electromagnetism (ML) theory, the decisive new features introduced are the consideration of magnetic flux as quantized, consideration of quasiextension, i.e., structuralization of the source model, and consideration of closure of the flux loopforms which implies their Seifert fibration topology.

The clarification of the electron and muon issues is a necessary prerequisite for the formulation of the quark-loop program.

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APPENDIX A: STEREOGRAPHIC PROJECTION OF CAYLEY-KLEIN SPACE

We may give a short résumé about the stereographic projection of the S_3 sphere,

$$\tilde{x}_0^{\,2}+\tilde{x}_1^{\,2}+\tilde{x}_2^{\,2}+\tilde{x}_3^{\,2}=1\,, \tag{A1}$$

where $\tilde{x}_i$ represent the Euler angles β, θ, α according to Eq. (2.1). A point on that unit sphere S_3 may be represented by the point $\tilde{x}_0, \tilde{x}_1, \tilde{x}_2, \tilde{x}_3$ on the unit circle of Fig. 2, whose north pole, from which we project, may be denoted by $X_0=1$, $X_1=X_2=X_3=0$. The projection of that point $\tilde{x}_i$ from the north pole X_i onto the equatorial plane $x_0=0$ may be denoted by x_1, x_2, x_3. We should like to express the loci $\theta=\text{const}$, $\beta=\text{const}$, $\alpha=\text{const}$ [cf. Eq. (2.1)] in terms of x_1, x_2, x_3. To this effect we find x_1, x_2, x_3 as functions of $\tilde{x}_0, \tilde{x}_1, \tilde{x}_2, \tilde{x}_3$.

This stereographic projection implies the following proportionalities (cf. Fig. 2):

$$\frac{x_3}{\tilde{x}_3}=\frac{x_2}{\tilde{x}_2}=\frac{x_1}{\tilde{x}_1}=\frac{x_0-1}{\tilde{x}_0-1}=\frac{-1}{\tilde{x}_0-1}\,. \tag{A2}$$

Therefore,

$$x_1=\frac{\tilde{x}_1}{1-\tilde{x}_0}\,,\quad x_2=\frac{\tilde{x}_2}{1-\tilde{x}_0}\,,\quad x_3=\frac{\tilde{x}_3}{1-\tilde{x}_0} \tag{A3}$$

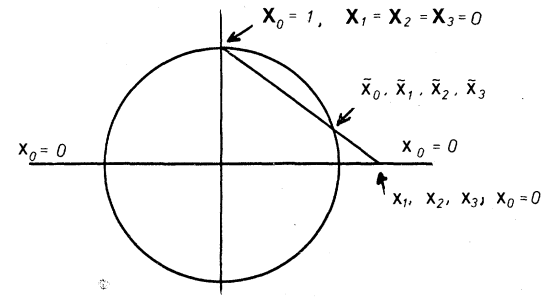

FIG. 2. Definition of the stereographic projection of the Cayley-Klein sphere $(\tilde{x}_0, \tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$ onto the three-dimensional space (x_1, x_2, x_3) of Fig. 1. The unit hypersphere S_3 in the Cayley-Klein space E_4 (Fig. 2) is represented by the circle. From the north pole $X_0=1$, $X_1=X_2=X_3=0$ of that unit hypersphere, its points $\tilde{x}_0$, $\tilde{x}_1$, $\tilde{x}_2$, $\tilde{x}_3$ are projected onto the equatorial hyperplane (x_1, x_2, x_3), which is represented in Fig. 2 by the abscissa axis $x_0=0$.

and, because of (A1),

$$x_1^{\,2}+x_2^{\,2}+x_3^{\,2}=\frac{1-\tilde{x}_0^{\,2}}{(1-\tilde{x}_0)^2} =\frac{1+\tilde{x}_0}{1-\tilde{x}_0}\,. \tag{A4}$$

Toroidal coordinates v, u, ϕ, on the other hand, are defined by

$$x_1+ix_2=\frac{\sinh v\, e^{i\phi}}{\cosh v-\cos u}\,,\qquad x_3=\frac{\sin u}{\cosh v-\cos u}\,. \tag{A5}$$

Therefore

$$x_1^{\,2}+x_2^{\,2}+x_3^{\,2}=\frac{\cosh v+\cos u}{\cosh v-\cos u}\,. \tag{A6}$$

We may then ask the following question: *If* the latter x_1, x_2, x_3 [(A5) and (A6)] can be identified with the former stereographic projections x_1, x_2, x_3 [(A2)–(A4)] of the $\tilde{x}_0, \tilde{x}_1, \tilde{x}_2, \tilde{x}_3$ [Eq. (2.1)] from the poles X_0, X_1, X_2, X_3, then what are the v, u, ϕ? Equation (A6) with (A4) gives

$$\tilde{x}_0=\frac{\cos u}{\cosh v}\,,\quad 1-\tilde{x}_0=\frac{\cosh v-\cos u}{\cosh v}\,, \tag{A7}$$

$$\tilde{x}_1=x_1(1-\tilde{x}_0) =\frac{\sinh v\cos\phi}{\cosh v} =\tanh v\cos\phi\,,\qquad \tilde{x}_2=\tanh v\sin\phi\,,\quad \tilde{x}_3=\frac{\sin u}{\cosh v}\,. \tag{A8}$$

Comparing (A7), (A8) with the definitions [Eq. (2.1)], we get

$$u=\tfrac{1}{2}(\beta+\alpha)\,,\quad \phi=\tfrac{1}{2}(\beta-\alpha)\,,$$
$$(\cosh v)^{-1}=\cos\tfrac{1}{2}\theta\,,\quad \tanh v=\sin\tfrac{1}{2}\theta\,. \tag{A9}$$

As all these equations are compatible, the toroidal coordinates can indeed be considered as coordinates resulting from stereographic projection, by the identifying equations (A5), (A1), and (A2).

Now, let us find, in x_1, x_2, x_3 space, the loci for given values of θ, β, α, i.e., given ϕ, u, v.

First, consider the *meridional* planes passing through the x_3 axis. For them

$$\begin{aligned}\text{const}&=\frac{x_1}{x_2}\\&=\frac{\tilde{x}_1}{\tilde{x}_2}\\&=\cot\phi\\&=\cot[\tfrac{1}{2}(\beta-\alpha)]\,,\end{aligned} \tag{A10}$$

i.e., they are the loci $\beta-\alpha=\text{const}$.

Second, consider *spherical surfaces* of radius R which pass through the circle $x_1^2+x_2^2=1$ and which therefore have their centers at $(x_3)_c=(R^2-1)^{1/2}$. For them

$$[x_3-(R^2-1)^{1/2}]^2+x_1^2+x_2^2=R^2\,, \tag{A11}$$

which, by (A4), (A2), gives

$$\frac{1+\tilde{x}_0}{1-\tilde{x}_0}-\frac{2(R^2-1)^{1/2}\tilde{x}_3}{(1-\tilde{x}_0)}=1\,, \tag{A12}$$

$$\begin{aligned}(R^2-1)^{1/2}&=\tilde{x}_0/\tilde{x}_3\\&=\cot u\\&=\cot[\tfrac{1}{2}(\beta+\alpha)]\,,\\ R&=\frac{1}{\sin u}\\&=\frac{1}{\sin[\frac{1}{2}(\beta+\alpha)]}\,,\end{aligned} \tag{A13}$$

i.e., these spherical surfaces are loci $\beta+\alpha=\text{const}$.

Third, (a) consider the circle $(x_1^2+x_2^2=1, x_3=0)$. By stereographic projection, Eq. (A4), this means also

$$\tilde{x}_0=0\,,\quad\text{i.e.,}\quad 1=\tilde{x}_1^2+\tilde{x}_2^2\,,\quad\text{i.e.,}\quad \theta=\pi\,. \tag{A14}$$

(b) Consider the x_3 axis, i.e., $x_1=x_2=0$. By stereographic projection, Eq. (A2) this means, for $-1\le\tilde{x}_0<+1$, $1-\tilde{x}_0>0$,

$$\tilde{x}_1=\tilde{x}_2=0\,,\quad\text{i.e.,}\quad \theta=0\,. \tag{A15}$$

(c) Consider torus surfaces which contain, inside of them, the circle $(x_1^2+x_2^2=1,\ x_3=0)$, and whose cross sections with meridional planes are circles of radius r whose centers are located a distance $(r^2+1)^{1/2}$ outside the $x_1^2+x_2^2=1$, i.e.,

$$[(x_1^2+x_2^2)^{1/2}-(r^2+1)^{1/2}]^2+x_3^2=r^2\,. \tag{A16}$$

By using (A4) we get

$$\frac{1+\tilde{x}_0}{1-\tilde{x}_0}-\frac{2(r^2+1)^{1/2}(\tilde{x}_1^2+\tilde{x}_2^2)^{1/2}}{1-\tilde{x}_0}=-1\,, \tag{A17}$$

$$\begin{aligned}(r^2+1)^{1/2}&=(\tilde{x}_1^2+\tilde{x}_2^2)^{-1/2}\\&=\coth v\\&=\frac{1}{\sin(\frac{1}{2}\theta)}\,,\\ r&=\frac{1}{\sinh v}\\&=\cot(\tfrac{1}{2}\theta)\,,\end{aligned} \tag{A18}$$

i.e., the toroidal surfaces characterize $\theta=\text{const}$; the thickness of the toroid (doughnut) is $2r=2\cot(\frac{1}{2}\theta)$. In Fig. 1 these circular cross sections of the toroids, with a meridional plane, are shown, as well as the circular cross sections of the spheres $\beta+\alpha=\text{const}$ with the same meridional plane.

APPENDIX B: WAVE EQUATION FOR THE PROBABILITY AMPLITUDES OF ELECTRON OR MUON LOOPFORMS[12]

The requirement of isotropy of the electric Coulomb field, generated by the spinning of magnetic flux loopforms, demanded that the spinning angular velocity $\vec{\Omega}$ of the loopforms (in ordinary three-space around their flux orientation axes $\hat{\xi}$, i.e., β, θ) be independent of these Euler angles $\beta, \theta, \alpha \equiv \hat{\xi}, \alpha$ and of the sizes σ of the loopforms. The motion of a manifold of those loopforms, when plotted with Cayley-Klein coordinates, amounts therefore, by Eq. (2.6), to a rigid, SO(4)-invariant motion of the S_3 sphere in itself. The wave functions should therefore be formed from representations of SO(4). The bases of irreducible representations of SO(4), i.e., $D^{j_L}_{n_L m_L} D^{j_K}_{n_K m_K}$ which correspond to resulting spin $\frac{1}{2}$, are characterized by $(j_K, j_L) = (0, \frac{1}{2}), \frac{1}{2}, 1)$, etc., and the n_L, m_L, n_K, m_K have been discussed above.

How does the wave equation enter here? The frequencies

$$\omega(j_a)=\omega_j \tag{B1}$$

are to be obtained from a wave equation for the D^j_{nm}. This relation $\omega(j)$ is needed for the definition of the statistical distribution (4.6) with an assumed T. The relation $\omega=\omega(j)$ together with the photon distribution then determine the coupling constant

$e^2/\hbar c$ through the reduction factor N, Eq. (6.20), and they determine also the mass ratio m_μ/m_e through the angular velocity ratios $\langle\Omega_\mu\rangle_{av}/\langle\Omega_e\rangle_{av}$, Eqs. (5.13), (5.14), (4.8), and (5.21).

We conjectured that

$$\omega \approx 2j+1 \qquad \text{(B2)}$$

is a fairly good approximation. Let us examine this relation in the present appendix.

In the appendix of paper B we discussed a wave equation of the original Casimir type (for a *single* top) in the Cayley-Klein variables $\tilde{x}_0, \tilde{x}_1, \tilde{x}_2, \tilde{x}_3$ [Eq. (2.1)] or, with

$$\xi_i = \mathfrak{r}\tilde{x}_i\,, \qquad \text{(B3)}$$

$$\left[\mathfrak{r}^2\left(\frac{\partial^2}{\partial\xi_0{}^2}+\frac{\partial^2}{\partial\xi_1{}^2}+\frac{\partial^2}{\partial\xi_2{}^2}+\frac{\partial^2}{\partial\xi_3{}^2}\right)-\frac{\partial^2}{\partial t^2}\right]\psi=0\,. \qquad \text{(B4)}$$

The essential point of the electron-muon dichotomy, mentioned in the Introduction and specified in Eqs. (5.7), (5.10) is that the $D^{jL}D^{jK}$ wave functions (3.3) are used both for the electron and for the muon, with $\omega_L-\omega_K$ and $\omega_L+\omega_K$ characterizing the two kinds of leptons. This means that the units of time t and of frequencies ω are the same, for electron as well as for muon wave functions.

If a Lorentz transformation is applied to examine the lepton in motion, this time t is, of course, to be considered as a proper time in a space-time system of the lepton's center of mass whose rectangular coordinates, not the $\tilde{x}_0, \tilde{x}_1, \tilde{x}_2, \tilde{x}_3$ or $\xi_0, \xi_1, \xi_2, \xi_3$, figure in the Lorentz transformation.

Whereas the variables $\tilde{x}_i$ confine us to the hypersphere S_3 of Cayley and Klein, the ξ_i cover the entire 4-dimensional space,

$$\xi_0{}^2+\xi_1{}^2+\xi_2{}^2+\xi_3{}^2=\mathfrak{r}^2(\tilde{x}_0{}^2+\tilde{x}_1{}^2+\tilde{x}_2{}^2+\tilde{x}_3{}^2) = \mathfrak{r}^2. \qquad \text{(B5)}$$

These variables ξ are useful because they permit us to formulate the angular momentum operator (homogeneous of degree zero in $\mathfrak{r}$)

$$\mathcal{J}^2=-\mathfrak{r}^2\left(\frac{\partial^2}{\partial\xi_0{}^2}+\frac{\partial^2}{\partial\xi_1{}^2}+\frac{\partial^2}{\partial\xi_2{}^2}+\frac{\partial^2}{\partial\xi_3{}^2}\right) + \mathfrak{r}^{-1}\left(\frac{\partial}{\partial\mathfrak{r}}\right)\mathfrak{r}^3\left(\frac{\partial}{\partial\mathfrak{r}}\right)\,. \qquad \text{(B6)}$$

It has the eigenvalues

$$4j(j+1) \qquad \text{(B7)}$$

for the eigenfunctions $D^j_{nm}(\beta\,,\theta,\alpha)$.

We point to the following argument (in the Appendix of paper B) concerning the question of the physical meaning of $\mathfrak{r}$: As the shape of the loopforms of a point-dipole field is not only the same for all values of $\hat{\xi},\alpha$ but also for all values of the size σ (i.e., aphelion distance of a loopform), we have to do with a type of scale invariance which suggests that we may extend the 3-parameter loopform characterization (by $\hat{\xi},\alpha$) to a 4-parameter characterization $(\hat{\xi},\alpha,\sigma)$. We may, furthermore, assume the interpretation

$$\mathfrak{r}=\frac{\sigma}{\hbar/mc} \qquad \text{(B8)}$$

so that we have (a) the same $\psi(\mathfrak{r})$ distribution for the muon ($m=m_\mu$) and for the electron ($m=m_e$). We may then (b) discuss how to express scale invariance with respect to loopform size σ as a scale invariance in the $\mathfrak{r}$ distribution, cf. Eq. (B5). In this way we associate with every point-dipole loopform a point in the 4-dimensional space of the variables $\xi_0, \xi_1, \xi_2, \xi_3$. (The magnetic field which corresponds to an extended source may then be visualized as corresponding to a point-source magnetization being spread out, by virtue of the *Zitterbewegung*, over a magnetized "core region" of the linear extension $\approx\hbar/mc$, cf. paper A, Sec. VC, where this spreading out was calculated in terms of a Gaussian distribution of magnetization over the core region.)

It may be appropriate to mention that the invariance (a) of the Coulomb field charge e was the first test of consistency of the flux loop model. Indeed, we found in paper A, Sec. VB (cf. also paper C, Sec. IIB), that the same equivalent electric charge e resulted for the electron and for the muon alike, because of the rigorous cancellation of mass in the calculation of the electric field.

To summarize: Whereas the units of time (t) and frequencies (ω) are the same for the muon and for the electron, their actual size parameters σ are to be expressed in terms of $\hbar/m_\mu c$ and $\hbar/m_e c$, respectively, Eq. (B8), i.e., $\mathfrak{r}=\sigma/(\hbar/mc)$, which appears in the wave equations (B4) and (B3).

The scale invariance (b) may be satisfied if $\psi(\mathfrak{r})$ is proportional to a power of $\mathfrak{r}$, i.e.

$$R(\mathfrak{r})\propto\mathfrak{r}^\kappa, \qquad \text{(B9)}$$

$$\psi\propto D^j_{nm}(\beta\,,\theta,\alpha)R(\mathfrak{r})^{-i\tilde{\omega}t} \qquad \text{(B10)}$$

(we denote by ω_L and by ω_K the absolute values of the frequencies $\tilde{\omega}_L$ and $\tilde{\omega}_K$). Inserting (B9) into (B4), (B6), and (B7),

$$\left[\mathfrak{r}^{-1}\left(\frac{\partial}{\partial\mathfrak{r}}\right)\mathfrak{r}^3\left(\frac{\partial}{\partial\mathfrak{r}}\right)-4j(j+1)+\tilde{\omega}^2\right]R(\mathfrak{r})=0, \qquad \text{(B11)}$$

which indeed can be satisfied with an $R(\mathfrak{r})$ of the power expression (B9) and yields

$$\omega^2=\tilde{\omega}^2=4j(j+1)-\kappa(\kappa+2). \qquad \text{(B12)}$$

In three respects we need to supplement this discussion of the wave equation.

First, we have recognized that a mode of muon or electron wave functions, in order to represent the SO(4) group, should be a product of *two* Wigner D^j_{nm} functions, one of them (L) representing the dependence on the space-fixed coordinate system, the other (K) representing the dependence on the base- (loop-) fixed systems. This 6-parameter product of two functions, each of 3 variables, i.e., of $\beta_L, \theta_L, \alpha_L$ and of $\beta_K, \theta_K, \alpha_K$, is then to be projected onto the space of the 3 variables $\beta_L, \theta_L, \alpha_L$.

As we consider distributions in the space of the 4 variables $\hat{\xi}, \alpha, \sigma=\beta, \theta, \alpha, \sigma$, i.e., of $\xi_0, \xi_1, \xi_2, \xi_3$, we simply extend that projection operation to project the product of two functions, each of 4 variables, i.e., of $\beta_L, \theta_L, \alpha_L, \mathfrak{r}_L, \beta_K, \theta_K, \alpha_K, \mathfrak{r}_K$ onto the space of the 4 variables $\beta_L, \theta_L, \alpha_L, \mathfrak{r}_L$, where, or course, $\mathfrak{r}_K = \mathfrak{r}_L$ in the process of projection.

Using the $\Box$ for the all-positive-signature second-order differential operator $\sum_{i=0}^{3}\partial^2/\partial\xi_i^2$, we might then perhaps suspect some equation

$$0=\left(\mathfrak{r}_L{}^2\Box_L - \frac{\partial^2}{\partial t^2}\right)D^{j_L}_{n_L m L}(\beta_L, \theta_L, \alpha_L)R_L(\mathfrak{r}_L)e^{-i\tilde{\omega}_L t} \tag{B13}$$

to hold and also the same equation with the subscripts L replaced by the subscripts K. And one might assume that the analogy between the K and the L systems suggests the same $\mathfrak{r}$ dependence of ψ for L and for K:

$$R_L(\mathfrak{r}_L)=R(\mathfrak{r}_L),\quad R_K(\mathfrak{r}_K)=R(\mathfrak{r}_K). \tag{B14}$$

By projection of $\mathfrak{r}_L$ and $\mathfrak{r}_K$ onto each other, and calling them simply $\mathfrak{r}$, we get

$$R_L(\mathfrak{r}_L)R_K(\mathfrak{r}_K)=R^2(\mathfrak{r}) \tag{B15}$$

for the probability amplitude $|\psi|$ as a function of $\mathfrak{r}$, and therefore

$$R^4(\mathfrak{r}) \propto |\psi|^2 \tag{B16}$$

as the probability for the size $\mathfrak{r}=\sigma/(\hbar/mc)$, i.e., aphelion distance of a loopform to fall into a unit volume of the shell $(\mathfrak{r}, \mathfrak{r}+d\mathfrak{r})$ whose volume element is $\propto \mathfrak{r}^3 d\mathfrak{r}$.

That probability is to be $\propto B\mathfrak{r}d\mathfrak{r}$, so that

$$B\mathfrak{r}d\mathfrak{r} \propto |\psi|^2\mathfrak{r}^3 d\mathfrak{r} \propto R^4\mathfrak{r}^3 d\mathfrak{r}, \tag{B17}$$

i.e., for the point-dipole field under consideration, $B \propto \mathfrak{r}^{-3}$. Therefore

$$R \propto \mathfrak{r}^{-5/4},\ \text{i.e.,}\ \kappa=-\tfrac{5}{4}, \tag{B18}$$

which gives with Eq. (B11)

$$\omega^2 = \tilde{\omega}^2 = (2j+1)^2 - (\tfrac{1}{4})^2 \tag{B19}$$

or, indeed,

$$\omega \approx 2j+1. \tag{B20}$$

Second, we should recognize that it would be quite artificial to assume two wave equations, one for the L system, the other for the K system. We also should realize that addition of the L operators appearing in Eq. (B12) with the corresponding K operators should not be the proper procedure to construct a single wave equation, for the following reasons. Even though the coaxiality condition (3.10) of the angular momenta of L and of K might simplify angular momentum addition in the present case, the addition of the two L,K terms (without recourse to two time variables) should yield $(\omega_L \pm \omega_K)^2$ which is not equal to an $\omega_L{}^2 \pm \omega_K{}^2$ resulting from an addition or subtraction of the two operators, $J_L{}^2 \pm J_K{}^2$, i.e., of the two Casimir invariants. A linearization of the wave equation might not only be helpful in overcoming these kinds of problems; linearization is already imperative in order to satisfy the requirements of quantum mechanics, i.e., to have a wave equation which contains only a first-order derivative in time.

Third, as already pointed out at the end of Sec. IV, we have to consider spin *and* whirl. For electron or muon loops [of winding numbers (2, 1)] we have additive (cooperative) motion as regards spin and whirl. At least in the present parametrized version of the theory, and for the present case of leptons which have only one type of loop (one domain), unlike mesons or baryons (which have 2 or 3 loop domains), it is to be understood that, for these smooth torus loops, whirling motion is equivalent to a supplementary amount of spinning motion with commensurable angular velocity. For the winding numbers (2, 1) of electron or muon torus loops, the additive whirling motion is actually equivalent to an enhancement of the spinning motion by a factor $1\frac{1}{2}$ if spinning and whirling are naturally assumed to have the same angular velocity, so that the effective spinning occurs with $(1+\frac{1}{2})2mc^2/\hbar$. It was pointed out in paper C that lepton loopforms should be interlinked "so as not to fall apart," to speak in the Faraday-Maxwell terminology of field stresses; the simplest winding numbers to that effect are (2,+1) and (2,−1), i.e., the electron or muon case. It was also pointed out that it is exactly this feature of effective spinning angular velocities equal to $(1\pm\frac{1}{2})2mc^2/\hbar$, and + and − for electron and muon, and for $\mathfrak{N}$ quarks, respectively, which gives correct account of the charges e and $\frac{1}{3}e$, respectively.

One should note that the Casimir invariants are

$$\begin{aligned}|\vec{\mathrm{L}}^2 - \vec{\mathrm{K}}^2| &= |(j_L - j_K)(j_L + j_K + 1)|\\ &= |j_L - j_K|(2j+1)\\ &\approx \tfrac{1}{2}\omega_j\\ &= \tfrac{1}{4}\omega_\mu\omega_e,\end{aligned}$$

$$\vec{L}^2+\vec{K}^2=\tfrac{1}{2}[(2j+1)^2-\tfrac{3}{4}]$$
$$\approx\tfrac{1}{2}\omega_j^{\,2}.$$

A final obvious remark: A Lagrangian $\mathcal{L}$ = $\mathcal{L}$(loopform probability amplitudes) can be written down only when one has defined the parameters on which the probability amplitudes depend. Such a Lagrangian should be formed from functions which are invariants with respect to transformations of these parameters. Thus one has first to know the group of transformations of those parameters, the group with respect to which one then forms invariants, the group which therefore leaves the Lagrangian invariant and which therefore is admitted by the theory. In the present case the parameters are the Cayley-Klein parameters, and from physical arguments of isotropy of the electric field we know that the group is SO(4) in Cayley-Klein space. The solutions of the variational principle $\delta\mathcal{L}=0$ are then simply expressible in terms of the irreducible representations of that SO(4) group. The exact form of the wave equation which corresponds to that variational principle is therefore not always needed for steps in the development of the theory.

APPENDIX C: FOOTNOTES AND CORRECTIONS TO PAPER C

It is important to note that the loop configurations drawn in paper C represent the basic types of fibration (Seifert and Threlfall) and thus the basic types of topological structures of electromagnetic fields. In the pictures of paper B the 2 or 3 loops of mesons or baryons referred to 2 or 3 nonoverlapping toroidal regions of three-space, adjacent regions being bounded by 1 or 2 surfaces of discontinuity of fibration. Actually, each of the 2 or 3 loopform manifolds may cover the entire space, but are mainly confined to the 2 or 3 separate regions. In other words, as the 2 or 3 factors of a meson or baryon share their common torus axes, the probability amplitudes's dependence on the Euler angles is one and the same as that for the 2- or 3-quark-loop distributions; the composite character of 2 meson or 3 baryon loops of a hadron expresses itself in products of 2 or 3 functions characterizing the distribution of probability amplitudes over the *size* parameters σ.

The third and fourth paragraphs in the remarks on magnetic moments, page 451 of paper B, should be deleted: Magnetic moments of quarks are not proportional to their electric charges in the flux loop model, cf. Sec. VI of paper C.

Instead of the remarks about string models, given in Appendix C of paper C, we may give here some comments on strings.

It will be important to find the relationship of the flux loop model to the string models.[14] It is interesting to note that string models in which the strings are closed loops, are able to represent dual amplitudes. It might thus be suggested that one may relate the concept of two merging closed-loop strings to the concept of the two outer quark loops which merge in a strong interaction between two hadrons [cf. below in Appendix E, part (E)]. Whereas, in conventional string models, the strings are sometimes considered as open-ended, and where internal quantum numbers of a quark are represented by boundary conditions at the open ends of the string, it is now, with closed strings, suggested that the internal quantum numbers of a quark are represented by the topology of the quark loop, in particular by its winding numbers (2,1), (3,1), (3,2) for the three older quarks. The linearity of the Regge trajectory should not be handled by a straight string with two heavy ends attached to it, but by assuming a hadron as being formed from the qqq or $q\bar{q}$ characterizing the particular hadron, associated with several $q\bar{q}$ in a ring-shaped aggregate (as pointed out by Holger B. Nielsen). These $q\bar{q}$ should be considered as submits of the L excited hadron (an obvious assumption with our topological fluxloop model in which q and $\bar{q}$ are coaxial fibrations, cf. papers B and C). These $q\bar{q}$ subunits move like balls of a ball bearing. Whatever the number of these $q\bar{q}$ subunits, the close distances between nearest-neighbor subunits are expected to be approximately a standard distance, determined by the interaction between subunits. Their circumferential velocity is also expected to be a standard velocity, given by the condition that the de Broglie wavelength (corresponding to the mass of a $q\bar{q}$ subunit) should be equal to that nearest-neighbor distance. Accordingly, the total mass (sum of the $q\bar{q}$ subunit energies minus binding energies between subunits) is proportional to the number of subunits and, as the radius (like the circumference) of the ring-shaped aggregate is also proportional to that number, the total angular momentum is approximately proportional to the square of the number of subunits, the circumferential velocity being approximately standdard. The relativistic calculation shows the right order of the slope of this linear Regge trajectory.

The developments in paper C of the heuristic-model calculations are important to determine the basic structure of the model and to check on its consistency; reliable numerical results are, however, only obtained with the tools developed in the present paper.[15]

The higher quark[16] loops (4,3), etc. had been listed in the early 1974 prepublication report of the paper C; their effective spinning angular velocities (taking the contribution to spinning from whirling into account) had been correctly stated but mislabeled as "effective spin"; spin angular momentum $\frac{1}{2}$ and effective spinning angular velocity are different concepts and had been properly distinguished all along.

In order not to cause confusion, we do not use the word "charm" for these higher-winding-number quarks. Their charges and high strangeness numbers listed in the tables of paper C make them excellent candidates for very stable mesons. The leptons of winding numbers (5, 1), i.e., electric charge 2, and those of winding numbers (5, 4) of charge 3, both listed in Table II of paper C, are the next integer-charged leptons predicted in the loop model. The (5, 4) leptons [not the (5, 1) leptons] have a very high unwinding number, implying strange properties, and the existence of corresponding neutrinos. All these new leptons might perhaps occur in both the electron-type and in the muon-type version, but will occur in at least one of them.

As a final remark, we noted that a rest-system solution may be simply Lorentz-transformed to describe a particle in constant motion. Indeed, the starting point of the theory was the requirement of a Lorentz-covariant definition[17] of the magnetic and electric field through $A_k - (\hbar c/e)\partial_k\vartheta = 0$. Very different to this issue is the question of behavior of the model in accelerated motion, and still more different is the question as to a description of what happens in any situation which implies a change of the standard forms of the flux loops. This parametrized loopform model does not cover those general situations. It is for this reason that a proper connection to quantum electrodynamics[12] may be expected to be obtained only by a theory which formulates flux loopforms in terms of functionals and uses differential topology.

APPENDIX D: ESTIMATE OF RATES OF WEAK INTERACTIONS FROM ELECTROMAGNETIC (LOOP MODEL)

We have recognized that a lepton (and presumably also a quark of a hadron) may be represented by a large number of substates,[19] termed "modes" (i.e., the $D^{jL}D^{jK}$ functions). Each mode is understood as occupied by a number of (indistinguishable) noncoherent excitons, their expected number $\langle n_j\rangle_{ex}$ being given by Planck's law $(e^{\beta\omega_j}-1)^{-1}$, corresponding to a probability amplitude r_j of the mode given by $(e^{\beta\omega_j}-1)^{-1/2}$. Superpositions of these mode amplitudes over the spectrum ω_j form the probability-amplitude distribution on the loopform manifolds of a lepton or of a quark. Because of that statistical distribution one may speak of a statistical average of occupancy of a mode, i.e., $N \approx 2\times 119$, and of an effective total occupancy $\sum_j \langle n_j\rangle_{ex} \approx 2\times 2177$ obtained from a simple tabulation of $\langle n_j\rangle_{ex}, \langle n_j^2\rangle_{ex}, \omega_j\langle n_j\rangle_{ex}, (\omega_j\langle n_j\rangle)^2$.

When a "crosscutting" of loops is necessary in a strong interaction or in any other interaction, i.e., if the process involves a change of the topology of the loops, that process becomes a weak process,[20,13] as evidenced by an analysis of structure and linkage of quark loops in interactions known phenomenologically to be weak. We should now like to consider this statement also from the point of view of electromagnetic theory and thereby estimate the rate of weak interactions due to such "crosscutting."

(For some of the immediately following remarks we have to refer to paper C.) We do not distinguish *here* between weak interactions in which crosscutting is due to change of strangeness and those in which it is not, neither do we consider here the distinction between crosscutting of a loop over itself or over another loop with which it interacts, nor do we consider here whether the interacting torus loops share both their torus axes (i.e., belong to the same particle) or whether they do not share them, in which case the torus loops belong to two particles approaching each other. We do, however, distinguish between situations in which the topology (winding numbers, spin and flux orientations) of the interacting loops leads to essentially antiparallel alignment with the possibility of full merging of the interacting loops, or to essentially parallel alignment representing the case which excludes possible merging, while allowing crosscutting. We shall here consider the latter case.

With crosscutting, overlap of loopform manifolds is involved during that process, and that means that, virtually, the electromagnetic energy, formerly a sum of mc^2 of the two intersecting loopform manifolds before intersection, is now for a short moment increased (because the field intensities, but not the energies, superpose linearly), by some 20%, 30%, or 40%. That the increase in field energy might be of that order of magnitude may be inferred from the following consideration. Before (and after) the crosscutting, the flux loopforms (i.e., magnetic field lines) had been thickly concentrated in essentially different regions of x, y, z space with little overlap (these regions are of the order of linear extension of $\hbar/mc$). At crosscutting, the loopform manifolds share a substantial volume of the x, y, z space which they separately occupied before (and after), i.e., at crosscutting a sizable fraction of the interacting loopform regions overlap. Even though the present model gives only a parametrized description of loopform manifolds and thus, strictly speaking, is inadequate for a quantitative handling of overlap and of change of topological structure, we may arrive at reliable estimates if we handle the energy rise during overlap as if it were simply an energy rise of the regular parametrized loopform manifolds.

The crosscutting, thus characterized by crossing of an energy barrier, is not calculated with the formalism of a quantum-mechanical tunnel effect (we even have no formalism yet available for the calculation of interacting loopforms). But we may calculate it as an energy fluctuation as in Eyring's and in Kramers's models of reaction kinetics, asking what the probability would be to get over that hump of 20%, 30%, or 40% excess energy, per unit time.

To calculate this fluctuation effect it should be remarked that here we do *not* consider virtual, short changes from random-phased superposition of exciton amplitudes to coherent superposition. We consider the fluctuations of mode occupancies n_j in a Bose-Einstein photon distribution, characterized by the moments $\langle n_j\rangle_{ex}$, $\langle n_j^2\rangle_{ex}$, $\langle n_j^3\rangle_{ex}$, etc. and remember that the distribution of n_j may be formulated in terms of random-phasedly superposed exciton amplitudes $|\vec{r}_j|^2 = r_j^2 = n_j$, according to the normalized distribution law (cf. Sec. VI)

$$1 = \int_0^\infty e^{-(r_j/l_j)^2}\, d(r_j/l_j)^2 , \tag{D1}$$

where

$$\langle n_j\rangle_{ex} = \langle r_j^2\rangle_{ex} = l_j^2 . \tag{D2}$$

Correspondingly, the expectation value of the jth mode's energy contribution is

$$\langle \epsilon_j\rangle_{ex} = \omega_j \langle r_j^2\rangle_{ex} = \omega_j\, l_j^2 . \tag{D3}$$

With

$$\epsilon_j \equiv \omega_j r_j^2 , \tag{D4}$$

we may consider the total energy

$$\mathcal{E} = \sum_j \epsilon_j \tag{D5}$$

as a one-dimensional statistical superposition of positive-definite ϵ_j steps. The normalized probability distributions for these steps is

$$\tau_j\, d\epsilon_j = \langle \epsilon_j\rangle_{ex}^{-1}\, e^{-\epsilon_j/\langle\epsilon_j\rangle_{ex}}\, d\epsilon_j \tag{D6}$$

and is used in the Markoff method to get the product of their Fourier transforms,

$$\begin{aligned} A(\rho) &= \prod_j \int_0^\infty d(\epsilon_j/\langle\epsilon_j\rangle) e^{-\epsilon_j/\langle\epsilon_j\rangle + i\rho\epsilon_j} \\ &= \prod_j (1 - i\langle\epsilon_j\rangle\rho)^{-1} \\ &= 1 + i\rho\langle\mathcal{E}\rangle - \rho^2\Big(\tfrac{1}{2}\langle\mathcal{E}\rangle^2 + \tfrac{1}{2}\sum_j \langle\epsilon_j\rangle^2\Big) + \cdots . \end{aligned} \tag{D7}$$

In order to Fourier transform this back, we need

$$\begin{aligned} \mathcal{E} &= \langle\mathcal{E}\rangle + (\mathcal{E} - \langle\mathcal{E}\rangle) \\ &= \sum_j \langle\epsilon_j\rangle + (\mathcal{E} - \langle\mathcal{E}\rangle), \\ e^{-i\rho\mathcal{E}} &= (1 - i\rho\langle\mathcal{E}\rangle - \tfrac{1}{2}\rho^2\langle\mathcal{E}\rangle^2 + \cdots) \\ &\quad \times e^{-i\rho(\mathcal{E}-\langle\mathcal{E}\rangle)} . \end{aligned} \tag{D8}$$

We therefore obtain

$$\begin{aligned} W(\mathcal{E})\, d\mathcal{E} &= \left(\frac{d\mathcal{E}}{2\pi}\right) \int d\rho\, A(\rho) e^{-i\rho\mathcal{E}} \\ &= \left(\frac{d\mathcal{E}}{2\pi}\right) \int d\rho \Big(1 - \tfrac{1}{2}\rho^2 \sum_j \langle\epsilon_j\rangle^2 + \cdots\Big) \\ &\quad \times e^{-i\rho(\mathcal{E}-\langle\mathcal{E}\rangle)} \\ &\approx \Big(2\pi \sum_j \langle\epsilon_j\rangle_{ex}^2\Big)^{-1/2} \\ &\quad \times \exp\Big[-\tfrac{1}{2}\Big(\mathcal{E} - \sum_j \langle\epsilon_j\rangle_{ex}\Big)^2 \Big/ \sum_j \langle\epsilon_j\rangle_{ex}^2\Big]\, d\mathcal{E} \end{aligned} \tag{D9}$$

[The numerical value of $\langle\mathcal{E}\rangle$, etc. corresponds to $\langle\mathcal{E}\rangle = \sum_j \langle\epsilon_j\rangle_{ex} = \sum_j \omega_j \langle n_j\rangle_{ex}$ so that $2\langle\mathcal{E}\rangle/\sum_j\langle n_j\rangle_{ex}$ would represent the muon mass in units of electron mass, if we would talk about the energy fluctuations corresponding to a muon participating in the crossing. Cf. Eq. (6.18).]

The important number here is the variance in the last Gaussian distribution. From the numerical evaluation of the $\langle\epsilon_j\rangle$ and the $\langle\epsilon_j\rangle^2$ we get a standard deviation of about $\frac{1}{23}$ of the expectation value $\langle\mathcal{E}\rangle = \sum_j \langle\epsilon_j\rangle_{ex}$.

Accordingly, the probabilities for relative excess energy

$$(\mathcal{E} - \langle\mathcal{E}\rangle)/\langle\mathcal{E}\rangle = 0.2, 0.3, 0.4 \tag{D10}$$

are $10^{-4.6}, 10^{-10.4}, 10^{-18.5}$, respectively. These probabilities, in order to relate to rates of reactions, should be considered in units of frequency $2mc^2/\hbar$, where m is of the order of the masses of the hadrons or leptons involved in the interaction. That amounts to a very crude assessment of weak-interaction rates, but this qualitative model fits both the requirements of fluctuation theory and those of the phenomenology of the quark loop model.

It is therefore seen that the electromagnetic coupling constant, the muon-electron mass ratio, and also the rate of weak interactions are interrelated, all of these depending on those numbers $\langle n_j\rangle$, $\langle n_j^2\rangle$, $\langle n_j\rangle^2$ and their products with the mode frequencies, numbers which define the quantitative aspects of lepton or quark loopform distributions. One should note that the concept of substates (modes) and of excitons does not imply observability of them, nor, still less, does it imply that a quantum state be superposed from such excitons by probabilities, only probability-amplitude superposition provides for a consistent model.

It should be noted that even the present parametrized formulation of the flux loop model might perhaps account for interactions between particles, i.e., for scattering cross sections, before even a formulation of loop amplitudes in terms of functionals is available: The method applied in this appendix might provide some answers to scattering problems, and one might not have to wait for far-reaching mathematical tools to establish the link between the flux loop model and quantum electrodynamics.

APPENDIX E: BRIEF SUMMARY OF PAPERS A–D ON MAGNETIC FLUX QUANTIZATION AND PARTICLE PHYSICS

(A) Quantized flux is to be considered as the basic unit in relation to a theory of the electron. A heuristic model is used to show the consistency of that program and to show that the occurrence and size of the coupling constant $e^2/\hbar c$ may be understood in terms of the flux quantization model, without recourse to magnetic monopoles. While in the present paper (D) we simply interrelate this electromagnetic coupling constant to the muon-electron mass ratio, to the weak-interaction constant, and all of them to the internal temperature T, the heuristic papers A and C make the even more ambitious attempt to calculate $e^2/\hbar c$ on the basis of a simple assumption about grouping loopform manifolds together into "bundles" which differ by about one radian in their orientations, as regards the "distance" between neighboring loopform manifolds.

(B) Whereas in paper A a continuous distribution of the "alternative forms"[19] of one quantized flux loop (a Seifert fibration of space) determines the magnetic (and thereby also the electric) field of a lepton, in paper B a $\bar{q}q$ meson is represented by a fibration of space in terms of two domains of fibration (one inside a toroidal surface, the other outside) and a qqq baryon in terms of three domains. The fibrations have the structure of coaxial torus knots of winding numbers (2,1) for $\mathfrak{N}$ quarks, (3,1) for $\mathcal{P}$ quarks, and (3,2) for λ quarks. The topology of fibration [(3,2) of a λ quark] is made responsible for the strangeness (unknotting number) of the λ quark and a change of topology characterizes weak interactions. The circumstance—that the interpretation of quarks as domains of fibration makes quarks localized objects—obviates the conflict between symmetric spin-isospin functions of baryon quarks and the Pauli principle. Furthermore, it is seen that the assignment of torus knots because of their interlinkage obviates the confinement problem of quarks.

(C) A short review (and appended footnotes) of the entire program is followed by a demonstration that integer electric charges of leptons and fractional charges of quarks arise simply from the two possible types of flux loop motions of the model, i.e., the two possible types of internal motions of fibrated space. The model shows the existence of a sequel of new types of quarks, i.e., loops of higher winding numbers, following the conventional quarks of winding numbers (2,1), (3,1), (3,2). It also shows that the magnetic moments of quarks are no longer exactly proportional to their electric charges.[21]

(D) In this paper the heuristic model of characterizing the probability-amplitude distribution of a quantized flux loop (for leptons) is reformulated in terms of wave functions characterizing that distribution. It becomes obvious that muons as well as electrons have to arise and it becomes possible to estimate their mass ratio, as well as the value of the electromagnetic coupling constant $e^2/\hbar c$. The present form of the flux quantization project characterizes the electromagnetic field by a few parameters, essentially Euler angles; for a theory which properly connects this theory with quantum electrodynamics, one needs a formulation of the loopform manifolds by functionals instead of parameters and one needs to proceed to a differential topological formulation of the loop model.

(E) In a following paper some rules will be given for the interaction of quarks. The basic rule is that a change of topology of the fibration, implied in any "crosscutting" (in order to avoid entanglement) of loops over themselves or over those which are coaxial with them or over those with which they interact, e.g., in the process of merging and of transfer of quark loops or of pair creation, represents a weak process. It is particularly interesting to note that some outer torus loop (quark loop) of a baryon may merge with the corresponding outer antiquark loop of a meson without entanglement even though the q of the baryon is linked with two other q (all three coaxial), and the $\bar{q}$ of the meson is linked with another q (coaxial with it). Such a pair merging process may occur without crosscutting of loops and may result in the q of the meson neatly transferred to the outer region surrounding the two q loops of the baryon. In that process the meson torus and the baryon torus come first to lie side by side, sharing their central symmetry axes. The joining of the meson's $\bar{q}$ with the baryon's q results in the intermediary configuration in which 2 or 3 loops which are formed from those merging $q, \bar{q}$ (those loops are not interlinked with each other), and form 2 or 3 links which chain the meson torus to the baryon torus. These chain links join with this meson's q and thereby transfer it to the outer region of the baryon. This process therefore follows Zweig's rule. With the topological interpretation of weak interactions, parity violations are to be expected there.

(F) Heavy leptons. It seems that in principle all those types of loops are represented in nature, which correspond to integer electric charge in case of leptons (in which case spinning and whirling cooperate additively to effective spinning) and which satisfy the requirement of being linked together. We listed so far only types (2, 1) for electrons and muons, (3, 2) for neutrinos, and (5, 1) and perhaps (5, 4) for leptons of charge 2 and 3, respectively (paper C, Table II). In particular, the latter type, as it is heavily knotted in addition to having a densely packed magnetic field, is expected to be difficult to find.

The loops of type $(1, \pm 2)$ have, compared with $(2, \pm 1)$, half as many core traverses of the magnetic flux, and thus a *magnetic moment* $\frac{1}{2}(e\hbar/2mc)$, where m is their mass. They have an effective spinning angular velocity $(1+2)2mc^2/\hbar$, compared with $(1+\frac{1}{2})2mc^2/\hbar$; therefore the electric charge, proportional to the product of the number of core traverses and the effective angular velocity, is the same, i.e., e.

Notes added in proof. To Appendix E: If the handedness of motion of a loopform manifold (fibration) is opposite to the handedness of its structure, we have an additive, "cooperative," contribution of spin and whirl (leptons); if the handednesses are the same, we have a subtractive, "sliding" relationship between spin and whirl (quarks).

Fibrations (2, 1) or (1, 2) imply loopforms which are interlinked with each other; besides, they go around the central torus axis. Because they represent quantized flux, the Faraday-Maxwell stresses are expected to keep the structure together. That does not hold for (0, 1) or (1, 0) loops which therefore should be counted in the category of photons.

The older quarks $\mathcal{N}, \mathcal{P}, \lambda$ are represented by the simplest fibrations (2, 1), (3, 1), (3, 2), presumably superposed with the corresponding (1, 2), (1, 3), (2, 3) which have the same electric charges. These are the simplest fibrations [fibrations (1, 0) or (0, 1) are of the type of noninterlinked, photon type, and fibrations (1, 1) quark loops have electric charge zero]. The next simple fibrations with fractional charge are (4, 3) or (3, 4) [listed in Univ. of Maryland report, 1974 (unpublished); cf. paper C], their high unwinding number 3 (strangeness 3) shows that they are the evident candidates to form, with their antiquarks, the J/ψ particle. These (4, 3) or (3, 4) quarks have electric charge $\pm e/3$, not $\pm 2e/3$.

The role of the various fibrations is to be assessed when the rules for superposition of lepton and quark fibrations and the rules for their interactions have been all worked out. With the same comments we refer also to the interesting (4, 1) loop shown in those tables of paper C. This (4, 1) or (1, 4) fibration of the sliding type motion (as quarks usually have) has integer electric charge $\pm e$. Because of this type of motion it might be considered as a peculiar hadronic constituent, occurring singly, not in a $q\bar{q}$ or qqq combination. But it would, as it consists of a single loop (fibration), and because of its integer charge, show up somewhat like a lepton. The same remarks apply also to the interesting (1, 1) fibration which carries electric charge zero.

Notes to the Introduction: As in quantum electrodynamics we assume interactions to be local, but we use the *Zitterbewegung* to justify an averaging of the position of the source over a volume of the order of $r_0^3 \approx (\hbar/mc)^3$, cf. Eq. (6.22); r_0 has nothing to do with the complex amplitudes $\vec{r}_j$ used in the next sections.

The reduction states in which manner global (integrals over the loopform parameters) quantities, such as total magnetic flux, magnetic moment, electromagnetic energy, and electromagnetic angular momentum, get reduced to effective flux etc. by $\Phi_{\text{eff}} = 2N^{-1}\Phi_q$, $(\Phi^2)_{\text{eff}} = 2N^{-1}(\Phi_q)^2$: For a fibration of loopforms (2, 1) which pass twice through the "core" of the source and thus have two "wings," the magnetic moments (linear in Φ_q), and the electromagnetic energy or electromagnetic angular momentum (quadratic in Φ_q), add up to terms proportional to $2(\Phi_q)$ and to $2(\Phi_q)^2$, respectively. These get reduced to the effective quantities Φ_{eff} and $(\Phi^2)_{\text{eff}}$, respectively, by the same factor N^{-1}.

Note to Eq. (2.7): α_4 implies whirling α_w, and α_3 implies spinning α_s.

Note to Eqs. (6.25) ff.: The indices j in these following paragraphs label modes, not mode pairs (i.e., not levels).

Note to the end of Appendix B: A fourth remark should be considered in regard to the form of the wave equation: Besides the Euler angles, representing freedom of motion of the loopforms (of the fibration), also internal energy, including vibrational energy, may be expected to be important terms in the wave equation.

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†On leave from University of Maryland; now at the University of Uppsala and the University of Amsterdam. Reprint requests to 1208 Sherwood Rd., Charlottesville, Virginia 22901 or to Munich.

[1]The references are grouped together under several general topics. They cover a broad spectrum of papers which have an indirect bearing on the present paper. It has been found useful to have this coverage, but we omit many repetitions by referring to the corresponding quotations in the previous papers, e.g., "Ref. 5 of B." The reference to preceding papers are H. Jehle, Phys. Rev. D 3, 306 (1971), quoted as A; 6, 441 (1972), quoted as B; 11, 2147 (1975), quoted as C. The present paper is paper D. Cf. Appendix D. Also see *Lectures in Theoretical Physics,* Proceedings of the 1971 Boulder Conference, edited by A. O. Barut and W. E. Brittin (Colorado Associated Univ. Press, Boulder, 1972), Vol. 14A, p. 399; W. C. Parke, Ph.D. thesis, George Washington Univ., 1967 (unpublished); Herbert Jehle, Int. J. Quant. Chem. 3, 269 (1969); Acta Phys. Austriaca (to be published).

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